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$p(x)$-HARMONIC FUNCTIONS WITH UNBOUNDED EXPONENT IN A SUBDOMAIN

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To the memory of Odded Schramm

Abstract: We study the Dirichlet problem $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0$ in Ω, with $u = f$ on $\partial\Omega$ and $p(x) = \infty$ in D, a subdomain of the reference domain Ω. The main issue is to give a proper sense to what a solution is. To this end, we consider the limit as $n \to \infty$ of the solutions u_n to the corresponding problem when $p_n(x) = p(x) \wedge n$, in particular, with $p = n$ in D. Under suitable assumptions on the data, we find that such a limit exists and that it can be characterized as the unique solution of a variational minimization problem. Moreover, we examine this limit in the viscosity sense and find an equation it satisfies.

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1. Introduction

The goal of this paper is to study the elliptic problem

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \subset \mathbb{R}^N, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u(x) := \operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right)$ is the $p(x)$-Laplacian operator and the variable exponent $p(x)$ verifies

$$p(x) = +\infty, \qquad x \in D, \tag{1.2}$$

for some subdomain $D \subset \Omega$. We assume that Ω and D are convex domains with smooth boundaries, at least of class C^1. On the complementary domain $\Omega \setminus \overline{D}$ we assume that $p(x)$ is a continuously differentiable bounded function.

On the variable exponent, apart from (1.2), we also require that

$$p_- := \inf_{x\in\Omega} p(x) > N, \tag{1.3}$$

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so that we will always be dealing with continuous solutions for (1.1); to fix notation, we define

$$p_+ := \sup_{x\in\Omega\setminus\overline{D}} p(x).$$

The boundary data f is taken to be Lipschitz continuous; this hypothesis simplifies the analysis but it could be relaxed.

Our strategy to solve (1.1) is to replace $p(x)$ by a sequence of bounded functions $p_n(x)$ such that $p_n(x)$ is increasing and converging to $p(x)$. For definiteness, we consider, for $n > N$,

$$p_n(x) := \min\{p(x), n\}.$$

We will use the notation $(1.1)_n$ to refer to problem (1.1) for the variable exponents $p_n(x)$.

Since $p(x)$ is bounded in $\Omega \setminus D$, we have, for large n, specifically for $n > p_+$,

$$p_n(x) = \begin{cases} p(x), & x \in \Omega \setminus D, \\ n, & x \in D. \end{cases}$$

Moreover, still for large n, the boundary of the set $\{p(x) > n\}$ coincides with the boundary of D and thus does not depend on n. This fact is important when passing to the limit.

Using a variational method, we solve $(1.1)_n$ obtaining solutions u_n; if the limit

$$\lim_{n\to\infty} u_n \tag{1.4}$$

exists, we call it u_∞. It is a natural candidate to be a solution to (1.1) with the original variable exponent $p(x)$. A crucial role in this process will be played by the set

$$S = \Big\{u \in W^{1,p^-}(\Omega) \quad : \quad u|_{\Omega\setminus\overline{D}} \in W^{1,p(x)}(\Omega \setminus \overline{D}), \\ \|\nabla u\|_{L^\infty(D)} \leq 1 \quad \text{and} \quad u|_{\partial\Omega} = f\Big\}$$

and by the infinity Laplacian

$$\Delta_\infty u := \big(D^2 u\, \nabla u\big) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i x_j}.$$

Our main results are condensed in the following theorem.

Theorem. *There exists a unique solution* u_n *to* $(1.1)_n$. *Moreover, if* $S \neq \emptyset$ *then the uniform limit*

$$u_\infty := \lim_{n\to\infty} u_n$$

exists and is characterized as the unique solution of the variational problem

$$\min_{u\in S} \int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\, dx.$$

In addition, u_∞ *is a viscosity solution of*

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_\infty u(x) = 0, & x \in D, \\ \operatorname{sgn}(|\nabla u|(x) - 1) \operatorname{sgn}\left(\frac{\partial u}{\partial \nu}(x)\right) = 0, & x \in \partial D \cap \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases}$$

where ν *is the exterior unit normal vector to* ∂D *in* Ω.

Finally, if $S = \emptyset$ *then*

$$\lim_{n\to\infty} \left(\int_D \frac{|\nabla u_n|^n}{n}\, dx \right)^{1/n} = \lambda > 1,$$

where λ *is the best Lipschitz constant in* D *of any possible extension to* D *of* $f|_{\partial\Omega\cap\overline{D}}$.

We remark that the value of λ that appears in the last statement of the theorem is related to the problem of finding the best absolutely minimizing Lispchitz extension (the so-called AMLE) of $f|_{\partial\Omega\cap\overline{D}}$ to D. This problem has been extensively studied in the literature, see [2], [11], the survey [3], and the recent approach using tug-of-war games, [6], [13], [14]. In the rest of the paper we will use the notation "v is the AMLE of f in D" to denote this best absolutely minimizing Lipschitz extension problem (v is the solution) and refer to [3] for further details.

Partial differential equations involving variable exponents became popular a few years ago in relation to applications to elasticity and electrorheological fluids. Meanwhile, the underlying functional analytical tools have been extensively developed and new applications, *e.g.* to image processing, have kept the subject as the focus of an intensive research activity. For general references on the $p(x)$-Laplacian we refer to [8], that includes a thorough bibliography, and [12], a seminal paper where many of the basic properties of variable exponent spaces were established.

In the literature, the variable exponent $p(x)$ is always assumed to be bounded, a necessary condition to define a proper norm in the corresponding Lebesgue spaces. To the best of our knowledge, this paper is the first attempt at analyzing a problem where the exponent $p(\cdot)$ becomes infinity in some part of the domain. For constant exponents, limits as $p \to \infty$ in $p-$Laplacian type problems have been widely studied, see for example [5], and are related to optimal transport problems (*cf.* [1]).

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we show existence and uniqueness of solutions with $p(x) = p_n(x) = p \wedge n$ using a variational argument; moreover we find the equation that they verify in the viscosity sense and prove some useful estimates independent of n; in Section 3 we pass to the limit in the variational formulation of the problem and we deal with the limit in the viscosity sense. Finally, in Section 4 we present a detailed analysis of the one-dimensional case.

2. Weak and viscosity approximate solutions

To start with, let us establish the existence and uniqueness of the approximations u_n in the weak sense.

Lemma 2.1. *There exists a unique weak solution u_n to $(1.1)_n$, which is the unique minimizer of the functional*

$$F_n(u) = \int_\Omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\,dx = \int_D \frac{|\nabla u|^n}{n}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\,dx \tag{2.1}$$

in

$$S_n = \left\{ u \in W^{1,p_n(\cdot)}(\Omega) \;:\; u|_{\partial\Omega} = f \right\}. \tag{2.2}$$

Proof: Although the exponent $p_n(\cdot)$ might be discontinuous, functions in the variable exponent Sobolev space $W^{1,p_n(\cdot)}(\Omega)$ are continuous thanks to assumption (1.3). Indeed, for n sufficiently large, we have $p_n(\cdot) \geq (p_n)_- \geq p_- > N$ and the continuous embedding in

$$W^{1,p_n(\cdot)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \subset C\left(\overline{\Omega}\right) \tag{2.3}$$

follows from [12, Theorem 2.8 and (3.2)]. That the boundedness away from the dimension is not superfluous when the exponent is not continuous is shown by a counter-example in [9, Example 3.3].

We can then take the boundary condition $u|_{\partial\Omega} = f$ in the classical sense (recall that f is assumed to be Lipschitz) and the results of [10] apply since the jump condition (cf. [10, (4.1)-(4.2)]) is trivially satisfied by the variable exponent because $p_n(\cdot) \geq N$. This is a sufficient condition for a $p_n(\cdot)$-Poincaré inequality to hold in $W_0^{1,p_n(\cdot)}(\Omega)$ which, in turn, is instrumental in obtaining the coercivity of the functional. The lower semicontinuity is standard as is the strict convexity, that also gives the uniqueness.

It is also standard that the minimizer of F_n in S_n is the unique weak solution of $(1.1)_n$, *i.e.*, $u_n = f$ on $\partial\Omega$ and it satisfies the weak form of the equation, namely,

$$\int_\Omega |\nabla u_n|^{p_n(x)-2}\nabla u_n \cdot \nabla\varphi \, dx = 0, \qquad \forall\, \varphi \in C_0^\infty(\Omega). \tag{2.4}$$

■

Lemma 2.2. *Problem* $(1.1)_n$ *can be rewritten as*

$$\begin{cases} -\Delta_{p(x)} u_n(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_n u_n(x) = 0, & x \in D, \\ |\nabla u_n(x)|^{n-2}\dfrac{\partial u_n}{\partial \nu}(x) = |\nabla u_n(x)|^{p(x)-2}\dfrac{\partial u_n}{\partial \nu}(x), & x \in \partial D \cap \Omega, \\ u_n(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where ν *is the exterior unit normal to* ∂D *in* Ω.

Proof: Just notice that the weak form of this problem is exactly the same as the one that holds for $(1.1)_n$. This follows since after multiplying by a test function and integrating by parts one arrives at (2.4) for both problems. ■

Next, we investigate the problem satisfied by u_n from the point of view of viscosity solutions.

Let us recall the definition of viscosity solution (see [7] and [4]) for a problem like (2.5), which involves a transmission condition across the boundary $\partial D \cap \Omega$. Assume we are given a family of continuous functions

$$F_i : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N\times N} \to \mathbb{R}.$$

The associated equations

$$F_i(x, \nabla u, D^2 u) = 0$$

are called (degenerate) elliptic if

$$F_i(x, \xi, X) \leq F_i(x, \xi, Y) \quad \text{whenever } X \geq Y.$$

Definition 2.3. *Consider the problem*

$$\begin{aligned} F_1(x, \nabla u, D^2 u) = 0, \qquad & in\ \ \Omega \setminus \overline{D}, \\ F_2(x, \nabla u, D^2 u) = 0, \qquad & in\ \ D, \end{aligned} \tag{2.6}$$

with a transmission condition

$$B(x, u, \nabla u) = 0, \qquad on\ \ \partial D \cap \Omega, \tag{2.7}$$

and a boundary condition

$$u = f, \qquad on\ \ \partial\Omega. \tag{2.8}$$

A lower semi-continuous function u is a viscosity supersolution of (2.6)–(2.8) *if $u \geq f$ on $\partial\Omega$ and for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \Omega$, with $u(x_0) = \phi(x_0)$, we have*

$$\begin{aligned} F_1(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \ &\geq\ 0 \ \ if\ \ x_0 \in \Omega \setminus \overline{D}, \\ F_2(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \ &\geq\ 0 \ \ if\ \ x_0 \in D, \\ \max \left\{ \begin{array}{l} F_1\left(x_0, \nabla\phi(x_0), D^2\phi(x_0)\right) \\ F_2(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \\ B(x_0, \phi(x_0), \nabla\phi(x_0)) \end{array} \right\} \ &\geq\ 0 \ \ if\ \ x_0 \in \partial D \cap \Omega. \end{aligned}$$

An upper semi-continuous function u is a viscosity subsolution of (2.6)–(2.8) *if $u \leq f$ on $\partial\Omega$ and for every $\psi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \Omega$, with $u(x_0) = \psi(x_0)$, we have*

$$\begin{aligned} F_1(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \ &\leq\ 0 \ \ if\ \ x_0 \in \Omega \setminus \overline{D}, \\ F_2(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \ &\leq\ 0 \ \ if\ \ x_0 \in D, \\ \min \left\{ \begin{array}{l} F_1\left(x_0, \nabla\psi(x_0), D^2\psi(x_0)\right) \\ F_2(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \\ B(x_0, \psi(x_0), \nabla\psi(x_0)) \end{array} \right\} \ &\leq\ 0 \ \ if\ \ x_0 \in \partial D \cap \Omega. \end{aligned}$$

Finally, u is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

In the sequel, we will use the notation as in the definition: ϕ will always stand for a test function touching the graph of u from below and ψ for a test function touching the graph of u from above.

Proposition 2.4. *Let u_n be a continuous weak solution of* $(1.1)_n$. *Then u_n is a viscosity solution of* (2.5) *in the sense of Definition 2.3.*

Proof: To simplify, we omit in the proof the subscript n. Let $x_0 \in \Omega \setminus \overline{D}$ and a let ϕ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$\begin{aligned} -\Delta_{p(x_0)}\phi(x_0) &= -|\nabla\phi(x_0)|^{p(x_0)-2}\Delta\phi(x_0) \\ &\quad -(p(x_0)-2)|\nabla\phi(x_0)|^{p(x_0)-4}\Delta_\infty\phi(x_0) \\ &\quad -|\nabla\phi(x_0)|^{p(x_0)-2}\ln(|\nabla\phi|)(x_0)\,\langle\nabla\phi(x_0),\nabla p(x_0)\rangle \\ &\geq 0. \end{aligned}$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega \setminus \overline{D}$ and

$$\begin{aligned} -\Delta_{p(x)}\phi(x) &= -|\nabla\phi(x)|^{p(x)-2}\Delta\phi(x) - (p(x)-2)|\nabla\phi(x)|^{p(x)-4}\Delta_\infty\phi(x) \\ &\quad -|\nabla\phi(x)|^{p(x)-2}\ln(|\nabla\phi|)(x)\langle\nabla\phi(x),\nabla p(x)\rangle \\ &< 0, \end{aligned}$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u-\phi)(x)$ and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$ and

$$-\Delta_{p(x)}\Phi = -\mathrm{div}(|\nabla\Phi|^{p(x)-2}\nabla\Phi) < 0 \qquad \text{in } B(x_0, r). \tag{2.9}$$

Multiplying (2.9) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx < 0.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of $(1.1)_n$, we obtain

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u)\,dx = 0,$$

since $p_n(x) = p(x)$ in $\Omega \setminus \overline{D}$. Upon subtraction and using a well know inequality, we conclude

$$\begin{aligned} 0 &> \int_{B(x_0,r)\cap\{\Phi>u\}} \Big(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(\Phi-u)\,dx \\ &\geq c\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi-\nabla u|^{p(x)}\,dx, \end{aligned}$$

a contradiction. Here c is a constant that depends on N, p^- and $\sup_{x\in B(x_0,r)} p(x)$.

If $x_0 \in D$ the proof is entirely analogous, albeit simpler due to the absence of the logarithmic term, and we obtain

$$-\Delta_n\phi(x_0) = -|\nabla\phi(x_0)|^{n-2}\Delta\phi(x_0) - (n-2)|\nabla\phi(x_0)|^{n-4}\Delta_\infty\phi(x_0) \geq 0.$$

The constant c in this case depends on N and n.

If $x_0 \in \partial D \cap \Omega$ we want to prove that

$$\max\left\{\begin{array}{l} -\Delta_{p(x_0)}\phi(x_0) \\ -\Delta_n\phi(x_0) \\ |\nabla\phi(x_0)|^{n-2}\frac{\partial\phi}{\partial\nu}(x_0) - |\nabla\phi(x_0)|^{p(x_0)-2}\frac{\partial\phi}{\partial\nu}(x_0) \end{array}\right\} \geq 0.$$

If this is not the case, there exists a radius $r > 0$ such that

$$-\Delta_{p(x)}\phi(x) < 0 \qquad \text{and} \qquad -\Delta_n\phi(x) < 0,$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u-\phi)(x)$ and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$,

$$-\Delta_{p(x)}\Phi < 0 \qquad \text{in } \ B(x_0,r) \cap (\Omega \setminus \overline{D}) \tag{2.10}$$

and

$$-\Delta_n\Phi < 0 \qquad \text{in } \ B(x_0,r) \cap D. \tag{2.11}$$

Moreover, we can assume (taking r smaller if necessary) that

$$|\nabla\Phi(x)|^{n-2}\frac{\partial\Phi}{\partial\nu}(x) - |\nabla\Phi(x)|^{p(x)-2}\frac{\partial\Phi}{\partial\nu}(x) < 0 \qquad \text{in } \ B(x_0,r) \cap \partial D. \tag{2.12}$$

Multiplying both (2.10) and (2.11) by $(\Phi - u)^+$, integrating by parts and adding, we obtain

$$\int_{B(x_0,r)\cap\Omega\setminus\overline{D}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)^+\,dx + \int_{B(x_0,r)\cap D} |\nabla\Phi|^{n-2}\nabla\Phi\cdot\nabla(\Phi-u)^+\,dx$$
$$< \int_{B(x_0,r)\cap\partial D} \left(|\nabla\Phi|^{n-2}\frac{\partial\Phi}{\partial\nu} - |\nabla\Phi|^{p(x)-2}\frac{\partial\Phi}{\partial\nu}\right)(\Phi-u)^+\,dS,$$

taking also into account that the test function vanishes on the boundary of $B(x_0, r)$. Using (2.12), we finally get

$$\int_{B(x_0,r)\cap(\Omega\setminus\overline{D})\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx$$

$$+\int_{B(x_0,r)\cap D\cap\{\Phi>u\}} |\nabla\Phi|^{n-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx<0.$$

On the other hand, taking $(\Phi-u)^+$, extended by zero outside $B(x_0,r)$, as test function in the weak formulation of $(1.1)_n$, we reach a contradiction as in the previous cases. This proves that u is a viscosity supersolution.

The proof that u is a viscosity subsolution runs as above and we omit the details. ■

We next obtain uniform estimates (independent of n) for the sequence of approximations $(u_n)_n$.

Proposition 2.5. *Assume the set*

$$S=\Big\{u\in W^{1,p^-}(\Omega)\quad:\quad u|_{\Omega\setminus\overline{D}}\in W^{1,p(x)}(\Omega\setminus\overline{D}),\\ \|\nabla u\|_{L^\infty(D)}\le 1\quad and\quad u|_{\partial\Omega}=f\Big\}$$

is nonempty. Then u_n, the minimizer of F_n in S_n, satisfies

$$F_n(u_n)=\int_\Omega\frac{|\nabla u_n|^{p_n(x)}}{p_n(x)}\,dx\le\int_D\frac{|\nabla v|^n}{n}\,dx+\int_{\Omega\setminus\overline{D}}\frac{|\nabla v|^{p(x)}}{p(x)}\,dx,$$

for every $v\in S$. Hence, the sequence $(F_n(u_n))_n$ is uniformly bounded and the sequence $(u_n)_n$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and equicontinuous.

Proof: Recalling (2.2), the definition of S_n, observe that $S\subset S_n$, for every n. Since u_n is a minimizer, we have

$$F_n(u_n)\le F_n(v),\quad\forall\ v\in S.$$

Hence, picking an element $v\in S\neq\emptyset$,

$$\begin{aligned}F_n(u_n)=\int_\Omega\frac{|\nabla u_n|^{p_n(x)}}{p_n(x)}\,dx &\le \int_\Omega\frac{|\nabla v|^{p_n(x)}}{p_n(x)}\,dx\\ &= \int_D\frac{|\nabla v|^n}{n}\,dx+\int_{\Omega\setminus\overline{D}}\frac{|\nabla v|^{p(x)}}{p(x)}\,dx\\ &\le |D|+\int_{\Omega\setminus\overline{D}}\frac{|\nabla v|^{p(x)}}{p(x)}\,dx\equiv C_*.\end{aligned}$$

In order to estimate the Sobolev norm, we first use Poincaré inequality and the boundary data, to obtain

$$\begin{aligned}\|u_n\|_{W^{1,p_-}(\Omega)} &\leq \|u_n - f\|_{W_0^{1,p_-}(\Omega)} + \|f\|_{W^{1,p_-}(\Omega)}\\ &\leq C\,\|\nabla(u_n - f)\|_{L^{p_-}(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)}\\ &\leq C\,\|\nabla u_n\|_{L^{p_-}(\Omega)} + (C+1)\|f\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

We proceed, using Hölder inequality and elementary computations, to get

$$\begin{aligned}\|\nabla u_n\|_{L^{p_-}(\Omega)} &= \left(\int_\Omega |\nabla u_n|^{p_-}\,dx\right)^{1/p_-}\\ &\leq \left(\int_D |\nabla u_n|^{p_-}\,dx\right)^{1/p_-} + \left(\int_{\Omega\setminus\overline{D}} |\nabla u_n|^{p_-}\,dx\right)^{1/p_-}\\ &\leq |D|^{\frac{1}{p_-}-\frac{1}{n}}\left(\int_D |\nabla u_n|^n\right)^{1/n} + |\Omega| + \left(\int_{\Omega\setminus\overline{D}} |\nabla u_n|^{p(x)}\,dx\right)^{1/p_-}.\end{aligned}$$

Since we have the bounds

$$\left(\int_D |\nabla u_n|^n\right)^{1/n} = n^{1/n}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{1/n} \leq n^{1/n}\left(F_n(u_n)\right)^{1/n} \leq 2C_*$$

and

$$\int_{\Omega\setminus\overline{D}} |\nabla u_n|^{p(x)}\,dx \leq p_+ \int_{\Omega\setminus\overline{D}} \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx \leq p_+ F_n(u_n) \leq p_+ C_*,$$

we conclude that the sequence $(u_n)_n$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and, recalling the embedding in (2.3), that it is equicontinuous. ■

3. Variational and viscosity limit

We first analyze the case $S = \emptyset$.

Theorem 3.1. *Assume $S = \emptyset$, i.e., that the Lipschitz constant in D of the AMLE of $f|_{\partial\Omega\cap\overline{D}}$ to D, call it λ, is greater than one. Then*

$$(F_n(u_n))^{1/n} \to \lambda.$$

Hence, $F_n(u_n) \to \infty$ and the natural energy associated to u_n is unbounded.

Remark 3.2. *Note that if $\partial\Omega \cap \overline{D} = \emptyset$ then $\lambda = 0$, since any constant can play the role of a best Lipschitz extension. However, when $\partial\Omega \cap \overline{D} \neq \emptyset$ the condition is meaningful and it can happen that $\lambda > 1$. As a trivial example* *consider a function f with $f|_{\partial\Omega\cap\overline{D}}$ having a Lipschitz constant greater than one.*

Proof: Let v be the AMLE of f in D. Since u_n is a minimizer, we have

$$\limsup_{n\to\infty}(F_n(u_n))^{1/n} \le \limsup_{n\to\infty}\left(\int_D \frac{|\nabla v|^n}{n}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx\right)^{1/n} = \lambda > 1.$$

Now suppose that

$$\liminf_{n\to\infty}(F_n(u_n))^{1/n} = \beta < \lambda$$

and consequently that

$$\liminf_{n\to\infty}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{1/n} \le \beta.$$

Fix $m \ge p_-$ and take $n > m$. By Hölder's inequality,

$$\begin{aligned}\left(\int_D |\nabla u_n|^m\right)^{1/m} &\le |D|^{\frac{1}{m}-\frac{1}{n}}\left(\int_D |\nabla u_n|^n\right)^{1/n} \\ &\le |D|^{\frac{1}{m}-\frac{1}{n}} n^{1/n}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{1/n}.\end{aligned}$$

Taking the limit in n, we conclude

$$\left(\int_D |\nabla u_n|^m\right)^{1/m} \le |D|^{\frac{1}{m}}\beta,$$

so, for a subsequence, there exists a weak limit in $W^{1,m}(D)$, that we denote by u_∞. This weak limit has to verify the inequality

$$\left(\int_D |\nabla u_\infty|^m\right)^{1/m} \le |D|^{\frac{1}{m}}\beta$$

for every m. Thus, taking the limit $m \to \infty$, we get that $u_\infty \in W^{1,\infty}(D)$ and, moreover,

$$|\nabla u_\infty| \le \beta, \qquad \text{a.e. } x \in D.$$

But this is a contradiction since λ is the best possible Lipschitz constant of any extension of f to D. We conclude that

$$\liminf_{n\to\infty}(F_n(u_n))^{1/n} = \lambda,$$

and the result follows. ■

We now focus on the main case $S \neq \emptyset$. Recall that solutions to $(1.1)_n$ are minima of the functional

$$F_n(u) = \int_\Omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\, dx$$

in

$$S_n = \left\{ u \in W^{1,p_n(x)} \ : \ u|_{\partial\Omega} = f \right\}.$$

The limit of these variational problems is given by minimizing

$$F(u) = \int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\, dx \tag{3.1}$$

in

$$S = \Big\{ u \in W^{1,p^-}(\Omega) \quad : \quad u|_{\Omega\setminus\overline{D}} \in W^{1,p(x)}(\Omega \setminus \overline{D}), \\ \|\nabla u\|_{L^\infty(D)} \le 1 \quad \text{and} \quad u|_{\partial\Omega} = f \Big\}.$$

Theorem 3.3. *Assume that $S \neq \emptyset$ and let u_n be minimizers of F_n in S_n. Then, along subsequences, $(u_n)_n$ converges uniformly in $\overline{\Omega}$, weakly in $W^{1,m}(D)$, for every $m \ge p_-$, and weakly in $W^{1,p(x)}(\Omega \setminus \overline{D})$ to u_∞, a minimizer of F in S. Moreover, the limit u_∞ is ∞-harmonic in D, i.e.,*

$$-\Delta_\infty u_\infty = 0 \qquad in \ \ D,$$

in the viscosity sense.

Proof: We use the estimates obtained in the previous section.

Since the sequence $(u_n)_n$ is equicontinuous and uniformly bounded, by Arzelà-Ascoli theorem it converges (along subsequences) uniformly in $\overline{\Omega}$; the weak convergence in the space $W^{1,m}(D)$, for every $m \ge p_-$, was obtained in the proof of Proposition 3.1 and the weak convergence in $W^{1,p(x)}(\Omega \setminus \overline{D})$ follows from the estimates in Proposition 2.5.

Also as before, we get that $u_\infty \in W^{1,\infty}(D)$, with $|\nabla u_\infty| \le 1$, a.e. $x \in D$, thus concluding that $u_\infty \in S$. On the other hand, also from Proposition 2.5, we get

$$\int_{\Omega\setminus\overline{D}} \frac{|\nabla u_n|^{p(x)}}{p(x)}\, dx \le F_n(u_n) \le F_n(v) \longrightarrow \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\, dx$$

and we conclude that

$$F(u_\infty) = \int_{\Omega\setminus\overline{D}} \frac{|\nabla u_\infty|^{p(x)}}{p(x)}\, dx \le \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\, dx = F(v), \quad \forall\, v \in S$$

so that u_∞ is a minimizer for F in S.

That a uniform limit of n-harmonic functions is ∞-harmonic is a well known fact (*cf.*, for example, [5] or [11]). ■

Remark 3.4. *We stress that the minimization of the functional F in S gives a variational meaning to being a solution of* (1.1).

The next result gives the uniqueness of the limit u_∞ and therefore we may conclude that the whole sequence u_n converges uniformly in $\overline{\Omega}$.

Proposition 3.5. *There exists a unique minimizer of F in S that verifies*

$$-\Delta_\infty u_\infty = 0 \qquad \text{in } D. \tag{3.2}$$

Proof: Suppose we have two minimizers in S, u_1 and u_2. Then, considering

$$v = \frac{u_1 + u_2}{2} \in S,$$

we obtain that they coincide in $\Omega \setminus \overline{D}$. Using the uniqueness of solutions of the Dirichlet problem for the ∞-Laplacian in D (note that u_1 coincides with u_2 on the whole of ∂D), we conclude that $u_1 = u_2$ also in D. ■

Our next task is to pass to the limit in (2.5), the problem satisfied by u_n in the viscosity sense, to identify the equation solved by u_∞. We are under the assumption $S \neq \emptyset$ and we recall that

$$u_n \to u_\infty$$

uniformly in $\overline{\Omega}$.

Theorem 3.6. *Every uniform limit of a sequence $\{u_n\}$ of solutions of* $(1.1)_n$ *is a viscosity solution of*

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_\infty u(x) = 0, & x \in D, \\ \operatorname{sgn}\left(|\nabla u|(x) - 1\right)\, \operatorname{sgn}\left(\dfrac{\partial u}{\partial \nu}(x)\right) = 0, & x \in \partial D \cap \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{3.3}$$

Proof: Since $u_n(x) = f(x)$, for $x \in \partial\Omega$, it is clear that $u(x) = f(x)$, for $x \in \partial\Omega$.

Let u_∞ be a uniform limit of $\{u_n\}$ and let ϕ be a test function such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$. Depending on the location of the point x_0 we have different situations.

If $x_0 \in D$, we encounter the standard fact the the uniform limit of n-harmonic functions is ∞-harmonic.

If $x_0 \in \Omega \setminus \overline{D}$, consider a sequence of points x_n such that $x_n \to x_0$ and $u_n - \phi$ has a minimum at x_n, with $x_n \in \Omega \setminus \overline{D}$ for n large. Using the fact that u_n is a viscosity solution of (2.5), we obtain

$$-\Delta_{p_n(x_n)}\phi(x_n) \geq 0.$$

Now we observe that $p_n(x) = p(x)$ is a neighborhood of x_0 and hence, taking the limit as $n \to \infty$, we get

$$-\Delta_{p(x_0)}\phi(x_0) \geq 0.$$

That is, u_∞ is a viscosity supersolution of $-\Delta_{p(x)}u_\infty = 0$ in $\Omega \setminus \overline{D}$.

If $x_0 \in \partial D \cap \Omega$, we have to show that

$$\max \left\{ \begin{array}{l} -\Delta_{p(x_0)}\phi(x_0) \\ -\Delta_\infty \phi(x_0) \\ \operatorname{sgn}(|\nabla\phi|(x_0) - 1)\operatorname{sgn}\left(\frac{\partial\phi}{\partial\nu}(x_0)\right) \end{array} \right\} \geq 0.$$

Again, since u_n converges to u uniformly, there exists a sequence of points x_n converging to x_0 such that $u_n - \phi$ has a minimum at x_n. We distinguish several cases.

Case 1. There exists infinitely many n such that $x_n \in D$.

Then we have, by Lemma 2.4,

$$-\Delta_n\phi(x_n) = -|\nabla\phi(x_n)|^{n-2}\Delta\phi(x_n) - (n-2)|\nabla\phi(x_n)|^{n-4}\Delta_\infty\phi(x_n) \geq 0.$$

If $\nabla\phi(x_0) = 0$, we get $-\Delta_\infty\phi(x_0) = 0$. If this is not the case, we have that $\nabla\phi(x_n) \neq 0$, for large n, and then

$$-\Delta_\infty\phi(x_n) \geq \frac{1}{n-2}|\nabla\phi(x_n)|^2\Delta\phi(x_n) \to 0, \quad \text{as } n \to \infty.$$

We conclude that

$$-\Delta_\infty\phi(x_0) \geq 0.$$

Case 2. There exists infinitely many n such that $x_n \in \Omega \setminus \overline{D}$.

Then we have, by Lemma 2.4,

$$-\Delta_{p_n(x_n)}\phi(x_n) \geq 0.$$

Proceeding as before, we get

$$-\Delta_{p(x_0)}\phi(x_0) \geq 0.$$

Case 3. There exists infinitely many n such that $x_n \in \partial D \cap \Omega$.
In this case, we have

$$|\nabla\phi(x_n)|^{n-2}\frac{\partial\phi}{\partial\nu}(x_n) - |\nabla\phi(x_n)|^{p(x_n)-2}\frac{\partial\phi}{\partial\nu}(x_n) \geq 0.$$

Hence, we get

$$\frac{\partial\phi}{\partial\nu}(x_n) \leq |\nabla\phi(x_n)|^{n-p(x_n)}\frac{\partial\phi}{\partial\nu}(x_n).$$

Taking $n \to \infty$, we deduce that

$$|\nabla\phi|(x_0) > 1 \Rightarrow \frac{\partial\phi}{\partial\nu}(x_0) \geq 0,$$

and

$$|\nabla\phi|(x_0) < 1 \Rightarrow \frac{\partial\phi}{\partial\nu}(x_0) \leq 0.$$

That is

$$\operatorname{sgn}(|\nabla\phi|(x_0) - 1)\operatorname{sgn}\left(\frac{\partial\phi}{\partial\nu}(x_0)\right) \geq 0.$$

This concludes the proof that u_∞ is a viscosity supersolution.

The proof that u is a viscosity subsolution runs as above and we omit the details. ∎

4. The one-dimensional case

In this section, we analyze with some detail the one-dimensional case, which is easier since the equation reduces to an ODE.

Let $\Omega = (0,1)$ and assume $p(x) \equiv \infty$ for $x \in (0,\xi)$. Then the problem at level n reads

$$\begin{cases} (|u_n'|^{p_n(x)-2}u_n')'(x) = 0, \\ u_n(0) = f(0), \\ u_n(1) = f(1). \end{cases}$$

To simplify, we assume that $f(0) = 0$ and $f(1) > 0$. Then, integrating the equation, we get

$$|u_n'|^{p_n(x)-2}u_n'(x) = C_1.$$

Assuming that $u_n' \geq 0$, we get

$$u_n'(x) = (C_1)^{\frac{1}{p_n(x)-1}}.$$

Thus

$$u_n(x) = \int_0^x (C_1)^{\frac{1}{p_n(s)-1}}\, ds$$

and the constant C_1 (that must be positive and depends on n) verifies

$$f(1) = \int_0^1 (C_1)^{\frac{1}{p_n(s)-1}}\, ds.$$

Since $f(1)$ is finite, we conclude that C_1 must be bounded; if not,

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) = +\infty$$

in the whole interval $(\xi, 1]$ and this contradicts $u_n(1) = f(1)$. Therefore, we can assume (taking a subsequence if necessary) that

$$\lim_{n\to\infty} C_1(n) = C_\infty.$$

Case 1. When $C_\infty > 0$, we conclude that the limit of u_n is given by

$$u_\infty(x) = \lim_{n\to\infty} u_n(x) = \begin{cases} x & x \in [0,\xi], \\ \xi + \displaystyle\int_\xi^x (C_\infty)^{\frac{1}{p(s)-1}}\, ds & x \in [\xi, 1]. \end{cases}$$

As $u_n(1) = f(1)$, we realize that the constant C_∞ is determined by

$$\xi + \int_\xi^1 (C_\infty)^{\frac{1}{p(s)-1}}\, ds = f(1).$$

This case, $C_\infty > 0$, actually happens when $f(1) > \xi$. Since C_∞ is uniquely determined, we obtain the convergence of the whole sequence u_n.

Note that in this case we can verify that u_∞ is a minimizer of the functional F given by (3.1). Indeed, since $|u_\infty'|(x) \leq 1$, for $x \in [0,\xi]$, we have that $u_\infty \in S$ and since u_∞ is a solution of

$$(|u'|^{p(x)-2}u')'(x) = 0, \qquad u(\xi) = \xi, \quad u(1) = f(1),$$

we have that it minimizes the functional F, which in this case is given by

$$F(u_\infty) = \int_\xi^1 \frac{(C_\infty)^{\frac{p(s)}{p(s)-1}}}{p(s)}\, ds,$$

among functions that verify $u(\xi) = \xi$ and $u(1) = f(1)$.

Now, for any function $w \in S$, we have $|w'|(x) \le 1$, for $x \in [0,\xi]$, and we get $w(\xi) \le \xi$. Let z be the solution of

$$(|z'|^{p(x)-2}z')'(x) = 0, \qquad z(\xi) = w(\xi) \le \xi, \quad z(1) = f(1).$$

Then we have

$$F(w) \ge F(z) \ge F(u_\infty).$$

To see that the last inequality is true just use the monotonicity of the function

$$C \mapsto \int_\xi^1 \frac{(C)^{\frac{p(s)}{p(s)-1}}}{p(s)}\, ds$$

with respect to C.

Case 2. When $C_\infty = 0$, we have that

$$\lim_{n\to\infty} u_n(x) = \begin{cases} Kx & x \in [0,\xi], \\ K\xi & x \in [\xi,1]. \end{cases}$$

Here $K \le 1$ is given by

$$K = \lim_{n\to\infty} (C_1(n))^{1/n}$$

(recall that we are taking $p_n(x) = p(x) \wedge n$).

As $u_n(1) = f(1)$ we get that the constant K is given by

$$K\xi = f(1).$$

This case actually happens when $f(1) \le \xi$. Since K is uniquely determined, we obtain the convergence of the whole sequence u_n.

Note that in this case the limit u_∞ is not differentiable, but it is Lipschitz. Also note that it is easy to verify that u_∞ is a minimizer of the functional F given by (3.1). Indeed, $F(u_\infty) = 0$ and $F(w) \ge 0$, for every $w \in S$.

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