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ENTROPY BOUNDS FOR HIERARCHICAL MOLECULAR NETWORKS

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ABSTRACT: In this paper we derive entropy bounds for hierarchical networks. More precisely, starting from a recently introduced measure to determine the topological entropy of non-hierarchical networks, we provide bounds for estimating the entropy of hierarchical graphs. Apart from bounds to estimate the entropy of a single hierarchical graph, we see that the derived bounds can also be used for characterizing graph classes. Our contribution is an important extension to previous results about the entropy of non-hierarchical networks because for practical applications hierarchical networks are playing an important role in chemistry and biology. In addition to the derivation of the entropy bounds, we provide a numerical analysis for two special graph classes, rooted trees and generalized trees, and demonstrate hereby not only the computational feasibility of our method but also learn about its characteristics and interpretability with respect to data analysis.

KEYWORDS: Hierarchical networks; molecular networks; graph entropy; graph measures.

1. Introduction

The investigation of topological aspects of chemical structures concerns a major part of the research in chemical graph theory and mathematical chemistry [12, 27, 36, 70]. Following, e.g., [5, 14, 8, 12, 27, 58, 66], classical and current research topics in chemical graph theory involve, e.g., modeling of chemical molecules by means of graphs, graph polynomials, graph-theoretical matrices, enumeration of chemical structures, and aspects of quantitative structure analysis like measuring the structural similarity of graphs and structural information. Further, a lot of the above mentioned contributions can be integrated under the following thematic categories which are well know in chemistry: QSAR and QSPR. QSAR (Quantitative structure-activity relationship) deals with descripting pharmacokinetic processes as well as biological activity or chemical reactivity [6, 25]. In contrast, QSPR (Quantitative Structure-Property Relationship) generally addresses the problem to convert chemical structures into molecular descriptors which are relevant to a physico-chemical property or a biological activity [25, 26]. However, a main

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problem in QSPR is to investigate relationships between molecular structure and physicochemical properties, e.g., the topological complexity of chemical structures [8, 10, 11, 25].

This paper mainly deals with a challenging problem of quantitative graph analysis: Deriving bounds for the entropies of hierarchical graphs. An important application area of information-theoretic methods applied to networks is, e.g., QSPR where our main focus lies on the examination of graph classes which are widely used in chemical graph theory and computational biology. Generally, there are two main directions in quantitative graph analysis: (i) Comparing and (ii) characterizing networks. Network comparison addresses the problem of measuring their structural similarity or distance, see, e.g., [17, 18, 19, 67, 68, 41, 73, 74]. In contrast, to characterize a network means that one has to infer structural network statistics which capture certain structural information of the networks [16, 28, 4, 46]. For giving a short review on information-theoretic methods to characterize graphs [14, 8, 11, 13, 23, 59], we want to emphasize that the problem of quantifying certain structural information of systems was a starting point of an emerging field that deals with applying information-theoretic techniques to networks, e.g., for investigating living systems [50, 56, 21, 44, 57, 71]. As a fundament, Shannon [62] extended the concept of entropy that was known in thermodynamics for transmitting information. For this, he considered a message transmitted through information channels as a certain set of symbols denoted as an outcome which was selected from the ensemble of all k such sets containing the same total number of symbols N [13]. By assigning probabilities $p_1, p_2, \ldots, p_k$ to each i-th outcome based on the quantities $p_i = \frac{N_i}{N}$ where N_i denotes the number of symbols of the i-th outcome, Shannon characterized the entropy H as the uncertainty of the expected outcome [13]. Then, the classical Shannon-entropy formula to measure the average entropy of information per communication symbol can be expressed by

$$H_m = -\sum_{i=1}^{k} p_i \log(p_i) = -\sum_{i=1}^{k} \frac{N_i}{N} \log\left(\frac{N_i}{N}\right) \text{ bits/symbol.} \tag{1}$$

H_m is often called the mean information. Additionally, Brillouin [15] defined the total information as

$$H = N \log(N) - \sum_{i=1}^{k} N_i \log(N_i) \text{ bits.} \tag{2}$$

Now, the topics we just mentioned [50, 56, 21, 44, 57, 71] have been mainly influenced by the, at that time, novel insight that an inferred or constructed graph structure can be considered as the result of a certain information process or communication between the elements of the underlying system [11, 62]. As a consequence [8, 54], Equation (1) and Equation (2) can be now interpreted as the mean information content

$$I_m(G) = -\sum_{i=1}^{k} p_i \log(p_i), \tag{3}$$

and the total information content

$$I(G) = |V|\log(|V|) - \sum_{i=1}^{k} |V_i|\log(|V_i|), \tag{4}$$

of a graph G. Here, $|V|$ denotes the number of vertices of a graph G, k denotes the number of different (obtained) sets of vertices, $|V_i|$ is the number of elements in the i-th set of vertices, and it holds $p_i = \frac{|V_i|}{|V|}$. The first attempt in this direction was given by [57] who developed a technique to determine the structural information content of a graph. This technique is based on the principle of finding distinguishable vertices of a graph to apply SHANNON's entropy (Equation (3) and Equation (4)) for determining the information content of such a graph-based system. Also, [54, 51, 52, 53] investigated this problem by using algebraic methods, i.e., determining the automorphism groups of graphs. We remark that the mentioned methods, e.g., [54, 51, 52, 53, 57, 71] for measuring the structural information content of a graph-based system are based on the following principle: Starting from a certain equivalence criterion, a graph-based system with n elements can be partitioned into k classes, see, e.g., [11]. As a consequence, a probability distribution can be obtained that leads directly to the definition of an entropy of the system under consideration (Equation (3) and Equation (4)). Following [11, 54, 23], the structural information content of such a system is interpreted as the entropy of the underlying graph topology. As a remark, we note that graph entropy definitions which are rooted in information theory can be found in [34, 42, 43, 64].

A major contribution of this paper addresses the problem of finding bounds for the entropies of hierarchical graphs, which often occurs in chemical graph

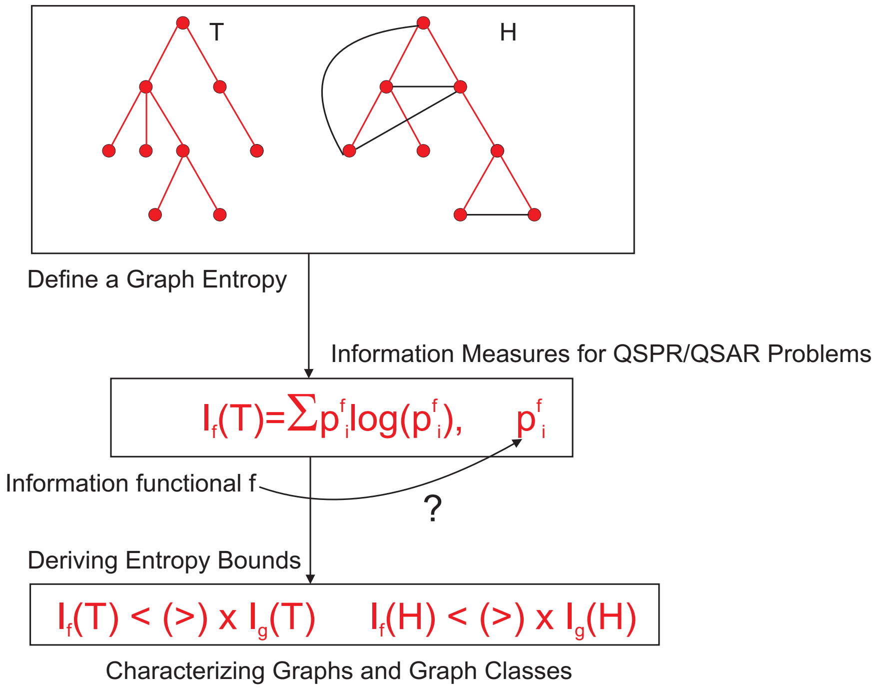

FIGURE 1. Overall approach to derive entropy bounds for hierarchical graphs.

theory and computational and systems biology. Here, the term "hierarchical" means that we deal with graphs having a distinct vertex that is called a root. To achieve this goal, we use an approach for determining the entropy of undirected and connected graphs that has been recently presented in [23]. In contrast to the classical methods which we have already outlined above, this method is based on assigning a probability value to each vertex in a graph by using a special information functional. The information functional we have presented in [23] is based on metrical properties of graphs, more precisely, on so-called j-spheres. In terms of practical applications, we want to point that the task of deriving bounds for the entropies of graphs is crucial because the exact entropy value can often not be calculated concretely, especially regarding large graphs. For this reason, entropy bounds for special graph classes help to reduce the complexity of such problems and can be also used for characterizing graphs or graph classes by using information-theoretic measures, e.g., see Section (2.3). As mentioned, hierarchical (rooted) graph structures do have a large application potential in chemical graph theory and computational biology. Therefore, we restrict our analysis on such graph structures. A further reason for focusing on rooted graphs is, to our knowledge, such a study does not exist. Another contribution of this paper deals with demonstrating the practical ability of the used graph entropy approach [23] by interpreting the produced numerical results. Starting from two graph classes, ordinary rooted trees and so-called generalized trees [38, 48], we show that our entropy measure captures important structural information meaningfully. To summarize the main contribution of this paper, Figure (1) shows the overall approach exemplarily.

This paper is organized as follows: Before starting with our main contribution in Section (2.3), we express some relevant application areas and problems dealing with hierarchical graphs in Section (2.1). Further, in Section (2.2) we briefly repeat the construction of the graph entropy measure [23] that is the fundament of our theoretical analysis. In Section (2.3), we derive bounds for the entropies of rooted trees and generalized trees. Here, we focus on so-called implicit bounds, i.e., the entropy of a graph is characterized by another graph entropy expression based on an estimation. Section (3.1) presents numerical results. The paper finishes in Section 3.2 with a summary and conclusion.

2. Analysis

2.1. Applications of Hierarchical Graphs. In this section, we briefly outline some applications of hierarchical graphs in chemical graph theory and computational biology.

2.1.1. *Mathematical Chemistry.* There is a universe of problems dealing with trees for modeling and analyzing chemical structures [1, 12, 27, 36, 70]. However, also rooted tree structures are of particular interest because, e.g., considering such graph classes often helps to solve more general graph problems. In the following, we state some interesting applications of rooted trees in chemical graphs theory:

- Enumeration and coding problems of chemical structures by using rooted trees [2, 55, 47, 29].
- Describing so-called signatures as molecular descriptors for problems in QSAR [72].
- Graph polynomials of hierarchical graphs [75].
- Chemical graph analysis by using algebraic and metrical graph properties [7, 20, 45, 65].

2.1.2. *Biology.* Tree structures have been intensely investigated for solving and modeling biological problems. In particular, rooted trees often serve as an important graph representation for many biological classification problems as well as for problems in evolutionary biology [60]. To summarize some known approaches involving hierarchical graph structures, we state the following listing:

- Reconstruction problems and so-called supertree methods in phylogenetics [32, 33, 69, 61, 60].
- Modeling and analyzing RNA structures [39, 63].
- Supervised and unsupervised graph classification problems in computational biology [30, 40].
- Clustering problems in computational biology [35, 49].

2.2. A Method for Determining the Entropy of Graphs. In this section, we briefly repeat the method to measure the entropy of arbitrary undirected and connected networks, see [23]. As mentioned, we will interpret and define the structural information content as the entropy of the underlying graph topology [23]. The method we want to use is mainly based on the principle to assign a probability value to each vertex in a graph by using a certain information functional for quantifying structural information in a graph and, hence, for determining its entropy. The information functional that has been used [23] is based on determining the so-called j-spheres of a graph. Before outlining the main construction steps of this approach, we want to mention that [9] also used so-called vertex distance degree sequences (DDS) to develop the idea of a graph center for chemical structures. Interestingly, the derived DDS-distributions correspond to vertex distributions by using j-spheres. Similarly to the just described idea, one main idea of the approach of [23] to determine the entropy of a graph was to use a connectivity concept to express neighborhood relations of its vertices. Finally, it turned out that a natural procedure for expressing such relations is to calculate the number of the first neighboring vertices, the number of the second neighboring vertices, etc. and, hence, this just corresponds to the definition of the j-sphere. As an example, Figure (2) shows the process of determining j-spheres visually.

In order to repeat the main construction step of the above mentioned graph entropy method, we first express some mathematical preliminaries [3, 37, 23]. We define an undirected, finite and connected graph by $G = (V, E)$, $|V| < \infty$,

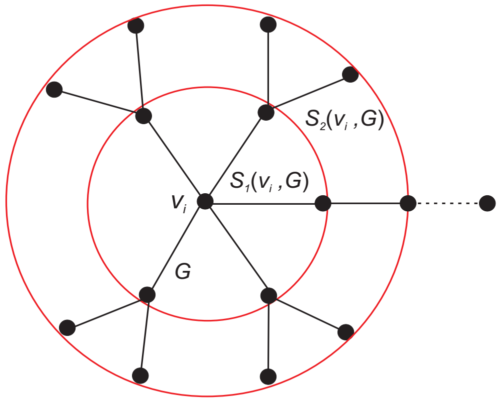

FIGURE 2. G represents an undirected and connected graph. For example, we get $|S_1(v_i, G)| = 5$ and $|S_2(v_i, G)| = 9$.

$E \subseteq \binom{V}{2}$. G is called connected if for arbitrary vertices v_i and v_j there exists an undirected path from v_i to v_j. Otherwise, we call G unconnected. $\mathcal{G}_{UC}$ denotes the set of finite, undirected and connected graphs. The degree of a vertex $v \in V$ is denoted by $\delta(v)$ and equals the number of edges $e \in E$ which are incident with v. In order to measure distances between vertices in a graph, we denote $d(u, v)$ as distance between $u \in V$ and $v \in V$ expressed as the minimum length of a path between u, v. We notice that $d(u, v)$ is a metric. We call the quantity $\sigma(v) = \max_{u \in V} d(u, v)$ the eccentricity of $v \in V$. Further, $\rho(G) = \max_{v \in V} \sigma(v)$ is called the diameter of G. The j-sphere of a vertex v_i regarding $G \in \mathcal{G}_{UC}$ is defined as the set,

$$S_j(v_i, G) := \{v \in V |\, d(v_i, v) = j,\ j \geq 1\}. \quad (5)$$

Now, we state the definition of a special information functional that has been introduced in [23] to define the entropy of a graph. Here, the information functional f^V quantifies structural information of a graph G by using the cardinalities of the corresponding j-spheres.

Definition 2.1. *Let $G \in \mathcal{G}_{UC}$ with arbitrary vertex labels. For the vertex $v_i \in V$, the information functional f^V is defined as*

$$f^V(v_i) := \alpha^{c_1|S_1(v_i,G)|+c_2|S_2(v_i,G)|+\cdots+c_\rho|S_\rho(v_i,G)|},$$
$$c_k > 0, 1 \leq k \leq \rho, \alpha > 0. \quad (6)$$

$f^V(v_i)$ captures structural information of G by using metrical properties of G. The parameters α and c_k are introduced to weight structural characteristics or differences of G in each sphere, e.g., a vertex with a large degree.

As a remark, we generally see that it always

$$\begin{aligned} &|S_1(v_1,G)| + |S_2(v_1,G)| + \cdots + |S_\rho(v_1,G)|, && (7)\\ =&|S_1(v_2,G)| + |S_2(v_2,G)| + \cdots + |S_\rho(v_2,G)|, && (8)\\ =&\cdots \qquad\qquad\qquad\qquad \cdots \\ =&|S_1(v_{|V|},G)| + |S_2(v_{|V|},G)| + \cdots + |S_\rho(v_{|V|},G)|, && (9)\end{aligned}$$

holds [23]. Hence, the c_k have to be chosen such that they are not equal, e.g, $c_1 > c_2 > \cdots > c_\rho$. Finally, we observe that the variation of c_k and α aims to study the local information spread in a network.

Definition 2.2. *The vertex probabilities are defined by the quantities*

$$p^V(v_i) := \frac{f^V(v_i)}{\sum_{j=1}^{|V|} f^V(v_j)}. \tag{10}$$

Definition 2.3. *Let $G = (V,E) \in \mathcal{G}_{UC}$. Then, we define the entropy of G by*

$$I_{f^V}(G) := -\sum_{i=1}^{|V|} p^V(v_i)\log(p^V(v_i)). \tag{11}$$

As outlined in [23], we recall that the process of defining information functionals and, hence, the entropy of a graph by using structural properties or graph-theoretical quantities is not unique. Consequently, each information functional captures structural information of a given graph differently. Further, we pointed out [23] that the parameter α can always be determined via an optimization procedure based on a given data set and, hence, is uniquely defined for a given classification problem.

2.3. Bounds for the Entropies of Hierarchical Graphs. In this section, we derive bounds for the entropies of hierarchical graphs. For this, we use the entropy measure explained in Section (2.2). As mentioned, in this paper we choose the class of rooted trees and so-called generalized trees [48]. We notice that a generalized tree contains an ordinary rooted tree as a special case [48]. Further, it turned out that generalized trees can be very useful for solving current problems in applied discrete mathematics, computer science and systems biology [48, 24, 31, 30]. To start with the problem of finding entropy bounds, we first define the mentioned graph classes. Directed generalized trees have already been defined in [48].

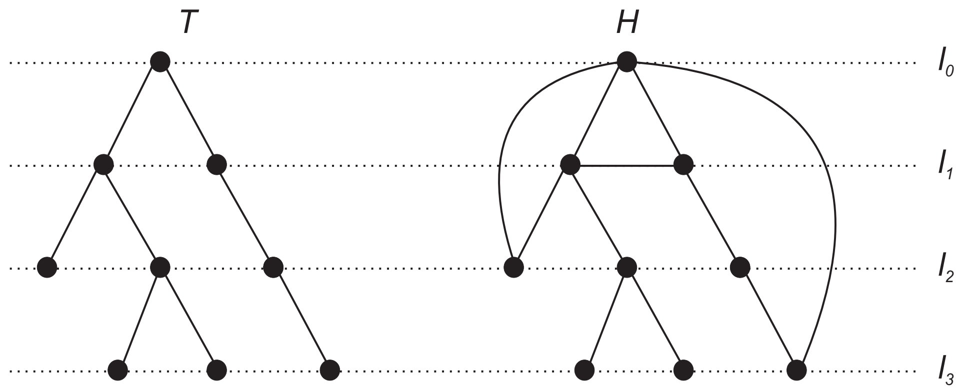

FIGURE 3. An undirected tree T and its corresponding undirected generalized tree H. It holds $|L| = 4$ and $h = |L| - 1 = 3$.

Definition 2.4. *An undirected graph is called undirected tree if this graph is connected and cycle free. An undirected rooted tree $T = (V, E)$ is an undirected graph which has exactly one vertex $r \in V$ for which every edge is directed away from the root r. Then, all vertices in T are uniquely accessible from r. The level of a vertex v in a rooted tree T is simply the length of the path from r to v. The path with the largest path length from the root to a leaf is denoted as h.*

Definition 2.5. *As a special case of $T = (V, E)$ we also define an ordinary ϑ-tree denoted as T_ϑ where ϑ is a natural number. For the root vertex r, it holds $\delta(r) = \vartheta$ and for all internal vertices $v \in V$ holds $\delta(v) = \vartheta + 1$. Leaves are vertices without successors. A ϑ-tree is fully occupied, denoted by T_ϑ^o, if all leaves possess the same height h.*

Definition 2.6. *Let $T = (V, E_1)$ be an undirected finite rooted tree. $|L|$ denotes the cardinality of the level set $L := \{l_0, l_1, \cdots, l_h\}$. The longest length of a path in T is denoted as h. It holds $h = |L| - 1$. $\mathcal{L} : V \longrightarrow L$ is a surjective mapping and it is called a multi level function if it assigns to each vertex an element of the level set L. A graph $H = (V, E_{GT})$ is called a finite, undirected generalized tree if its edge set can be represented by the union $E_{GT} := E_1 \cup E_2 \cup E_3$, where*

- *E_1 forms the edge set of the underlying undirected rooted tree T.*
- *E_2 denotes the set of horizontal Across-edges. A horizontal Across-edge does not change a level i.*
- *E_3 denotes the set of edges which change at least one level.*

As an example, Figure (3) shows an undirected rooted tree T and its corresponding undirected generalized tree H.

2.3.1. *Entropy Bounds for Rooted Trees.* Starting from the definition of the information functional f^V (see Equation (6)), we first express a technical assertion proven in [22] that states a relationship between certain vertex probabilities. Starting from the definition of f^V, this assertion expresses that it is always possible to infer inequalities between the corresponding vertex probabilities. In order to achieve this, we also use simple estimations of parameters which we introduce in Lemma (2.1). Finally, we will see that by applying this lemma, we can easily derive entropy bounds for the graph classes under consideration. Hence, the following lemma serves as a fundament for the proofs of some theorems we want state in this section.

Lemma 2.1. *Let T be a rooted tree with a certain height h and let f^V be the information functional represented by Equation (6). Further, we define the quantities*

$$\omega(v_{ik}) := \max_{1\leq j\leq \rho} |S_j(v_{ik}, T)|,\ \omega := \max_{\substack{0\leq i\leq h\\ 1\leq k\leq \sigma_i}} \omega(v_{ik}),\ \phi := \max_{1\leq j\leq\rho} c_j,$$
$$\text{and } \varphi := \min_{1\leq j\leq\rho} c_j. \tag{12}$$

It holds

$$p^V(v_{ik}) < \alpha^{\rho[\phi\cdot\omega-\varphi]}\cdot p^g(v_{ik}), \quad \rho[\phi\cdot\omega-\varphi] > 0 \quad \forall\, \alpha > 1, \tag{13}$$

where

$$p^g(v_{ik}) := \frac{g(v_{ik})}{g(v_{01}) + \sum_{i=1}^{h}\sum_{k=1}^{\sigma_i} g(v_{ik})}, \tag{14}$$

and $g(v_{ik}) = \alpha^{\rho\cdot\phi\cdot\omega(v_{ik})}$. $p^V(v_{ik})$ denotes the vertex probability of v_{ik} regarding f^V. Further, v_{ik} denotes the k-th vertex on the i-th level, $1\leq i\leq h, 1\leq k\leq \sigma_i$. σ_i denotes the number of vertices on level i.

In the following, we derive entropy bounds for hierarchical networks by applying Lemma (2.1). Because Lemma (2.1) provides inequalities between vertex probabilities for each vertex in a graph, the main idea for inferring entropy bounds is to add up the obtained inequalities. As a result, we get relations between graph entropy measures for hierarchical networks which can be interpreted as entropy bounds. Also, the conclusion of Lemma (2.1) implies that by varying the Inequalities (13), special entropy bounds can be obtained.

Theorem 2.2. *Let T be a rooted tree. For the entropy of T, it holds the inequality*

$$I_{f^V}(T) > \alpha^{\rho[\phi\cdot\omega-\varphi]}\left[I_g(T) - \log\left(\alpha^{\rho[\phi\cdot\omega-\varphi]}\right)\right] \quad \forall\, \alpha > 1, \tag{15}$$

where

$$I_g(T) := -\left[g(v_{01}) + \sum_{i=1}^{h}\sum_{k=1}^{\sigma_i} g(v_{ik})\log(g(v_{ik}))\right]. \tag{16}$$

Proof: To start the proof, we consider Inequality (13) in Lemma (2.1). If we multiply this inequality by -1, we get

$$-p^V(v_{ik}) > -\alpha^{\rho[\phi\cdot\omega-\varphi]}\cdot p^g(v_{ik}). \tag{17}$$

Now, by using the assertion of Lemma (2.1) and the monotonicity property of the logarithm function, we obtain

$$\begin{aligned} -\,p^V(v_{ik})\log(p^V(v_{ik})) > -\alpha^{\rho[\phi\cdot\omega-\varphi]}\cdot p^g(v_{ik})\cdot\log(p^g(v_{ik})) \\ -\,\alpha^{\rho[\phi\omega-\varphi]}\cdot p^g(v_{ik})\cdot\log\left(\alpha^{\rho[\phi\omega-\varphi]}\right). \end{aligned} \tag{18}$$

If we perform this step for each vertex $v_{ik} \in V$ and then add up the obtained inequalities, we get

$$\begin{aligned} &-\,p^V(v_{01})\log(p^V(v_{01})) - p^V(v_{11})\log(p^V(v_{11})) - \cdots - p^V(v_{h\sigma_h})\log(p^V(v_{h\sigma_h})) \\ &\quad > \alpha^{\rho[\phi\cdot\omega-\varphi]}\big[-p^g(v_{01})\log(p^g(v_{01})) - p^g(v_{11})\log(p^g(v_{11})) - \cdots \\ &-p^g(v_{h\sigma_h})\log(p^g(v_{h\sigma_h}))\big] - \alpha^{\rho[\phi\cdot\omega-\varphi]}\log\left(\alpha^{\rho[\phi\cdot\omega-\varphi]}\right)\left[p^g(v_{01}) + \sum_{i=1}^{h}\sum_{k=1}^{\sigma_i} p^g(v_{ik})\right]. \end{aligned}$$

Because by definition it holds

$$p^g(v_{01}) + \sum_{i=1}^{h}\sum_{k=1}^{\sigma_i} p^g(v_{ik}) = 1,$$

we obviously get

$$
\begin{aligned}
&- p^V(v_{01})\log(p^V(v_{01})) - p^V(v_{11})\log(p^V(v_{11})) - \cdots - p^V(v_{h\sigma_h})\log(p^V(v_{h\sigma_h})) \\
&\quad > \alpha^{\rho[\phi\cdot\omega-\varphi]}\big[- p^g(v_{01})\log(p^g(v_{01})) - p^g(v_{11})\log(p^g(v_{11})) - \cdots \\
&\qquad - p^g(v_{h\sigma_h})\log(p^g(v_{h\sigma_h}))\big] - \alpha^{\rho[\phi\cdot\omega-\varphi]}\log\left(\alpha^{\rho[\phi\cdot\omega-\varphi]}\right). \quad (19)
\end{aligned}
$$

Now, by using the definition of the graph entropy (see Definition (2.3)), Inequality (19) finally becomes to

$$I_{f^V}(T) > \alpha^{\rho[\phi\cdot\omega-\varphi]}\left[I_g(T) - \log\left(\alpha^{\rho[\phi\cdot\omega-\varphi]}\right)\right].$$

This completes the proof of the theorem. □

By considering special classes of rooted trees, we obviously get special bounds for the corresponding entropies.

Theorem 2.3. *Let T_ϑ^o be a fully occupied ϑ-tree. For the graph entropy of T_ϑ^o holds*

$$I_{f^V}(T_\vartheta^o) > \alpha^{2h[\phi\cdot\vartheta^h-\varphi]}\left[I_g(T_\vartheta^o) - \log\left(\alpha^{2h[\phi\cdot\vartheta^h-\varphi]}\right)\right], \quad \forall\,\alpha > 1. \qquad (20)$$

Proof: Let T_ϑ^o be a fully occupied ϑ-tree. Therefore, it holds $\rho = 2h$. Starting from the root vertex v_{01}, all other vertices are reachable. Hence, we obtain $|S_h(v_{01}, T_\vartheta^o)| = \vartheta^h$. Then, we clearly get $|S_j(v_{ik}, T_\vartheta^o)| < \vartheta^h$, $1 \leq j \leq 2h$. Hence, we can set $\omega = \vartheta^h$. Now, the proof of the Theorem (2.3) can be obtained by analogously applying the same technique and steps of the proof of Theorem (2.2). □

Theorem 2.4. *Let T_ϑ be an ordinary ϑ-tree. For the graph entropy of T_ϑ holds*

$$I_{f^V}(T_\vartheta) > \alpha^{\rho[\phi\cdot\vartheta^h-\varphi]}\left[I_g(T_\vartheta) - \log\left(\alpha^{\rho[\phi\cdot\vartheta^h-\varphi]}\right)\right], \quad \forall\,\alpha > 1. \qquad (21)$$

Proof: Let T_ϑ be an ordinary ϑ-tree. Actually, it holds $\omega \leq \vartheta^h$. From this, and by applying Lemma (2.1), we yield

$$p^V(v_{ik}) < \alpha^{\rho[\phi\cdot\vartheta^h-\varphi]} \cdot p^g(v_{ik}). \qquad (22)$$

Finally, we obtain the assertion of the theorem by applying the same technique and steps performed in the proof of Theorem (2.2). □

We emphasize that each information functional captures structural information of a graph differently. Obviously, the resulting graph entropies are also different. If we now apply Theorem (2.2) and additionally assume an abstract information functional f^*, we find as a consequence of the previous theorems that one can infer a statement that expresses a relationship between the resulting graph entropies. These kind of inequalities can be used to study the influence of an information functional on the final graph entropies.

Corollary 2.5. *Let T be a rooted tree and let $f^*(v_{ik})$ be an information functional such that*

$$p^V(v_{ik}) < \alpha^{\rho[\phi^* \cdot \omega^* - \varphi]} \cdot p^*(v_{ik}), \quad \rho[\phi^* \cdot \omega^* - \varphi] > 0,\ \alpha > 1. \tag{23}$$

$p^V(v_{ik})$ and $p^*(v_{ik})$ denotes the vertex probability value (k-th vertex on the i-th level) regarding f^V and f^*. Then, it holds

$$I_{f^V}(T) > \alpha^{\rho[\phi^* \cdot \omega^* - \varphi]} \left[I_{f^*}(T) - \log\left(\alpha^{\rho[\phi^* \cdot \omega^* - \varphi]}\right)\right]. \tag{24}$$

2.3.2. *Entropy Bounds for Generalized Trees.* In this section, we give a first attempt to state entropy bounds for certain classes of generalized trees. By only allowing generalized trees with specific edge sets, we get bounds for the entropies of special classes of generalized trees. The assertion of the next theorem means the following: The entropy of a specific generalized tree can be characterized by the entropy of another generalized tree that is extremal with respect to a certain structural property.

Theorem 2.6. *Let $H = (V, E_{GT})$ be a generalized tree with $E_{GT} = E_1 \cup E_2$, i.e., H possesses Across-edges only. Starting from H, we define $H^\star$ as the generalized tree with the maximal number of Across-Edges on each level $i, 1 \leq i \leq h$.*

- *First, there exist positive real coefficients c_k which satisfy the inequality system*

$$\begin{aligned} c_1|S_1(v_{ik}, H^\star)| + c_2|S_2(v_{ik}, H^\star)| + \cdots + c_\rho|S_\rho(v_{ik}, H^\star)| \\ > c_1|S_1(v_{ik}, H)| + c_2|S_2(v_{ik}, H)| + \cdots + c_\rho|S_\rho(v_{ik}, H)|, \\ 0 \leq i \leq h,\ 1 \leq k \leq \sigma_i,\ c_j > 0,\ 1 \leq j \leq \rho. \end{aligned} \tag{25}$$

- *Second, it holds*

$$I_{f^V}(H) > \alpha^{\rho[\phi^\star \cdot \omega^\star - \varphi]} \left[I_{f^V}(H^\star) - \log\left(\alpha^{\rho[\phi^\star \cdot \omega^\star - \varphi]}\right)\right] \quad \forall\, \alpha > 1. \tag{26}$$

Proof: We assume $H = (V, E_{GT})$ such that $E_{GT} = E_1 \cup E_2$. Besides edges $e \in E_1$, H possesses Across-edges $e \in E_2$ only. Then, we first determine

$$|S_1(v_{ik}, H)|, |S_2(v_{ik}, H)|, \dots, |S_\rho(v_{ik}, H)|.$$

Now, we consider $H^\star$ and find that the total number of Across-edges for each level equals $\frac{\sigma_i(\sigma_i-1)}{2}$, $i \geq 1$. Except for the root vertex v_{01}, we further see that in particular $|S_1(v_{ik}, H^\star)| \geq |S_1(v_{ik}, H)|$ holds. This corresponds to the fact that $H^\star$ has normally more connections than H. Finally, the cardinalities of the remaining j-spheres of $H^\star$ increase correspondingly. Therefore, we conclude that we can find coefficients $c_k > 0$ such that the Inequality System (25) holds. But from this, we directly obtain

$$\begin{aligned} f^V_H(v_{ik}) &:= \alpha^{c_1|S_1(v_{ik},H)|+c_2|S_2(v_{ik},H)|+\cdots+c_\rho|S_\rho(v_{ik},H)|} \\ &< \alpha^{c_1|S_1(v_{ik},H^\star)|+c_2|S_2(v_{ik},H^\star)|+\cdots+c_\rho|S_\rho(v_{ik},H^\star)|}, \\ &=: f^V_{H^\star}(v_{ik}), \end{aligned} \tag{27}$$

if $\alpha > 1,\ 1 \leq i \leq h,\ 1 \leq k \leq \sigma_i$. $f^V_H(v_{ik})$ and $f^V_{H^\star}(v_{ik})$ denotes the information functional f^V regarding H and $H^\star$, respectively. We want to emphasize that it holds $f^V_H(v_{01}) = f^V_{H^\star}(v_{01})$. Similarly as in Lemma (2.1), by using the quantities

$$\omega^\star(v_{ik}) := \max_{1\leq j\leq\rho} S_j(v_{ik}, H^\star),\ \omega^\star := \max_{\substack{0\leq i\leq h \\ 1\leq k\leq\sigma_i}} \omega^\star(v_{ik}),\ \phi^\star := \max_{1\leq j\leq\rho} c_j,$$

$$\text{and } \varphi := \min_{1\leq j\leq\rho} c_j,$$

we yield

$$p^V_H(v_{ik}) < p^V_{H^\star}(v_{ik}) \cdot \alpha^{\rho[\phi^\star\cdot\omega^\star-\varphi]}. \tag{28}$$

Finally, Equation (26) can be obtained by applying the assertion of Theorem (2.2). □

We want to remark that by using the main argument of Theorem (2.6), one can easily express similar assertions for other specific generalized tree classes. To finalize this section, we state a simple lemma concerning the maximum entropy of a graph. Then, we apply this assertion to generalized trees.

Lemma 2.7. *Let $K_{|V|,|V|}$ be the complete graph with $|V|$ vertices. $K_{|V|,|V|}$ maximizes the graph entropy with respect to the information functional f^V,* *i.e.*,

$$I_{f^V}(K_{|V|,|V|}) = -\sum_{i=1}^{|V|} \frac{1}{|V|} \log\left(\frac{1}{|V|}\right) = -\log\left(\frac{1}{|V|}\right). \tag{29}$$

Theorem 2.8. *Let $H = (V^H, E)$ be an arbitrary generalized tree and let $H_{|V|,|V|}$ be the complete generalized tree such that $|V^H| \leq |V|$. It holds*

$$I_{f^V}(H) \leq I_{f^V}(H_{|V|,|V|}). \tag{30}$$

Proof: The proof follows directly by using the monotonicity property of the logarithm function and the assertion of Lemma (2.7).

Corollary 2.9. *Let $H^\star = (V^\star, E^\star)$ and it holds $|V^\star| \leq |V|$.*

$$I_{f^V}(H^\star) \leq I_{f^V}(H_{|V|,|V|}). \tag{31}$$

3. Discussion

3.1. Numerical Results for Hierarchical Graphs. This section aims to demonstrate that our entropic measure is able to distinguish certain graph classes of hierarchical graphs structurally by comparing the resulting cumulative entropy distributions. As a result of our numerical analysis, we will find that the calculated entropy distributions can be clearly distinguished and, hence, also the graph classes under consideration. Thus, this proves that the entropy measure captures significant structural information. To start, we give a short overview on the key steps we performed to carry out our numerical analysis:

- Generate the data classes C_α^{RT} and C_α^{GT}. For this, we randomly create rooted trees with a fixed height h. Further, we use these trees to generate generalized trees (see also below).
- Choose the parameters c_k.
- Vary α to compute I_{f^V} for different classes C_α^{RT} and C_α^{GT}.
- Compute the mean of the entropies for each such class denoted by $\bar{m}$ and the variances σ^2.
- Compute and interpret the cumulative entropy distributions for C_α^{RT} and C_α^{GT}.

We remark that the intuitive meaning of the entropy $I_{f^V}(G)$ has been already explained in [23]. Now, we start our numerical section with defining some data classes. These data classes emerge from starting with fixed sets of hierarchical graphs and by varying certain parameters.

$h = 7$										
	C_1^{RT}	C_1^{GT}	C_2^{RT}	C_2^{GT}	C_3^{RT}	C_3^{GT}	C_4^{RT}	C_4^{GT}	C_5^{RT}	C_5^{GT}
$\bar{m}$	6.029	6.218	1.766	2.846	1.123	1.987	0.895	1.629	0.775	1.423
σ^2	0.595	0.614	0.828	1.221	0.611	1.110	0.506	0.955	0.444	0.841
	C_6^{RT}	C_6^{GT}	C_7^{RT}	C_7^{GT}	C_8^{RT}	C_8^{GT}	C_9^{RT}	C_9^{GT}	C_{10}^{RT}	C_{10}^{GT}
$\bar{m}$	0.701	1.287	0.649	1.188	0.610	1.113	0.580	1.054	0.556	1.005
σ^2	0.402	0.758	0.373	0.696	0.351	0.648	0.333	0.611	0.320	0.580

$h = 8$										
	C_1^{RT}	C_1^{GT}	C_2^{RT}	C_2^{GT}	C_3^{RT}	C_3^{GT}	C_4^{RT}	C_4^{GT}	C_5^{RT}	C_5^{GT}
$\bar{m}$	6.713	6.767	2.087	3.434	1.287	2.389	1.009	1.959	0.867	1.720
σ^2	0.546	0.672	1.027	1.571	0.711	1.792	0.581	1.686	0.511	1.559
	C_6^{RT}	C_6^{GT}	C_7^{RT}	C_7^{GT}	C_8^{RT}	C_8^{GT}	C_9^{RT}	C_9^{GT}	C_{10}^{RT}	C_{10}^{GT}
$\bar{m}$	0.780	1.566	0.721	1.456	0.677	1.372	0.643	1.306	0.616	1.253
σ^2	0.467	1.449	0.436	1.355	0.412	1.277	0.394	1.211	0.379	1.153

TABLE 1. $\bar{m}$ represent the means of the entropies for each class C_α^{RT} and C_α^{GT}. σ^2 denotes the corresponding variance. It holds $|C_\alpha^{RT}| = |C_\alpha^{GT}| = 100$. α varies in natural numbers from 1 to 10, the step size is equal to 1.

Definition 3.1. *The class C_α^{RT} denotes a certain set of rooted trees whose entropies have been computed by using the value α and the coefficient vector $(c_1, c_2, \ldots, c_{\rho_m})$. We set*

$$\rho_m := \max(\rho(T_1), \rho(T_2), \ldots, \rho(T_{|C_\alpha^{RT}|})).$$

Correspondingly, C_α^{GT} denotes a certain set of generalized trees whose entropies have been computed by also using the value α and $(c_1, c_2, \ldots, c_{\rho_m})$.

In order to compute the graph entropies concretely, we choose the c_k values such that

$$c_1 > c_2 > c_3 > c_4 > c_5 > c_6,$$

holds, and set $c_1 := 6, c_2 := 5, c_3 := 4, c_4 = 3, c_5 = 2, c_6 = 1$. A class C_α^{RT} was generated by providing a fixed value h as the height of each tree $T \in C_\alpha^{RT}$. Further, each $T \in C_\alpha^{RT}$ has an unique root vertex and the remaining vertices and edges were created randomly. To generate a class C_α^{GT}, we first

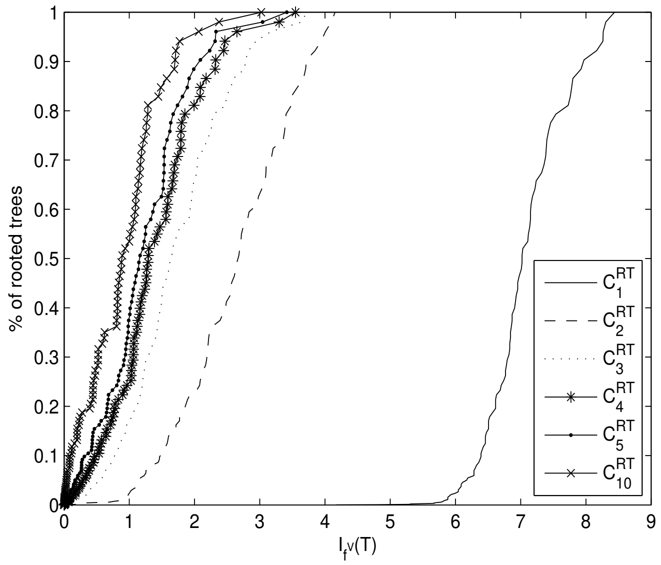

FIGURE 4. Cumulative entropy distributions of the classes C_α^{RT} for $h = 8$. The x-axis corresponds to the entropy $I_{f^V}(T)$ and the y-axis represents the cumulative entropy distribution for C_1^{RT}-C_5^{RT} and C_{10}^{RT}.

compute an arbitrary random tree with height h as mentioned and, then, a certain number of additional edges of a generalized tree randomly. The numerical results of our study are summarized in Table (1). As we have already mentioned, we computed the entropies of certain classes of rooted and generalized trees with a fixed height h by varying the α-value. We notice that by providing a fixed height h, the number of vertices of T or H can be nevertheless extremely different. Now, from Table (1) we see that the resulting entropies of generalized trees are in average larger than the entropies of rooted trees, depending on α. This corresponds to our intuition that a generalized tree can be generally considered as structurally more complex than an ordinary rooted tree. To argue in this way, we apply a definition due to [11] that states, the higher the information content (entropy) of a system is, the more complex is the system. Further, one finds that the variances of the generated tree and generalized tree classes can be clearly distinguished. This can be also explained by the fact that a set of generalized trees is in average more structurally complex and diverse than a set of rooted trees with the same height h. Also, we observe that the larger the α-value of C_α^{RT} and C_α^{GT} is, the smaller is the resulting mean and variance. Additionally, we also

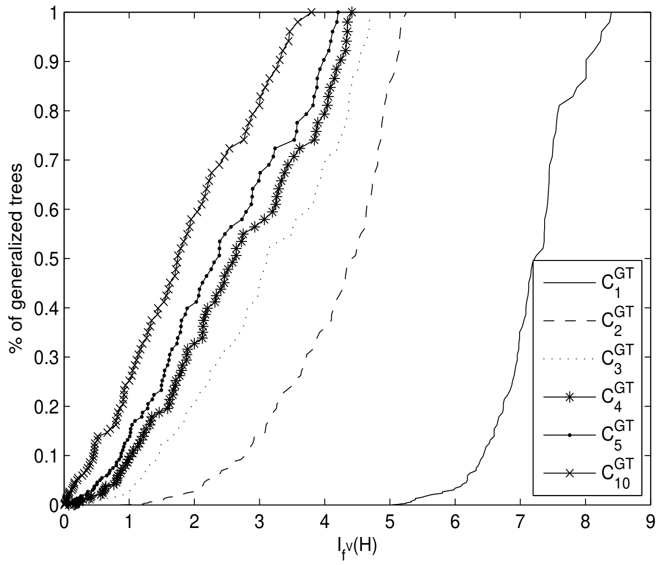

FIGURE 5. Cumulative entropy distributions of the classes C_α^{GT} for $h = 8$. The x-axis corresponds to the entropy $I_{f^V}(H)$ and the y-axis represents the cumulative entropy distribution for C_1^{GT}-C_5^{GT} and C_{10}^{GT}.

find that the entropy of a graph decreases with an increasing α-value. In the following, we interpret the cumulative entropy distributions (for $h = 8$) which are shown in Figure (4) and Figure (5). Such a distribution expresses the percentage rate of graphs (of the cardinality of C_α^{RT} or C_α^{GT}) which possess an entropy value less or equal $I_{f^V}(T)$ or $I_{f^V}(H)$. As an important observation, we find that for $\alpha \in \{2, 3, 4, 5, 10\}$ the cumulative entropy distributions of C_α^{RT} (see Figure (4)) are clearly different from the corresponding cumulative distributions of C_α^{GT} (see Figure (5)). Hence, we interpret this result such that the entropy measure (by incorporating the information functional f^V) is able to detect that we deal with different graph classes. The reason why the distribution for C_1^{RT} and C_1^{GT} seems to be almost equal is related to the fact that our entropy measure has always a maximum at $\alpha = 1$. For this case, the entropies of trees- and generalized trees are almost equal. We remark that we have already been proven that the entropy functional (by using f^V) possesses for every graph a maximum at $\alpha = 1$, see [23]. As the main result of this section, we find that our entropy measure captures important structural information meaningfully and, hence, detects that rooted and generalized trees manifest structurally different graph classes.

3.2. Summary and Conclusion. In this paper, we investigated the problem of finding entropy bounds for hierarchical graphs. Based on an entropic measure to determine the entropy of graphs, we derived certain estimations for the corresponding entropies. We now summarize the main contributions and arguments of our paper as follows:

In Section (2.3), we defined two classes of hierarchical graphs, rooted trees and generalized trees. A generalized tree is structurally more complex than an ordinary rooted tree because it contains a rooted tree as a special case. As a main result of Section (2.3), we proved entropy bounds for rooted trees as well as for generalized trees. Also, assuming specific structural properties of the graph classes under consideration led us to characteristic bounds. It is important to note that we presented only one method for finding those entropy bounds, different bounds can be derived by using different entropy measures and techniques. To classify these bounds, we call the derived bounds *implicit bounds* because the entropy of a graph was estimated by a quantity that contains another graph entropy expression. Generally, bounds to estimate the entropy of graphs are very useful for practical applications because the real entropy value is often difficult to obtain. Particularly, an interesting result represents Corollary (2.5). From this assertion, we found that an information functional (e.g., f^V or f^*) has an influence on the resulting graph entropy because each such functional quantifies structural information differently. Hence, Corollary (2.5) can be used for describing relations of the resulting entropies by using different information functional.

In Section (3.1), we performed a numerical study to demonstrate the practical ability of our graph entropy measure. Based on two generated graph classes of rooted and generalized trees, we computed the entropies of each such class by varying the free parameter α. Then, we calculated the cumulative entropy distributions for these classes. From the obtained results we could conclude that our entropy measure can distinguish between rooted trees and generalized trees. This implied that the used entropy measure captures significant structural information because we know that rooted trees and generalized trees are different graph classes.

References

[1] R. Aringhieri, P. Hansen, and F. Malucelli. A linear algorithm for the hyper-wiener index of chemical trees. *Journal of Chemical Information and Computer Science*, 41(4):958–963, 2001.

[2] R. Aringhieri, P. Hansen, and F. Malucelli. Chemical trees enumeration algorithms. *4OR, A Quarterly Journal of Operations Research*, 1(1):67–83, 2003.

[3] J. Bang-Jensen and G. Gutin. *Digraphs. Theory, Algorithms and Applications*. Springer, London, Berlin, Heidelberg, 2002.

[4] A. L. Barabási. *How Everything Is Connected to Everything Else and What It Means*. Plume, Reissue edition, 2003.

[5] V. Batagelj. Similarity measures between structured objects. In *Proceedings of an International Course and Conference on the Interfaces between Mathematics, Chemistry and Computer Sciences*, Dubrovnik, Yugoslavia, 1988.

[6] R. Benigni. *Quantitative Structure-Activity Relationship (QSAR) Models of Mutagens and Carcinogens*. CRC Press, 2003.

[7] S. Bohanec and M. Perdih. Symmetry of chemical structures: A novel method of graph automorphism group determination. *J. Chem. Inf. Comput. Sci.*, 33:719–726, 1993.

[8] D. Bonchev. *Information Theoretic Indices for Characterization of Chemical Structures*. Research Studies Press, Chichester, 1983.

[9] D. Bonchev. The concept for the center of a chemical structure and its applications. *Journal of Molecular Structure: Theochem*, 185:155–168, 1989.

[10] D. Bonchev. Overall connectivities and topological complexities: A new powerful tool for QSPR/QSAR. *J. Chem. Inf. Comput. Sci.*, 40(4):934–941, 2000.

[11] D. Bonchev. *Complexity in Chemistry. Introduction and Fundamentals*. Taylor and Francis, 2003.

[12] D. Bonchev and D. H. Rouvray. *Chemical Graph Theory. Introduction and Fundamentals*. Abacus Press, 1991.

[13] D. Bonchev and D. H. Rouvray. *Complexity in Chemistry, Biology, and Ecology*. Mathematical and Computational Chemistry. Springer, 2005.

[14] D. Bonchev and N. Trinajstić. Information theory, distance matrix and molecular branching. *Journal of Chemical Physics*, 67:4517–4533, 1977.

[15] L. Brillouin. *Science and Information Theory*. Academic Press, New York, 1956.

[16] M. Brinkmeier and T. Schank. Network statistics. In U. Brandes and T. Erlebach, editors, *Network Analysis*, Lecture Notes of Computer Science, pages 293–317. Springer, 2005.

[17] H. Bunke. What is the distance between graphs ? *Bulletin of the EATCS*, 20:35–39, 1983.

[18] H. Bunke. Recent developments in graph matching. In *15-th International Conference on Pattern Recognition*, volume 2, pages 117–124, 2000.

[19] H. Bunke and M. Neuhaus. Graph matching. exact and error-tolerant methods and the automatic learning of edit costs. In D. Cook and L. B. Holder, editors, *Mining Graph Data*, pages 17–32. Wiley-Interscience, 2007.

[20] V. Chepoi. On distances in benzenoid systems. *J. Chem. Inf. Comput. Sci.*, 36:1169–1172, 1996.

[21] S. M. Dancoff and H. Quastler. Information content and error rate of living things. In H. Quastler, editor, *Essays on the Use of Information Theory in Biology*, pages 263–274. University of Illinois Press, 1953.

[22] M. Dehmer. Information processing in complex networks: Graph entropy and information functionals. *Applied Mathematics and Computation*, 201:82–94, 2008.

[23] M. Dehmer and F. Emmert-Streib. Structural information content of networks: Graph entropy based on local vertex functionals. *Computational Biology and Chemistry*, 32:131–138, 2008.

[24] M. Dehmer, F. Emmert-Streib, A. Mehler, and J. Kilian. Measuring the structural similarity of web-based documents: A novel approach. *International Journal of Computational Intelligence*, 3(1):1–7, 2006.

[25] J. Devillers and A. T. Balaban. *Topological Indices and Related Descriptors in QSAR and QSPAR*. CRC Press, 2000.

[26] M. V. Diudea. *QSPR / QSAR Studies by Molecular Descriptors.* Nova Publishing, 2001.
[27] M. V. Diudea, I. Gutman, and L. Jäntschi. *Molecular Topology.* Nova Publishing, 2001.
[28] S. N. Dorogovtsev and J. F. F. Mendes. *Evolution of Networks. From Biological Networks to the Internet and WWW.* Oxford University Press, 2003.
[29] S. B: Elk. A canonical ordering of polybenzenes and polyadamantanes using a prime factorization technique. *J. Math. Chem.*, 4:55–68, 1990.
[30] F. Emmert-Streib, M. Dehmer, and J. Kilian. Classification of large graphs by a local tree decomposition. In H. R. Arabnia and A. Scime, editors, *Proceedings of DMIN'05, International Conference on Data Mining, Las Vegas, June 20-23*, pages 200–207, 2005.
[31] F. Emmert-Streib, M. Dehmer, J. Liu, and M. Mühlhäuser. Ranking genes from DNA microarray data of cervical cancer by a local tree comparison. *International Journal of Biomedical Science*, 1(1):17–22, 2006.
[32] J. Felsenstein. *Inferring Phylogenies.* Sinauer Associates, 2003.
[33] L. R. Foulds. *Graph Theory Applications.* Springer, 1992.
[34] J. I. Fujii and S. Yuki. Entropy and coding for graphs. *Int. J. Math. Stat. Sci.*, 6(1):63–77, 1997.
[35] A. Gambin and P. P. Slonimski. Hierarchical clustering based upon contextual alignment of proteins: A different way to approach phylogeny. *Comptes Rendus Biologies*, 328(1):1–22, 2005.
[36] I. Gutman and O. E. Polansky. *Mathematical Concepts in Organic Chemistry.* Springer, 1986.
[37] R. Halin. *Graphentheorie.* Akademie Verlag, 1989.
[38] F. Harary. *Graph Theory.* Addison Wesley Publishing Company, 1969.
[39] M. Höchstmann, T. Töller, R. Giegerich, and S. Kurtz. Local similarity in RNA secondary structures. In *Proceedings of the IEEE Computational Systems Bioinformatics Conference (CSB'03)*, pages 159–168, 2003.
[40] T. Horváth, T. Gärtner, and S. Wrobel. Cyclic pattern kernels for predictive graph mining. In *Proceedings of the 2004 ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 158–167, 2004.
[41] F. Kaden. Graphmetriken und Distanzgraphen. *ZKI-Informationen, Akad. Wiss. DDR*, 2(82):1–63, 1982.
[42] J. Kieffer and E. Yang. Ergodic behavior of graph entropy. *Electronic Research Announcements of the American Mathematical Society*, 3:11–16, 1997.
[43] J. Körner. Coding of an information source having ambiguous alphabet and the entropy of graphs. *Transactions of the 6-th Prague Conference on Information Theory*, pages 411–425, 1973.
[44] H. Linshitz. The information content of a battery cell. In H. Quastler, editor, *Essays on the Use of Information Theory in Biology.* University of Illinois Press, 1953.
[45] S. C. Liu, L. D. Tong, and Y. N. Yeh. Trees with the minimum wiener number. *International Journal of Quantum Chemistry*, 78(5):331–340, 2000.
[46] O. Mason and M. Verwoerd. Graph theory and networks in biology. *IET Systems Biology*, 1(2):89–119, 2007.
[47] D. W. Matula. A natural rooted tree enumeration by prime factorization. *SIAM Review*, 10:273, 1968.
[48] A. Mehler, M. Dehmer, and R. Gleim. Towards logical hypertext structure — A graph-theoretic perspective. In Thomas Böhme and Gerhard Heyer, editors, *Proceedings of the Fourth International Workshop on Innovative Internet Computing Systems (I2CS '04)*, Lecture Notes in Computer Science 3473, pages 136–150., Berlin/New York, 2004. Springer.

[49] H. Morihiro, A. Tatsuya, and N. Hitoshi. A novel clustering method for analysis of biological networks using maximal components of graphs. *IPSJ SIG Technical Reports*, 99:1–8, 2006.
[50] H. Morowitz. Some order-disorder considerations in living systems. *Bull. Math. Biophys.*, 17:81–86, 1953.
[51] A. Mowshowitz. Entropy and the complexity of graphs II: The information content of digraphs and infinite graphs. *Bull. Math. Biophys.*, 30:225–240, 1968.
[52] A. Mowshowitz. Entropy and the complexity of graphs III: Graphs with prescribed information content. *Bull. Math. Biophys.*, 30:387–414, 1968.
[53] A. Mowshowitz. Entropy and the complexity of graphs IV: Entropy measures and graphical structure. *Bull. Math. Biophys.*, 30:533–546, 1968.
[54] A. Mowshowitz. Entropy and the complexity of the graphs I: An index of the relative complexity of a graph. *Bull. Math. Biophys.*, 30:175–204, 1968.
[55] W. R. Müller, K. Szymanski, J. V. Knop, and N. Trinajstić. A comparison between the matula numbers and bit-tuple notation for rooted trees. *J. Chem. Inf. Comput. Sci.*, 35(2):211–213, 1995.
[56] H. Quastler. *Information Theory in Biology.* University of Illinois Press, 1953.
[57] N. Rashevsky. Life, information theory, and topology. *Bull. Math. Biophys.*, 17:229–235, 1955.
[58] M. Rupp, E. Proschak, and G. Schneider. Kernel approach to molecular similarity based on iterative graph similarity. *J. Chem. Inf. Comput. Sci.*, 47:2280–2286, 2007.
[59] P. K. Sahu and S.-L. Lee. Net-sign identity information index: A novel approach towards numerical characterization of chemical signed graph theory. *Chemical Physics Letters, in press*, 2008.
[60] C. Semple and M. Steel. A supertree method for rooted trees. *Discrete Appl. Math.*, 105(1-3), 2000.
[61] C. Semple and M. Steel. *Phylogenetics.* Graduate Series Mathematics and its Applications. Oxford University Press, 2003.
[62] C. E. Shannon and W. Weaver. *The Mathematical Theory of Communication.* University of Illinois Press, 1997.
[63] B. A. Shapiro and K. Zhang. Comparing multiple RNA secondary structures using tree comparisons. *Comp. Appl. Biosci.*, 6(4):309–318, 1990.
[64] G. Simonyi. Graph entropy: A survey. In W. Cook, L. Lovász, and P. Seymour, editors, *Combinatorial Optimization*, volume 20, pages 399–441. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 1995.
[65] V. A. Skorobogatov and A. A. Dobrynin. Metrical analysis of graphs. *MATCH*, 23:105–155, 1988.
[66] M. I. Skvortsova, I. I. Baskin, I. V. Stankevich, V. A. Palyulin, and N. S. Zefirov. Molecular similarity. 1. analytical description of the set of graph similarity measures. *J. Chem. Inf. Comput. Sci.*, 38:785–790, 1998.
[67] F. Sobik. Graphmetriken und Klassifikation strukturierter Objekte. *ZKI-Informationen, Akad. Wiss. DDR*, 2(82):63–122, 1982.
[68] F. Sobik. Modellierung von Vergleichsprozessen auf der Grundlage von Ähnlichkeitsmaßen für Graphen. *ZKI-Informationen, Akad. Wiss. DDR*, 4:104–144, 1986.
[69] M. Steel, A. Dress, and S. Boker. Simple but fundamental limitations on supertree and consensus tree methods. *Systematic Biology*, 49:363–368, 2000.
[70] N. Trinajstić. *Chemical Graph Theory.* CRC Press, 1992.
[71] E. Trucco. A note on the information content of graphs. *Bulletin of Mathematical Biology*, 18(2):129–135, 1956.

[72] D. P. Visco, R. S. Pophale, M. D. Rintoul, and J.-L. Faulon. Developing a methodology for an inverse quantitative structure-activity relationship using the signature molecular descriptor. *Journal of Molecular Graphics and Modelling*, 20:429–438, 2002.

[73] B. Zelinka. On a certain distance between isomorphism classes of graphs. *Časopis pro pěst. Mathematiky*, 100:371–373, 1975.

[74] P. Zhu and R. C. Wilson. A study of graph spectra for comparing graphs. The 16-th British Machine Vision Conference, Oxford Brookes University, 2005.

[75] B. Zmazeka and J. Žerovnik. The hosoya-wiener polynomial of weighted trees. *Croatica Chemica Acta*, 80(1):75–80, 2007.

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