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ON DIFFERENTIAL EQUATIONS FOR ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract: In this paper we characterize sequences of orthogonal polynomials on the unit circle whose corresponding Carathéodory function satisfies a Riccati differential equation with polynomial coefficients, in terms of second order matrix differential equations.

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1. Introduction

In this work we present a study about orthogonal polynomials on the unit circle whose Carathéodory function, F, satisfies a differential equation with polynomial coefficients

$$zAF' = BF^2 + CF + D\,, \; A \not\equiv 0\,. \tag{1}$$

The set of such polynomials (equivalently, the set of such Carathéodory functions) was defined in [3, 4] as the *Laguerre-Hahn class on the unit circle*. As particular cases some well known classes appear, for example: the *Laguerre-Hahn affine class on the unit circle*, when $B = 0$ in (1); the *semi-classical class on the unit circle*, when $B = 0$ and C, D are specific polynomials in (1) (see [2, 3, 5, 18]).

Our goal is to give a characterization of the Laguerre-Hahn class on the unit circle via second order differential equations.

The motivation for our work were some results due to W. Hahn, concerning differential equations for orthogonal polynomials on the real line. In [10, 11], W. Hahn establishes the equivalence between second order linear differential equations with polynomial coefficients $A_nP_n'' + B_nP_n' + C_nP_n = 0\,,\; n \geq 1\,,$ for sequences of orthogonal polynomials on a subset of the real line, $\{P_n\}$, and

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structure relations with polynomial coefficients for $\{P_n\}$,

$$\phi(x)P'_{n+1}(x) = S_n(x)P_{n+1}(x) + T_n(x)P_n(x)\,,\ n \geq 0\,. \tag{2}$$

Moreover, using results about quasi-orthogonality on the real line, P. Maroni establishes that the structure relations of type (2) give the semi-classical character of $\{P_n\}$, equivalently, the Laguerre-Hahn affine character of $\{P_n\}$, since, on the real line, these two classes coincide (see [14, 15]). In [9] W. Hahn establishes that the minimal order of a linear differential equation with polynomial coefficients for sequences of orthogonal polynomials on subsets of the real line can only take the values two or four. In addition, it is proved that such solutions of a fourth order linear differential equation can be constructed by means of the solutions of second order linear differential equations.

In [4] it is established that Laguerre-Hahn sequences of orthogonal polynomials on the unit circle are *factorized* in terms of well characterized semi-classical sequences of orthogonal polynomials. In this paper we will obtain some results for Laguerre-Hahn sequences on the unit circle which are analogous with the referred results on the real line. For the particular case of semi-classical orthogonal polynomials on the unit circle, we will obtain a characterization in terms of second order linear differential equations that can be regarded as a generalization of Hahn's results from [9] on the real line.

Still in the issue of second order differential equations we remark the paper [12], where the authors, using a different method from ours, deduce second order differential equations with analytic coefficients for sequences of orthonormal polynomials on the unit circle, $\{\varphi_n\}$,

$$\varphi''_n(z) + M_n(z)\varphi'_n(z) + N_n(z)\varphi_n(z) = 0\,. \tag{3}$$

This is done by deriving raising and lowering operators for orthogonal polynomials on the unit circle, $L_{n,1} = \frac{d}{dz} + D_n(z)\,,\ L_{n,2} = -\frac{d}{dz} - D_{n-1}(z) + E_n(z)\,,$ with analytic functions D_n, E_n, such that $L_{n,1}$ and $L_{n,2}$ satisfy $L_{n,1}(\varphi_n(z)) = F_n(z)\varphi_{n-1}(z)\,,\ \ L_{n,2}(\varphi_{n-1}(z)) = G_n(z)\varphi_n(z)\,,$ thus obtaining a second order differential equation with analytic coefficients, $L_{n,2}\left(\dfrac{1}{F_n(z)}L_{n,1}\right)\varphi_n(z) = G_n(z)\varphi_n(z)\,,$ which is written in the equivalent way (3).

Let us return to the Laguerre-Hahn class on the unit circle. Given a Carathéodory function, F, let $\{\phi_n\}, \{\Omega_n\}$ and $\{Q_n\}$ be the corresponding sequences of monic orthogonal polynomials, of associated polynomials of the second kind, and functions of the second kind, respectively (cf. section 2). For our purposes, we define the vectors $\psi_n^1 = [\phi_n \;\; -\Omega_n]^T$, $\psi_n^2 = [\phi_n^* \; \Omega_n^*]^T$, $n \geq 0$, where $[\cdot]^T$ denotes the transpose. Firstly, by reinterpreting a result of [4, Theorem 3], we establish the equivalence between (1) and the following structure relations,

$$\begin{cases} zA(\psi_n^1)' = M_{n,1}\psi_n^1 + N_{n,1}\psi_n^2 \\ zAQ_n' = (l_{n,1} + C/2 + BF)Q_n + \Theta_{n,1}Q_n^* \, , \quad n \in \mathbb{N} \, , \end{cases} \tag{4}$$

where $M_{n,1}$ and $N_{n,1}$ are matrices with polynomial elements and $l_{n,1}, \Theta_{n,1}$ are polynomials with degrees not depending on n (cf. Theorem 1). Secondly, with the help of equations (4), we establish the equivalence between (1) and the following second order differential equations,

$$\begin{cases} \tilde{\mathcal{A}}_{n,1}\, I(\psi_n^1)'' + \mathcal{B}_{n,1}(\psi_n^1)' + \mathcal{C}_{n,1}\psi_n^1 = 0_{2\times 1} \\ \tilde{\mathcal{A}}_{n,1}(Q_n)'' + \tilde{\mathcal{B}}_{n,1}Q_n' + \tilde{\mathcal{C}}_{n,1}Q_n = 0 \, , \quad n \in \mathbb{N} \, , \end{cases} \tag{5}$$

where $\mathcal{B}_{n,1}, \mathcal{C}_{n,1}$ are matrices with polynomial elements whose degrees do not depend on n, $\tilde{\mathcal{B}}_{n,1}, \tilde{\mathcal{C}}_{n,1}$ are analytic functions on the unit disk, $\tilde{\mathcal{A}}_{n,1}$ is a polynomial whose degree do not depend on n, and I is the identity matrix of order two (cf. Theorem 2).

As a consequence, we obtain that Laguerre-Hahn polynomials on the unit circle, as well the sequences $\{\phi_n^*\}$, $\{\Omega_n\}$ and $\{\Omega_n^*\}$, satisfy fourth order linear differential equations with polynomial coefficients and, in the Laguerre-Hahn affine case, we obtain second order linear differential equations with polynomial coefficients for $\{\phi_n\}$ as well for $\{\phi_n^*\}$, $\{Q_n\}$, and $\{Q_n^*\}$ (cf. Corollary 1 and 2). Furthermore, taking into account the referred second order linear differential equations, we prove that F is associated with a semi-classical measure (whose absolutely continuous part we denote by w) if, and only if, $\{\phi_n\}$ and $\{Q_n/w\}$ satisfy well determined second order linear differential equations with polynomial coefficients,

$$A_n(\Upsilon)'' + B_n\Upsilon' + C_n\Upsilon = 0 \, , \; n \in \mathbb{N} \, ,$$

where the polynomials A_n, B_n, C_n depend on the polynomials A, C, D from the corresponding differential equation for F, and on the reflection coefficients of $\{\phi_n\}$ (cf. Theorem 3).

This paper is organized as follows: in section 2 we give the definitions and state the basic results which will be used in the forthcoming sections; in section 3 we establish the equivalence between (1) and (5); in section 4 we establish a characterization of the semi-classical class on the unit circle through second order linear differential equations with polynomial coefficients.

2. Preliminary results and notations

Let $\Lambda = \operatorname{span}\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with complex coefficients, Λ' its algebraic dual space, $\mathbb{P} = \operatorname{span}\{z^k : k \in \mathbb{N}\}$ the space of complex polynomials, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. Let $u \in \Lambda'$ be a linear functional. We denote by $\langle u, f\rangle$ the action of u over $f \in \Lambda$.

Given the sequence of moments (c_n) of u, $c_n = \langle u, \xi^{-n}\rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, the minors of the Toeplitz matrix are defined by

$$\Delta_{-1} = 1\,,\ \Delta_0 = c_0\,,\ \Delta_k = \begin{vmatrix} c_0 & \cdots & c_k \\ \vdots & & \vdots \\ c_{-k} & \cdots & c_0 \end{vmatrix}\,,\ \ k \in \mathbb{N}\,.$$

The linear functional u is *Hermitian* if $c_{-n} = \overline{c}_n, \forall n \in \mathbb{N}$, and *regular* (*positive definite*) if $\Delta_n \neq 0$ ($\Delta_n > 0$), $\forall n \in \mathbb{N}$.

In this work we shall consider Hermitian linear functionals which are positive definite. If u is such a linear functional, then there exists a non-trivial probability measure μ supported on $\mathbb{T}$ such that

$$\langle u, \xi^{-n}\rangle = \frac{1}{2\pi}\int_0^{2\pi} \xi^{-n} d\mu(\theta)\,, \quad \xi = e^{i\theta}\,, \quad n \in \mathbb{Z}\,.$$

Given a probability measure μ on $\mathbb{T}$, the function F defined by

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta) \tag{6}$$

is a *Carathéodory function*, i.e., is an analytic function in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = 1$ and $\Re e(F) > 0$ for $|z| < 1$. The converse result also holds, since any Carathéodory function has a representation (6) for a unique probability measure μ on $\mathbb{T}$ (see, for example, [16]).

Definition 1. Let $\{\phi_n\}$ be a sequence of complex polynomials with $\deg(\phi_n) = n$ and μ a probability measure on $\mathbb{T}$. We say that $\{\phi_n\}$ is a *sequence of orthogonal polynomials* with respect to μ (or $\{\phi_n\}$ is a *sequence of orthogonal* *polynomials on the unit circle*) if

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})\overline{\phi}_m(e^{-i\theta})\, d\mu(\theta) = h_n\delta_{n,m}\,,\ \ h_n \neq 0\,,\ \ n,m \in \mathbb{N}\,.$$

If the leading coefficient of each ϕ_n is 1, then $\{\phi_n\}$ is said to be a *sequence of monic orthogonal polynomials* and will be denoted by MOPS.

Given a MOPS, $\{\phi_n\}$, with respect to μ, the *sequence of associated polynomials of the second kind*, $\{\Omega_n\}$, is defined by

$$\Omega_0(z) = 1,\ \ \Omega_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\, d\mu(\theta)\,,\ \ \forall n \in \mathbb{N}\,,$$

and the *functions of the second kind* associated with $\{\phi_n\}$ are given by

$$Q_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\, d\mu(\theta)\,,\ \ \ n = 0,1,\dots\,.$$

Given an analytic function f and $p \in \mathbb{N}$, the function f^{*_p} is defined by $f^{*_p}(z) = z^p\overline{f}(1/z)$. Thus, if $f(z) = \sum_{k=0}^{+\infty} b_k z^k$, then $f^{*_p}(z) = \sum_{k=0}^{+\infty} \overline{b}_k z^{-k+p}$. For matrices, X, with elements $x_{i,j}\,, i = 1,\dots,m\,,\ j = 1,\dots,n$, with $n,m \geq 0$, X^{*_p} is the matrix whose elements are $x_{i,j}^{*_p}\,, i = 1,\dots,m\,,\ j = 1,\dots,n$. Throughout the paper we will omit the index p, i.e., we will write f^* (or X^*) instead of f^{*_p} (or X^{*_p}), whenever f is a polynomial of degree p and whenever X is a matrix whose elements are polynomials whose degrees are all equal to p. Moreover, throughout the paper, we will write Q_n^* instead of $Q_n^{*_n}$.

Given a probability measure μ on $\mathbb{T}$, let $\{\phi_n\}$, $\{\Omega_n\}$, $\{Q_n\}$ be the corresponding MOPS, the sequence of associated polynomials of the second kind, and the sequence of functions of the second kind, respectively. We define the following matrices, for $n \geq 0$,

$$\psi_n^1 = \begin{bmatrix}\phi_n\\ -\Omega_n\end{bmatrix}\,,\ \ \psi_n^2 = \begin{bmatrix}\phi_n^*\\ \Omega_n^*\end{bmatrix}\,,\ \ \mathcal{Q}_n = \begin{bmatrix}-Q_n\\ Q_n^*\end{bmatrix}\,,\ \ Y_n = \begin{bmatrix}\phi_n & -Q_n\\ \phi_n^* & Q_n^*\end{bmatrix}\,. \tag{7}$$

Throughout the text, the matrices I and J are the matrices defined as follows:

$$I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \ J = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\,. \tag{8}$$

The following relations hold, $\forall n \geq 0$:

$$(\psi_n^1)^* = J\,\psi_n^2\,,\ \ (\psi_n^2)^* = J\,\psi_n^1\,.$$

Next we present the Szegő recurrence relations (see [7, 8, 17]) in the matrix form.

Lemma 1. *Let F be a Carathéodory function and $\{\psi_n^1\}$, $\{\psi_n^2\}$, $\{\mathcal{Q}_n\}$, $\{Y_n\}$ the corresponding sequences defined in (7), and $a_n = \phi_n(0)$, $\forall n \geq 0$. Then:*
a) ψ_n^1 and ψ_n^2 satisfy, for all $n \geq 0$,

$$\psi_n^1(z) = z\psi_{n-1}^1(z) + a_n\psi_{n-1}^2(z)$$
$$\psi_n^2(z) = \overline{a}_n z\psi_{n-1}^1(z) + \psi_{n-1}^2(z)\,;$$

b) $\varphi_n = \begin{bmatrix}\psi_n^1\\ \psi_n^2\end{bmatrix}$ satisfies, for all $n \in \mathbb{N}$,

$$\varphi_n = \mathcal{K}_n^1 \varphi_{n-1}, \quad \mathcal{K}_n^1 = \begin{bmatrix} zI & a_n I \\ \overline{a}_n z I & I \end{bmatrix}, \tag{9}$$

with initial conditions $\varphi_0 = \begin{bmatrix}1 & -1 & 1 & 1\end{bmatrix}^T$;
c) $\mathcal{Q}_n$ satisfies, for all $n \in \mathbb{N}$,

$$\mathcal{Q}_n = \mathcal{K}_n \mathcal{Q}_{n-1}, \quad \mathcal{K}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix}, \tag{10}$$

with initial conditions $\mathcal{Q}_0 = [-F \ \ -F]^T$;
d) Y_n satisfies (10), for all $n \in \mathbb{N}$, with initial conditions $Y_0 = \begin{bmatrix} 1 & -F \\ 1 & -F \end{bmatrix}$.

Remark . Since $\det[\psi_n^1 \ \psi_n^2] \neq 0$, $\forall z \neq 0$, $\forall n \in \mathbb{N}$, the vectors ψ_n^1, ψ_n^2 are linearly independent.

Throughout this paper we will use the relations that come next.

Lemma 2. *Let F be a Carathéodory function, let $\{Q_n\}$ be the sequence of functions of the second kind, and $\{\psi_n^1\}$, $\{\psi_n^2\}$, the corresponding sequences defined in (7). The following relations hold, for all $n \in \mathbb{N}$,*

$$\left[(\psi_n^1)'\right]^{*_{n-1}} = nJ\,\psi_n^2 - zJ\,(\psi_n^2)', \tag{11}$$
$$\left[\left(\psi_n^1\right)''\right]^{*_{n-2}} = z^2\,J\,(\psi_n^2)'' - 2(n-1)z\,J\,(\psi_n^2)' + n(n-1)J\,\psi_n^2, \tag{12}$$
$$(Q_n')^{*_{n-1}} = nQ_n^* - z(Q_n^*)', \tag{13}$$
$$\left[(Q_n)''\right]^{*_{n-2}} = z^2(Q_n)'' - 2(n-1)zQ_n' + n(n-1)Q_n\,. \tag{14}$$

The following theorem is a reinterpretation of Theorem 3 of [4] (cf. equations (15) and (16) therein). Also, it is an extension of Theorem 4 of [2].

Theorem 1. *Let F be a Carathéodory function and $\{\psi_n^1\}$, $\{\psi_n^2\}$ be the corresponding sequences defined in (7), and $\{Q_n\}$ the sequence of functions of the second kind. The following statements are equivalent:*
a) F satisfies

$$zAF' = BF^2 + CF + D\,,\ \ A, B, C, D \in \mathbb{P}\,.$$

b) $\{\psi_n^1\}$ and $\{Q_n\}$ satisfy

$$\begin{cases} zA(\psi_n^1)' = M_{n,1}\psi_n^1 + N_{n,1}\psi_n^2 \\ zAQ_n' = (l_{n,1} + C/2 + BF)Q_n + \Theta_{n,1}Q_n^* \end{cases} \tag{15}$$

where $M_{n,1}, N_{n,1}$ are matrices with polynomial elements whose degree do not depend on n,

$$M_{n,1} = \begin{bmatrix} l_{n,1} - C/2 & -B \\ D & l_{n,1} + C/2 \end{bmatrix}, \ \ N_{n,1} = -\Theta_{n,1}\, I.$$

c) $\{\psi_n^2\}$ and $\{Q_n^*\}$ satisfy

$$\begin{cases} zA(\psi_n^2)' = N_{n,2}\psi_n^1 + M_{n,2}\psi_n^2 \\ zA(Q_n^*)' = (l_{n,2} + C/2 + BF)\, Q_n^* + \Theta_{n,2}Q_n \end{cases} \tag{16}$$

where $M_{n,2}, N_{n,2}$ are matrices with polynomial elements whose degree do not depend on n,

$$M_{n,2} = \begin{bmatrix} l_{n,2} - C/2 & -B \\ D & l_{n,2} + C/2 \end{bmatrix}, \ \ N_{n,2} = -\Theta_{n,2}\, I.$$

Remark . If we take $B = 0$ in previous theorem we obtain the differential relations for $\{\phi_n\}$ in the Laguerre-Hahn affine class:

$$\begin{cases} zA(\phi_n)' = (l_{n,1} - C/2)\phi_n - \Theta_{n,1}\phi_n^* \\ zA(\phi_n^*)' = -\Theta_{n,2}\phi_n + (l_{n,2} - C/2)\phi_n^*\,, \ \ n \in \mathbb{N}\,. \end{cases}$$

Using results on quasi-orthogonality on the unit circle, one can give sufficient conditions on the degrees of the polynomials $l_{n,1} - C/2, \Theta_{n,1}, \Theta_{n,2}$ and $l_{n,2} - C/2$, for $\{\phi_n\}$ to be semi-classical (cf. [1, Theorem 3]).

In what comes next we discuss the equations (15) and (16) of Theorem 1.

Lemma 3. *Let* $\{\psi_n^1\}$, $\{\psi_n^2\}$ *be the sequence of vectors defined in (7) and* $\{Q_n\}$ *the sequence of functions of the second kind. Then:*
a) $\{\psi_n^1\}$ *and* $\{Q_n\}$ *satisfy (15) if, and only if,* $\{\psi_n^2\}$ *and* $\{Q_n^*\}$ *satisfy*

$$\begin{cases} zA^*(\psi_n^2)' = -N_{n,1}^*\psi_n^1 + (nA^*I - J\,M_{n,1}^*J\,)\psi_n^2 \\ zA^*(Q_n^*)' = (nA^* - l_{n,1}^* - C^*/2 + B^*F)Q_n^* - \Theta_{n,1}^*Q_n\,. \end{cases} \tag{17}$$

b) $\{\psi_n^2\}$ *and* $\{Q_n^*\}$ *satisfy (16) if, and only if,* $\{\psi_n^1\}$ *and* $\{Q_n\}$ *satisfy*

$$\begin{cases} zA^*(\psi_n^1)' = -N_{n,2}^*\psi_n^2 + (nA^*I - J\,M_{n,2}^*J\,)\psi_n^1 \\ zA^*(Q_n)' = (nA^* - l_{n,2}^* - C^*/2 + B^*F)Q_n - \Theta_{n,2}^*Q_n^*\,. \end{cases} \tag{18}$$

Proof: If we apply the operator $*_{n+p}$ to equation (15), with

$p = \max\{\deg(A), \deg(l_{n,1} - C/2), \deg(l_{n,1} + C/2), \deg(B), \deg(D), \deg(\Theta_{n,1})\}$,

and use (11) we obtain

$$A^{*_p}\left(nJ\,\psi_n^2 - zJ\,(\psi_n^2)'\right) = M_{n,1}^{*_p}(\psi_n^1)^* + N_{n,1}^{*_p}(\psi_n^2)^*\,,$$

thus

$$zA^{*_p}J\,(\psi_n^2)' = nA^{*_p}J\,\psi_n^2 - M_{n,1}^{*_p}(\psi_n^1)^* - N_{n,1}^{*_p}(\psi_n^2)^*\,.$$

If we multiply previous equation (on the left) by J we get

$$zA^{*_p}(\psi_n^2)' = nA^{*_p}\psi_n^2 - J\,M_{n,1}^{*_p}(\psi_n^1)^* - J\,N_{n,1}^{*_p}(\psi_n^2)^*\,.$$

Taking into account that $(\psi_n^1)^* = J\,\psi_n^2$, $(\psi_n^2)^* = J\,\psi_n^1$, there follows the equation

$$zA^{*_p}(\psi_n^2)' = (nA^*I - J\,M_{n,1}^{*_p}J)\,\psi_n^2 - J\,N_{n,1}^{*_p}J\,\psi_n^1\,.$$

Since $N_{n,1}^{*_p}$ is diagonal we obtain the first equation in (17).

Analogously, by applying $*_{n+p}$ to $zAQ_n' = (l_{n,1} + C/2 + BF)Q_n + \Theta_{n,1}Q_n^*$ and using (13), we obtain

$$A^{*_p}(nQ_n^* - z(Q_n^*)') = (l_{n,1}^{*_p} + C^{*_p}/2 + B^{*_p}\overline{F}(1/z))Q_n^* + \Theta_{n,1}^{*_p}Q_n\,.$$

Using $\overline{F}(1/z) = -F(z)$ in previous equation we get

$$zA^{*_p}(Q_n^*)' = (nA^{*_p} - (l_{n,1}^{*_p} + C^{*_p}/2) + B^{*_p}F)Q_n^* - \Theta_{n,1}^{*_p}Q_n\,.$$

By an analogue manner one proves assertion b). ∎

Remark . Taking into account Theorem 1 and previous lemma we conclude that equations (15) and (18) are equivalent, as well as (16) and (17). Hence, the following relations take place:

$$\frac{A^*}{A}=\frac{B^*}{B}=\frac{-C^*}{C}=\frac{D^*}{D}=\frac{-\Theta^*_{n,2}}{\Theta_{n,1}}=\frac{nA^*-l^*_{n,2}}{l_{n,1}}\,.$$

The above relations are consistent with the the results of [3, Lemma 4], concerning the distributional equation for Laguerre-Hahn hermitian functionals.

3. Second order matrix differential equation

In this section we will denote the element of a matrix X in the position (i,j) by $[\,X\,]_{i,j}$. Our goal is to establish the following theorem.

Theorem 2. *Let F be a Carathéodory function and $\{\psi^1_n\}$, $\{\psi^2_n\}$ be the corresponding sequences defined in (7), and $\{Q_n\}$ the sequence of functions of the second kind. The following statements are equivalent:*
a) F satisfies

$$zAF'=BF^2+CF+D\,,\quad A,B,C,D\in\mathbb{P}\,.$$

b) $\{\psi^1_n\}$ and $\{Q_n\}$ satisfy

$$\mathcal{A}_{n,1}(\psi^1_n)''+\mathcal{B}_{n,1}(\psi^1_n)'+\mathcal{C}_{n,1}\psi^1_n=0_{2\times1}\,, \tag{19}$$

$$\tilde{\mathcal{A}}_{n,1}Q''_n+\tilde{\mathcal{B}}_{n,1}Q'_n+\tilde{\mathcal{C}}_{n,1}Q_n=0\,, \tag{20}$$

where $\mathcal{A}_{n,1},\mathcal{B}_{n,1},\mathcal{C}_{n,1}$ are matrices, with polynomial elements, given by

$$\begin{cases}\mathcal{A}_{n,1}=(zA)^2\Theta_{n,1}\,I\,,\\ \mathcal{B}_{n,1}=-zA\Theta_{n,1}\left(M_{n,1}-(zA)'\,I\right)+zA\left(zAN'_{n,1}+N_{n,1}M_{n,2}\right)\,,\\ \mathcal{C}_{n,1}=-\Theta_{n,1}(zAM'_{n,1}+\Theta_{n,1}\Theta_{n,2}\,I)-\left(zAN'_{n,1}+N_{n,1}M_{n,2}\right)M_{n,1}\,,\end{cases} \tag{21}$$

$\tilde{\mathcal{A}}_{n,1}\in\mathbb{P}$, $\tilde{\mathcal{B}}_{n,1},\tilde{\mathcal{C}}_{n,1}$ are analytic functions given by

$$\begin{cases}\tilde{\mathcal{A}}_{n,1}=(zA)^2\Theta_{n,1}\,,\\ \tilde{\mathcal{B}}_{n,1}=[\mathcal{B}_{n,1}]_{2,2}-2zA\Theta_{n,1}BF\,,\\ \tilde{\mathcal{C}}_{n,1}=[\mathcal{C}_{n,1}]_{2,2}-\Theta_{n,1}F\left(zAB'-B(l_{n,1}+l_{n,2})\right)+zA\Theta'_{n,1}BF\,;\end{cases} \tag{22}$$

the polynomials $\Theta_{n,1}$ and the matrices $N_{n,1},N_{n,2},M_{n,1}M_{n,2}$ are given in Theorem 1.

c) $\{\psi_n^2\}$ and $\{Q_n^*\}$ satisfy

$$\mathcal{A}_{n,2}(\psi_n^2)'' + \mathcal{B}_{n,2}(\psi_n^2)' + \mathcal{C}_{n,2}\psi_n^2 = 0_{2\times 1}\,, \tag{23}$$

$$\tilde{\mathcal{A}}_{n,2}(Q_n^*)'' + \tilde{\mathcal{B}}_{n,2}(Q_n^*)' + \tilde{\mathcal{C}}_{n,2}Q_n^* = 0\,, \tag{24}$$

where $\mathcal{A}_{n,2}, \mathcal{B}_{n,2}, \mathcal{C}_{n,2}$ are matrices, with polynomial elements, given by

$$\begin{cases} \mathcal{A}_{n,2} = (zA)^2\Theta_{n,2}\, I\,, \\ \mathcal{B}_{n,2} = -zA\Theta_{n,2}\left(M_{n,2} - (zA)'\, I\right) + zA\left(zAN'_{n,2} + N_{n,2}M_{n,1}\right)\,, \\ \mathcal{C}_{n,2} = -\Theta_{n,2}\left(zAM'_{n,2} + \Theta_{n,2}\Theta_{n,1}\, I\right) - \left(zAN'_{n,2} + N_{n,2}M_{n,1}\right)M_{n,2}\,, \end{cases} \tag{25}$$

$\tilde{\mathcal{A}}_{n,2} \in \mathbb{P}$, $\tilde{\mathcal{B}}_{n,2}, \tilde{\mathcal{C}}_{n,2}$ are analytic functions given by

$$\begin{cases} \tilde{\mathcal{A}}_{n,2} = (zA)^2\Theta_{n,2}\,, \\ \tilde{\mathcal{B}}_{n,2} = [\mathcal{B}_{n,2}]_{2,2} - 2zA\Theta_{n,2}BF\,, \\ \tilde{\mathcal{C}}_{n,2} = [\mathcal{C}_{n,2}]_{2,2} - \Theta_{n,2}F\left(zAB' - B(l_{n,1} + l_{n,2})\right) + zA\Theta'_{n,2}BF\,; \end{cases} \tag{26}$$

the polynomials $\Theta_{n,2}$ and the matrices $N_{n,1}, N_{n,2}, M_{n,1}M_{n,2}$ are given in Theorem 1.

The proof of the previous theorem will use the lemmas that follow.

Lemma 4. *Let F be a Carathéodory function, and let $\{\psi_n^1\}$, $\{\psi_n^2\}$, $\{\mathcal{Q}_n\}$ be the corresponding sequences defined in (7). If F satisfies $zAF' = BF^2 + CF + D$, then: $\{\psi_n^1\}$ satisfies (19) with coefficients (21); $\{\psi_n^2\}$ satisfies (23) with coefficients (25); $\{Q_n\}$ satisfies (20) with coefficients (22); $\{Q_n^*\}$ satisfies (24) with coefficients (26).*

Proof: If F satisfies $zAF' = BF^2 + CF + D$, then, $\forall n \in \mathbb{N}$, ψ_n^1 and ψ_n^2 satisfy (15) and (16), respectively.
Step 1. Taking derivatives on (15) we obtain

$$zA(\psi_n^1)'' = \left(M_{n,1} - (zA)'\right)(\psi_n^1)' + M'_{n,1}\psi_n^1 + N'_{n,1}\psi_n^2 + N_{n,1}(\psi_n^2)'\,.$$

Step 2. To eliminate $(\psi_n^2)'$ in previous equation, we multiply previous equation by zA, which commutes with the coefficients of the equation, and using (16), there follows

$$(zA)^2(\psi_n^1)'' = zA\left(M_{n,1} - (zA)'\right)(\psi_n^1)' + \left(zAM'_{n,1} + N_{n,1}N_{n,2}\right)\psi_n^1 + \left(zAN'_{n,1} + N_{n,1}M_{n,2}\right)\psi_n^2\,.$$

Step 3. To eliminate ψ_n^2 in previous equation we multiply previous equation by $\Theta_{n,1}$, thus obtaining

$$
\begin{aligned}
(zA)^2\Theta_{n,1}(\psi_n^1)'' &= zA\Theta_{n,1}\left(M_{n,1}-(zA)'\right)(\psi_n^1)' \\
&\quad + \Theta_{n,1}\left(zAM_{n,1}'+N_{n,1}N_{n,2}\right)\psi_n^1 - \left(zAN_{n,1}'+N_{n,1}M_{n,2}\right)(-\Theta_{n,1})\psi_n^2 \,.
\end{aligned}
$$

Using (15),

$$-\Theta_{n,1}\psi_n^2 = zA(\psi_n^1)' - M_{n,1}\psi_n^1\,,$$

we get $\mathcal{A}_{n,1}(\psi_n^1)''+\mathcal{B}_{n,1}(\psi_n^1)'+\mathcal{C}_{n,1}\psi_n^1 = 0_{2\times 1}$ with coefficients (21).

By an analogue manner we obtain that $\{\psi_n^2\}$ satisfies (23) with coefficients (25), $\{Q_n\}$ satisfies (20) with coefficients (22), and $\{Q_n^*\}$ satisfies (24) with coefficients (26). ■

Remark . Equations (20) and (24) can be written as

$$\tilde{\mathcal{A}}_n\mathcal{Q}_n''+\tilde{\mathcal{B}}_n\mathcal{Q}_n'+\tilde{\mathcal{C}}_n\mathcal{Q}_n = 0_{2\times 1} \tag{27}$$

with

$$
\begin{cases}
\tilde{\mathcal{A}}_n = (zA)^2\begin{bmatrix}\Theta_{n,1} & 0\\ 0 & \Theta_{n,2}\end{bmatrix}, \\
\tilde{\mathcal{B}}_n = \begin{bmatrix}[\mathcal{B}_{n,1}]_{2,2}-2zA\Theta_{n,1}BF & 0\\ 0 & [\mathcal{B}_{n,2}]_{2,2}-2zA\Theta_{n,2}BF\end{bmatrix}, \\
\tilde{\mathcal{C}}_n = \begin{bmatrix}[\mathcal{C}_{n,1}]_{2,2}+\alpha_{n,1} & 0\\ 0 & [\mathcal{C}_{n,2}]_{2,2}+\alpha_{n,2}\end{bmatrix},
\end{cases} \tag{28}
$$

where

$$
\begin{aligned}
\alpha_{n,1} &= -\Theta_{n,1}F\left(zAB'+BC-B(l_{n,1}+l_{n,2}+C)\right)+zA\Theta_{n,1}'BF\,,\\
\alpha_{n,2} &= -\Theta_{n,2}F\left(zAB'+BC-B(l_{n,1}+l_{n,2}+C)\right)+zA\Theta_{n,2}'BF\,.
\end{aligned}
$$

If we take $B=0$ in previous lemma we obtain second order linear differential equations for sequences of Laguerre-Hahn affine orthogonal polynomials on the unit circle. This is an extension of the result from [18] about semi-classical families on the unit circle, to the Laguerre-Hahn affine case.

Corollary 1. *Let $\{\phi_n\}$ be a MOPS on the unit circle, $\{Q_n\}$ the sequence of functions of the second kind, and F be the corresponding Carathéodory function. If F satisfies $zAF'=CF+D$, $A,C,D\in\mathbb{P}$, then $\{\phi_n\}$, $\{\phi_n^*\}$, $\{Q_n\}$,* *and* $\{Q_n^*\}$ *satisfy second order linear differential equations with polynomial coefficients,* $\forall n \in \mathbb{N}$,

$$(zA)^2\Theta_{n,1}\phi_n'' + [\mathcal{B}_{n,1}]_{1,1}\phi_n' + [\mathcal{C}_{n,1}]_{1,1}^0\phi_n = 0\,, \tag{29}$$

$$(zA)^2\Theta_{n,2}(\phi_n^*)'' + [\mathcal{B}_{n,2}]_{1,1}(\phi_n^*)' + [\mathcal{C}_{n,2}]_{1,1}^0\phi_n^* = 0\,, \tag{30}$$

$$(zA)^2\Theta_{n,1}Q_n'' + [\mathcal{B}_{n,1}]_{2,2}Q_n' + [\mathcal{C}_{n,1}]_{2,2}^0 Q_n = 0\,, \tag{31}$$

$$(zA)^2\Theta_{n,2}(Q_n^*)'' + [\mathcal{B}_{n,2}]_{2,2}(Q_n^*)' + [\mathcal{C}_{n,2}]_{2,2}^0 Q_n^* = 0\,. \tag{32}$$

where $[\mathcal{C}_{n,i}]_{k,k}^0$, $i,k = 1,2$ *is the corresponding* $[\mathcal{C}_{n,i}]_{k,k}$ *with* $B = 0$.

Proof: Taking $B = 0$ in previous lemma we obtain, from (21) and (25), that $[\mathcal{B}_{n,1}]_{1,2} = [\mathcal{C}_{n,1}]_{1,2} = [\mathcal{B}_{n,2}]_{1,2} = [\mathcal{C}_{n,2}]_{1,2} = 0$, thus we get (29) and (30) with the referred coefficients. Also, from (22) and (26), we obtain the equations (31) and (32) with the referred coefficients. ∎

Analogously, in the Laguerre-Hahn case one can obtain fourth order linear differential equations for $\{\phi_n\}$, $\{\phi_n^*\}$, $\{\Omega_n\}$, and $\{\Omega_n^*\}$.

Corollary 2. *Let* $\{\phi_n\}$ *be a MOPS on the unit circle and* F *be the corresponding Carathéodory function. If* F *satisfies* $zAF' = BF^2 + CF + D$, $A, B, C, D \in \mathbb{P}$, *then* $\{\phi_n\}, \{\phi_n^*\}, \{\Omega_n\}, \{\Omega_n^*\}$ *satisfy fourth order linear differential equations with polynomial coefficients.*

Lemma 5. *Let* F *be a Carathéodory function and* $\{\psi_n^1\}$, $\{\psi_n^2\}$ *be the corresponding sequences defined in (7), and* $\{Q_n\}$ *the sequence of functions of the second kind. Then,* $\{\psi_n^1\}$ *and* $\{Q_n\}$ *satisfy*

$$\begin{cases} \mathcal{A}_{n,1}(\psi_n^1)'' + \mathcal{B}_{n,1}(\psi_n^1)' + \mathcal{C}_{n,1}\psi_n^1 = 0_{2\times 1} \\ \tilde{\mathcal{A}}_{n,1}Q_n'' + \tilde{\mathcal{B}}_{n,1}Q_n' + \tilde{\mathcal{C}}_{n,1}Q_n = 0\,, \end{cases}$$

with $\mathcal{A}_{n,1}, \mathcal{B}_{n,1}, \mathcal{C}_{n,1}$ *matrices of order two with polynomial elements and* $\tilde{\mathcal{A}}_{n,1}$, $\tilde{\mathcal{B}}_{n,1}$, $\tilde{\mathcal{C}}_{n,1}$ *analytic functions, if, and only if,* $\{\psi_n^2\}$ *and* $\{Q_n^*\}$ *satisfy*

$$\begin{cases} \mathcal{A}_{n,2}(\psi_n^2)'' + \mathcal{B}_{n,2}(\psi_n^2)' + \mathcal{C}_{n,2}\psi_n^2 = 0_{2\times 1} \\ \tilde{\mathcal{A}}_{n,2}(Q_n^*)'' + \tilde{\mathcal{B}}_{n,2}(Q_n^*)' + \tilde{\mathcal{C}}_{n,2}Q_n^* = 0\,, \end{cases}$$

with

$$\begin{cases} \mathcal{A}_{n,2} = z^4 \mathcal{A}_{n,1}^{*_p} J\,, \quad \mathcal{B}_{n,2} = -2(n-1) z^3 \mathcal{A}_{n,1}^{*_p} J - z^2 \mathcal{B}_{n,1}^{*_p} J\,, \\ \mathcal{C}_{n,2} = n(n-1) z^2 \mathcal{A}_{n,1}^{*_p} J + nz \mathcal{B}_{n,1}^{*_p} J + \mathcal{C}_{n,1}^{*_p} J\,, \\ \tilde{\mathcal{A}}_{n,2} = z^4 \tilde{\mathcal{A}}_{n,1}^{*_p}\,, \quad \tilde{\mathcal{B}}_{n,2} = -2(n-1) z^3 \tilde{\mathcal{A}}_{n,1}^{*_p} - z^2 \tilde{\mathcal{B}}_{n,1}^{*_p}\,, \\ \tilde{\mathcal{C}}_{n,2} = n(n-1) z^2 \tilde{\mathcal{A}}_{n,1}^{*_p} + nz \tilde{\mathcal{B}}_{n,1}^{*_p} + \tilde{\mathcal{C}}_{n,1}^{*_p}\,. \end{cases} \tag{33}$$

Moreover, if the matrices $\mathcal{A}_{n,1}$ *are diagonal, then* $\mathcal{A}_{n,2}, \mathcal{B}_{n,2}, \mathcal{C}_{n,2}$ *are given by*

$$\begin{aligned} &\mathcal{A}_{n,2} = z^4 \mathcal{A}_{n,1}^{*_p}\,, \ \mathcal{B}_{n,2} = -2(n-1) z^3 \mathcal{A}_{n,1}^{*_p} - z^2\, J \mathcal{B}_{n,1}^{*_p} J\,, \\ &\mathcal{C}_{n,2} = n(n-1) z^2 \mathcal{A}_{n,1}^{*_p} + nz\, J \mathcal{B}_{n,1}^{*_p} J + J \mathcal{C}_{n,1}^{*_p} J\,, \end{aligned}$$

where J *is given in (8).*

Proof: If we apply $*_{n+p}$, with

$$p = \max\{\deg([\mathcal{A}_{n,1}]_{i,j}), \deg([\mathcal{B}_{n,1}]_{i,j}), \deg([\mathcal{C}_{n,1}]_{i,j}),\ i,j = 1,2\}$$

to

$$\mathcal{A}_{n,1} (\psi_n^1)'' + \mathcal{B}_{n,1} (\psi_n^1)' + \mathcal{C}_{n,1} \psi_n^1 = 0_{2\times 1}$$

we obtain

$$\mathcal{A}_{n,1}^{*_{p+2}} \left[(\psi_n^1)'' \right]^{*_{n-2}} + \mathcal{B}_{n,1}^{*_{p+1}} \left[(\psi_n^1)' \right]^{*_{n-1}} + \mathcal{C}_{n,1}^{*_p} (\psi_n^1)^{*_n} = 0_{2\times 1}\,.$$

Using (11) and (12) in previous equation, there follows

$$\begin{aligned} z^2 \mathcal{A}_{n,1}^{*_{p+2}} J (\psi_n^2)'' + \left\{ -2(n-1) z \mathcal{A}_{n,1}^{*_{p+2}} J - z \mathcal{B}_{n,1}^{*_{p+1}} J \right\} (\psi_n^2)' \\ + \left\{ n(n-1) \mathcal{A}_{n,1}^{*_{p+2}} J + n \mathcal{B}_{n,1}^{*_{p+1}} J + \mathcal{C}_{n,1}^{*_p} J \right\} \psi_n^2 = 0_{2\times 1}\,. \end{aligned}$$

Since $z^2 \mathcal{A}_{n,1}^{*_{p+2}} = z^4 \mathcal{A}_{n,1}^{*_p}\,,\ z \mathcal{A}_{n,1}^{*_{p+2}} = z^3 \mathcal{A}_{n,1}^{*_p}\,,\ z \mathcal{B}_{n,1}^{*_{p+1}} = z^2 \mathcal{B}_{n,1}^{*_p}\,,$ we get

$$\mathcal{A}_{n,2} (\psi_n^2)'' + \mathcal{B}_{n,2} (\psi_n^2)' + \mathcal{C}_{n,2} \psi_n^2 = 0_{2\times 1}\,,$$

with $\mathcal{A}_{n,2}, \mathcal{B}_{n,2}, \mathcal{C}_{n,2}$ given in (33).

Moreover, if $\mathcal{A}_{n,1}$ is diagonal, $\forall n \in \mathbb{N}$, after multiplying the previous equation by J, we obtain the required $\mathcal{A}_{n,2}, \mathcal{B}_{n,2}, \mathcal{C}_{n,2}$.

The equivalence between $\tilde{\mathcal{A}}_{n,1} Q_n'' + \tilde{\mathcal{B}}_{n,1} Q_n' + \tilde{\mathcal{C}}_{n,1} Q_n = 0$ and $\tilde{\mathcal{A}}_{n,2} (Q_n^*)'' + \tilde{\mathcal{B}}_{n,2} (Q_n^*)' + \tilde{\mathcal{C}}_{n,2} Q_n^* = 0$, with $\tilde{\mathcal{A}}_{n,2}, \tilde{\mathcal{B}}_{n,2}, \tilde{\mathcal{C}}_{n,2}$ given in (33), follows in the same way as above by using (13) and (14). ■

Lemma 6. *Let F be a Carathéodory function and $\{\psi_n^1\}$, $\{\psi_n^2\}$ be the corresponding sequences defined in (7). If $\{\psi_n^1\}$ satisfies the differential equation (19) with coefficients (21) and $\{\psi_n^2\}$ satisfies (23) with coefficients (25), then the following equations hold, $\forall n \in \mathbb{N}$,*

$$zA_{n,1}(z)(\psi_n^1)'(z) = \mathcal{M}_{n,1}\psi_n^1(z) + \mathcal{N}_{n,1}\psi_n^2(z)\,, \tag{34}$$
$$zA_{n,2}(z)(\psi_n^2)'(z) = \mathcal{N}_{n,2}\psi_n^1(z) + \mathcal{M}_{n,2}\psi_n^2(z)\,, \tag{35}$$

with $A_{n,1}, A_{n,2} \in \mathbb{P}$, $\mathcal{M}_{n,1}, \mathcal{N}_{n,1}, \mathcal{M}_{n,2}, \mathcal{N}_{n,2}$ matrices or order 2 with polynomial elements, and $\mathcal{N}_{n,1}, \mathcal{N}_{n,2}$ scalar matrices.

Proof: Step 1. Let us write equations (19) and (23) in the form

$$\mathcal{A}_n\varphi_n'' + \mathcal{B}_n\varphi_n' + \mathcal{C}_n\varphi_n = 0_{4\times 1}\,, \tag{36}$$

where, for all $n \in \mathbb{N}$, $\varphi_n = \begin{bmatrix}\psi_n^1\\ \psi_n^2\end{bmatrix}$ and $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ are block matrices given by

$$\mathcal{A}_n = (zA)^2\begin{bmatrix}\Theta_{n,1}I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n,2}I\end{bmatrix}, \; \mathcal{B}_n = \begin{bmatrix}\mathcal{B}_{n,1} & 0_{2\times 2}\\ 0_{2\times 2} & \mathcal{B}_{n,2}\end{bmatrix}, \; \mathcal{C}_n = \begin{bmatrix}\mathcal{C}_{n,1} & 0_{2\times 2}\\ 0_{2\times 2} & \mathcal{C}_{n,2}\end{bmatrix}.$$

Taking $n+1$ in (36) we obtain, for $n \in \mathbb{N}$,

$$\mathcal{A}_{n+1}\varphi_{n+1}'' + \mathcal{B}_{n+1}\varphi_{n+1}' + \mathcal{C}_{n+1}\varphi_{n+1} = 0_{4\times 1}\,.$$

Using the recurrence relations (9) in previous equation there follows, for $n \in \mathbb{N}$,

$$\mathcal{A}_{n+1}\mathcal{K}_{n+1}^1\varphi_n'' + \left\{2\mathcal{A}_{n+1}\left(\mathcal{K}_{n+1}^1\right)' + \mathcal{B}_{n+1}\mathcal{K}_{n+1}^1\right\}\varphi_n' + \left\{\mathcal{B}_{n+1}(\mathcal{K}_{n+1}^1)' + \mathcal{C}_{n+1}\mathcal{K}_{n+1}^1\right\}\varphi_n = 0_{4\times 1}\,, \tag{37}$$

with $\mathcal{K}_{n+1}^1 = \begin{bmatrix} zI & a_{n+1}I\\ \overline{a}_{n+1}zI & I\end{bmatrix}$.

Step 2. If we multiply (37) on the left by $\Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2}I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n+1,1}I\end{bmatrix}$, then we get, for $n \in \mathbb{N}$,

$$\begin{aligned}
&\Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}^1_{n+1}\begin{bmatrix}\Theta_{n,2}\,I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n,1}\,I\end{bmatrix}\mathcal{A}_n\varphi_n'' \\
&\quad + \Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2}\,I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n+1,1}\,I\end{bmatrix}\left\{2\mathcal{A}_{n+1}\left(\mathcal{K}^1_{n+1}\right)' + \mathcal{B}_{n+1}\mathcal{K}^1_{n+1}\right\}\varphi_n' \\
&\quad + \Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2}\,I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n+1,1}\,I\end{bmatrix}\left\{\mathcal{B}_{n+1}\left(\mathcal{K}^1_{n+1}\right)' + \mathcal{C}_{n+1}\mathcal{K}^1_{n+1}\right\}\varphi_n = 0_{4\times 1}\,.
\end{aligned}$$

Using (36) to n in previous equation it follows that

$$\hat{\mathcal{A}}_n\varphi_n' = \hat{\mathcal{M}}_n\varphi_n\,,\ n\in\mathbb{N}\,, \tag{38}$$

with

$$\begin{aligned}
\hat{\mathcal{A}}_n = &-\Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}^1_{n+1}\begin{bmatrix}\Theta_{n,2}I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n,1}I\end{bmatrix}\mathcal{B}_n \\
&\quad + \Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2}I & 0_{2\times 2}\\ 0_{2\times 2} & \Theta_{n+1,1}I\end{bmatrix}\left\{2\mathcal{A}_{n+1}\left(\mathcal{K}^1_{n+1}\right)' + \mathcal{B}_{n+1}\mathcal{K}^1_{n+1}\right\}\,,
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{M}}_n = &\,\Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}^1_{n+1}\begin{bmatrix}\Theta_{n,2}I & 0\\ 0 & \Theta_{n,1}I\end{bmatrix}\mathcal{C}_n \\
&\quad - \Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2}I & 0\\ 0 & \Theta_{n+1,1}I\end{bmatrix}\left\{\mathcal{B}_{n+1}\left(\mathcal{K}^1_{n+1}\right)' + \mathcal{C}_{n+1}\mathcal{K}^1_{n+1}\right\}\,.
\end{aligned}$$

Step 3. If we multiply (38) by the adjoint matrix of $\hat{\mathcal{A}}_n$, $\operatorname{adj}(\hat{\mathcal{A}}_n)$, we obtain

$$\det(\hat{\mathcal{A}}_n)\varphi_n' = \mathcal{M}_n\varphi_n\,, n\in\mathbb{N}\,, \tag{39}$$

with $\mathcal{M}_n = \operatorname{adj}(\hat{\mathcal{A}}_n)\hat{\mathcal{M}}_n$. Moreover, for all $n\in\mathbb{N}$, we have that $\det(\hat{\mathcal{A}}_n)$ is a polynomial (with a zero at $z=0$) and, writing $\mathcal{M}_n = \begin{bmatrix}\mathcal{M}_{n,1} & \mathcal{N}_{n,1}\\ \mathcal{N}_{n,2} & \mathcal{M}_{n,2}\end{bmatrix}$, we have that $\mathcal{N}_{n,1}$ and $\mathcal{N}_{n,2}$ are scalar matrices. Thus we obtain, in the matrix form (39), equations (34) and (35). ■

Lemma 7. *Let F be a Carathéodory function and $\{\mathcal{Q}_n\}$ be the corresponding sequence given in (7). If $\{\mathcal{Q}_n\}$ satisfies the differential equation (27),*

$$\tilde{\mathcal{A}}_n\mathcal{Q}_n'' + \tilde{\mathcal{B}}_n\mathcal{Q}_n' + \tilde{\mathcal{C}}_n\mathcal{Q}_n = 0_{2\times 1}\,,$$

with $\tilde{\mathcal{A}}_n, \tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_n$ *given by (28), then*

$$z A_n \mathcal{Q}_n' = \mathcal{M}_n \mathcal{Q}_n \, , \; \forall n \in \mathbb{N}, \tag{40}$$

with $A_n \in \mathbb{P}$ *and* $\mathcal{M}_n$ *a matrix of order 2 with analytic elements.*

Proof: Step 1. Taking $n+1$ in (27) we obtain, for all $n \in \mathbb{N}$,

$$\tilde{\mathcal{A}}_{n+1} \mathcal{Q}_{n+1}'' + \tilde{\mathcal{B}}_{n+1} \mathcal{Q}_{n+1}^{'} + \tilde{\mathcal{C}}_{n+1} \mathcal{Q}_{n+1} = 0_{2\times 1} \, .$$

Using the recurrence relations (10) we obtain

$$\tilde{\mathcal{A}}_{n+1} \mathcal{K}_{n+1} \mathcal{Q}_n'' + \left\{ 2\tilde{\mathcal{A}}_{n+1} \mathcal{K}_{n+1}' + \tilde{\mathcal{B}}_{n+1} \mathcal{K}_{n+1} \right\} \mathcal{Q}_n' \\ + \left\{ \tilde{\mathcal{B}}_{n+1} \mathcal{K}_{n+1}' + \tilde{\mathcal{C}}_{n+1} \mathcal{K}_{n+1} \right\} \mathcal{Q}_n = 0_{2\times 1} \, ,$$

where $\mathcal{K}_{n+1} = \begin{bmatrix} z & a_{n+1} \\ \overline{a}_{n+1} z & 1 \end{bmatrix}$.

Step 2. If we multiply previous equation by $\Theta_{n,1}\Theta_{n,2} \begin{bmatrix} \Theta_{n+1,2} & 0 \\ 0 & \Theta_{n+1,1} \end{bmatrix}$, on the left, we get, for all $n \in \mathbb{N}$,

$$\Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}_{n+1} \begin{bmatrix} \Theta_{n,2} & 0 \\ 0 & \Theta_{n,1} \end{bmatrix} \tilde{\mathcal{A}}_n \mathcal{Q}_n'' \\ + \Theta_{n,1}\Theta_{n,2} \begin{bmatrix} \Theta_{n+1,2} & 0 \\ 0 & \Theta_{n+1,1} \end{bmatrix} \left\{ 2\tilde{\mathcal{A}}_{n+1} \mathcal{K}_{n+1}' + \tilde{\mathcal{B}}_{n+1} \mathcal{K}_{n+1} \right\} \mathcal{Q}_n' \\ + \Theta_{n,1}\Theta_{n,2} \begin{bmatrix} \Theta_{n+1,2} & 0 \\ 0 & \Theta_{n+1,1} \end{bmatrix} \left\{ \tilde{\mathcal{B}}_{n+1} \mathcal{K}_{n+1}' + \tilde{\mathcal{C}}_{n+1} \mathcal{K}_{n+1} \right\} \mathcal{Q}_n = 0_{2\times 1} \, .$$

Using (27) to n in previous equation, there follows

$$\hat{\mathcal{A}}_n \mathcal{Q}_n' = \hat{\mathcal{M}}_n \mathcal{Q}_n \, , \forall n \in \mathbb{N} \, , \tag{41}$$

with

$$\hat{\mathcal{A}}_n = -\Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}_{n+1} \begin{bmatrix} \Theta_{n,2} & 0 \\ 0 & \Theta_{n,1} \end{bmatrix} \tilde{\mathcal{B}}_n \\ + \Theta_{n,1}\Theta_{n,2} \begin{bmatrix} \Theta_{n+1,2} & 0 \\ 0 & \Theta_{n+1,1} \end{bmatrix} \left\{ 2\tilde{\mathcal{A}}_{n+1} \mathcal{K}_{n+1}' + \tilde{\mathcal{B}}_{n+1} \mathcal{K}_{n+1} \right\} \, ,$$

$$\hat{\mathcal{M}}_n = \Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}_{n+1}\begin{bmatrix}\Theta_{n,2} & 0\\ 0 & \Theta_{n,1}\end{bmatrix}\tilde{\mathcal{C}}_n \\ - \Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2} & 0\\ 0 & \Theta_{n+1,1}\end{bmatrix}\left\{\tilde{\mathcal{B}}_{n+1}\mathcal{K}'_{n+1} + \tilde{\mathcal{C}}_{n+1}\mathcal{K}_{n+1}\right\}.$$

Furthermore, $\hat{\mathcal{A}}_n$ is a matrix with polynomial elements,

$$\hat{\mathcal{A}}_n = -\Theta_{n+1,1}\Theta_{n+1,2}\mathcal{K}_{n+1}\begin{bmatrix}\Theta_{n,2} & 0\\ 0 & \Theta_{n,1}\end{bmatrix}\mathcal{B}_n \\ + \Theta_{n,1}\Theta_{n,2}\begin{bmatrix}\Theta_{n+1,2} & 0\\ 0 & \Theta_{n+1,1}\end{bmatrix}\left\{2\tilde{\mathcal{A}}_{n+1}\mathcal{K}'_{n+1} + \mathcal{B}_{n+1}\mathcal{K}_{n+1}\right\},$$

where $\mathcal{B}_n$ is given by (cf. (28))

$$\mathcal{B}_n = \begin{bmatrix}[\mathcal{B}_{n,1}]_{2,2} & 0\\ 0 & [\mathcal{B}_{n,2}]_{2,2}\end{bmatrix}, n \in \mathbb{N}.$$

Step 3. If we multiply (41) by the adjoint matrix of $\hat{\mathcal{A}}_n$ and take into account that $\det(\hat{\mathcal{A}}_n)$ is a polynomial with a zero at $z = 0$, $\forall n \in \mathbb{N}$, then we obtain the structure relations (40),

$$zA_n\mathcal{Q}'_n = \mathcal{M}_n\mathcal{Q}_n\,,\ n \in \mathbb{N}\,,$$

with $zA_n = \det(\hat{\mathcal{A}}_n)$ and $\mathcal{M}_n = \operatorname{adj}(\hat{\mathcal{A}}_n)\hat{\mathcal{M}}_n$. ■

Now we study the coefficients of the structure relations previously obtained, (34), (35) and (40) (a similar technique was used in [13]).

Lemma 8. *Let F be a Carathéodory function and $\{\psi_n^1\}$, $\{\psi_n^2\}$ be the corresponding sequences defined in (7). Let $\varphi_n = \begin{bmatrix}\psi_n^1\\ \psi_n^2\end{bmatrix}$, $\forall n \in \mathbb{N}$. If $\{\varphi_n\}$ satisfies*

$$zA_n\varphi'_n = \widehat{\mathcal{M}}_n\varphi_n \tag{42}$$

with $A_n \in \mathbb{P}$ and $\widehat{\mathcal{M}}_n = \begin{bmatrix}\mathcal{M}_{n,1} & \mathcal{N}_{n,1}\\ \mathcal{N}_{n,2} & \mathcal{M}_{n,2}\end{bmatrix}$ where $\mathcal{M}_{n,1}, \mathcal{N}_{n,1}, \mathcal{N}_{n,2}, \mathcal{M}_{n,2}$ are matrices of order 2, then, $\forall n \in \mathbb{N}$,

$$\begin{cases} A_n = A_1\,, \\ zA_1 = z\mathcal{M}_{n+1,1} - z\mathcal{M}_{n,1} - a_{n+1}\mathcal{N}_{n,2} + \overline{a}_{n+1}z\mathcal{N}_{n+1,1}\,, \\ \overline{a}_{n+1}zA_1 = z\mathcal{N}_{n+1,2} + \overline{a}_{n+1}z\mathcal{M}_{n+1,2} - \overline{a}_{n+1}z\mathcal{M}_{n+1,1} - \mathcal{N}_{n,2}\,, \\ a_{n+1}\mathcal{M}_{n+1,1} + \mathcal{N}_{n+1,1} - z\mathcal{N}_{n,1} - a_{n+1}\mathcal{M}_{n,2} = 0_{2\times 2}\,, \\ a_{n+1}\mathcal{N}_{n+1,2} + \mathcal{M}_{n+1,2} - \overline{a}_{n+1}z\mathcal{N}_{n,1} - \mathcal{M}_{n,2} = 0_{2\times 2}\,. \end{cases} \tag{43}$$

Proof: Taking $n+1$ in (42) we get, for $n \in \mathbb{N}$,

$$zA_{n+1}\varphi'_{n+1} = \widehat{\mathcal{M}}_{n+1}\varphi_{n+1}\,.$$

Using the recurrence relations (9) we get

$$zA_{n+1}\left\{\left(\mathcal{K}^1_{n+1}\right)'\varphi_n + \mathcal{K}^1_{n+1}\varphi'_n\right\} = \widehat{\mathcal{M}}_{n+1}\mathcal{K}^1_{n+1}\varphi_n\,,\ \mathcal{K}^1_{n+1} = \begin{bmatrix} zI & a_{n+1}I \\ \overline{a}_{n+1}zI & I \end{bmatrix}.$$

Thus we have

$$zA_{n+1}\mathcal{K}^1_{n+1}\varphi'_n = \left(\widehat{\mathcal{M}}_{n+1}\mathcal{K}^1_{n+1} - zA_{n+1}\left(\mathcal{K}^1_{n+1}\right)'\right)\varphi_n\,,\quad \forall n \in \mathbb{N}\,.$$

Therefore,

$$zA_{n+1}\varphi'_n = \left(\mathcal{K}^1_{n+1}\right)^{-1}\left(\widehat{\mathcal{M}}_{n+1}\mathcal{K}^1_{n+1} - zA_{n+1}\left(\mathcal{K}^1_{n+1}\right)'\right)\varphi_n\,,\quad \forall n \in \mathbb{N}\,. \tag{44}$$

Now, comparing equations (44) and (42) it follows that, $\forall n \in \mathbb{N}$,

$$\begin{cases} A_{n+1} = A_n \\ \left(\mathcal{K}_{n+1}\right)^{-1}\left(\widehat{\mathcal{M}}_{n+1}\mathcal{K}^1_{n+1} - zA_{n+1}\left(\mathcal{K}^1_{n+1}\right)'\right) = \widehat{\mathcal{M}}_n\,. \end{cases}$$

Thus we get, $\forall n \in \mathbb{N}$,

$$\begin{cases} A_n = A_1\,, \\ \widehat{\mathcal{M}}_{n+1}\mathcal{K}^1_{n+1} - \mathcal{K}^1_{n+1}\widehat{\mathcal{M}}_n = zA_1\left(\mathcal{K}^1_{n+1}\right)'\,, \end{cases}$$

and equations (43) follow. ∎

Lemma 9. *Let F be a Carathéodory function and $\{\mathcal{Q}_n\}$ the corresponding sequence defined in (7). If*

$$zA_n\mathcal{Q}'_n = \mathcal{M}_n\mathcal{Q}_n,\quad \forall n \in \mathbb{N}$$

with $A_n \in \mathbb{P}$ and $\mathcal{M}_n$ a matrix or order 2 with analytic elements, then A_n does not depend on n.

Proof: Analogous with the proof of previous lemma. ∎

Lemma 10. *The coefficients of the structure relations (34) and (35) satisfy, for all* $n \in \mathbb{N}$,

$$A_{n,1} = A_{n,2} = A_1 \,, \tag{45}$$

$$\mathcal{M}_{n,1} = \begin{bmatrix} l_{n,1} - \gamma/2 & -\beta \\ \delta & l_{n,1} + \gamma/2 \end{bmatrix} , \tag{46}$$

$$\mathcal{M}_{n,2} = \begin{bmatrix} l_{n,2} - \gamma/2 & -\beta \\ \delta & l_{n,2} + \gamma/2 \end{bmatrix} , \tag{47}$$

$$\mathcal{N}_{n,1} = h_{n,1}\, I \,, \tag{48}$$

$$\mathcal{N}_{n,2} = h_{n,2}\, I \,, \tag{49}$$

where $A_1, \beta, \gamma, \delta, l_{n,1}, l_{n,2}, h_{n,1}, h_{n,2}$ *are polynomials* (A_1, β, γ, *and* δ *are independent of* n).
Moreover, with respect to the relations (40), the following hold, $\forall n \in \mathbb{N}$,

$$A_n = A_1, \;\; [\mathcal{M}_n]_{1,1} = [\mathcal{M}_{n,1}]_{2,2} - [\mathcal{M}_{n,1}]_{1,2} F, \;\; [\mathcal{M}_n]_{1,2} = -[\mathcal{N}_{n,1}]_{2,2} \tag{50}$$
$$[\mathcal{M}_n]_{2,1} = -[\mathcal{N}_{n,2}]_{2,2}, \;\; [\mathcal{M}_n]_{2,2} = [\mathcal{M}_{n,2}]_{2,2} - [\mathcal{M}_{n,2}]_{1,2} F \,.$$

Proof: From Lemma 8 we get (45). From Lemma 6 we obtain that the matrices $\mathcal{N}_{n,1}$ and $\mathcal{N}_{n,2}$ are scalar matrices, thus (48) and (49) follow, with polynomials $h_{n,1}, h_{n,2}$.

We now establish (46) and (47).

Taking into account that $[\mathcal{N}_{n,1}]_{1,2} = [\mathcal{N}_{n,1}]_{2,1} = 0$, $\forall n \in \mathbb{N}$, from (43) we get, $\forall n \in \mathbb{N}$,

$$[\mathcal{M}_{n+1,1}]_{1,2} = [\mathcal{M}_{n,2}]_{1,2} \,, \; [\mathcal{M}_{n+1,1}]_{2,1} = [\mathcal{M}_{n,2}]_{2,1} \,, \tag{51}$$

$$[\mathcal{M}_{n+1,2}]_{1,2} = [\mathcal{M}_{n,2}]_{1,2} \,, \; [\mathcal{M}_{n+1,2}]_{2,1} = [\mathcal{M}_{n,2}]_{2,1} \,. \tag{52}$$

From (52) we conclude that the elements $[\mathcal{M}_{n,2}]_{1,2}$ and $[\mathcal{M}_{n,2}]_{2,1}$ do not depend on n, and we write

$$[\mathcal{M}_{n,2}]_{1,2} = -\beta \,, \;\; [\mathcal{M}_{n,2}]_{2,1} = \delta \,, \;\; \forall n \in \mathbb{N} \,. \tag{53}$$

Consequently, from (51), there follows

$$[\mathcal{M}_{n,1}]_{1,2} = -\beta, \;\; [\mathcal{M}_{n+1,1}]_{2,1} = \delta \,, \;\; \forall n \in \mathbb{N} \,. \tag{54}$$

From (43) it follows that, $\forall n \in \mathbb{N}$,

$$[\mathcal{M}_{n+1,2}]_{1,1} - [\mathcal{M}_{n,2}]_{1,1} = \overline{a}_{n+1} z [\mathcal{N}_{n,1}]_{1,1} - a_{n+1} [\mathcal{N}_{n+1,2}]_{1,1} \,,$$
$$[\mathcal{M}_{n+1,2}]_{2,2} - [\mathcal{M}_{n,2}]_{2,2} = \overline{a}_{n+1} z [\mathcal{N}_{n,1}]_{2,2} - a_{n+1} [\mathcal{N}_{n+1,2}]_{2,2} \,.$$

Taking into account that $\mathcal{N}_{n,1}$ and $\mathcal{N}_{n,1}$ are scalar matrices, there follows

$$[\mathcal{M}_{n+1,2}]_{2,2} - [\mathcal{M}_{n+1,2}]_{1,1} = [\mathcal{M}_{n,2}]_{2,2} - [\mathcal{M}_{n,2}]_{1,1} \,, \quad \forall n \in \mathbb{N} \,.$$

Thus $[\mathcal{M}_{n,2}]_{2,2} - [\mathcal{M}_{n,2}]_{1,1}$ do not depend on n, and we write

$$[\mathcal{M}_{n,2}]_{2,2} - [\mathcal{M}_{n,2}]_{1,1} = \gamma \,, \quad \forall n \in \mathbb{N} \,. \tag{55}$$

Analogously, from (43) we obtain

$$[\mathcal{M}_{n+1,1}]_{2,2} - [\mathcal{M}_{n+1,1}]_{1,1} = [\mathcal{M}_{n,2}]_{2,2} - [\mathcal{M}_{n,2}]_{1,1}, \quad \forall n \in \mathbb{N} \,.$$

Consequently, from (55) there follows

$$[\mathcal{M}_{n,1}]_{2,2} - [\mathcal{M}_{n,1}]_{1,1} = \gamma \,, \quad \forall n \in \mathbb{N} \,. \tag{56}$$

Taking into account (53)-(56), we obtain (46) and (47).

Finally, to establish (50) we take into account Lemmas 4, 7 and 9 and the computations therein done with $\{\mathcal{Q}_n\}$. ■

Lemma 11. *Let F be a Carathéodory function, let $\{\psi_n^1\}$, $\{\psi_n^2\}$ be the corresponding sequences defined in (7), and $\{Q_n\}$ the sequence of functions of the second kind. If $\{\psi_n^1\}$ satisfies (19) with coefficients (21) and $\{Q_n\}$ satisfies (20) with coefficients (22), then F satisfies $zAF' = BF^2 + CF + D$.*

Proof: Let $\{\psi_n^1\}$ satisfy (19) and $\{Q_n\}$ satisfy (20). From Lemma 5 it follows that $\{\psi_n^2\}$ and $\{Q_n^*\}$ satisfy

$$\mathcal{A}_{n,2} (\psi_n^2)'' + \mathcal{B}_{n,2} (\psi_n^2)' + \mathcal{C}_{n,2} \psi_n^2 = 0_{2\times 1} \,,$$
$$\tilde{\mathcal{A}}_{n,2} (Q_n^*)'' + \tilde{\mathcal{B}}_{n,2} (Q_n^*)' + \tilde{\mathcal{C}}_{n,2} Q_n^* = 0 \,.$$

Hence, from Lemmas 6 and 7 we obtain the structure relations (34), (35) and (40) and, from Lemma 10, equations (45), (46), (48) and (50) follow. Hence, taking into account Theorem 1, we conclude that F satisfies $zA_1F' = \beta F^2 + \gamma F + \delta \,,\ A_1, \beta, \gamma, \delta \in \mathbb{P}$. Furthermore, since the coefficients of (19) and (20) are given by (21) and (22), respectively, then we obtain $A_1 = A,\ \beta = B,\ \gamma = C,\ \delta = D$. ■

Finally, we give the proof of Theorem 2: Lemma 4 establishes $a) \Rightarrow b)$ and $a) \Rightarrow c)$. Using Lemma 5 we establish the equivalence between $b)$ and $c)$. Lemma 11 establishes $b) \Rightarrow a)$.

4. The semi-classical case

In this section we take a closer look at the semi-classical class on the unit circle and its characterization in terms of second order linear differential equations. We shall consider Carathéodory functions associated to measures, μ, of the following type:

$$d\mu = w\, d\theta + \sum_{k=1}^{N} \lambda_k \delta_{z_k}\,,\ \lambda_k > 0\,,\ N \in \mathbb{N}\,, \tag{57}$$

where w is the absolutely continuous part with respect to the Lebesgue measure $d\theta$, and δ_{z_k} is the Dirac measure at z_k, with $z_k \in \mathbb{T}\,,\ k = 1, \ldots, N$.

Lemma 12 (cf. [2, 3, 5]). *A measure defined by (57) is semi-classical and its absolutely continuous part satisfies* $\dfrac{w'(z)}{w(z)} = \dfrac{C(z)}{zA(z)}$, *if and only if, the corresponding Carathéodory function,* F, *satisfies* $zAF' = CF + D$, *with*

$$D(z) = -zA'(z) - 2z \sum_{k=2}^{\deg(A)} \frac{A^{(k)}(z)}{2\pi k!} \int_0^{2\pi} e^{i\theta}(e^{i\theta} - z)^{k-2} d\mu(\theta) - \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} (C(e^{i\theta}) - C(z)) d\mu(\theta)\,. \tag{58}$$

Taking into account Theorem 2 (cf. also Corollary 1), the semi-classical class on the unit circle is characterized in terms of the second order linear differential equations (29) and (31) for $\{\phi_n\}$ and $\{Q_n\}$, respectively, and second order differential equations for $\{\Omega_n\}$ of type

$$(zA)^2 \Theta_{n,1} (\Omega_n)'' + [\mathcal{B}_{n,1}]_{2,2} \Omega_n' + [\mathcal{C}_{n,1}]_{2,2}^0 \Omega_n = [\mathcal{C}_{n,1}]_{2,1} \phi_n + [\mathcal{B}_{n,1}]_{2,1} \phi_n'$$

(take $B = 0$ and D as given in (58) in equations (19)-(26)). However, we will see that if we consider suitable second order linear differential equations for $\{Q_n/w\}$ instead of $\{Q_n\}$, then the referred equations for $\{\Omega_n\}$ are superfluous.

We will need the lemma that follows.

Lemma 13 (see [6]). *Let X and M be matrix functions of order two such that $X' = M\,X$. Then,*

$$(\det(X))' = \operatorname{tr}(M)\,\det(X)\,, \tag{59}$$

where $\operatorname{tr}(M)$ denotes the trace of the matrix M.

Our main result is the following.

Theorem 3. *Let F be a Carathéodory function associated with a measure μ of type (57), let $\{\phi_n\}$ be the corresponding MOPS, and $\{Q_n\}$ the sequence of functions of the second kind. The following statements are equivalent:*
a) F satisfies a first order differential equation with polynomial coefficients

$$zAF' = CF + D$$

with D defined in (58).
b) $\{\phi_n\}$ and $\{Q_n/w\}$ are solutions of second order linear differential equations

$$(zA)^2\Theta_{n,1}(\Upsilon)'' + B_{n,1}\Upsilon' + C_{n,1}\Upsilon = 0\,,\ n \in \mathbb{N}\,, \tag{60}$$

with polynomial coefficients $B_{n,1} = [\mathcal{B}_{n,1}]_{1,1}$, $C_{n,1} = [\mathcal{C}_{n,1}]^0_{1,1}$.
c) $\{\phi_n^*\}$ and $\{Q_n^*/w\}$ are solutions of second order linear differential equations

$$(zA)^2\Theta_{n,2}(\Upsilon)'' + B_{n,2}(\Upsilon)' + C_{n,2}(\Upsilon) = 0\,,\ n \in \mathbb{N}\,, \tag{61}$$

with polynomial coefficients $B_{n,2} = [\mathcal{B}_{n,2}]_{1,1}$, $C_{n,2} = [\mathcal{C}_{n,2}]^0_{1,1}$.

Proof: $a) \Rightarrow b)$ and $a) \Rightarrow c)$:
Taking into account Theorem 1 we have

$$\begin{aligned} zA\begin{bmatrix}\phi_n\\ \phi_n^*\end{bmatrix}' &= (\mathcal{B}_n - C/2\,I)\begin{bmatrix}\phi_n\\ \phi_n^*\end{bmatrix}\,,\ \mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1}\\ -\Theta_{n,2} & l_{n,2}\end{bmatrix}\,,\\ zA\begin{bmatrix}Q_n'/w\\ -(Q_n^*)'/w\end{bmatrix} &= (\mathcal{B}_n + C/2\,I)\begin{bmatrix}Q_n/w\\ -Q_n^*/w\end{bmatrix}\,. \end{aligned} \tag{62}$$

Moreover, as $w'/w = C/(zA)$, we obtain

$$zA\begin{bmatrix}Q_n/w\\ -Q_n^*/w\end{bmatrix}' = zA\begin{bmatrix}Q_n'/w\\ -(Q_n^*)'/w\end{bmatrix} - C\,I\begin{bmatrix}Q_n/w\\ -Q_n^*/w\end{bmatrix}\,. \tag{63}$$

If we substitute (62) in (63) we get

$$zA\begin{bmatrix}Q_n/w\\ -Q_n^*/w\end{bmatrix}' = (\mathcal{B}_n - C/2\,I)\begin{bmatrix}Q_n/w\\ -Q_n^*/w\end{bmatrix}\,.$$

Hence, $\begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}$ and $\begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}$ satisfy the same structure relations. Taking into account Corollary 1 (cf. (29) and (30)), the referred equations follow.
b) $\Leftrightarrow$ *c)* :
Using the same technique as in Lemma 5 there follows that $\{\phi_n\}$ and $\{Q_n/w\}$ satisfy (60) if, and only if, $\{\phi_n^*\}$ and $\{Q_n^*/w\}$ satisfy (61).
b) $\Rightarrow$ *a)* :
Let $\{\phi_n\}$ and $\{Q_n/w\}$ satisfy (60). Then, $\{\phi_n^*\}$ and $\{Q_n^*/w\}$ satisfy (61). Hence, if we write $\tilde{Y}_n^1 = \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}$, $\tilde{Y}_n^2 = \begin{bmatrix} -Q_n/w \\ Q_n^*/w \end{bmatrix}$, $n \in \mathbb{N}$, both $\tilde{Y}_n^1$ and $\tilde{Y}_n^2$ satisfy the following second order differential equation:

$$\mathcal{A}_n(\Lambda)'' + \mathcal{B}_n\Lambda' + \mathcal{C}_n\Lambda' = 0_{2\times 1}\,,\ n \in \mathbb{N}\,,$$

with

$$\mathcal{A}_n = (zA)^2 \begin{bmatrix} \Theta_{n,1} & 0 \\ 0 & \Theta_{n,2} \end{bmatrix}, \ \mathcal{B}_n = \begin{bmatrix} B_{n,1} & 0 \\ 0 & B_{n,2} \end{bmatrix}, \ \mathcal{C}_n = \begin{bmatrix} C_{n,1} & 0 \\ 0 & C_{n,2} \end{bmatrix}.$$

Using the same technique as in Lemmas 6, 7 and 8, we obtain the structure relations

$$zA_1\left(\tilde{Y}_n^1\right)' = \Xi_n\tilde{Y}_n^1\,,\ zA_1\left(\tilde{Y}_n^2\right)' = \Xi_n\tilde{Y}_n^2\,,\ n \in \mathbb{N}\,,$$

where $A_1 \in \mathbb{P}$ and Ξ_n are matrices of order two with polynomial elements. Hence, we get the differential system

$$zA_1\tilde{Y}_n' = \Xi_n\tilde{Y}_n,\ \ \tilde{Y}_n = \begin{bmatrix} \phi_n & -Q_n/w \\ \phi_n^* & Q_n^*/w \end{bmatrix}, \forall n \in \mathbb{N}\,.$$

Taking into account Lemma 13 (cf. (59)) we get $\det(\tilde{Y}_n)' = \dfrac{\operatorname{tr}(\Xi_n)}{zA_1}\det(\tilde{Y}_n)$. Since $\det(\tilde{Y}_n) = 2h_n z^n/w$, it follows that

$$\frac{w'}{w} = \frac{nA_1 - \operatorname{tr}(\Xi_n)}{zA_1}\,. \tag{64}$$

Next we compute the trace of Ξ_n. Let us write $\Xi_n = \begin{bmatrix} M_{n,1} & N_{n,1} \\ N_{n,2} & M_{n,2} \end{bmatrix}$. Taking into account Lemma 8 and the equations (43) for the polynomials $M_{n,1}$, $N_{n,1}$,

$N_{n,2}$, $M_{n,2}$, (from the first and fourth equations in (43)) there follows

$$\begin{aligned} z(M_{n+1,1}+M_{n+1,2}) &= zA_1 + z(M_{n,1}+M_{n,2}) \\ &\quad + \overline{a}_{n+1}z^2N_{n,1} + a_{n+1}N_{n,2} - a_{n+1}zN_{n+1,2} - \overline{a}_{n+1}zN_{n+1,1}\,. \end{aligned} \tag{65}$$

Using the third and the second equations of (43) in (65) it follows that

$$M_{n+1,1}+M_{n+1,2} = M_{n,1}+M_{n,2}+A_1+|a_{n+1}|^2\left(2M_{n+1,2}-M_{n,2}-M_{n+1,1}-A_1\right)\,,$$

and we obtain

$$M_{n+1,1}+M_{n+1,2} = nA_1 + L_n\,,\ n\in\mathbb{N}\,,$$

with

$$L_n = M_{1,1}+M_{1,2}+\sum_{k=1}^{n}|a_k|^2\left(2M_{k+1,2}-M_{k,2}-M_{k+1,1}-A_1\right)\,.$$

Hence,

$$\mathrm{tr}(\Xi_{n+1}) = nA_1 + L_n\,. \tag{66}$$

If we use (66) for $n-1$ in (64) we obtain

$$\frac{w'}{w} = \frac{A_1 - L_{n-1}}{zA_1}\,,$$

and we conclude that L_n does not depend on n. Using Lemma 12 we get the the semi-classical character of the corresponding measure, and the required differential equation for F follows. ∎

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