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COMPLETELY NORMAL FRAMES AND REAL-VALUED FUNCTIONS

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Abstract: Up to now point-free insertion results have been obtained only for semicontinuous real functions. Notably, there is now available a setting for dealing with arbitrary, not necessarily (semi-)continuous, point-free real functions, due to Gutiérrez García, Kubiak and Picado, that gives point-free topology the freedom to deal with general real functions only available before to point-set topology. As a first example of the usefulness of that setting, we apply it to characterize completely normal frames in terms of an insertion result for general real functions. This characterization extends a well known classical result of T. Kubiak about completely normal spaces. In addition, characterizations of completely normal frames that extend results of H. Simmons for topological spaces are presented. In particular, it follows that complete normality is a lattice-invariant property of spaces, correcting an erroneous conclusion in [Y.-M. Wong, Lattice-invariant properties of topological spaces, *Proc. Amer. Math. Soc.* 26 (1970) 206-208].

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1. Introduction

If X is a topological space, the partially ordered set $\mathcal{O}X$ of open subsets of X is a complete lattice, in which the infinite distributive law

$$U \wedge \bigvee \mathcal{S} = \bigvee\{U \wedge S \mid S \in \mathcal{S}\}$$

holds for all open subsets U and collections of open subsets $\mathcal{S}$ in X. We recall that a *frame* is an abstract lattice with these properties; like inverse image along a continuous mapping, a *frame homomorphism* is taken to preserve

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arbitrary joins and finite meets. We write Frm for the category of frames and frame homomorphisms.

The above representation is contravariant (continuous maps $f : X \to Y$ are represented by frame homomorphisms $h : \mathcal{O}Y \to \mathcal{O}X$). This is easily mended, in order to keep the geometric (topological) motivation, by considering, instead of Frm simply its opposite category. It is called the category Loc of *locales* and *localic maps*, and we have "generalized continuous maps" $f : L \to M$ that are precisely frame homomorphisms $h : L \leftarrow M$.

In the whole paper we keep the algebraic (frame) approach and reasoning. The reader should keep in mind that the geometric (localic) motivation reads backwards.

Several insertion theorems for semicontinuous real functions (most notably, Katětov-Tong Theorem) have been obtained recently in the point-free setting of frames and locales [7, 3, 4, 5, 6] (see also [12, 13]) using the point-free description of semicontinuity of [7]. They were also obtained equivalently by the more general setting of [6] describing the ring $\mathsf{F}(L)$ of arbitrary real functions on a frame L.

This paper was prompted by the latter paper. The possibility provided by that paper of considering arbitrary not necessarily semicontinuous real functions opens new horizons and naturally addresses the question of the extension to the point-free setting of insertion theorems classically formulated for general real functions. The first obvious choice appears to be the complete normality separation axiom; completely normal spaces X were characterized by T. Kubiak [10] by the following insertion condition for general functions $f_1, f_2 : X \to \mathbb{R}$:

> *If $f_1{}^- \leq f_2$ and $f_1 \leq f_2{}^\circ$, then there exists a lower semicontinuous $f : X \to \mathbb{R}$ such that $f_1 \leq f \leq f^- \leq f_2$ (where $f_1{}^-$ denotes the upper regularization of f_1 and $f_2{}^\circ$ denotes the lower regularization of f_2).*

Our purpose with the present paper is to study complete normality in the setting of point-free topology, with the goal of obtaining an insertion-type characterization for completely normal frames that extends the classical one of Kubiak quoted above.

We start by recalling the notion of a completely normal frame due to Isbell ([8]). Then, by making some straightforward observations, we obtain several characterizations of completely normal frames that extend results of H. Simmons for topological spaces [16]. In particular, we conclude that complete normality is a lattice-invariant property of spaces, correcting an erroneous conclusion in [17].

Finally, with the help of generalized characteristic maps, we present the insertion theorem for completely normal frames and a few nice consequences of it.

For general background regarding frames and locales we refer to Johnstone [9] and Picado, Pultr and Tozzi [15], and for details concerning the ring $\mathfrak{R}L$ of continuous real functions to Banaschewski [1].

2. Background on sublocales ([9], [14], [15])

A sublocale S of a locale L is defined to be a regular subobject of L in Loc, that is, a localic map $j_S : S \to L$ for which the corresponding frame homomorphism $L \to S$ is onto. We have a natural order in the class of all sublocales of L:

$$j_1 \sqsubseteq j_2 \text{ if and only if there is an } f \text{ such that } j_2 f = j_1.$$

The sublocales j_1 and j_2 are equivalent if $j_1 \sqsubseteq j_2$ and $j_2 \sqsubseteq j_1$. The partially ordered set obtained is a *co-frame* (that is, a complete lattice satisfying the dual of the frame distributive law).

There are various equivalent ways in the literature of describing the sublocales of L. Here we prefer to use the following [14]:

¿From the frame distribution law it follows that any frame L is precisely a complete Heyting algebra with implication $\to$ satisfying the standard equivalence $a \wedge b \leq c$ if and only if $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element $a^* = a \to 0 = \bigvee\{b \in L : a \wedge b = 0\}$. A *sublocale set* (briefly, a sublocale) S in a frame L is a subset $S \subseteq L$ such that

(S1) for every $A \subseteq S$, $\bigwedge A$ is in S, and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

In the co-frame of sublocale sets of L the least element is $\{1\}$ and the largest one is L. The meets coincide with intersections and the joins are given by the formula

$$\bigvee_{i \in I} S_i = \Big\{ \bigwedge A \mid A \subseteq \bigcup_{i \in I} S_i \Big\}.$$

Among the important examples of sublocales are, for each $a \in L$, the *closed sublocales*

$$\mathfrak{c}(a) = {\uparrow} a = \{b \in L : a \leq b\}$$

and the *open sublocales*

$$\mathfrak{o}(a) = \{a \to b : b \in L\}.$$

Each sublocale $S \subseteq L$ is also determined by the frame surjection $c_S : L \to S$ given by $c_S(x) = \bigwedge\{s \in S \mid s \geq x\}$ for all $x \in L$. E.g. the quotients $c_{\mathfrak{c}(a)}$ and $c_{\mathfrak{o}(a)}$ are given by

$$c_{\mathfrak{c}(a)}(x) = a \vee x \quad \text{and} \quad c_{\mathfrak{o}(a)}(x) = a \to x, \text{ respectively.}$$

Further, each sublocale S of L is itself a frame with the same meets as in L, and since the Heyting operation $\to$ depends on the meet structure only, with the same Heyting operation. However the joins in S and L will not necessarily coincide:

$$\overset{S}{\bigvee_{i \in I}} a_i = \bigwedge \Big\{ s \in S \mid s \geq \bigvee_{i \in I} a_i \Big\} \geq \bigvee_{i \in I} a_i.$$

It follows that $1_S = 1$ but in general $0_S \neq 0$. In particular

$$0_{\mathfrak{c}(a)} = a, \quad x \overset{\mathfrak{c}(a)}{\vee} y = x \vee y, \quad 0_{\mathfrak{o}(a)} = a^* \quad \text{and} \quad x \overset{\mathfrak{o}(a)}{\vee} y = a \to (x \vee y).$$

We shall denote the closed and open sublocales of a sublocale S of L by $\mathfrak{c}^S(a)$ and $\mathfrak{o}^S(a)$, respectively.

Convention 2.1. For notational reasons, we shall make the co-frame of all sublocales of L into a frame $\mathfrak{S}L$ by considering the opposite ordering:

$$S_1 \leq S_2 \quad \Leftrightarrow \quad S_2 \subseteq S_1.$$

Thus, given $\{S_i \in \mathfrak{S}L : i \in I\}$, we have

$$\bigvee_{i \in I} S_i = \bigcap_{i \in I} S_i \quad \text{and} \quad \bigwedge_{i \in I} S_i = \Big\{ \bigwedge A : A \subseteq \bigcup_{i \in I} S_i \Big\}.$$

Then $\{1\}$ is the top element and L is the bottom element in $\mathfrak{S}L$ that we just denote by 1 and 0, respectively. Contrarily to the spatial case, sublocales do not necessarily have complements. But there is a natural substitute, given by the pseudocomplement S^* of $S \in \mathfrak{S}L$ described by $S^* = \bigvee\{T \in \mathfrak{S}L \mid S \wedge T = 0\}$. When S^* is a complement of S we denote it by $\neg S$ as usual.

The *interior* S° of a sublocale $S \in \mathfrak{S}L$ is the smallest open sublocale bigger than S. In particular, $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$. The *closure* S of a sublocale $S \in \mathfrak{S}L$, that is, the largest closed sublocale smaller than S, is given by the formula $\overline{S} = \uparrow(\bigwedge S)$ and satisfies:

(1) $\overline{0} = 0$, $\overline{S} \leq S$ and $\overline{\overline{S}} = \overline{S}$,
(2) $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$,
(3) $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.

We shall freely use the following properties of sublocales.

Proposition 2.2. *For every $a, b \in L$, $A \subseteq L$ and $S \in \mathfrak{S}L$, we have:*

(1) $\mathfrak{c}(a) \leq \mathfrak{c}(b)$ *if and only if* $a \leq b$,
(2) $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$,
(3) $\mathfrak{c}(\bigvee A) = \bigvee_{a \in A} \mathfrak{c}(a)$,
(4) $\mathfrak{c}(\bigwedge A) \leq \bigwedge_{a \in A} \mathfrak{c}(a)$,
(5) $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ *and* $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$,
(6) $\mathfrak{o}(a) \geq \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$,
(7) $\mathfrak{o}(a) \leq \mathfrak{c}(b)$ *if and only if* $a \vee b = 1$,
(8) $S \vee \mathfrak{c}(a) = 1$ *if and only if* $S \geq \mathfrak{o}(a)$,
(9) $S \vee \mathfrak{o}(a) = 1$ *if and only if* $S \geq \mathfrak{c}(a)$,
(10) $S \wedge \mathfrak{c}(a) = 0$ *if and only if* $S \leq \mathfrak{o}(a)$,
(11) $S \wedge \mathfrak{o}(a) = 0$ *if and only if* $S \leq \mathfrak{c}(a)$.

Note that the map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathfrak{S}L$. The subframe of $\mathfrak{S}L$ consisting of all closed sublocales will be denoted by $\mathfrak{c}L$. Clearly, L and $\mathfrak{c}L$ are isomorphic. We denote by $\mathfrak{o}L$ the subframe of $\mathfrak{S}L$ generated by all $\mathfrak{o}(a)$, $a \in L$.

We shall also need the following

Proposition 2.3. *Let $S \in \mathfrak{S}L$. Then:*

(1) *For every $a \in L$, $\mathfrak{c}(a) \vee S$ is the closed sublocale $\mathfrak{c}^S(c_S(a))$ of S.*
(2) *For every $a \in L$, $\mathfrak{o}(a) \vee S$ is the open sublocale $\mathfrak{o}^S(c_S(a))$ of S.*
(3) *If T is a closed sublocale of S then $T = \mathfrak{c}(a) \vee S$ for some $a \in S$.*
(4) *If T is an open sublocale of S then $T = \mathfrak{o}(a) \vee S$ for some $a \in S$.*

3. Complete normality

One of the classical separation axioms of topology is complete normality (also known as relative normality). A topological space X is *completely* *normal* if for every pair of subsets A and B of X which are separated (i.e. $\overline{A}\cap B=\varnothing=A\cap\overline{B}$) there are disjoint open sets containing A and B respectively. A standard exercise is to show that this is equivalent to hereditary normality.

Accordingly (recall Convention 2.1), two sublocales S and T of a frame L are *separated* [12] if $S\vee\overline{T}=1=\overline{S}\vee T$. We say that S and T are *separated by open sublocales* if there exist open sublocales U and V of L such that $U\vee V=1$, $S\geq U$ and $T\geq V$; a frame L is *completely normal* ([8]) if every pair of separated sublocales of L is separated by open sublocales.

Since the lattice of sublocales of a topological space can be much larger than the Boolean algebra of its subspaces, it is not obvious that this definition provides a conservative extension from spaces to frames of complete normality, that is, whether a space X is completely normal if and only if the corresponding frame $\mathcal{O}X$ of open sets is completely normal. We will see in the sequel that this is indeed the case.

Remark 3.1. We point out that a different concept of complete normality for frames (and distributive lattices), not directly related with the classical concept, has been introduced by B. Banaschewski in [2].

In [16] H. Simmons proved that a space X is completely normal if and only if $L=\mathcal{O}X$ satisfies the following condition:

$$\forall\, a,b\in L\ \ \exists\, x,y\in L:\ x\wedge y=0,\ x\leq b\leq a\vee x,\ y\leq a\leq b\vee y. \tag{CN}$$

Remarks 3.2. (a) For any frame L, (CN) is equivalent to

$$\forall\, a,b\in L\ \ \exists\, x\in L:\ x\leq b\leq a\vee x,\ x^*\wedge(a\vee b)\leq a\leq b\vee x^*. \tag{CN*}$$

Proof: (CN) $\Rightarrow$ (CN*): For each $a,b\in L$ let $x\in L$ given by (CN). Then:

- $x^*\wedge(a\vee b)=(x^*\wedge a)\vee(x^*\wedge b)\leq a$ since $x^*\wedge b\leq x^*\wedge(a\vee x)=x^*\wedge a$.
- $a\leq b\vee y\leq b\vee x^*$.

(CN*) $\Rightarrow$ (CN): For each $a,b\in L$ let $x\in L$ given by (CN*) and take $y=x^*\wedge(a\vee b)$. Then $x\wedge y=0$, $y\leq a$ and $b\vee y=b\vee(x^*\wedge(a\vee b))=(b\vee x^*)\wedge(b\vee a)\geq a$. ∎

(b) The conditions $x\leq b$ and $y\leq a$ in (CN) are redundant because condition

$$\forall\, a,b\in L\ \ \exists\, x,y\in L:\ x\wedge y=0,\ b\leq a\vee x,\ a\leq b\vee y \tag{CN**}$$

implies (CN). Indeed, given (CN**), the elements $\tilde{x}:=x\wedge b$ and $\tilde{y}:=y\wedge a$ satisfy immediately the conditions $\tilde{x}\wedge\tilde{y}=0$, $\tilde{x}\leq b\leq a\vee\tilde{x}$ and $\tilde{y}\leq a\leq b\vee\tilde{y}$.

The following proposition shows, in particular, that Simmons characterization above may be extended to a general frame.

Proposition 3.3. *The following are equivalent for a frame L:*

(1) *L is completely normal.*
(2) *For every $a, b \in L$ there exist $x, y \in L$ such that $x \wedge y = 0$, $b \leq a \vee x$ and $a \leq b \vee y$.*
(3) *For every $S, T \in \mathfrak{S}L$ such that $S \leq \overline{T}$ and $S^\circ \leq T$ there exist an open sublocale U and a closed sublocale F such that $S \leq F \leq U \leq T$.*
(4) *For every $S, T \in \mathfrak{S}L$ such that $S \wedge T^\circ = 0 = S^\circ \wedge T$ there exist closed sublocales F and G such that $F \wedge G = 0$, $S \leq F$ and $T \leq G$.*

Proof: (1) $\Rightarrow$ (2): Given $a, b \in L$ let $S = \mathfrak{o}(a) \vee \mathfrak{c}(b)$ and $T = \mathfrak{c}(a) \vee \mathfrak{o}(b)$. The sublocales S and T are separated: $S \vee \overline{T} \geq S \vee \mathfrak{c}(a) = 1$ and $\overline{S} \vee T \geq \mathfrak{c}(b) \vee T = 1$. Thus, by complete normality there exist $x, y \in L$ such that $\mathfrak{o}(x) \vee \mathfrak{o}(y) = 1$, $S \geq \mathfrak{o}(y)$ and $T \geq \mathfrak{o}(x)$. These are the elements x and y we are looking for. Indeed:

- $\mathfrak{o}(x) \vee \mathfrak{o}(y) = 1$ means that $x \wedge y = 0$.
- $S \geq \mathfrak{o}(y)$ means that $\mathfrak{o}(a) \vee \mathfrak{c}(b) \geq \mathfrak{o}(y)$, that is, $\mathfrak{c}(y) \vee \mathfrak{o}(a) \vee \mathfrak{c}(b) = 1$. Equivalently, $\mathfrak{o}(a) \vee \mathfrak{c}(y \vee b) = 1$, that is, $\mathfrak{c}(y \vee b) \geq \mathfrak{c}(a)$. Hence $a \leq y \vee b$.
- Similarly, $T \geq \mathfrak{o}(x)$ implies that $b \leq x \vee a$.

(2) $\Rightarrow$ (3): Let $S, T \in \mathfrak{S}L$ such that $S \leq \overline{T}$ and $S^\circ \leq T$, with $\overline{T} = \mathfrak{c}(a)$ and $S^\circ = \mathfrak{o}(b)$ for some $a, b \in L$. By hypothesis, there exist $x, y \in L$ satisfying $x \wedge y = 0$, $b \leq a \vee x$ and $a \leq b \vee y$. Then:

- $\mathfrak{c}(y) \leq \mathfrak{o}(x)$.
- $S \leq S^\circ = \mathfrak{o}(b)$. On the other hand, $S \leq \overline{T} = \mathfrak{c}(a)$ implies $S \leq \mathfrak{c}(b) \vee \mathfrak{c}(y)$ (since $a \leq b \vee y$). Hence $S \leq \mathfrak{o}(b) \wedge (\mathfrak{c}(b) \vee \mathfrak{c}(y)) = \mathfrak{o}(b) \wedge \mathfrak{c}(y) \leq \mathfrak{c}(y)$.
- $T \geq \overline{T} = \mathfrak{c}(a)$. On the other hand, $\mathfrak{o}(b) = S^\circ \leq T$ implies $\mathfrak{o}(a \vee x) \leq T$ (since $b \leq a \vee x$). Hence $T \geq \mathfrak{c}(a) \vee (\mathfrak{o}(a) \wedge \mathfrak{o}(x)) = \mathfrak{c}(a) \vee \mathfrak{o}(x) \geq \mathfrak{o}(x)$.

(3) $\Rightarrow$ (4): Let $S, T \in \mathfrak{S}L$ such that $S \wedge T^\circ = 0 = S^\circ \wedge T$. Then $S \leq T^{\circ *} = \overline{T^*}$ and $S^\circ \leq T^*$ and so there exist an open sublocale U and a closed sublocale F such that $S \leq F \leq U \leq T^*$. Consequently F and $\neg U$ are closed sublocales such that $F \wedge \neg U = 0$, $S \leq F$ and $T \leq T^{**} \leq \neg U$.
(4) $\Rightarrow$ (1): Let $S, T \in \mathfrak{S}L$ such that $S \vee \overline{T} = 1 = \overline{S} \vee T$. The sublocales $S_1 = \overline{T} \wedge \neg \overline{S}$ and $T_1 = \overline{S} \wedge \neg \overline{T}$ satisfy $S_1 \wedge {T_1}^\circ = 0$ and $T_1 \wedge {S_1}^\circ = 0$. Hence there exist closed sublocales F and G such that $F \wedge G = 0$, $S_1 \leq F$ and

$T_1 \leq G$. Thus $\neg F$ and $\neg G$ are open sublocales such that $\neg F \vee \neg G = 1$, $\neg F = (\neg F \wedge \overline{S}) \vee (\neg F \wedge \neg\overline{S}) \leq \overline{S} \vee ((\neg\overline{T} \vee \overline{S}) \wedge \neg\overline{S}) \leq \overline{S} \vee (\neg\overline{T} \wedge \neg\overline{S}) \leq S \vee \neg\overline{T} = (S \vee \neg\overline{T}) \wedge (S \vee \overline{T}) = S$. Similarly, $\neg G \leq T$. ■

Remarks 3.4. (a) In particular, Proposition 3.3 (together with Simmons characterization) shows that complete normality is a conservative extension from spaces to frames: a space X is completely normal if and only if the frame $\mathcal{O}X$ is completely normal. (Compare this with Simmons proof in the spatial case ([16], Theorem 5).)

In [17] it is asserted that complete normality is not lattice-invariantly, which contradicts the equivalence above. However a glance to the counter-example provided there (p. 208) reveals a mistake (σ to be a topology must contain also the empty set and then it is no longer lattice-isomorphic to 2^X). Hence, complete normality is, like many other separation properties, lattice-invariant.

(b) Recall that a frame L is *normal* if for every $a, b \in L$ with $a \vee b = 1$ there exist $x, y \in L$ such that $x \wedge y = 0$, $a \vee x = 1 = b \vee y$. By condition (2) in Proposition 3.3, every completely normal frame is normal. There is a result for normal frames parallel to Proposition 3.3 that makes visible the difference between normality and the stronger concept of complete normality. After the calculations done in the proof of Proposition 3.3 we feel free to avoid its proof.

Proposition 3.5. *The following are equivalent for a frame L:*

1. *L is normal.*
2. *For every closed $S, T \in \mathfrak{S}L$ such that $S \vee T = 1$ there exist open sublocales U and V such that $U \vee V = 1$, $U \leq S$ and $V \leq T$.*
3. *For every open S and closed T in $\mathfrak{S}L$ such that $S \leq T$ there exist an open sublocale V and a closed sublocale F such that $S \leq F \leq V \leq T$.*
4. *For every open $S, T \in \mathfrak{S}L$ satisfying $S \wedge T = 0$ there exist closed sublocales F and G such that $F \wedge G = 0$, $S \leq F$ and $T \leq G$.*

(c) In [13] normal frames were characterized by the condition that, for any countable subsets $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ of L, satisfying $a_i \vee (\bigwedge_{j\in\mathbb{N}} b_j) = 1$ and $b_i \vee (\bigwedge_{j\in\mathbb{N}} a_j) = 1$ for every $i \in \mathbb{N}$, there exists $u \in L$ such that $a_i \vee u = 1$ and $b_i \vee u^* = 1$ for every $i \in \mathbb{N}$. Similarly, one can show that

Proposition 3.6. *A frame L is completely normal if and only if for any countable subsets $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}} \subseteq L$ there exists $u \in L$ such that $\bigwedge_{i\in\mathbb{N}} b_i \leq a_k \vee u$ and $\bigwedge_{i\in\mathbb{N}} a_i \leq b_k \vee u^*$ for every $k \in \mathbb{N}$.*

Recall from [3] that a frame is *hereditarily normal* if every its sublocale is normal. This is the same as complete normality:

Theorem 3.7. *For each frame L the following are equivalent:*
(1) *L is completely normal.*
(2) *L is hereditarily normal.*
(3) *Each open sublocale of L is normal.*

Proof: (1) $\Rightarrow$ (2): Let S be a sublocale of L and let $a \overset{S}{\vee} b = 1$. Then $\mathfrak{c}^S(a) = S \vee \mathfrak{c}(a)$, $\mathfrak{c}^S(b) = S \vee \mathfrak{c}(b)$, and

$$\overline{\mathfrak{c}^S(a)} \vee \mathfrak{c}^S(b) = \overline{S \vee \mathfrak{c}(a)} \vee \mathfrak{c}^S(b) \geq \overline{S} \vee \overline{\mathfrak{c}(a)} \vee \mathfrak{c}^S(b) = \overline{S} \vee \mathfrak{c}(a) \vee \mathfrak{c}^S(b).$$

But $\overline{\mathfrak{c}^S(a)} \vee \mathfrak{c}^S(b) \geq \mathfrak{c}^S(b) \geq S$. Therefore

$$\overline{\mathfrak{c}^S(a)} \vee \mathfrak{c}^S(b) \geq S \vee \mathfrak{c}(a) \vee \mathfrak{c}^S(b) = \mathfrak{c}^S(a) \vee \mathfrak{c}^S(b) = \mathfrak{c}^S(a \overset{S}{\vee} b) = 1.$$

Similarly, $\mathfrak{c}^S(a) \vee \overline{\mathfrak{c}^S(b)} = 1$. Hence, by hypothesis, there exist $u, v \in L$ such that $u \wedge v = 0$, $\mathfrak{c}^S(a) \geq \mathfrak{o}(u)$ and $\mathfrak{c}^S(b) \geq \mathfrak{o}(v)$. Consider the open sublocales $\mathfrak{o}^S(c_S(u))$ and $\mathfrak{o}^S(c_S(v))$ of S. Then, clearly,

$$\mathfrak{c}^S(a) \geq \mathfrak{o}^S(c_S(u)) \quad \text{and} \quad \mathfrak{c}^S(b) \geq \mathfrak{o}^S(c_S(v)),$$

that is, $a \overset{S}{\vee} c_S(u) = 1$ and $b \overset{S}{\vee} c_S(v) = 1$. Moreover, $c_S(u) \wedge c_S(v) = c_S(u \wedge v) = c_S(0) = 0_S$. This shows that S is normal.
(2) $\Leftrightarrow$ (3): It is proved in ([3], Proposition 3.3).
(2) $\Rightarrow$ (1): If $S \vee \overline{T} = 1 = \overline{S} \vee T$ with $\overline{T} = \mathfrak{c}(t)$ and $\overline{S} = \mathfrak{c}(s)$ then $S \geq \mathfrak{o}(t)$ and $T \geq \mathfrak{o}(s)$. Let $U = \mathfrak{o}(s) \wedge \mathfrak{o}(t) = \mathfrak{o}(s \vee t)$. By hypothesis, U is normal. Further, $\overline{S} \cap U = \mathfrak{c}(s) \vee \mathfrak{o}(t)$ and $\overline{T} \cap U = \mathfrak{c}(t) \vee \mathfrak{o}(s)$. By (1) of Proposition 2.3, $\overline{S} \cap U = \mathfrak{c}^U(c_S(s))$ and $\overline{T} \cap U = \mathfrak{c}^U(c_T(t))$. These are disjoint closed sublocales of U so

$$\mathfrak{c}^U(c_S(s) \overset{U}{\vee} c_T(t)) = (\overline{S} \cap U) \cap (\overline{T} \cap U) = 1.$$

Thus $c_S(s) \overset{U}{\vee} c_T(t) = 1$. Then, by the normality of U, there exist $u, v \in U$ satisfying $u \wedge v = 0_U$, $c_S(s) \overset{U}{\vee} u = 1 = c_T(t) \overset{U}{\vee} v$. In particular,

$$u \wedge v = 0_U \Leftrightarrow u \wedge v = (t \vee s) {\to} 0 = (t {\to} 0) \wedge (s {\to} 0) \Leftrightarrow u \wedge v \wedge (t \vee s) = 0. \quad (*)$$

On the other hand, by (7) of Proposition 2.2 and (4) of Proposition 2.3,

$$c_S(s) \overset{U}{\vee} u = 1 \Leftrightarrow \mathfrak{c}^U(c_S(s)) \geq \mathfrak{o}^U(u) = U \vee \mathfrak{o}(u) = \mathfrak{o}(u \wedge (t \vee s))$$

and

$$c_T(t) \overset{U}{\vee} v = 1 \Leftrightarrow \mathfrak{c}^U(c_T(t)) \geq \mathfrak{o}^U(v) = U \vee \mathfrak{o}(v) = \mathfrak{o}(v \wedge (t \vee s)).$$

Let $a = u \wedge (t \vee s)$ and $b = v \wedge (t \vee s)$. By $(*)$, $a \wedge b = 0$ thus $\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1$. Finally,

$$S \geq \overline{S} \vee \mathfrak{o}(t) = \overline{S} \vee U = \mathfrak{c}^U(c_S(s)) \geq \mathfrak{o}(a)$$

and, similarly,

$$T \geq \overline{T} \vee \mathfrak{o}(s) = \overline{T} \vee U = \mathfrak{c}^U(c_T(t)) \geq \mathfrak{o}(b).$$

■

4. Background on real-valued functions ([6])

We denote by $\mathfrak{L}(\mathbb{R})$ the frame of reals and by $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$, respectively, the lower and upper frames of reals (see [1, 4] for the details). There are also the *extended* variants of these frames: $\mathfrak{L}(\overline{\mathbb{R}})$, $\mathfrak{L}_l(\overline{\mathbb{R}})$ and $\mathfrak{L}_u(\overline{\mathbb{R}})$.

Let

$$\mathsf{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{S}L), \quad \overline{\mathsf{F}}(L) = \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathfrak{S}L).$$

An $f \in \mathsf{F}(L)$ is called an *arbitrary real function* on L. Further f is:

(1) *lower semicontinuous* if $f(r, \text{—})$ is a closed sublocale for all r.
(2) *upper semicontinuous* if $f(\text{—}, r)$ is a closed sublocale for all r.
(3) *continuous* if $f(p, q)$ is a closed sublocale for all p, q, i.e. $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$.

We denote by $\mathsf{LSC}(L), \mathsf{USC}(L)$ and $\mathsf{C}(L)$ the collections of all lower semicontinuous, upper semicontinuous, and continuous members of $F(L)$. If we replace $f \in \mathsf{F}(L)$ by $f \in \overline{\mathsf{F}}(L)$ in, respectively, (1), (2), and (3) above, we get the collections $\overline{\mathsf{LSC}}(L), \overline{\mathsf{USC}}(L),$ and $\overline{\mathsf{C}}(L)$ of all *extended* lower semicontinuous, upper semicontinuous, and continuous members of $\overline{\mathsf{F}}(L)$. Evidently, one has

$$\mathsf{C}(L) = \mathsf{LSC}(L) \cap \mathsf{USC}(L) \quad \text{and} \quad \overline{\mathsf{C}}(L) = \overline{\mathsf{LSC}}(L) \cap \overline{\mathsf{USC}}(L).$$

Remark 4.1. All the above collections of morphisms are partially ordered by

$$f \leq g \;\Leftrightarrow\; f(r, \text{—}) \leq g(r, \text{—}) \text{ for all } r \in \mathbb{Q} \;\Leftrightarrow\; g(\text{—}, r) \leq f(\text{—}, r) \text{ for all } r \in \mathbb{Q}.$$

Let $f \in \mathsf{F}(L)$. It follows that $f \geq \mathbf{0}$ if $f(\text{—}, 0) = 0$. Similarly, $f \leq \mathbf{1}$ means that $f(1, \text{—}) = 0$.

The set $\mathsf{LSC_b}(L) = \{f \in \mathsf{LSC}(L) : \mathbf{0} \le f \le \mathbf{1}\}$ has arbitrary joins and finite meets. Indeed, given $\mathcal{F} \subseteq \mathsf{LSC_b}(L)$ one has

$$(\bigvee \mathcal{F})(r,\text{—}) = \bigvee_{f\in\mathcal{F}} f(r,\text{—}) \text{ and } (\bigvee \mathcal{F})(\text{—},s) = \bigvee_{r<s} \neg\,(\bigvee \mathcal{F})(r,\text{—}) = \bigvee_{r<s} \bigwedge_{f\in\mathcal{F}} \neg\, f(r,\text{—})$$

and, for $\mathcal{F}$ finite,

$$(\bigwedge \mathcal{F})(r,\text{—}) = \bigwedge_{f\in\mathcal{F}} f(r,\text{—}) \text{ and } (\bigwedge \mathcal{F})(\text{—},s) = \bigvee_{f\in\mathcal{F}} f(\text{—},s)$$

for every $r,s \in \mathbb{Q}$.

On the other hand, $\mathsf{USC_b}(L) = \{f \in \mathsf{USC}(L) : \mathbf{0} \le f \le \mathbf{1}\}$ is closed under arbitrary meets and finite joins. Given $\mathcal{G} \subseteq \mathsf{USC_b}(L)$ one has

$$(\bigwedge \mathcal{G})(r,\text{—}) = \bigvee_{r<s} \bigwedge_{g\in\mathcal{G}} \neg\, g(\text{—},s) \text{ and } (\bigwedge \mathcal{G})(\text{—},s) = \bigvee_{g\in\mathcal{G}} g(\text{—},s)$$

and, for $\mathcal{G}$ finite,

$$(\bigvee \mathcal{G})(r,\text{—}) = \bigvee_{g\in\mathcal{G}} g(r,\text{—}) \text{ and } (\bigvee \mathcal{G})(\text{—},s) = \bigwedge_{g\in\mathcal{G}} g(\text{—},s)$$

for every $r,s \in \mathbb{Q}$.

The *lower regularization* f° of $f \in \overline{\mathsf{F}}(L)$ is defined by

$$f^\circ(r,\text{—}) = \bigvee_{s>r} \overline{f(s,\text{—})} \quad \text{and} \quad f^\circ(\text{—},s) = \bigvee_{r<s} \neg\, \overline{f(r,\text{—})}.$$

and, dually, the *upper regularization* f^- of f is defined by

$$f^-(r,\text{—}) = \bigvee_{s>r} \neg\, \overline{f(\text{—},s)} \quad \text{and} \quad f^-(\text{—},s) = \bigvee_{r<s} \overline{f(\text{—},r)}.$$

The following properties ([4], [6]) of the operators $(\cdot)^\circ : \overline{\mathsf{F}}(L) \to \overline{\mathsf{LSC}}(L)$ and $(\cdot)^- : \overline{\mathsf{F}}(L) \to \overline{\mathsf{USC}}(L)$ will be useful in the sequel:

Proposition 4.2. *For every $f,g \in \overline{\mathsf{F}}(L)$ we have:*

1. $f^\circ \le f$.
2. $(f \wedge g)^\circ = f^\circ \wedge g^\circ$.
3. $\overline{\mathsf{LSC}}(L) = \{f \in \overline{\mathsf{F}}(L) : f = f^\circ\}$.
4. $f^\circ = \bigvee\{g \in \overline{\mathsf{LSC}}(L) : g \le f\}$.
5. *If $f \in \mathsf{F}(L)$ and $\bigvee_{r\in\mathbb{Q}} \overline{f(r,\text{—})} = 1$, then $f^\circ \in \mathsf{LSC}(L)$.*
6. $f \le f^-$.
7. $(f \vee g)^- = f^- \vee g^-$.
8. $\overline{\mathsf{USC}}(L) = \{f \in \overline{\mathsf{F}}(L) : f = f^-\}$.

(9) $f^- = \bigwedge\{g \in \overline{\mathsf{USC}(L) : g \geq f}\}$.
(10) *If* $f \in \mathsf{F}(L)$ *and* $\bigvee_{r\in\mathbb{Q}} f(—, r) = 1$, *then* $f^- \in \mathsf{USC}(L)$.

5. Characteristic maps

Given a complemented sublocale S of L and $0 \leq s < r \leq 1$, the *generalized characteristic map* [6]

$$\chi_S^{r,s} = (\chi_S \wedge \mathbf{r}) \vee \mathbf{s} : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L$$

is defined by

$$\chi_S^{r,s}(p, —) = \begin{cases} 1 & \text{if } p < s, \\ \neg S & \text{if } s \leq p < r, \\ 0 & \text{if } p \geq r, \end{cases} \quad \text{and} \quad \chi_S^{r,s}(—, q) = \begin{cases} 0 & \text{if } q \leq s, \\ S & \text{if } s < q \leq r, \\ 1 & \text{if } q > r, \end{cases}$$

for each $p, q \in \mathbb{Q}$. (Note that in [6] we only considered the case $\chi_S^{1,0} = \chi_S$.) Then, as in the classical context, we have:

(1) $\chi_S^{r,s} \in \mathsf{LSC_b}(L)$ if and only if S is open,
(2) $\chi_S^{r,s} \in \mathsf{USC_b}(L)$ if and only if S is closed,
(3) $\chi_S^{r,s} \in \mathsf{C_b}(L) = \mathsf{LSC_b}(L) \cap \mathsf{USC_b}(L)$ if and only if S is clopen.
(4) $\left(\chi_S^{r,s}\right)^- = \chi_{\overline{S}}^{r,s}$ and $\left(\chi_S^{r,s}\right)^\circ = \chi_{S^\circ}^{r,s}$.

Lemma 5.1. *Let* $\mathbf{0} \leq f \leq \mathbf{1}$ *be such that for each* $r \in \mathbb{Q} \cap [0, 1]$ *there exists* $x_r \in L$ *satisfying* $f(r, —) \leq \mathfrak{c}(x_r)$. *Then*

$$f \leq \bigvee_{r\in\mathbb{Q}\cap[0,1]} \chi_{\mathfrak{o}(x_r)}^{r,0} \in \mathsf{LSC_b}(L).$$

Proof: First note that $\bigvee_{r\in\mathbb{Q}\cap[0,1]} \chi_{\mathfrak{o}(x_r)}^{r,0} \in \mathsf{LSC_b}(L)$ by Remark 4.1. It suffices to observe that

$$\bigvee_r \chi_{\mathfrak{o}(x_r)}^{r,0}(p, —) = \bigvee_{p<r} \mathfrak{c}(x_r) \geq \bigvee_{p<r} f(r, —) = f(p, —)$$

for each $0 \leq p < 1$. ■

Similarly we have

Lemma 5.2. *Let* $\mathbf{0} \leq g \leq \mathbf{1}$ *be such that for each* $r \in \mathbb{Q} \cap [0, 1]$ *there exists* $y_r \in L$ *satisfying* $g(—, r) \leq \mathfrak{c}(y_r)$. *Then*

$$\mathsf{USC_b}(L) \ni \bigwedge_{r\in\mathbb{Q}\cap[0,1]} \chi_{\mathfrak{c}(y_r)}^{1,r} \leq g.$$

■

6. The insertion theorem

The Normalization Lemma of Kubiak ([10], Lemma 2.1) cannot be translated immediately to the point-free setting since joins of upper semicontinuous functions (and meets of lower semicontinuous ones) do not necessarily exist. Nevertheless we can get the following which suffices for the insertion result.

Lemma 6.1. *Let L be a frame and $\mathbf{0} \leq h_1 \leq h_2 \leq \mathbf{1}$ in $\mathsf{F}(L)$. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathsf{LSC_b}(L)$ and $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathsf{USC_b}(L)$ be such that $h_1 \leq \bigvee_n f_n$, $\bigwedge_n g_n \leq h_2$, ${f_n}^- \leq h_2$ and $h_1 \leq {g_n}^\circ$ for every $n \in \mathbb{N}$. Then there exists an $f \in \mathsf{LSC_b}(L)$ such that $h_1 \leq f \leq f^- \leq h_2$.*

Proof: Define $\tilde{f}_1 = f_1$ and $\tilde{f}_n = f_n \wedge \bigwedge_{i<n} {g_i}^\circ$ for each $n \geq 2$. Now take $f = \bigvee_{n\in\mathbb{N}} \tilde{f}_n$. It follows from Remark 4.1 that $f \in \mathsf{LSC_b}(L)$. It remains to show that $h_1 \leq f \leq f^- \leq h_2$.

We have for each $p \in \mathbb{Q}$ (and by Remark 4.1)

$$\begin{aligned} f(p,\text{—}) &= f_1(p,\text{—}) \vee \bigvee_{n\geq 2} (f_n(p,\text{—}) \wedge \bigwedge_{i<n} {g_i}^\circ(p,\text{—})) \\ &\geq (f_1(p,\text{—}) \wedge h_1(p,\text{—})) \vee \bigvee_{n\geq 2} (f_n(p,\text{—}) \wedge h_1(p,\text{—})) \\ &= \big(\bigvee_{n\in\mathbb{N}} f_n(p,\text{—})\big) \wedge h_1(p,\text{—}) = h_1(p,\text{—}). \end{aligned}$$

Hence $f \geq h_1$.

On the other hand, since $\tilde{f}_m \leq f_m \leq {f_m}^-$, then $\tilde{f}_m \leq \bigvee_{i\leq n} {f_i}^-$ for every $m \leq n$, and $\tilde{f}_m \leq g_n^\circ \leq g_n$ if $m > n$. Hence $\tilde{f}_m \leq g_n \vee \bigvee_{i\leq n} f_i^-$ for all $m, n \in \mathbb{N}$. Since $\mathbf{0} \leq g_n \vee \bigvee_{i\leq n} {f_i}^- \leq \mathbf{1}$ for each $n \in \mathbb{N}$, by Remark 4.1 it follows that $\bigwedge_{n\in\mathbb{N}}(g_n \vee \bigvee_{i\leq n} {f_i}^-) \in \mathsf{USC_b}(L)$ and therefore $f^- \leq \bigwedge_{n\in\mathbb{N}}(g_n \vee \bigvee_{i\leq n} f_i^-)$. Finally, we have for each $q \in \mathbb{Q}$ (and by Remark 4.1)

$$\begin{aligned} f^-(\text{—},q) &\geq \bigvee_{n\in\mathbb{N}} \Big(g_n(\text{—},q) \wedge \bigwedge_{i\leq n} {f_i}^-(\text{—},q)\Big) \\ &\geq \bigvee_{n\in\mathbb{N}} (g_n(\text{—},q) \wedge h_2(\text{—},q)) \\ &= \Big(\bigvee_{n\in\mathbb{N}} g_n(\text{—},q)\Big) \wedge h_2(\text{—},q) \\ &= \Big(\bigwedge_{n\in\mathbb{N}} g_n\Big)(\text{—},q) \wedge h_2(\text{—},q) = h_2(\text{—},q). \end{aligned}$$

Hence $f^- \le h_2$. Consequently, $\mathbf{0} \le h_1 \le f \le f^- \le h_2 \le \mathbf{1}$. ■

Proposition 6.2. *Let L be a frame. For $\mathbf{0} \le h_1 \le h_2 \le \mathbf{1}$ in $\mathsf{F}(L)$, the following are equivalent:*

1. *There exists an $f \in \mathsf{LSC_b}(L)$ such that $h_1 \le f \le f^- \le h_2$.*
2. *For every r in $\mathbb{Q}\cap[0,1]$, there exist $x_r, y_r \in L$ such that $x_r \wedge y_r = 0$, $h_1(r,\text{—}) \le \mathfrak{c}(x_r)$ and $h_2(\text{—},r) \le \mathfrak{c}(y_r)$.*

Proof: (1) $\Rightarrow$ (2): For each $r \in \mathbb{Q}$ take $x_r, y_r \in L$ such that $f(r,\text{—}) = \mathfrak{c}(x_r)$ and $f^-(\text{—},r) = \mathfrak{c}(y_r)$. Since $f \le f^-$, it follows that $\mathfrak{c}(x_r) \wedge \mathfrak{c}(y_y) = f(r,\text{—}) \wedge f^-(\text{—},r) = 0$. On the other hand, $h_1 \le f$ implies that $h_1(r,\text{—}) \le \mathfrak{c}(x_r)$ and $f^- \le h_2$ implies that $\mathfrak{c}(y_r) \ge h_2(\text{—},r)$.
(2) $\Rightarrow$ (1): By Lemmas 5.1 and 5.2 we have that

$$h_1 \le \bigvee_{r\in\mathbb{Q}\cap[0,1]} \chi^{r,0}_{\mathfrak{o}(x_r)} \quad \text{and} \quad \bigwedge_{r\in\mathbb{Q}\cap[0,1]} \chi^{1,r}_{\mathfrak{c}(y_r)} \le h_2.$$

Further

$$(\chi^{r,0}_{\mathfrak{o}(x_r)})^- = \chi^{r,0}_{\mathfrak{c}(x_r^*)} \le h_2 \quad \text{and} \quad (\chi^{1,r}_{\mathfrak{c}(y_r)})^\circ = \chi^{1,r}_{\mathfrak{o}(y_r^*)} \le h_1$$

since $h_2(\text{—},r) \le \mathfrak{c}(y_r) \le \mathfrak{c}(x_r^*)$ and $h_1(r,\text{—}) \le \mathfrak{c}(x_r) \le \mathfrak{c}(y_r^*)$ for each $r \in \mathbb{Q}\cap[0,1]$. Then Lemma 6.1 implies the existence of f. ■

Remark 6.3. The result above can be extended to any (not necessarily bounded) $h_1 \le h_2$ by the following general procedure:

Take any continuous increasing bijection $\varphi : (0,1) \to \mathbb{R}$ that maps rationals into rationals. Given $h \in \mathsf{F}(L)$, define $g_h \in \mathsf{F}(L)$ by $g_h(\text{—},q) = 0$ if $q \le 0$, $g_h(\text{—},q) = h(\text{—},\varphi(q))$ if $0 < q < 1$, $g_h(\text{—},q) = 1$ if $q \ge 1$ and $g_h(p,\text{—}) = 0$ for $p \ge 1$, $g_h(p,\text{—}) = h(\varphi(p),\text{—})$ for $0 < p < 1$ and $g_h(p,\text{—}) = 1$ in case $p \le 0$. Let $h_1 \le h_2$ in $\mathsf{F}(L)$. Clearly $\mathbf{0} \le g_{h_1} \le g_{h_2} \le \mathbf{1}$ and by Proposition 6.2 there exists $f \in \mathsf{LSC}(L)$ such that $g_{h_1} \le f \le f^- \le g_{h_2}$. Then $f_\varphi : \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}L$ given by $f_\varphi(\text{—},q) = f(\text{—},\varphi^{-1}(q))$ and $f_\varphi(p,\text{—}) = f(\varphi^{-1}(p),\text{—})$ is also in $\mathsf{LSC}(L)$ and is easily seen to satisfy $h_1 \le f_\varphi \le f_\varphi^- \le h_2$.

Theorem 6.4. *For each frame L the following are equivalent:*

1. *L is completely normal.*
2. *For each $h_1, h_2 \in \mathsf{F}(L)$, if ${h_1}^- \le h_2$ and $h_1 \le {h_2}^\circ$, then there exists an $f \in \mathsf{LSC}(L)$ such that $h_1 \le f \le f^- \le h_2$.*

Proof: (1) $\Rightarrow$ (2): Let $h_1, h_2 \in \mathsf{F}(L)$ such that ${h_1}^- \leq h_2$ and $h_1 \leq {h_2}^\circ$. Then $h_1(r,\text{—}) \wedge h_2(\text{—},r)^\circ = 0 = h_1(r,\text{—})^\circ \wedge h_2(\text{—},r)$ for any r in $\mathbb{Q}$. Indeed:

Clearly enough $h_1(r,\text{—}) \leq {h_2}^\circ(r,\text{—})$ and ${h_2}^\circ(r,\text{—}) \wedge h_2(\text{—},r) = 0$ for any $r \in \mathbb{Q}$. Hence $h_2(\text{—},r) \leq \neg {h_2}^\circ(r,\text{—})$ and so $h_2(\text{—},r)^\circ \leq \neg {h_2}^\circ(r,\text{—})$ (since $\neg {h_2}^\circ(r,\text{—})$ is an open sublocale). It follows that $h_1(r,\text{—}) \wedge h_2(\text{—},r)^\circ \leq {h_2}^\circ(r,\text{—}) \wedge \neg {h_2}^\circ(r,\text{—}) = 0$. Similarly $h_1(r,\text{—})^\circ \wedge h_2(\text{—},r) = 0$.

Hence, by Proposition 3.3, there exist $x_r, y_r \in L$ such that $\mathfrak{c}(x_r) \wedge \mathfrak{c}(y_r) = 0$, $h_1(r,\text{—}) \leq \mathfrak{c}(x_r)$ and $h_2(\text{—},r) \leq \mathfrak{c}(y_r)$. Then, by Proposition 6.2 and Remark 6.3, there exists an $f \in \mathsf{LSC}(L)$ such that $h_1 \leq f \leq f^- \leq h_2$.

(2) $\Rightarrow$ (1): For each $a, b \in L$ let $S = \mathfrak{c}(a) \wedge \mathfrak{o}(b)$ and $T = \mathfrak{c}(a) \vee \mathfrak{o}(b)$. Since both S and T are complemented, we have $\chi_S, \chi_T \in \mathsf{F}(L)$. Also $S^\circ \leq \mathfrak{o}(b) \leq T$ and $S \leq \mathfrak{c}(a) \leq \overline{T}$, hence $\chi_T \leq \chi_{S^\circ}$ and $\chi_{\overline{T}} \leq \chi_S$. By hypothesis it follows that there exists an $f \in \mathsf{LSC}(L)$ such that $\chi_T \leq f \leq f^- \leq \chi_S$. Take $\mathfrak{c}(x) = f(\frac{1}{2},\text{—})$ and $\mathfrak{c}(y) = f^-(\text{—},\frac{1}{2})$. Then:

- $f \leq f^-$ implies that $\mathfrak{c}(x \wedge y) = f(\frac{1}{2},\text{—}) \wedge f^-(\text{—},\frac{1}{2}) = 0$ and so $x \wedge y = 0$.
- $\chi_T \leq f$ implies that

$$\mathfrak{c}(x) = f(\tfrac{1}{2},\text{—}) \geq \chi_T(\tfrac{1}{2},\text{—}) = \neg T = \mathfrak{o}(a) \wedge \mathfrak{c}(b)$$

  and so $\mathfrak{o}(x) \wedge \mathfrak{o}(a) \wedge \mathfrak{c}(b) = \mathfrak{o}(x \vee a) \wedge \mathfrak{c}(b) = 0$. Hence $b \leq x \vee a$.
- Similarly $f^- \leq \chi_S$ implies that $a \leq y \vee b$.

Hence, by condition (2) of Proposition 3.3, L is completely normal. ■

Remark 6.5. In a similar way, it may be proved (we omit the details), more generally, that

> *A frame L is normal if and only if for every $h_1 = \bigvee_n h_n^1$ with $h_n^1 \in \mathsf{USC}(L)$ and $h_2 = \bigwedge_n h_n^2$ with $h_n^2 \in \mathsf{LSC}(L)$ such that ${h_1}^- \leq h_2$ and $h_1 \leq {h_2}^\circ$, there exists an $f \in \mathsf{LSC}(L)$ satisfying $h_1 \leq f \leq f^- \leq h_2$.*

In particular, when $h_1 = \chi_A$ for any F_σ-sublocale A (i.e. $h_1 = \chi_{\bigvee_n \mathfrak{c}(a_n)} = \bigvee_n \chi_{\mathfrak{c}(a_n)}$) and $h_2 = \chi_B$ for any G_δ-sublocale B (i.e. $h_2 = \chi_{\bigwedge_n \mathfrak{o}(b_n)} = \bigwedge_n \chi_{\mathfrak{o}(b_n)}$) we may conclude that for any normal frame L, if $\chi_A^- \leq \chi_B$ and $\chi_A \leq \chi_B^\circ$, then there is a lower semicontinuous f on L such that $\chi_A \leq f \leq f^- \leq \chi_B$. In other words, this means that in any normal frame L every two separated F_σ-sublocales of L are separated by open sublocales (and evidently the converse is also true). This is the point-free counterpart of the characterization of normal spaces due to Urysohn that each two separated F_σ-sets have disjoint open neighbourhoods.

Theorem 6.4 shows that there exists a lower semicontinuous function f such that $h_1 \le f \le f^- \le h_2$ if and only if L is completely normal. When and only when can one insert a continuous function f between such h_1 and h_2? As for spaces (see [11, Theorem 2]) this can be answered immediately. For that recall that a frame L is *extremally disconnected* if $a^* \vee a^{**} = 1$ for every $a \in L$ and that in any extremally disconnected frame L,

$$f \in \mathsf{LSC}(L) \text{ implies } f^- \in \mathsf{C}(L) \text{ and } f \in \mathsf{USC}(L) \text{ implies } f^\circ \in \mathsf{C}(L) \text{ [4]}. \quad (**)$$

Further, the point-free Stone-type insertion theorem from [4] asserts that extremally disconnected frames are precisely the ones where one can insert a continuous function in between $h_1 \in \mathsf{LSC}(L)$ and $h_2 \in \mathsf{USC}(L)$ satisfying $h_1 \le h_2$.

Corollary 6.6. *For each frame L the following are equivalent:*

(1) *L is completely normal and extremally disconnected.*
(2) *If $h_1, h_2 \in \mathsf{F}(L)$, ${h_1}^- \le h_2$ and $h_1 \le {h_2}^\circ$, then there exists an $f \in \mathsf{C}(L)$ such that $h_1 \le f \le h_2$.*
(3) *L is normal and if $h_1, h_2 \in \mathsf{F}(L)$ are such that ${h_1}^- \le h_2$ and $h_1 \le {h_2}^\circ$, then ${h_1}^- \le {h_2}^\circ$.*

Proof: (1) $\Rightarrow$ (2): Obvious by Theorem 6.4 and property $(**)$.
(2) $\Rightarrow$ (3): That L is normal follows from Katětov-Tong theorem (see e.g. [6]). Further, $h_1 \le f \le h_2$ with $f \in \mathsf{C}(L)$ implies that ${h_1}^- \le f^- = f = f^\circ \le {h_2}^\circ$.
(3) $\Rightarrow$ (1): That L is completely normal follows from normality and Theorem 6.4. Let $h_1 \in \mathsf{LSC}(L)$ and $h_2 \in \mathsf{USC}(L)$ with $h_1 \le h_2$. Then, by properties 4.2 (4) and (9), ${h_1}^- \le h_2$ and $h_1 \le {h_2}^\circ$ and therefore there exists $f \in \mathsf{C}(L)$ such that $h_1 \le f \le h_2$. Hence, by the Stone insertion theorem (see e.g. [6]), L is extremally disconnected. ∎

Corollary 6.7. *For each frame L the following are equivalent:*

(1) *L is completely normal and extremally disconnected.*
(2) *Every sublocale of L is C^*-embedded.* ∎

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