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LINEAR TIME EQUIVALENCE OF LITTLEWOOD–RICHARDSON COEFFICIENT SYMMETRY MAPS

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Abstract: Benkart, Sottile, and Stroomer have completely characterized by Knuth and dual Knuth equivalence a bijective proof of the conjugation symmetry of the Littlewood–Richardson coefficients, i.e. $c^{\lambda}_{\mu,\nu} = c^{\lambda^t}_{\mu^t,\nu^t}$. Tableau–switching provides an algorithm to produce such a bijective proof. Fulton has shown that the White and the Hanlon–Sundaram maps are versions of that bijection. In this paper one exhibits explicitly the Yamanouchi word produced by that conjugation symmetry map which on its turn leads to a new and very natural version of the same map already considered independently. A consequence of this latter construction is that using notions of Relative Computational Complexity we are allowed to show that this conjugation symmetry map is linear time reducible to the Schützenberger involution and reciprocally. Thus the Benkart–Sottile–Stroomer conjugation symmetry map with the two mentioned versions, the three versions of the commutative symmetry map, and Schützenberger involution, are linear time reducible to each other. This answers a question posed by Pak and Vallejo.

Keywords: Symmetry maps of Littlewood–Richardson coefficients; conjugation symmetry map; linearly time reduction of Young tableaux bijections; tableau–switching; Schützenberger involution.

1. Introduction

Given partitions μ and ν, the product $s_\mu s_\nu$ of the corresponding Schur functions is a non-negative integral linear combination of Schur functions

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\,\nu} s_\lambda,$$

where λ runs over all partitions, and $c^{\lambda}_{\mu\,\nu}$ are called Littlewood–Richardson coefficients [LiRi, Mac, Sa, St]. Let λ^t denote the conjugate or transpose of the partition λ. Applying, for instance, to that product the involutive algebra

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automorphism ω [LiRi, Mac, Sa, St], defined by $\omega(s_\lambda(x)) = s_{\lambda^t}(x)$, it follows that $c_{\mu\,\nu}^{\lambda} = c_{\mu^t\,\nu^t}^{\lambda^t}$, called the *conjugation symmetry* of Littlewood-Richardson coefficients. There are several combinatorial models for the Littlewood-Richardson coefficients besides the one given originally by Littlewood and Richardson in terms of tableaux [LiRi]. However in all of them the conjugation symmetry is somewhat hidden. In the tableau model, one denotes by $\mathrm{LR}(\lambda/\mu,\nu)$ the set of Littlewood-Richardson (LR for short) tableaux of shape λ/μ and content ν, and $c_{\mu\,\nu}^{\lambda}$ counts the number of elements of this set. The boundary data of a LR tableau of shape λ/μ and content or weight ν is $(\mu,\nu,\lambda^\vee)$, with $\lambda^\vee$ the complement partition of λ regarding the smallest rectangle containing λ. We write $c_{\mu\,\nu}^{\lambda} = c_{\mu\,\nu\,\lambda^\vee}$. The Littlewood-Richardson coefficients $c_{\mu\,\nu\,\lambda^\vee}$ are invariant under the following action of $\mathbb{Z}_2 \oplus S_3$: the non–identity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν and $\lambda^\vee$, and S_3 permutes μ, ν and $\lambda^\vee$ [BSS]. In this model, the conjugation symmetry map is a bijection [PV]

$$\varrho : LR(\lambda/\mu,\nu) \longrightarrow LR(\lambda^t/\mu^t,\nu^t).$$

The Berenstein-Zelevinsky interpretation of the Littlewood-Richardson coefficients [BZ] makes clear that these coefficients are symmetric with respect to the action of S_3. But the invariance of the Littlewood-Richardson coefficients under the conjugation of partitions is hidden. Postnikov shows in [GP] that this symmetry can be revealed from a bijection between web diagrams and Berenstein-Zelevinsky patterns. The Knutson-Tao puzzles [KTW] manifest partially the conjugation symmetry through the so–called *puzzle duality*, *viz.* $c_{\mu\,\nu}^{\lambda} = c_{\nu^t\,\mu^t}^{\lambda^t}$. However a bijection between hives and puzzles can be used to reveal the commutative symmetry [KTW, K1, K2] and thus get $c_{\mu\,\nu}^{\lambda} = c_{\mu^t\,\nu^t}^{\lambda^t}$. More recently, Purbhoo [Pu] introduced a new tool *mosaics* (puzzles with extra rhombi) naturally in bijection with puzzles and with LR tableaux, revealing the hidden symmetries of puzzles.

Let T be a tableau and $\widehat{T}$ its standardization. The Benkart-Sottile-Stroomer conjugation symmetry map [BSS] is the bijection

$$\begin{array}{rcl} \varrho^{BSS} : LR(\lambda/\mu,\nu) & \longrightarrow & LR(\lambda^t/\mu^t,\nu^t) \\ T & \mapsto & \varrho^{BSS}(T) = [Y(\nu^t)]_K \cap [(\widehat{T})^t]_d \end{array},$$

where $[Y(\nu^t)]_K$ is the Knuth class of all tableaux with rectification the Yamanouchi tableau $Y(\nu^t)$ of shape the conjugate of ν, and $[\widehat{T}^t]_d$ is the dual Knuth class of all tableaux of shape λ^t/μ^t with Q-symbol the the transpose of $\widehat{T}$. The image of T by the BSS-bijection is the unique tableau of shape λ^t/μ^t, rectification $Y(\nu^t)$ and where the Q-symbol of its column reading word is the transpose of the evacuation of the Q-symbol of the word of T. Fulton showed in [F] that the White-Hanlon-Sundaram map ϱ^{WHS} [W, HS] coincides with ϱ^{BSS}. Thus $\varrho^{BSS}(T)$ can be obtained either by tableau-switching or by the White-Hanlon-Sundaram transformation ϱ^{WHS}. In the BSS-bijection, $\widehat{T}$ constitutes a set of instructions telling where *jeu de taquin* expanding slides can be applied to $Y(\mu)$; thus changing the orientation of $\widehat{T}$ by transposition, $\widehat{T}^t$ is a set of instructions telling where *jeu de taquin* expanding slides can be applied to $Y(\mu)^t$. Similar procedure is used in mosaics: operations on mosaics, called *migration*, are moves of *flocks*, *i.e.* rhombi arranged in the shape of a Young diagram, which correspond to some sequence of *jeu de taquin* operations. Giving to the *flock* two orientations, transposed of each other, a mosaic is simultaneously in bijection with a LR-tableau of shape λ/μ and content ν, and with a LR-tableau of shape λ^t/μ^t and content ν^t.

As words are in bijection with pairs of tableaux, we may determine explicitly the Yamanouchi word of $\varrho^{BSS}(T)$: the column word of $\varrho^{BSS}(T)$ is a Yamanouchi word of weight ν^t, with the Q-symbol of the column word of $\widehat{T}^t$. The following transformation ϱ_3 [Z, A1, A2, ACM] makes clear the construction of that word and affords a simple way to construct $\varrho^{BSS}(T)$

$$\begin{array}{cccc} \varrho_3: & LR(\lambda/\mu,\nu) & \longrightarrow & LR(\lambda^t/\mu^t,\nu^t) \\ & \underset{\text{with word } w}{T} & \mapsto & \underset{\text{with column word } (\sigma_0 w)^{*\diamond}}{\varrho_3(T)} \end{array}$$

$$\left\{\begin{array}{l} \mathrm{T} = \ytableaushort{\none\none1111,\none122,233} \xrightarrow[\text{reversal}]{e} \mathrm{T}^e = \ytableaushort{\none\none1133,\none222,333} \xrightarrow[\text{of } \lambda/\mu]{\text{transposition}} \ytableaushort{\none\none3,\none23,123,12,3,3} \longrightarrow \\ w = 1111221332 \xrightarrow{\sigma_0} \sigma_0 w = 3311222333 \xrightarrow[\text{the word}]{\text{reverse}} 3332221133 \longrightarrow \end{array}\right.$$

$$\left\{\begin{array}{l} \longrightarrow \ytableaushort{\none\none1,\none12,123,23,4,5} = \varrho_3(\mathrm{T}) \\ \longrightarrow 1231231245 \qquad \text{column word of } \varrho_3(T) = \varrho^{BSS}(T) \end{array}\right.$$

where $\sigma_0 = \sigma_i \cdots \sigma_j \cdots \sigma_k$ is such that $s_i \cdots s_j \cdots s_k$, with s_l the transposition $(l, l+1)$, is the longest permutation of $S_{\nu_1^t}$, and σ_i is the reflection crystal operator acting on the subword over the alphabet $\{i, i+1\}$, for all i [LS, Loth]; $*$ denotes the dualization of a word; and $\diamond$ is the operator which transforms a Yamanouchi word of weight ν, into a Yamanouchi word of weight ν^t, by replacing the subword i^{ν_i} with $12\ldots\nu_i$, for all i. The action of the operator $\diamond$ is defined analogously on dual Yamanouchi words. More precisely, the $\diamond$ operator is a bijection between the Knuth classes of $Y(\nu)$ and $Y(\nu^t)$, and also between their dual. Indeed the operators $*$ and $\diamond$ are involutive and commute. The reversal e of a LR tableau can be computed by the action of σ_0 on its word. The image of a LR or dual LR tableau U under rotation of the skew-diagram by 180 degrees, with the dualization $*$ of its word is denoted by $U^{\bullet}$; and the image of U under the rotation and transposition of the skew-diagram , with the action of the operation $\diamond$ on its word is denoted by $U^{\blacklozenge}$. Again $\bullet$ and $\blacklozenge$ are involutive maps. Then

$$\varrho_3(T) = T^{e\,\bullet\blacklozenge} = T^{\blacklozenge\bullet\,e} = T^{\bullet\blacklozenge\,e}$$

and $(\sigma_0 w)^{*\diamond} = (\sigma_0 w)^{\diamond *} = \sigma_0(w^{\diamond *})$ is the column word of $T^{e\bullet\blacklozenge} = [Y(\nu^t)]_K \cap [(\widehat{T})^t]_d$.

This is easily seen recalling that two tableaux of the same shape are dual Knuth equivalent if and only if their words have the same Q-symbol. On the other hand crystal reflection operators preserve the Q-symbol.

Following the ideas established in [PV], we also address the problem of studying the computational cost of the conjugation symmetry map ϱ^{BSS} utilizing what is known as Relative Complexity, an approach based on reduction of combinatorial problems, see § 4. To this aim we use the version ϱ_3. We consider only linear time reductions; since the bijections we consider require subquadratic time the reductions have to preserve that. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. We say that δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$, where $\langle A \rangle$ is the bit–size of A. The transposition of the recording matrix of a LR tableau is the recording matrix of a tableau of normal shape. We have then a linear map τ which defines a bijection between tableaux of normal shape and LR tableaux [Lee1, Lee2, PV, O]. The rotation map $\bullet$ and the bijection τ are linear maps and so clearly operations of linear cost. Therefore reversal T^e of a LR tableau T can be linearly reduced to the evacuation E of the corresponding tableau

$\tau(T) = P$ of normal shape, *i.e.* $\tau(P^E) = T^{e\bullet}$. Additionally we prove that the bijection $\blacklozenge$ is of linear cost. This means that ϱ_3 is linearly time reducible to the evacuation operation E. The following scheme shows that the conjugation symmetry map ϱ_3, and therefore ϱ^{BSS} and ϱ^{WHS}, is linear equivalent to the Schützenberger involution or evacuation map on tableaux of normal shape,

$$\begin{array}{ccccc} T & \xleftrightarrow{e\bullet} & T^{e\bullet} & \xleftrightarrow{\blacklozenge} & T^{e\bullet\blacklozenge} \\ \tau \updownarrow & & \tau \updownarrow & & \\ P & \underset{E}{\overset{\text{evacuation}}{\longleftrightarrow}} & P^E. & & \end{array}$$

We may now extend the list of linear equivalent maps on tableaux in [PV], Theorem 1, which includes the Young tableau symmetry maps showing the action of $\mathbb{Z}_2 \oplus S_3$ on Littlewood Richardson coefficients.

Theorem 1.1. *The following maps are linearly equivalent:*

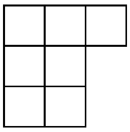
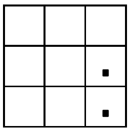
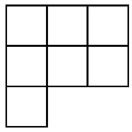
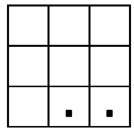
(1) [PV] *RSK correspondence.*
(2) [PV] *Jeu de taquin map.*
(3) [PV] *Littlewood–Robinson map.*
(4) [PV] *Tableau switching map s.*
(5) [PV] *Evacuation (Schützenberger involution) E for normal shapes.*
(6) [PV] *Reversal e.*
(7) [PV]*First and second fundamental symmetry maps.*
(8) [A3] *Third fundamental symmetry map ρ_3.*
(9) ϱ^{BSS}, ϱ^{WHS} *and* ϱ_3 *conjugation symmetry maps.*

In particular, first and second fundamental symmetry maps are identical [DK2]*; first and third fundamental symmetry maps are identical* [A3]*; ϱ^{BSS}, ϱ^{WHS} and ϱ_3 are identical conjugation symmetry maps.*

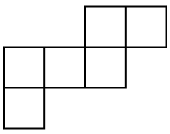
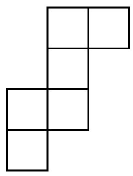
2. Preliminaries

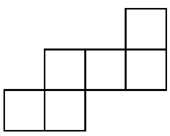
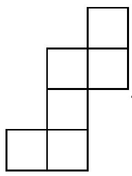
2.1. Young diagrams and transformations. A *partition* (or normal shape) λ is a sequence of non–negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. The number of parts is $\ell(\lambda) = \ell$ and the weight is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. The Young diagram of λ is the collection of boxes $\{(i,j) \in \mathbb{Z}^2 | \ 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$. The English convention is adopted in drawing such a diagram. Throughout the paper we do not make distinction between a partition λ and its Young diagram. If (r^ℓ) is a rectangle containing λ, that is $r \geq \lambda_1$, the *complement* of λ regarding r is the partition $\lambda_r^\vee = (r - \lambda_\ell, \ldots, r - \lambda_1)$. When $r = \lambda_1$ we omit the r as subindex. We define

λ^t the *conjugation* or *transposition* of λ as the image of λ under the transposition $(i,j) \rightarrow (j,i)$. For example, the Young diagram of $\lambda = (3,2,2)$ and its transpose $\lambda^t = (3,3,1)$ are depicted below; and $\lambda^\vee = (3,2,2)^\vee = (1,1,0)$, $(\lambda^t)^\vee = (\lambda^\vee)^t = (2,0,0)$ are depicted by dotted boxes

$\lambda =$  ,  $= \lambda^\vee$, $\lambda^t =$  ,  $= (\lambda^\vee)^t$.

Given partitions λ, μ, we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i > 0$. A skew-diagram (skew-shape) λ/μ is $\{(i,j) \in \mathbb{Z}^2 |\ (i,j) \in \lambda, (i,j) \notin \mu\}$ the collection of boxes in λ which are not in μ. When μ is the null partition, the skew-diagram λ/μ equals the Young diagram λ. The number of boxes in λ/μ is $|\lambda/\mu| = |\lambda| - |\mu|$. The *transpose* (conjugate shape) $(\lambda/\mu)^t$ is the skew-diagram λ^t/μ^t obtained by transposing the skew-diagram λ/μ. Let $r = \lambda_1$. The *rotation* (dual shape) $(\lambda/\mu)^*$ is the image of λ/μ by rotation of 180 degrees, or the image of λ/μ under $(i,j) \longrightarrow (\ell - i + 1, r - j + 1)$. Equivalently $(\lambda/\mu)^* = \mu_r^\vee/\lambda^\vee$. In particular, λ^* is the skew-diagram $r^\ell/\lambda^\vee$. The dual conjugate shape $(\lambda/\mu)^\diamondsuit$ is the image of λ/μ under $(i,j) \longrightarrow (r-j+1, l-i+1)$. The map $\diamondsuit$ is the composition of the transposition with the rotation maps $\diamondsuit = *t = t*$. In particular, $\lambda^\diamondsuit = l^r/(\lambda^\vee)^t$. For instance, if $\mu = (2) \subset \lambda = (4,3,1)$, we have

$\lambda/\mu =$  $(\lambda/\mu)^t =$ 

$(\lambda/\mu)^* =$  $(\lambda/\mu)^\diamondsuit =$  .

2.2. Tableaux and words. The Littlewood–Richardson (LR for short) numbering (reading) of the boxes of a skew-diagram λ/μ is an assignment of the labels $1, 2, \ldots$ which sorts the boxes of λ/μ in increasing order from right to left along each row, starting in the top row and moving downwards; and the column LR numbering of the boxes sorts in increasing order, from right to left along each column, starting in the rightmost column and moving downwards. Analogously the reverse LR numbering and the column LR numbering of λ/μ are defined.

Example 2.1. *If* $\lambda/\mu = \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & & \\ \hline & & & \\ \hline \end{array}$*, the LR–numbering, column LR–numbering and the corresponding reverse LR–numberings of* λ/μ *are, respectively,*

$$\begin{array}{cccc} & & 2 & 1 \\ 6 & 5 & 4 & 3 \end{array} \qquad \begin{array}{cccc} & & 3 & 1 \\ 6 & 5 & 4 & 2 \end{array} \qquad \begin{array}{cccc} & & 5 & 6 \\ 1 & 2 & 3 & 4 \end{array} \qquad \begin{array}{cccc} & & 4 & 6 \\ 1 & 2 & 3 & 5 \end{array}.$$

Clearly, the column LR–numbering of λ/μ is the LR–numbering of $(\lambda/\mu)^{\diamondsuit}$, and the reverses of LR–numbering and column LR–numbering of λ/μ define, respectively, the LR–numbering of $(\lambda/\mu)^{*}$ and $(\lambda/\mu)^{t}$.

A *Young tableau* T of shape λ/μ is a filling of the boxes of the skew-diagram λ/μ with positive integers in $\{1,\dots,t\}$ which is increasing in columns from top to bottom and non-decreasing in rows from left to right. When μ is the empty partition we say that T has normal shape λ. The weight of a tableau T is a sequence $m = (m_1, m_2, \dots, m_t)$, where m_i denotes the number of integers i in T, for all i, and put $\ell(m) = t$. The word $w(\mathrm{T})$ of a Young tableau T is the sequence obtained by reading the entries of T according to its LR numbering, that is, reading right-to-left the rows of T, from top to bottom. The column word $w_{col}(T)$ is the word obtained according the column LR numbering. The weight of w is the weight of T. Denote by $\mathrm{YT}(\lambda/\mu, m)$ the set of Young tableaux of shape λ/μ and weight m.

Example 2.2. $\mathrm{T} = \begin{array}{|c|c|c|c|c|c|} \cline{3-6} \multicolumn{2}{c|}{} & 1 & 1 & 1 & 1 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1 & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-4} 2 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$, $w(T) = 1111221332$ *and* $w_{col}(T) = 1112123132$.

A Young tableau with ℓ boxes is *standard* if it is filled with $\{1,\dots,\ell\}$ without repetitions. Given a tableau T, the *standardization* of T is denoted by $\widehat{\mathrm{T}}$. If $(m_1,\dots,m_t)$ is the weight of T, then $\widehat{\mathrm{T}}$ is obtained by replacing, west to east, the letters 1 in T with $1, 2, \dots, m_1$; the letters 2 with $m_1+1, \dots, m_1+m_2$; and so on. The standardization $\widehat{w}$ of a word w is defined accordingly, from right to left. For instance, the standardization of the tableau T in the previous example is $\widehat{\mathrm{T}} = \begin{array}{|c|c|c|c|c|c|} \cline{3-6} \multicolumn{2}{c|}{} & 2 & 3 & 4 & 5 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1 & 7 & 8 & \multicolumn{2}{c}{} \\ \cline{1-4} 6 & 9 & 10 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$, and $\widehat{w(T)} = w(\widehat{T}) = 54328711096$.

If T is a standard tableau of normal shape λ, with $|\lambda| = \ell$, and α is a permutation in the symmetric group $\mathcal{S}_\ell$, we let αT denote the filling of the diagram λ obtained replacing in T the letter i by $\alpha(i)$, for all $i = 1,\dots,\ell$. Clearly, αS does not need to be a tableau. If $w = w_1 w_2 \dots w_\ell$ is a word, define

$\alpha w = w_{\alpha(1)} \dots w_{\alpha(\ell)}$. In the case T is standard we have $w_{col}(\widehat{T}) = rev\, w(\widehat{T}^t)$, with rev the reverse permutation.

A Young tableau T is said a *Littlewood–Richardson* (LR for short) tableau if its word, when read from the beginning to any letter, contains at least as many letters i as letters $i+1$, for all i. More generally, a word such that every prefix satisfies this property is called a *lattice permutation* or a Yamanouchi word. Notice that the column word of a LR–tableau is also a Yamanouchi word of the same weight. Denote by LR$(\lambda/\mu, \nu)$ the set of LR tableaux of shape λ/μ and weight ν. When $\mu = 0$ we get the Yamanouchi tableau $Y(\nu)$, the unique tableau of shape and weight ν. In example 2.2, T is a LR tableau with Yamanouchi word $w(T) = 1111221332$ and column word $w_{col}(T) = 1112123132$.

There is an one–to–one correspondence between Yamanouchi words of weight ν and standard tableaux of shape ν. Let $w = w_1 w_2 \cdots w_\ell$ be a Yamanouchi word and put the number k in the w_kth row of the diagram ν. The labels of the ith row are the k's such that $w_k = i$, thus the length is ν_i and the shape is ν. We denote this standard tableau by $U(w)$. In example 2.2, $w = 1111221332$, $U(w) = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&7\\ \hline 5&6&10\\ \cline{1-3} 8&9\\ \cline{1-2}\end{array}$ where the entries of the ith row are the positions of the i's in the LR reading of T .

2.3. Matrices and tableaux. Given $T \in Y(\lambda/\mu, m)$, let $M = (M_{ij})_{1\le i\le \ell(\lambda), 1\le j\le \ell(m)}$ be a matrix with non–negative entries such that M_{ij} is the number of $j's$ in the ith row of T, called the recording matrix of T [Lee1, Lee2, PV]. The recording matrix of a tableau of normal shape is an upper triangular matrix, and the recording matrix of an LR tableau is a lower triangular matrix. Thus we have an one–to–one correspondence between LR tableaux and tableaux of normal shape as follows. Considering T in example 2.2, the recording matrix of T is $M = \begin{pmatrix} 4&0&0\\ 1&2&0\\ 0&1&2 \end{pmatrix}$. On the other hand, the transposition $M^t = \begin{pmatrix} 4&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$ encodes the tableau $B = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\ \hline 2&2&3\\ \cline{1-3} 3&3\\ \cline{1-2}\end{array}$ of normal shape ν, weight $\lambda - \mu$. For two Young diagrams μ and ν, define $\nu \circ \mu = (\nu_1+\mu_1, \dots, \nu_1+\mu_\ell, \nu_1, \dots, \nu_r)/(\mu_1+\nu_1, \dots, \mu_\ell+\nu_1)$, $\ell = \ell(\mu)$, $r = \ell(\nu)$. Then with $\mu = (1)$, $B \circ Y(\mu) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 1 \\ \hline 1 & 1 & 1 & 1 & 2 & \\ \hline 2 & 2 & 3 & & & \\ \hline 3 & 3 & & & & \\ \hline \end{array} \in LR(\nu \circ \mu, \lambda)$. Given partitions λ, μ, ν such that $|\lambda| = |\mu| + |\nu|$, define $CF(\nu, \mu, \lambda) = \{B \in YT(\nu, \lambda - \mu) : B \circ Y(\mu) \in \mathrm{LR}(\nu \circ \mu, \lambda)\}$ [PV]. The map $\tau : LR(\lambda/\mu, \nu) \to CF(\nu, \mu, \lambda)$ such that $\tau(M)$ is the tableau of normal shape with recording matrix M^t, where M is the recording matrix of T, is a bijection. Taking again example 2.2, we have $\tau(T) = B$.

2.4. Rotation and transposition of LR tableaux. Given an integer i in $\{1, \ldots, t\}$, let $i^* := t - i + 1$. Given a word $w = w_1 w_2 \cdots w_\ell$, over the alphabet $\{1, \ldots, t\}$, of weight $m = (m_1, \ldots, m_t)$, $w^* := w_\ell^* \cdots w_2^* w_1^*$ is the *dual word* of w and $m^* = (m_t, \ldots, m_1)$ its weight. Indeed $w^{**} = w$. A dual Yamanouchi word is a word whose dual word is Yamanouchi. Given a Young tableau T of shape λ/μ and weight $(m_1, \ldots, m_t)$, $\mathrm{T}^\bullet$ denotes the Young tableau of shape $(\lambda/\mu)^*$ and weight m^*, obtained from T by replacing each entry i with i^*, and then rotating the result by 180 degrees. The word of $\mathrm{T}^\bullet$ is $w(\mathrm{T})^*$, and $\mathrm{T}^{\bullet\bullet} = \mathrm{T}$. A dual LR tableau is a tableau whose word is a dual Yamanouchi word. $\mathrm{LR}(\lambda/\mu, \nu^*)$ denotes the set of dual LR tableaux of shape λ/μ and weight ν^*, and it is the image of $\mathrm{LR}((\lambda/\mu)^*, \nu)$ under the rotation map $\bullet$. Thus the rotation map $\bullet$ defines a bijection between $\mathrm{LR}((\lambda/\mu)^*, \nu^*)$ and $\mathrm{LR}(\lambda/\mu, \nu)$. Given a Yamanouchi word w of weight ν, define the standard tableau $U(w^*)$ of shape ν^* such that the label k is in row i if and only if $w_{\ell-i+1} = k^*$. Thus $U(w^*) = U(w)^\bullet$ and this affords a bijection between dual Yamanouchi words of weight ν^* and standard tableaux of shape ν^*. The rotation map $\bullet$ can also be easily defined using the notion of recording matrix: $M = (M_{ij})$ is the recording matrix of T if and only if the recording matrix of $T^\bullet$ is $M^\bullet = (M_{\ell(\lambda)+1-i, \ell(m)-j+1})$.

There is another natural bijection, denoted by $\blacklozenge$, between LR tableaux of conjugate weight and dual conjugate shape, see also [Z, A1, A2]. Given a Yamanouchi word w of weight $\nu = (\nu_1, \ldots, \nu_t)$, write $\nu^t = (\nu_1^t, \ldots, \nu_k^t)$ and observe that w is a shuffle of the words $12 \ldots \nu_i^t$ for all i, and its dual word is a shuffle of the words $t\, t-1 \cdots t - \nu_i^t + 1$, for all i. Thus, we define $w^\diamond$ as the Yamanouchi word of weight ν^t obtained by replacing the subword consisting only on the letters i with the subword $12 \cdots \nu_i$, for each i. The operation

$\diamond$ is defined similarly on dual Yamanouchi words, giving rise to a dual Yamanouchi word of weight ν^{*t}. Clearly, $w^{\diamond *} = w^{*\diamond}$ is a dual Yamanouchi word of weight ν^{t*}. The word $w^{\diamond *}$ can be obtained in only one step: replace the subword of w consisting only on the letters i with the subword $\nu_1\, \nu_1{-}1 \cdots \nu_1{-}\nu_i{+}1$, for all i. Clearly, $U(w^{\diamond}) = U(w)^t$ is of shape ν^t, and $U(w^{*\diamond}) = U(w)^{\bullet t}$ is of shape ν^*. Given $\mathrm{T} \in \mathrm{LR}(\lambda/\mu, \nu)$ $(\mathrm{LR}(\lambda/\mu, \nu^*))$ with word w, define $\mathrm{T}^{\blacklozenge}$ as the LR tableau of shape $(\lambda/\mu)^{\diamondsuit}$ and weight ν^t obtained from T by replacing the word w with $w^{\diamond}$, and then rotating the result by 180 degrees and transposing. Then $\blacklozenge : \mathrm{LR}(\lambda/\mu, \nu)(\mathrm{LR}(\lambda/\mu, \nu^*)) \longrightarrow \mathrm{LR}((\lambda/\mu)^{\diamondsuit}, \nu^t)\mathrm{LR}((\lambda/\mu)^{\diamondsuit}, \nu^{*t})$ is a bijection such that $T^{\blacklozenge}$ has column word $w^{\diamond}$ and $T^{\blacklozenge\blacklozenge} = T$. Since $\blacklozenge\bullet = \bullet\blacklozenge$, $T^{\blacklozenge\bullet} = T^{\bullet\blacklozenge} \in \mathrm{LR}((\lambda/\mu)^t, \nu^{t*})$ $(\mathrm{LR}((\lambda/\mu)^t, \nu^t))$ has column word $w^{*\diamond}$.

Example 2.3. $\mathrm{T} = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \hline 1 & 3 & & \\ \hline \end{array}$ *is a LR tableau with word* $w = 1122131$ *of weight* $\nu = (4, 2, 1)$. *Then* $\mathrm{T} = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \hline 1 & 3 & & \\ \hline \end{array} \quad \underset{\textit{by } w(\mathrm{T})^{\diamond}}{\overset{\textit{replace } w(\mathrm{T})}{\longleftrightarrow}} \quad \begin{array}{|c|c|c|c|} \hline & & 2 & 1 \\ \hline & 3 & 2 & 1 \\ \hline 4 & 1 & & \\ \hline \end{array} \quad \underset{\textit{transpose}}{\overset{\textit{rotate}}{\longleftrightarrow}}$

$\begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline 1 & 3 & \\ \hline 4 & & \\ \hline \end{array} = \mathrm{T}^{\blacklozenge} \underset{\bullet}{\longleftrightarrow} \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & 4 \\ \hline 3 & 3 & \\ \hline 4 & 4 & \\ \hline \end{array} = \mathrm{T}^{\blacklozenge\bullet}$. $\mathrm{T}^{\blacklozenge}$ *is a LR tableau with shape* $(\lambda/\mu)^{\diamondsuit}$ *and column word* $w^{\diamond} = 1212314$ *of weight* ν^t. $\mathrm{T}^{\blacklozenge\bullet}$ *is a dual LR tableau with shape* $(\lambda/\mu)^t$ *and column word* $w^{\diamond *} = 1423434$ *of weight* ν^{t*}, *where* $U(w) =$ $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & & \\ \hline 6 & & & \\ \hline \end{array}$, $U(w^{\diamond}) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \hline 7 & & \\ \hline \end{array} = U(w)^t$, *and* $U(w^{\diamond *}) = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 3 \\ \hline & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline \end{array} = U(w)^{\bullet t}$.

3. Conjugation symmetry maps

3.1. Knuth equivalence and dual Knuth equivalence. Whenever partitions $\nu \subset \mu \subset \lambda$, we say that λ/μ extends μ/ν. An *inside corner* of λ/μ is a box in the diagram μ such that the boxes below and to the right are not in μ. When a box extends λ/μ, this box is called an *outside corner*. Let T be a Young tableau and let b be an inside corner for T. A *contracting slide* [Sch, BSS] of T into the box b is performed by moving the empty box at b through T, successively interchanging it with the neighboring integers to the south and east according to the following rules: (i) if the empty box has only one neighbor, interchange with that neighbor; (ii) if it has two unequal neighbors, interchange with the smaller one; and (*iii*) if it has two equal neighbors, interchange with that one to the south. The empty box moves in this fashion until it has become an outside corner. This contracting slide can be reversed by performing an analogous procedure over the outside corner, called an *expanding slide*. Performing a contracting slide over each inside corner of T reduces T to a tableau T^{n} of normal shape. This procedure is known as *jeu de taquin*. T^{n} is independent of the particular sequence of inside corners used [Th], and so T^{n} is called the *rectification* of T. A word w corresponds by RSK–correspondence to a pair $(P(w), Q(w))$ of tableaux of the same shape, with $Q(w)$ standard, called the Q–symbol of w. Here we consider a variation of RSK-correspondence known as the Burge correspondence [B, F]. Given $w = w_1 w_2 \cdots w_\ell$, $P(w)$ is the insertion tableau obtained by column insertion of the letters of w from left to right [F]. The corresponding recording tableau $Q(w)$ is obtained by placing $1, 2, \ldots, \ell$. If w is the word of T then $P(w) = \mathrm{T}^{\mathrm{n}}$. Insertion can be translated into the language of Knuth elementary transformations. Two words w and v are said Knuth equivalent if they have the same insertion tableau. Each Knuth class is in bijection with the set of standard tableaux with the shape of the unique tableau in that class. Two tableaux T and R are *Knuth equivalent*, written $\mathrm{T} \equiv \mathrm{R}$, if and only if $P(w(T)) = P(w(R))$. Equivalently, $T^{\mathrm{n}} \equiv R^{\mathrm{n}}$, i.e. one of them can be transformed into the other one by a sequence of *jeu de taquin* slides. The insertion tableau of a Yamanouchi word w with partition weight ν, is the Yamanouchi tableau $\mathrm{Y}(\nu)$. The recording tableau of a Yamanouchi word w is $U(w)$. By symmetry of Berge correspondence $Q(w^*) = P(U(w)^\bullet) = U(w)^{\bullet \mathrm{n}}$. Given $w \equiv \mathrm{Y}(\nu)$, we may now define the word $w^\diamond$ as being the unique word satisfying $w^\diamond \equiv \mathrm{Y}(\nu^t)$ such that $Q(w^\diamond) = Q(w)^T = U(w)^t$. Since $(w^\diamond)^\diamond = w$, the map $w \mapsto w^\diamond$ establishes a bijection between the Knuth classes of $\mathrm{Y}(\nu)$ and $\mathrm{Y}(\nu^t)$. The word w^* is the unique word satisfying $w \equiv \mathrm{Y}(\nu^*)$ such that $Q(w^*) = U(w)^{\bullet \mathrm{n}}$.

Two tableaux T and R of the same shape are *dual equivalent*, written $\mathrm{T} \overset{d}{\equiv} \mathrm{R}$, if any sequence of contracting slides and expanding slides that can be applied to one of them, can also be applied to the other, and the sequence of shape changes is the same for both [H, F]. Dual equivalence may also be characterized by recording tableaux: $\mathrm{T} \overset{d}{\equiv} \mathrm{R}$ if and only if $Q(w(\mathrm{T})) = Q(w(\mathrm{R}))$. Thus two tableaux of the same normal shape are dual equivalent. Let S and T be tableaux such that T extends S, and consider the set union

$\mathrm{S} \cup \mathrm{T}$. The *tableau switching* [BSS] is a procedure based on *jeu de taquin* elementary moves on two alphabets that transforms $\mathrm{S} \cup \mathrm{T}$ into $\mathrm{A} \cup \mathrm{B}$, where B is a tableau Knuth equivalent to T which extends A, and A is a tableau Knuth equivalent to S. We write $\mathrm{S} \cup \mathrm{T} \xrightarrow{\mathrm{s}} \mathrm{A} \cup \mathrm{B}$. In particular, if S is of normal shape, $\mathrm{A} = \mathrm{T}^{\mathrm{n}}$, and $\mathrm{S} = \mathrm{B}^{\mathrm{n}}$. Switching of S with T may be described as follows: $\widehat{T}$ is a set of instructions telling where expanding slides can be applied to S. Thus switching and dual equivalence are related as below and tableaux are completely characterized by dual and Knuth equivalence.

Theorem 3.1. [H] *Let* T *and* U *be tableaux with the same normal shape and let* W *be a tableau which extends* T*. (1) If* $\mathrm{T} \cup \mathrm{W} \xrightarrow{\mathrm{s}} \mathrm{Z} \cup \mathrm{X}$ *and* $\mathrm{U} \cup \mathrm{W} \xrightarrow{\mathrm{s}} \mathrm{Z} \cup \mathrm{Y}$*, then* $\mathrm{X} \overset{\mathrm{d}}{\equiv} \mathrm{Y}$*.*

(2) *Let $\mathcal{D}$ be a dual equivalence class and $\mathcal{K}$ be a Knuth equivalence class, both corresponding to the same normal shape. Then, there is a unique tableau in $\mathcal{D} \cap \mathcal{K}$.*

Algorithm to construct $\mathcal{D} \cap \mathcal{K}$: Let $U \in \mathcal{D}$ and let $V \in \mathcal{K}$ be the only tableau with normal shape in this class, and W any tableau that U extends:

$$\begin{array}{ccc} W \cup U & & W \cup X \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle s} \\ U^{\mathrm{n}} \cup Z & \rightarrow & V \cup Z. \end{array}$$

Thus $X \overset{d}{\equiv} U$, $X \overset{k}{\equiv} V$, and $\mathcal{D} \cap \mathcal{K} = \{X\}$. since two words in the same Knuth class can not have the same Q–symbol.

3.2. The transposition of the rotated reversal LR tableau. Given a tableau T of normal shape, the evacuation T^E is the rectification of $\mathrm{T}^\bullet$, that is $\mathrm{T}^{\mathrm{E}} = \mathrm{T}^{\bullet\mathrm{n}}$. T^E is also obtained either as the insertion tableau of the word $w(\mathrm{T})^*$, or according to the Schützenberger evacuation algorithm, or applying the reverse *jeu de taquin* slides to T, in the smallest rectangle containing T, to obtain T^{a} the anti-normal form T and then $T^{\mathrm{a}\bullet} = T^E$. If w is a Yamanouchi word, $Q(w^*) = U(w)^E$ and $Q(w^{\diamond *}) = U(w)^{E\,t}$. Given a tableau T of any shape, the reversal T^e is the unique tableau Knuth equivalent to $\mathrm{T}^\bullet$, and dual equivalent to T [BSS]. By Theorem 3.1, $T^e = [T^{\mathrm{n\,E}}]_K \cap [T]_d$, where $[\ ]_K$ denotes Knuth class and $[\ ]_d$ dual Knuth class. If T has normal shape $T^E = T^e$. If $\mathrm{T} \in \mathrm{LR}(\lambda/\mu, \nu)$, then T^e is the only tableau Knuth equivalent to $\mathrm{Y}(\nu^*)$ and dual equivalent to T. Since crystal reflection operators, for the definition see [LS, Loth], preserve the Q–symbol, we may in the case of LR tableaux characterize explicitly the word of T^e as follows. Let w be a Yamanouchi word of weight $\nu = (\nu_1, \ldots, \nu_t)$, and let σ_i denote the reflection crystal operator acting on the subword over the alphabet $\{i, i+1\}$, for all i. If $s_{i_1} \cdots s_{i_r}$ is the longest permutation in $\mathcal{S}_t$, put $\sigma_0 := \sigma_{i_1} \cdots \sigma_{i_r}$. Then $\sigma_0 w$ is a dual Yamanouchi word of weight ν^*. Moreover, $w \equiv w'$ if and only if $\sigma_i(w) \equiv \sigma_i(w')$, and $Q(w) = Q(\sigma_i(w))$. Thus, we have proven the following

Theorem 3.2. *Let* T *be a LR tableau with shape λ/μ and word w. Then* T^e *is the dual LR tableau of shape λ/μ and word $\sigma_0 w$, and $T^{e\blacklozenge\bullet}$ is the LR tableau of shape $(\lambda/\mu)^t$ and column word $(\sigma_0 w)^{\diamond *}$.*

Corollary 3.1. *$T^{e\blacklozenge\bullet}$ is the unique tableau Knuth equivalent to $Y(\nu^t)$ and dual equivalent to $\widehat{T}^t$.*

Proof: It is enough to see that the column words of $T^{e\blacklozenge\bullet}$ and $\widehat{T}^t$ have the same Q–symbol. Let $\widehat{w}$ be the word of $\widehat{T}$. As $rev\,\widehat{w}$ the reverse word of $\widehat{T}$ is the column word of $\widehat{T}^t$, then $Q(rev\,\widehat{w}) = Q(\widehat{w})^{E\,t} = Q(w)^{E\,t} = Q(w^{\diamond *}) = Q(\sigma_0(w^{\diamond *})) = Q((\sigma_0 w)^{\diamond *})$. ■

Let w be a Yamanouchi word of weight ν. There is a natural bijection between words $\sigma_0 w$ and $U(w)^{\mathrm{a}}$ of shape ν^*, the the anti-normal form of $U(w)$. The labels in row i of $U(w)^{\mathrm{a}}$ are the k's such that $(\sigma_0 w)_k = i^*$. If θ_i denotes the *jeu de taquin* action on consecutive rows i and $i+1$ of $U(w)$, then $\theta_i U(w) = U(\sigma_i w)$ is the tableau of skew-shape $(i\,i+1)\nu$, where the shorter row is adjusted on the right with the longer one if above, and on the left otherwise, having the label k in row j if and only if $w_k = j$. Put $\theta_0 := \theta_{i_1} \ldots \theta_{i_r}$ with $i_1, \ldots, i_r$ as in σ_0. Thus $\theta_0 U(w) = U(\sigma_0 w)$ and $Q(\sigma_0 w) = U(w)^{\mathrm{an}}$. This was the procedure in [A1]. Similarly, we get

Theorem 3.3. *Let T be a LR tableau and $\tau(T) = P$. Then,*

$$\begin{array}{ccccc} T & \stackrel{e}{\longleftrightarrow} & T^e & \stackrel{\bullet}{\longleftrightarrow} & T^{e\bullet} \\ \tau \updownarrow & & \tau \updownarrow & & \tau \updownarrow \\ P & \underset{\mathrm{a}}{\longleftrightarrow} & P^{\mathrm{a}} & \stackrel{\bullet}{\longleftrightarrow} & P^E. \end{array}$$

3.3. Main bijections. Let

$$\begin{array}{rccc} \varrho^{BSS}: & LR(\lambda/\mu, \nu) & \rightarrow & LR(\lambda^t/\mu^t, \nu^t) \\ & T & \mapsto & \varrho^{BSS}(T) = [Y(\nu^t)]_K \cap [\widehat{T}^t]_d \end{array} \qquad \text{[BSS].}$$

The image of T by the BSS-bijection is the unique tableau of shape λ^t/μ^t whose rectification is $Y(\nu^t)$ and the Q–symbol of the column reading word is $Q(T)^{Et}$. The idea behind this bijection can be told as follows: $\widehat{T}$ constitutes a set of instructions telling where expanding slides can be applied to $Y(\mu)$. Then $\widehat{T}^t$ is a set of instructions telling where expanding slides can be applied to $Y(\mu)^t$. Tableau–switching provides an algorithm to give way to those instructions:

$$Y(\mu)\cup T \overset{\text{standardization}}{\longrightarrow} Y(\mu)\cup\widehat{T} \overset{\text{transposition}}{\longrightarrow} Y(\mu^t)\cup\widehat{T}^t \overset{\text{s}}{\longrightarrow} (\widehat{T}^t)^{\text{n}}\cup Z \overset{\text{s}}{\longrightarrow}$$
$$\overset{\text{s}}{\longrightarrow} Y(\nu^t)\cup Z \overset{\text{s}}{\longrightarrow} Y(\mu^t)\cup\varrho^{BSS}(T).$$

Then $\varrho^{BSS}(T)\equiv Y(\nu^t)$ and $\varrho^{BSS}(T)\equiv_d \widehat{T}^t$.

Example 3.4.

$$T \quad = \quad \begin{array}{cccccc} & & 1 & 1 & 1 & 1\\ & 1 & 2 & 2 & & \\ 2 & 3 & 3 & & & \end{array} \;\rightarrow \widehat{T} = \begin{array}{cccccc} & & 2 & 3 & 4 & 5\\ & 1 & 7 & 8 & & \\ 6 & 9 & 10 & & & \end{array} \;\rightarrow \widehat{T}^t = \begin{array}{ccc} & & 6\\ & 1 & 9\\ 2 & 7 & 10\\ 3 & 8 & \\ 4 & & \\ 5 & & \end{array} \;\rightarrow$$

$$\rightarrow \quad Y(\mu^t)\cup\widehat{T}^t = \begin{array}{ccc} \mathbf{1} & \mathbf{1} & 6\\ \mathbf{2} & 1 & 9\\ 2 & 7 & 10\\ 3 & 8 & \\ 4 & & \\ 5 & & \end{array} \overset{s}{\longrightarrow} (\widehat{T}^t)^{\text{n}}\cup Z = \begin{array}{ccc} 1 & 6 & 9\\ 2 & 7 & 10\\ 3 & 8 & \mathbf{1}\\ 4 & \mathbf{2} & \\ 5 & & \\ \mathbf{1} & & \end{array} \overset{s}{\longrightarrow}$$

$$\overset{s}{\longrightarrow} \quad Y(\nu^t)\cup Z = \begin{array}{ccc} 1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & \mathbf{1}\\ 4 & \mathbf{2} & \\ 5 & & \\ \mathbf{1} & & \end{array} \overset{s}{\longrightarrow} Y(\mu^t)\cup\varrho^{BSS}(T) = \begin{array}{ccc} \mathbf{1} & \mathbf{1} & 1\\ \mathbf{2} & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & \\ 4 & & \\ 5 & & \end{array}.$$

Let

$$\begin{array}{rccc} \varrho_3: & LR(\lambda/\mu,\nu) & \rightarrow & LR(\lambda^t/\mu^t,\nu^t) \\ & T & \mapsto & \varrho_3(T)=T^{e\,*\,\blacklozenge} \qquad [\text{Z, A1, A2}].\\ & w & \mapsto & \sigma_0 w^{*\diamond} \end{array}$$

As $T^{e*\blacklozenge}$ is the unique tableau Knuth equivalent to $Y(\nu^t)$ and dual equivalent to $(\widehat{T})^t$, $\varrho^{BSS}=\varrho_3$.

Example 3.5.

$$\begin{cases} \mathrm{T} = \ytableaushort{\none\none1111,\none122,233} \xrightarrow[\textit{reversal}]{e} \mathrm{T}^e = \ytableaushort{\none\none1133,\none222,333} \xrightarrow{\textit{transpose}} \ytableaushort{\none\none3,\none23,123,12,3,3} \longrightarrow \\ w = 1111221332 \longrightarrow \sigma_0 w = 3311222333 \xrightarrow{\textit{reverse}} 3332221133 \longrightarrow \end{cases}$$

$$\begin{cases} \longrightarrow \ytableaushort{\none\none1,\none12,123,23,4,5} = \varrho^{BSS}(\mathrm{T}) \\ \longrightarrow 1231231245 \quad \textit{column word of } \varrho_3(T) = \varrho^{BSS}(T) \end{cases}$$

or

$$\mathrm{T} = \ytableaushort{\none\none1111,\none122,233} \longrightarrow \mathrm{T}^e = \ytableaushort{\none\none3311,\none222,111} \xrightarrow{\textit{rotate}} \mathrm{T}^{e*}$$

$$= \ytableaushort{\none\none\none111,\none\none222,1133} \xrightarrow[\textit{transpose}]{\textit{rotate}} \mathrm{T}^{e*\blacklozenge} = \ytableaushort{\none\none1,\none12,123,23,4,5}.$$

4. Computational complexity of bijection ♦ and conjugation symmetry map

We show that the computational complexity of bijection ♦ is linear on the input. We follow closely [PV] for this section. Using ideas and techniques of Theoretical Computer Science, see [AHO], each bijection can be seen as an algorithm having one type of combinatorial objects as *input*, and another as *output*. We define a *correspondence* as an one–to–one map established by a bijection; therefore, obviously several different defined bijections can produce the same correspondence. In this way one can think of a correspondence as a function which is computed by the algorithm, viz. the bijection. The computational complexity is, roughly, the number of steps in the bijection. Two bijections are *identical* if and only if they define the same correspondence. Obviously one task can be performed by several different algorithms, each one having its own computational complexity, see [AHO]. For example we recall that there are several ways to multiply large integers, from naive algorithms, e.g. the Russian peasant algorithm, to that ones using FFT (Fast Fourier Transform), e.g. Schönhage–Strassen algorithm; see e.g. [GG] for a comprehensive and update reference. Formally, a function f reduces linearly to g, if it is possible to compute f in time linear in the time it takes to compute g; f and g are linearly equivalent if f reduces linearly to g and vice versa. This defines an equivalence relation on functions, which can be translated into a linear equivalence on bijections.

Let $D = (d_1, \ldots, d_n)$ be an array of integers, and let $m = m(D) := \max_i d_i$. The *bit–size* of D, denoted by $\langle D \rangle$, is the amount of space required to store D; for simplicity from now on we assume that $\langle D \rangle = n \lceil log_2 m + 1 \rceil$. We view a bijection $\tau : \mathcal{A} \longrightarrow \mathcal{B}$ as an algorithm which inputs $A \in \mathcal{A}$ and outputs $B = \tau(A) \in \mathcal{B}$. We need to present Young tableaux as arrays of integers so that we can store them and compute their bit–size. Suppose $A \in YT(\lambda/\mu; m)$: a way to encode A is through its recording matrix $(c_{i,j})$, which is defined by $c_{i,j} = a_{i,j} - a_{i,j-1}$; in other words, $c_{i,j}$ is the number of j's in the i–th row of A; this is the way Young tableaux will be presented in the input and output of the algorithms. Finally, we say that a map $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ is *size–neutral* if the ratio $\frac{\langle \gamma(A) \rangle}{\langle A \rangle}$ is bounded for all $A \in \mathcal{A}$. Throughout the paper we consider only size–neutral maps, so we can investigate the linear equivalence of maps comparing them by the number of times other maps are used, without be bothered by the timing. In fact, if we drop the condition of being size–neutral, it can happen that a map increases the bit–size of combinatorial objects, when it transforms the input into the output, and this affects the timing of its subsequent applications. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. We say that δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$. There are many ways to construct new bijections out of existing ones: we call such algorithms *circuits* and we define below several of them that we need.

- : ○ Suppose $\delta_1 : \mathcal{A}_1 \longrightarrow \mathcal{X}_1$, $\gamma : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and $\delta_2 : \mathcal{X}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_2 have linear cost, and consider $\chi = \delta_2 \circ \gamma \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *trivial* and denote it by $I(\delta_1, \gamma, \delta_2)$.
- : ○ Suppose $\gamma_1 : \mathcal{A} \longrightarrow \mathcal{X}$ and $\gamma_2 : \mathcal{X} \longrightarrow \mathcal{B}$, and let $\chi = \gamma_2 \circ \gamma_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *sequential* and denote it by $S(\gamma_1, \gamma_2)$.
- : ○ Suppose $\delta_1 : \mathcal{A} \longrightarrow \mathcal{X}_1 \times \mathcal{X}_2$, $\gamma_1 : \mathcal{X}_1 \longrightarrow \mathcal{Y}_1$, $\gamma_2 : \mathcal{X}_2 \longrightarrow \mathcal{Y}_2$, and $\delta_1 : \mathcal{Y}_1 \times \mathcal{Y}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_1 have linear cost. Consider

$\chi = \delta_2 \circ (\gamma_1 \times \gamma_2) \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$: we call this circuit *parallel* and denote it by $P(\delta_1, \gamma_1, \gamma_2, \delta_2)$.

For a fixed bijection α, we say that $\sqsupset$ is an *α–based ps–circuit* if one of the following holds:

: • $\sqsupset = \delta$, where δ is a bijection having linear cost.
: • $\sqsupset = I(\delta_1, \alpha, \delta_2)$, where δ_1, δ_2 are bijections having linear cost.
: • $\sqsupset = P(\delta_1, \gamma_1, \gamma_2, \delta_2)$, where γ_1, γ_2 are α–based ps–circuits and δ_1, δ_2 are bijections having linear cost.
: • $\sqsupset = S(\gamma_1, \gamma_2)$, where γ_1, γ_2 are α–based ps–circuits.

In other words, $\sqsupset$ is an α–based ps–circuit if there is a parallel–sequential algorithm which uses only a finite number of linear cost maps and a finite number of application of map α. The α–cost of $\sqsupset$ is the number of times the map α is used; we denote it by $s(\sqsupset)$.

Let $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ be a map produced by the α–based ps–circuit $\sqsupset$. We say that $\sqsupset$ computes γ at cost $s(\sqsupset)$ of α. A map β is *linearly reducible* to α, write $\beta \hookrightarrow \alpha$, if there exist a finite α–based ps–circuit $\sqsupset$ which computes β. In this case we say that β can be computed in at most $s(\sqsupset)$ cost of α. We say that maps α and β are linearly equivalent, write $\alpha \sim \beta$, if α is linearly reducible to β, and β is linearly reducible to α. We recall, gluing together, results proved in §4.2 of [PV].

Proposition 4.1. *Suppose $\alpha_1 \hookrightarrow \alpha_2$ and $\alpha_2 \hookrightarrow \alpha_3$, then $\alpha_1 \hookrightarrow \alpha_3$. Moreover, if α_1 can be computed in at most s_1 cost of α_2, and α_2 can be computed in at most s_2 cost of α_3, then α_1 can be computed in at most $s_1 s_2$ cost of α_3. Suppose $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$, then $\alpha_1 \sim \alpha_3$ Suppose $\alpha_1 \hookrightarrow \alpha_2 \hookrightarrow \ldots \hookrightarrow \alpha_n \hookrightarrow \alpha_1$, then $\alpha_1 \sim \alpha_2 \sim \ldots \sim \alpha_n \sim \alpha_1$.*

We state now the computational complexity of bijection $\blacklozenge$ and conjugation symmetry map.

Algorithm 4.1 (Bijection $\blacklozenge$.)**.**
Input: *LR tableau T of skew shape λ/μ, with $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$, $\mu = (\mu_1 \geq \ldots \geq \mu_n)$, and filling $\nu = (\nu_1 \geq \ldots \geq \nu_n)$, having $A = (a_{i,j}) \in M_{n\times n}(\mathbb{N})$ $\quad$ ($a_{i,j} = 0$ if $j > i$) as (lower triangular) recording matrix.*

Write $\widetilde{A}$, a copy of the matrix A.
For $j := n$ down to 2 do
For $i := 1$ to n do
Begin

If $i = j$ *then* $\widetilde{a}_{i,i} := \widetilde{a}_{i,i} + \lambda_1 - \lambda_i$
else
If $j > i$ *then* $\widetilde{a}_{i,j} = 0$ *else* $\widetilde{a}_{i,j} := \widetilde{a}_{i,j} + \widetilde{a}_{i,j+1}$.
End

So far the computational cost is $O\left(n^2\right) = O\left(\langle A \rangle\right)$.

Remark: *For all* $1 \leq i \leq n$ *and* $0 \leq j \leq n - i + 1$, *we have*

$$\widetilde{a}_{i+j+1,i} - \widetilde{a}_{i+j,i} \geq a_{i+j+1,i}.$$

Set a matrix $B = (b_{i,j}) \in M_{\lambda_1 \times \lambda_1}(\mathbb{N})$ *such that* $b_{i,j} = 0$ *for all* i, j.
For $i := 1$ *to* n *do*
Begin
Set $c := 0$.
For $j := 0$ *to* n *do*
Begin
$r := \widetilde{a}_{i+j,i} - a_{i+j,i}$.
For $t := 1$ *to* $a_{i+j,i}$ *do* $b_{r+t,c+t} := 1$.
$c := c + a_{i+j,i}$.
End
End
This part has total computational cost at most equal to

$$O\left(\sum_{1 \leq i.j \leq n} a_{i,j}\right) = O\left(|\lambda \setminus \mu|\right) = O\left(|\lambda| - |\mu|\right) = O\left(\langle T \rangle\right).$$

Output: *B recording matrix of the output tableau.*

Theorem 4.2. *The conjugation symmetry maps* ϱ^{BSS}, ϱ^{WHS} *and* ϱ_3 *are identical, and linear equivalent to the Schützenberger involution* E,

$$\begin{array}{ccccc} T & \overset{e\,\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} \\ \tau \updownarrow & & \tau \updownarrow & & \\ P & \underset{E}{\overset{evacuation}{\longleftrightarrow}} & P^E. & & \end{array}$$

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