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THE NONLINEAR N-MEMBRANES EVOLUTION PROBLEM

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Dedicated to V.A. Solonnikov, on the occasion of his 75th birthday, with admiration and friendship

Abstract: The parabolic N-membranes problem for the p-Laplacian and the complete order constraint on the components of the solution is studied in what concerns the approximation, the regularity and the stability of the variational solutions. We extend to the evolutionary case the characterization of the Lagrange multipliers associated with the ordering constraint in terms of the characteristic functions of the coincidence sets. We give continuous dependence results, and study the asymptotic behavior as $t \to \infty$ of the solution and the coincidence sets, showing that they converge to their stationary counterparts.

Keywords: Variational inequality; p-Laplacian; stability; asymptotic behaviour; coincidence set.
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1. Introduction

The aim of this work is to analyze the quasilinear parabolic N-system associated with the scalar operator involving the p-Laplacian in the elliptic part

$$Pu_i \equiv \partial_t u_i - \nabla \cdot \left(|\nabla u_i|^{p-2} \nabla u_i \right) , \quad i = 1, \dots, N, \tag{1.1}$$

with $1 < p < \infty$, $\partial_t = \partial/\partial t$ and $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)$, in a space-time cylinder $\Omega_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^d$, in the case in which the solution $\boldsymbol{u} = \boldsymbol{u}(x,t) = (u_1, \dots, u_N)$ has all its components completely ordered

$$u_1 \geq u_2 \geq \ldots \geq u_N , \quad \text{a.e. } (x,t) \in \Omega_T, \tag{1.2}$$

and subjected to a given nonhomogeneous term $\boldsymbol{f} = \boldsymbol{f}(x,t) = (f_1, \dots, f_N)$ and given boundary conditions. For simplicity, we assume

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Sigma_T = \partial\Omega \times (0,T) \qquad \text{and} \qquad \boldsymbol{u} = \boldsymbol{h} \quad \text{on } \Omega_0 = \Omega \times \{0\}, \tag{1.3}$$

for given Cauchy data $\boldsymbol{h}$.

The time independent case corresponds to the classical N-membranes problem which can be formulated as an elliptic variational inequality. It has been

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studied for different types of operators (see [20, 21, 8, 2, 3]) associated with a convex subset of a Sobolev space determined by the constraint (1.2). In the recent papers [2, 3] it has been shown, in particular, that the N-membranes problem can be interpreted as a reaction-diffusion system with additional discontinuous nonlinearities. In the evolutionary case (1.1), it will be shown in this work that the solution $\boldsymbol{u}$ solves a parabolic system of the form

$$\boldsymbol{P}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{R}(x,t,\boldsymbol{u}) \quad \text{in } \Omega_T, \tag{1.4}$$

where $\boldsymbol{P}\boldsymbol{u} = (Pu_1, \ldots, Pu_N)$ and each of the components of the nonlinear reaction term $\boldsymbol{R}$ depends on (x,t) through linear combinations of the f_i, $1 \leq i \leq N$, and on $\boldsymbol{u}$ through the characteristic functions $\chi_{j,k} = \chi_{j,k}(x,t)$ of the $N(N-1)/2$ coincidence sets

$$I_{j,k} = \{(x,t) \in \Omega_T \, : \, u_j(x,t) = \ldots = u_k(x,t)\}\ , \quad 1 \leq j < k \leq N, \tag{1.5}$$

i.e., $\chi_{j,k}(x,t) = 1$ if $(x,t) \in I_{j,k}$ and $\chi_{j,k}(x,t) = 0$ otherwise.

We can illustrate the general form of the system (1.4) for $N = 3$ (see [2])

$$\left\{\begin{array}{lllllll} Pu_1 & = & f_1 & + \frac{1}{2}(f_2 - f_1)\chi_{1,2} & & + & \frac{1}{6}(2f_3 - f_2 - f_1)\chi_{1,3} \\ Pu_2 & = & f_2 & - \frac{1}{2}(f_2 - f_1)\chi_{1,2} & + \frac{1}{2}(f_3 - f_2)\chi_{2,3} & + & \frac{1}{6}(2f_2 - f_1 - f_3)\chi_{1,3} \\ Pu_3 & = & f_3 & & - \frac{1}{2}(f_3 - f_2)\chi_{2,3} & + & \frac{1}{6}(2f_1 - f_2 - f_3)\chi_{1,3} \end{array}\right. \tag{1.6}$$

which contains the simpler case $N = 2$, that corresponds to the first two equations with $\chi_{2,3} \equiv 0$ and $\chi_{1,3} \equiv 0$, in which case the third equation is independent of the first two. Noting that, in general, $\chi_{j,k} = \chi_{j,j+1}\chi_{j+1,j+2} \ldots \chi_{k-1,k}$, for $1 \leq j < k \leq N$, in (1.6) the last terms containing $\chi_{1,3} = \chi_{1,2}\chi_{2,3}$ are in fact doubly nonlinear in $\boldsymbol{u}$. This introduces additional difficulties in analyzing the stability of the system with respect to the perturbation of the data. In fact, in section 3, we show that the sufficient conditions on the averages of the components of $\boldsymbol{f}$, obtained in [3] for the stability of the coincidence sets $I_{j,k}$ in the stationary problem, extend to the parabolic case as well. In particular, for $N = 3$, they take the form

$$f_1 \neq f_2, \qquad f_2 \neq f_3, \qquad f_1 \neq \frac{f_2 + f_3}{2}, \qquad f_3 \neq \frac{f_1 + f_2}{2} \qquad \text{a.e. in } \Omega_T.$$

We notice that the stability result on the $\chi_{j,k}$ is not a direct consequence of the stability of the solution $\boldsymbol{u}$ with respect to the data $\boldsymbol{f}$ and $\boldsymbol{h}$, which can, however, be obtained by direct variational methods, as we also show in subsection 2.4.

Classical monotonicity methods (see [15], for example) or the theory of accretive operators and evolution inclusions in Banach spaces (see [12], [18], [22] or [1] and their bibliography) are directly applicable and yield general results on the existence of solutions to our problem, when formulated as a variational inequality in the convex set associated with the constraints (1.2). In section 2, we introduce an approximation of the variational inequality formulation and we obtain directly useful *a priori* estimates for the existence of solutions. We remark that we assume the p'-integrability of $\boldsymbol{f}$ and rely on the p-integrability of a compatible $\boldsymbol{h}$ and its derivatives, but we do not require the boundedness of $\boldsymbol{h}$ nor of the variational solution globally in $\overline{\Omega}_T$.

Considering the relation of the upper and lower membranes (in particular, the two-membrane problem) with the obstacle problem and of the inner membranes of the N-problem, with $N \geq 3$, with the two–obstacles problem, we apply the dual estimates for unilateral parabolic problems (see [6], [12] or [11]) to obtain Lewy--Stampacchia type inequalities

$$\bigwedge_{j=1}^{i} f_j \leq Pu_i \leq \bigvee_{j=i}^{N} f_j \qquad \text{a.e. in } \Omega_T, \quad i = 1, \dots, N, \tag{1.7}$$

for the parabolic operator (1.1). Here we use the notation

$$\bigvee_{i=1}^{k} \xi_i = \xi_1 \vee \ldots \vee \xi_k = \sup\{\xi_1, \dots, \xi_k\},$$

$$\bigwedge_{i=1}^{k} \xi_i = \xi_1 \wedge \ldots \wedge \xi_k = \inf\{\xi_1, \dots, \xi_k\}$$

and we also denote $\xi^+ = \xi \vee 0$ and $\xi^- = -(\xi \wedge 0)$.

We also show how the estimates on Pu_i imply that the variational solution to the N-membranes problem solves a.e. a system of the type (1.4), for an explicit $\boldsymbol{R}$ with the same p'-integrability as $\boldsymbol{f}$, extending the analogous result obtained in [3] for the stationary problem. This implies, in particular when $\boldsymbol{f}$ is bounded, the Hölder continuity of the solution and of its gradient. In fact, this is an immediate consequence of known estimates for the parabolic operator (1.1), even without knowing the explicit form of $\boldsymbol{R}$, as we observe in section 2. Even for the linear case $p = 2$, for which we can apply Solonnikov's estimates in $W^{2,1}_p(\Omega_T)$, the regularity obtained here for the solution of the evolutionary N-membranes problem is new.

In section 3, we study the asymptotic convergence, when $t \to \infty$, of the solution $\boldsymbol{u}(t)$ to the corresponding solution of the stationary problem of [3] in $\boldsymbol{L}^2(\Omega_T)$ (here we denote $\boldsymbol{L}^2(\Omega_T) = \left[L^2(\Omega_T)\right]^N$), in the case $p \geq 2$. We show how a modest convergence of the solution, obtained as in [19], also implies the asymptotic stabilization of the evolution coincidence sets towards the stationary ones, under a natural nondegeneracy assumption identified in [3].

Finally, we observe that most results still hold, with suitable adaptations, for more general quasilinear parabolic scalar operators

$$Pu = \partial_t u - \nabla \cdot (a(x,t,\nabla u)),$$

in particular, for strongly monotone vector fields $a(\cdot,\xi)$, with p-structure as in [3], as well as more general data $\boldsymbol{f}$ in $L^q(0,T;\boldsymbol{L}^r(\Omega))$ (see [4]).

For simplicity of presentation, we limit ourselves here to the case of the p--Laplacian with homogeneous Dirichlet data, i.e., we consider only variational solutions in the usual Sobolev space $\boldsymbol{W}_0^{1,p}(\Omega) = \left[W_0^{1,p}(\Omega)\right]^N$, for $1 < p < \infty$. The case of a time-dependent Dirichlet boundary condition is more delicate and will be considered in [17].

2. Approximation and regularity of variational solutions

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, let $T > 0$, and define the space-time domain $\Omega_T := \Omega \times (0,T)$, with parabolic boundary $\partial_p \Omega_T := \Sigma_T \cup \Omega_0$. We use N-vectorial notation for vector fields

$$\boldsymbol{w} := (w_1, \ldots, w_N) \in \mathbb{R}^N$$

and function spaces $\boldsymbol{F} := [F]^N$. For $1 < p < \infty$, define the differential operator

$$\nabla_p \boldsymbol{w} = (\nabla_p w_1, \ldots, \nabla_p w_N), \ \nabla_p w_i := |\nabla w_i|^{p-2} \nabla w_i \ \text{ and } \ \Delta_p \boldsymbol{w} = \nabla \cdot \nabla_p \boldsymbol{w}.$$

We assume the data satisfy

$$\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega_T) \qquad \text{and} \qquad \boldsymbol{h} \in \mathbb{K} \cap \boldsymbol{L}^2(\Omega), \tag{2.8}$$

where $p' = p/(p-1)$ and $\mathbb{K}$ is the closed convex subset of $\boldsymbol{W}_0^{1,p}(\Omega)$ defined by

$$\mathbb{K} = \left\{ \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega) \ : \ v_1 \geq \ldots \geq v_N, \ \text{a.e. in } \Omega \right\}. \tag{2.9}$$

2.1. Variational formulation of the problem. The evolutive N-membranes problem for the p-Laplace operator consists in finding a vector field $\boldsymbol{u} = \boldsymbol{u}(x,t)$ such that

$$\boldsymbol{u} \in L^p\left(0,T;\boldsymbol{W}_0^{1,p}(\Omega)\right) \cap C\left([0,T];\boldsymbol{L}^2(\Omega)\right), \tag{2.10}$$

$$\partial_t \boldsymbol{u} \in L^{p'}\big(0,T;\boldsymbol{W}^{-1,p'}(\Omega)\big), \tag{2.11}$$

$$\boldsymbol{u}(t) \in \mathbb{K},\ \text{a.e. } t \in (0,T), \quad \boldsymbol{u}(0) = \boldsymbol{h} \in \boldsymbol{L}^2(\Omega), \tag{2.12}$$

and, for a.e. $t \in (0,T)$ and all $\boldsymbol{v} \in \mathbb{K}$,

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + \int_\Omega \nabla_p \boldsymbol{u}(t) : \nabla\left(\boldsymbol{v} - \boldsymbol{u}(t)\right) \geq \int_\Omega \boldsymbol{f}(t)\cdot\left(\boldsymbol{v} - \boldsymbol{u}(t)\right). \tag{2.13}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the sum of the N duality pairings in $\boldsymbol{W}^{-1,p'}(\Omega)\times\boldsymbol{W}_0^{1,p}(\Omega)$ of the components of the vector fields, and $A : B$ denotes the scalar product of the matrices A and B.

We observe that, by a simple comparison argument, there exists at most one solution of (2.10)–(2.13), the variational inequality formulation of the evolutionary N-membranes problem.

2.2. The approximating problem. We approximate the variational inequality (2.13) using a bounded penalization. For that purpose, for each $\varepsilon > 0$, let θ_ε be the real function defined in $[-\infty,+\infty]$ by

$$\theta_\varepsilon(\theta) = \begin{cases} -1 & \text{if} \quad \theta \leq -\varepsilon \\ \theta/\varepsilon & \text{if} \quad -\varepsilon < \theta < 0 \\ 0 & \text{if} \quad \theta \geq 0. \end{cases}$$

The approximating penalized problem is the system of boundary value problems defined as follows:

$$\begin{cases} Pu_i^\varepsilon + \xi_i\theta_\varepsilon\left(u_i^\varepsilon - u_{i+1}^\varepsilon\right) - \xi_{i-1}\theta_\varepsilon\left(u_{i-1}^\varepsilon - u_i^\varepsilon\right) = f_i \quad \text{in } \Omega_T \\ u_i^\varepsilon = 0 \quad \text{on } \Sigma_T \qquad \text{and} \qquad u_i^\varepsilon = h_i \quad \text{on } \Omega_0 \end{cases} \tag{2.14}$$

with $i = 1,\ldots,N$, and the convention $u_0 \equiv +\infty$ and $u_{N+1} \equiv -\infty$, where for $i = 1,\ldots,N$,

$$\xi_0 = \max\left\{\frac{f_1+\cdots+f_i}{i} : i = 1,\ldots,N\right\}, \qquad \xi_i = i\,\xi_0 - (f_1+\cdots+f_i), \tag{2.15}$$

(see [3]). Notice that, for $i = 1,\ldots,N$, we have $\xi_i \geq 0$ and $\xi_i \in L^{p'}(\Omega)$.

Lemma 2.1. *Using the convention $v_0 = +\infty$ and $v_{N+1} = -\infty$, the operator*

$$\langle B\boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{i=1}^{N} \int_\Omega \left(\xi_i \theta_\varepsilon\left(v_i - v_{i+1}\right) - \xi_{i-1}\theta_\varepsilon\left(v_{i-1} - v_i\right)\right) w_i; \quad \boldsymbol{v},\, \boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega),$$

is T-monotone, i.e.,

$$\langle B\boldsymbol{v} - B\boldsymbol{w}, (\boldsymbol{v} - \boldsymbol{w})^+\rangle \geq 0, \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega).$$

Proof: Since we can rewrite

$$\langle B\boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{i=1}^{N-1} \int_\Omega \xi_i \theta_\varepsilon\left(v_i - v_{i+1}\right)\left(w_i - w_{i+1}\right),$$

it is enough to observe that

$$\langle B\boldsymbol{v} - B\boldsymbol{w}, (\boldsymbol{v} - \boldsymbol{w})^+\rangle = \\ \sum_{i=1}^{N-1} \int_\Omega \xi_i \Big(\theta_\varepsilon(v_i - v_{i+1}) - \theta_\varepsilon(w_i - w_{i+1})\Big)\left((v_i - w_i)^+ - (v_{i+1} - w_{i+1})^+\right).$$

As $\xi_i \geq 0$, for $i = 1, \ldots, N$ and θ_ε is monotone nondecreasing, the conclusion follows. ∎

Proposition 2.2. *Under assumption* (2.8)*, the approximating problem* (2.14) *has a unique solution* $(u_1^\varepsilon, \ldots, u_N^\varepsilon) \in L^p(0,T; \boldsymbol{W}_0^{1,p}(\Omega)) \cap C\left([0,T]; \boldsymbol{L}^2(\Omega)\right)$ *such that*

$$u_i^\varepsilon \leq u_{i-1}^\varepsilon + \varepsilon, \quad i = 2, \ldots, N. \tag{2.16}$$

Proof: The existence and uniqueness follow, respectively, from standard results concerning monotone operators and comparison (see [15] or [22]), for instance, using the Faedo-Galerkin approximation. We notice that, since $\boldsymbol{f} \in L^{p'}(\Omega_T) \subset L^{p'}(0,T; \boldsymbol{W}^{-1,p'}(\Omega))$, we obtain, in particular, that

$$\partial_t \boldsymbol{u}^\varepsilon \in L^{p'}(0,T; \boldsymbol{W}^{-1,p'}(\Omega)).$$

To prove inequality (2.16), multiply both the i-th and the $(i-1)$-th equations by $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$, subtract and integrate over Ω, obtaining

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+|^2 + \int_\Omega \Big(\nabla_p u_i^\varepsilon - \nabla_p u_{i-1}^\varepsilon\Big)\cdot \nabla(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \\
= \int_\Omega \Big[\big(f_i - \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) + \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon - \varepsilon)\big)(u_i^\varepsilon - u_{i-1}^\varepsilon)^+ \\
-\big(f_{i-1} - \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon) + \xi_{i-2}\theta_\varepsilon(u_{i-2}^\varepsilon - u_{i-1}^\varepsilon)\big)(u_i^\varepsilon - u_{i-1}^\varepsilon)^+\Big] \\
\leq \int_\Omega \big((f_i - f_{i-1}) + (\xi_i - \xi_{i-1}) - (\xi_{i-1} - \xi_{i-2})\big)(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \\
\leq 0.
\end{aligned}$$

Integrating between 0 and t, using the fact that $h_1 \geq \cdots \geq h_N$ and the inequality

$$\int_\Omega \Big(\nabla_p u_i^\varepsilon - \nabla_p u_{i-1}^\varepsilon\Big)\cdot \nabla(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \geq 0,$$

we get

$$\frac{1}{2}\int_\Omega \Big[\big(u_i^\varepsilon(t) - u_{i-1}^\varepsilon(t) - \varepsilon\big)^+\Big]^2 \leq 0, \tag{2.17}$$

and so $u_i^\varepsilon \leq u_{i-1}^\varepsilon + \varepsilon$ a.e. in Ω_T. ■

2.3. Existence of variational solutions. The proof of the existence of solution for the variational inequality (2.13) will be done passing to the limit in $\varepsilon \to 0$ on the sequence of approximating solutions $\boldsymbol{u}^\varepsilon$, by using the following *a priori* estimates that can be rigorously obtained through the respective Faedo-Galerkin approximations.

Proposition 2.3. *Under assumption* (2.8)*, the solution of the approximating problem* (2.14) *satisfies the following estimates, for a nonnegative constant* C*, independent of* ε*:*

$$\|u_i^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_i^\varepsilon\|_{L^p(\Omega_T)} \leq C, \tag{2.18}$$

$$\|\partial_t u_i^\varepsilon\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C, \tag{2.19}$$

$$\left\{\begin{array}{lclcl} f_1 & \leq & Pu_1^\varepsilon & \leq & f_1+\xi_1 \\ \vdots & & & & \vdots \\ f_i-\xi_{i-1} & \leq & Pu_i^\varepsilon & \leq & f_i+\xi_i \qquad (i=2,\dots,N-1) \\ \vdots & & & & \vdots \\ f_N-\xi_{N-1} & \leq & Pu_N^\varepsilon & \leq & f_N \qquad \textit{a.e. in } \Omega_T. \end{array}\right. \tag{2.20}$$

Proof: For each $i=1,\dots,N$, we easily conclude (2.20) from (2.14) in the form

$$Pu_i^\varepsilon = f_i + g_i^\varepsilon \qquad \text{in } \Omega_T,$$

where

$$g_i^\varepsilon = \xi_{i-1}\theta_\varepsilon\left(u_{i-1}^\varepsilon - u_i^\varepsilon\right) - \xi_i\theta_\varepsilon\left(u_i^\varepsilon - u_{i+1}^\varepsilon\right) \tag{2.21}$$

is uniformly bounded in $\boldsymbol{L}^{p'}(\Omega_T)$.

Then, multiplying each equation in (2.14) by u_i^ε and integrating on $\Omega_t = \Omega\times\,(0,t)$, we get

$$\frac{1}{2}\int_\Omega |u_i^\varepsilon(t)|^2 + \int_{\Omega_t}|\nabla u_i^\varepsilon|^p \leq \int_{\Omega_t}\big(f_i+g_i^\varepsilon\big)u_i^\varepsilon + \frac{1}{2}\int_\Omega |u_i^\varepsilon(0)|^2.$$

Using Poincaré inequality, we find

$$\int_\Omega |u_i^\varepsilon(t)|^2 + \int_{\Omega_t}|\nabla u_i^\varepsilon|^p \leq C_0, \tag{2.22}$$

where the constant C_0 only depends on $\|\boldsymbol{h}\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega_T)}$. Hence, from (2.22), we immediately obtain (2.18). So

$$\Delta_p u_i^\varepsilon \;\text{ is bounded in } L^{p'}(0,T;W^{-1,p'}(\Omega)) \text{ independently of } \varepsilon, \tag{2.23}$$

and we conclude (2.19) by recalling (2.20). ■

Theorem 2.4. *Under assumption* (2.8)*, the problem* (2.12)-(2.13) *has a unique variational solution* $\boldsymbol{u}$ *in the class* (2.10)-(2.11)*.*

In addition, $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ *strongly in* $L^p(0,T;\boldsymbol{W}_0^{1,p}(\Omega))$ *and*

$$\boldsymbol{Pu}^\varepsilon \rightharpoonup \boldsymbol{Pu} \qquad \textit{in} \qquad \boldsymbol{L}^{p'}(\Omega_T)-\textit{weak}. \tag{2.24}$$

Proof: If $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ is a sequence of solutions of the approximating problems (2.14), by the *a priori* estimates (2.18) and (2.19), we can extract a subsequence such that, as $\varepsilon\to 0$,

$$\boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{u} \qquad \text{in} \qquad L^p(0,T;\boldsymbol{W}_0^{1,p}(\Omega)) - \text{weak},$$

$$\partial_t\boldsymbol{u}^\varepsilon \rightharpoonup \partial_t\boldsymbol{u} \qquad \text{in} \qquad L^{p'}(0,T;\boldsymbol{W}^{-1,p'}(\Omega)) - \text{weak},$$

and, by compactness, also $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ strongly in $\boldsymbol{L}^p(\Omega_T)$.

Let $\boldsymbol{v} \in L^p(0,T;\boldsymbol{W}_0^{1,p}(\Omega))$ be such that $\partial_t \boldsymbol{v} \in L^{p'}(0,T;\boldsymbol{W}^{-1,p'}(\Omega))$, $\boldsymbol{v}(t) \in \mathbb{K}$, for a.e. $t \in (0,T)$, and $\boldsymbol{v}(0) = \boldsymbol{h}$. As $\langle B\boldsymbol{v}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle = 0$, we have

$$\langle \partial_t \boldsymbol{u}^\varepsilon, \boldsymbol{v} - \boldsymbol{u}^\varepsilon\rangle + \int_{\Omega_T} \nabla_p \boldsymbol{u}^\varepsilon : \nabla\big(\boldsymbol{v} - \boldsymbol{u}^\varepsilon\big) \geq \int_{\Omega_T} \boldsymbol{f} \cdot \big(\boldsymbol{v} - \boldsymbol{u}^\varepsilon\big)$$

It follows from integration by parts that

$$\langle \partial_t \boldsymbol{u}^\varepsilon, \boldsymbol{v} - \boldsymbol{u}^\varepsilon\rangle = \langle \partial_t \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}^\varepsilon\rangle - \frac{1}{2}\int_\Omega |\boldsymbol{u}^\varepsilon(T) - \boldsymbol{v}(T)|^2$$

and, using the monotonicity, we get

$$\begin{aligned}\langle \partial_t \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}^\varepsilon\rangle + \int_{\Omega_T} \nabla_p \boldsymbol{v} : \nabla\big(\boldsymbol{v} - \boldsymbol{u}^\varepsilon\big) &\geq \int_{\Omega_T} \boldsymbol{f} \cdot \big(\boldsymbol{v} - \boldsymbol{u}^\varepsilon\big)\\ &\quad + \frac{1}{2}\int_\Omega |\boldsymbol{u}^\varepsilon(T) - \boldsymbol{v}(T)|^2\\ &\geq \int_{\Omega_T} \boldsymbol{f} \cdot \big(\boldsymbol{v} - \boldsymbol{u}^\varepsilon\big).\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}\rangle + \int_{\Omega_T} \nabla_p \boldsymbol{v} : \nabla\big(\boldsymbol{v} - \boldsymbol{u}\big) \geq \int_{\Omega_T} \boldsymbol{f} \cdot \big(\boldsymbol{v} - \boldsymbol{u}\big). \tag{2.25}$$

Now, let $\boldsymbol{w} = \boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u})$, $\theta \in (0,1]$. The verification that $\boldsymbol{w}$ can be chosen as test function in (2.25) is immediate. So,

$$\begin{aligned}\langle \partial_t \boldsymbol{u} + \theta\partial_t(\boldsymbol{v} - \boldsymbol{u}), \theta(\boldsymbol{v} - \boldsymbol{u})\rangle + \int_{\Omega_T} \nabla_p\big(\boldsymbol{u} + \theta(\boldsymbol{v} - \boldsymbol{u})\big) : \theta\nabla(\boldsymbol{v} - \boldsymbol{u})\\ \geq \int_{\Omega_T} \theta\boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}).\end{aligned}$$

Dividing both members by θ and letting $\theta \to 0$, we see that $\boldsymbol{u}$ solves the problem

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \int_{\Omega_T} \nabla_p \boldsymbol{u} : \nabla(\boldsymbol{v} - \boldsymbol{u}) \geq \int_{\Omega_T} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}),$$

for all $\boldsymbol{v}$ such that $\boldsymbol{v} \in L^p(0,T;\boldsymbol{W}_0^{1,p}(\Omega))$, $\boldsymbol{v}(t) \in \mathbb{K}$ for a.e. $t \in (0,T)$ and $\boldsymbol{v}(0) = \boldsymbol{h}$. Using standard arguments (see [15]), also

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)\rangle + \int_\Omega \nabla_p \boldsymbol{u}(t) : \nabla(\boldsymbol{v}(t) - \boldsymbol{u}(t)) \geq \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{v}(t) - \boldsymbol{u}(t)),$$

for a.e. $t \in (0,T)$, for all $\boldsymbol{v}$ such that $\boldsymbol{v} \in L^p(0,T;\boldsymbol{W}_0^{1,p}(\Omega))$ and $\boldsymbol{v}(t) \in \mathbb{K}$.

In order to conclude (2.24) it is sufficient to recall the estimates (2.20) for Pu_i^ε and that $\nabla_p u_i^\varepsilon \to \nabla_p u_i$ in an appropriate sense. In fact, recalling (2.21) and using equation (2.14), we conclude that

$$\limsup_{\varepsilon\to 0} \int_{\Omega_T} \nabla_p \boldsymbol{u}^\varepsilon \cdot \nabla(\boldsymbol{u}^\varepsilon - \boldsymbol{u})$$

$$\leq \limsup_{\varepsilon\to 0} \left[\int_{\Omega_T} (\boldsymbol{f} + \boldsymbol{g}^\varepsilon) \cdot (\boldsymbol{u}^\varepsilon - \boldsymbol{u}) - \langle \partial_t \boldsymbol{u}^\varepsilon, \boldsymbol{u} - \boldsymbol{u}^\varepsilon \rangle \right] = 0.$$

By well-known results (see, for instance, [5]) this is sufficient to show that $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ strongly in $L^p(0,T;\boldsymbol{W}_0^{1,p}(\Omega))$ (notice that $\boldsymbol{g}^\varepsilon \rightharpoonup \boldsymbol{g}$ weakly in $\boldsymbol{L}^{p'}(\Omega_T)$, for some $\boldsymbol{g}$). ■

Remark 2.5. *If we assume also that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega_T)$, which is a consequence of* (2.8) *if $1 < p \leq 2$, the Faedo-Galerkin approach yields directly the regularity*

$$\boldsymbol{u} \in H^1(0,T;\boldsymbol{L}^2(\Omega_T)) \cap L^\infty(0,T;\boldsymbol{W}_0^{1,p}(\Omega)) \tag{2.26}$$

through multiplication of (2.14) *by $\partial_t u_i^\varepsilon$.*

2.4. Strong continuous dependence.

Theorem 2.6. *Let $\boldsymbol{u}^*$ be the variational solution to* (2.12)-(2.13) *corresponding to data $\boldsymbol{f}^*$ and $\boldsymbol{h}^*$ satisfying also* (2.8) *and denote*

$$\varepsilon^* \equiv \|\boldsymbol{f}^* - \boldsymbol{f}\|^q_{\boldsymbol{L}^q(\Omega_T)} + \|\boldsymbol{h}^* - \boldsymbol{h}\|^2_{\boldsymbol{L}^2(\Omega)},$$

with $q = p' \wedge 2$ (i.e. $q = p'$ if $p > 2$ and $q = 2$ if $p \leq 2$). Then there exists a positive constant $c = c(T,p)$ such that

$$\sup_{0<t<T} \int_\Omega |\boldsymbol{u}^*(t) - \boldsymbol{u}(t)|^2 + \int_{\Omega_T} |\nabla(\boldsymbol{u}^* - \boldsymbol{u})|^p \leq c\,\varepsilon^* \qquad \textit{if } p \geq 2, \tag{2.27}$$

$$\sup_{0<t<T} \int_\Omega |\boldsymbol{u}^*(t) - \boldsymbol{u}(t)|^2 + \Big(\int_{\Omega_T} |\nabla(\boldsymbol{u}^* - \boldsymbol{u})|^p \Big)^{\frac{2}{p}} \leq c\,\varepsilon^* \qquad \textit{if } 1 < p < 2. \tag{2.28}$$

Proof: Let $\boldsymbol{v} = \boldsymbol{u}^*(t)$ in (2.13) with data $\boldsymbol{f}$ and $\boldsymbol{h}$, and $\boldsymbol{v} = \boldsymbol{u}(t)$ in (2.13) with data $\boldsymbol{f}^*$ and $\boldsymbol{h}^*$. In the latter case, we have

$$\langle \partial_t \boldsymbol{u}^*(t), \boldsymbol{u}(t) - \boldsymbol{u}^*(t)\rangle + \int_\Omega \nabla_p \boldsymbol{u}^*(t) : \nabla(\boldsymbol{u}(t) - \boldsymbol{u}^*(t)) \geq \int_\Omega \boldsymbol{f}^*(t) \cdot (\boldsymbol{u}(t) - \boldsymbol{u}^*(t)). \quad (2.29)$$

From (2.13) and (2.29), integrating between 0 and t, we obtain

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}^*(t) - \boldsymbol{u}(t)|^2 + \int_0^t \int_\Omega (\nabla_p \boldsymbol{u}^* - \nabla_p \boldsymbol{u}) : \nabla(\boldsymbol{u}^* - \boldsymbol{u}) \leq \int_0^t (\boldsymbol{f}^* - \boldsymbol{f}) \cdot (\boldsymbol{u}^* - \boldsymbol{u}) + \frac{1}{2}\int_\Omega |\boldsymbol{h}^* - \boldsymbol{h}|^2. \quad (2.30)$$

In the case $p \geq 2$, since

$$\int_0^t \int_\Omega (\nabla_p \boldsymbol{u}^* - \nabla_p \boldsymbol{u}) : \nabla(\boldsymbol{u}^* - \boldsymbol{u}) \geq C_p \int_0^t \int_\Omega |\nabla(\boldsymbol{u}^* - \boldsymbol{u})|^p,$$

the conclusion follows easily by using Hölder and Poincaré inequalities.

In the case $1 < p < 2$, from (2.30) we find

$$\int_\Omega |\boldsymbol{u}^*(t) - \boldsymbol{u}(t)|^2 \leq \varepsilon^* + \int_0^t \int_\Omega |\boldsymbol{u}^* - \boldsymbol{u}|^2,$$

which, by Gronwall inequality yields, first

$$\sup_{0<t<T} \int_\Omega |\boldsymbol{u}^*(t) - \boldsymbol{u}(t)|^2 \leq e^T \varepsilon^*$$

and, afterwards

$$\int_0^T \int_\Omega (\nabla_p \boldsymbol{u}^* - \nabla_p \boldsymbol{u}) : \nabla(\boldsymbol{u}^* - \boldsymbol{u}) \leq \frac{1}{2}(1 + Te^T)\varepsilon^*. \quad (2.31)$$

Next we consider the following reverse Hölder inequality: given $0 < r < 1$ and $r' = \frac{r}{r-1}$, if $F \in L^r(\Omega)$, $FG \in L^1(\Omega)$ and $\displaystyle\int_\Omega |G(x)|^{r'} dx < \infty$ in Ω_T, one has

$$\left(\int_\Omega |F(x)|^r dx\right)^{\frac{1}{r}} \leq \left(\int_\Omega |F(x)G(x)| dx\right)\left(\int_\Omega |G(x)|^{r'} dx\right)^{-\frac{1}{r'}}.$$

Letting $r = \frac{p}{2}$ and, for $i = 1, \ldots, N$, $F = |\nabla(u_i^* - u_i)|^2$ we get

$$\int_{\hat{\Omega}_t^i} (\nabla_p u_i^* - \nabla_p u_i) \cdot \nabla(u_i^* - u_i) \geq \int_{\hat{\Omega}_t^i} \frac{|\nabla(u_i^* - u_i)|^2}{(|\nabla u_i^*| + |\nabla u_i|)^{2-p}}$$

$$\geq \left(\int_{\hat{\Omega}_t^i} |\nabla(u_i^* - u_i)|^p \right)^{\frac{2}{p}} \left(\int_{\hat{\Omega}_t^i} \big(|\nabla u_i^*| + |\nabla u_i| \big)^p \right)^{\frac{p-2}{p}},$$

where $\hat{\Omega}_t^i = \{(x,t) \in \Omega_t : |\nabla u_i^*| + |\nabla u_i| > 0\}$. Thus, if we denote

$$\alpha_p \geq \left(\int_{\Omega_T} \big(|\nabla u_i^*| + |\nabla u_i| \big)^p \right)^{\frac{2-p}{p}},$$

by (2.30), we conclude (2.28) from

$$\sum_{i=1}^N \left(\int_{\Omega_T} |\nabla(u_i^* - u_i)|^p \right)^{\frac{2}{p}} \leq \alpha_p \sum_{i=1}^N \int_{\Omega_T} (\nabla_p u_i^* - \nabla_p u_i) \cdot \nabla(u_i^* - u_i) \leq \alpha_p \, c \, \varepsilon^*.$$

■

2.5. Hölder continuity and further regularity of the solution. The regularity of the variational solutions of the evolution N-membranes problem does not, in general, yield their boundedness for $1 < p \leq d$; but, by Sobolev imbedding, the solutions are bounded for $p > d$ and even Hölder continuous in the space variables for each $t \in (0, T)$.

However, estimates (2.20) and (2.24) imply that, in fact, $P\boldsymbol{u}$ has the same regularity in Ω_T as the data $\boldsymbol{f}$. Then, if $\boldsymbol{f} \in \boldsymbol{L}^\infty(\Omega_T)$, local and global Hölder estimates for the evolution p-Laplace equation may be directly applied to bounded solutions of the N-membranes problem (see [9], [14] or [10]). In order to illustrate these results, we assume in addition that $\boldsymbol{h} \in \boldsymbol{L}^\infty(\Omega)$, which also implies that $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega_T)$, and consequently that $\boldsymbol{u}$ and $\nabla \boldsymbol{u}$ are locally Hölder continuous. Referring to [7] and [14] for the boundary and initial regularity in the space of Hölder continuous functions C^α, $0 < \alpha < 1$, with the standard parabolic norms, we may state the following result:

Theorem 2.7. *Suppose $\boldsymbol{f} \in \boldsymbol{L}^\infty(\Omega)$ and the initial data $\boldsymbol{h} \in \boldsymbol{C}^\alpha(\overline{\Omega}) \cap \mathbb{K}$, $0 < \alpha < 1$. Then the solution $\boldsymbol{u} \in \boldsymbol{C}^{\alpha'}(\overline{\Omega}_T)$, $0 < \alpha' \leq \alpha < 1$, and $\nabla \boldsymbol{u} \in \boldsymbol{C}^\beta_{\mathrm{loc}}(\overline{\Omega}_T)$, for some $0 < \beta < 1$. If, in addition, $\partial\Omega \in \boldsymbol{C}^{1,\beta}$ and $\nabla \boldsymbol{h} \in \boldsymbol{C}^\beta(\overline{\Omega})$, $0 < \beta < 1$, then also $\nabla \boldsymbol{u} \in \boldsymbol{C}^{\beta'}(\overline{\Omega}_T)$, for some $0 < \beta' \leq \beta < 1$.*

In case of a linear operator ($p = 2$) we can apply directly Solonnikov's parabolic estimates (see [13], Thm. 9.1 of page 341).

Theorem 2.8. *Let $p = 2$. Then, for any $\boldsymbol{f} \in \boldsymbol{L}^q(\Omega_T)$, $q \geq 2$, the solution $\boldsymbol{u}$ to* (2.12)-(2.13) *satisfies $\boldsymbol{u} \in \boldsymbol{W}^{2,1}_{q,\mathrm{loc}}(\Omega_T)$, which implies, by Sobolev imbeddings, that $\boldsymbol{u}$ and $\nabla \boldsymbol{u}$ are locally Hölder continuous, respectively for $q > \frac{d+2}{d}$ and $q > d + 2$. If, in addition, $\boldsymbol{h} \in \mathbb{K} \cap \boldsymbol{W}^{2-\frac{2}{q},q}(\Omega)$, those results can be extended up to the boundary $\partial\Omega \in C^2$ and up to $t = 0$, i.e., $\boldsymbol{u} \in \boldsymbol{W}^{2,1}_q(\Omega_T)$ and $\boldsymbol{u}$, $\nabla \boldsymbol{u}$ are Hölder continuous on $\overline{\Omega}_T$.*

3. The N-system and its stability

The N-membranes problem can, a posteriori, be regarded as a lower obstacle problem for u_1, a double obstacle problem for u_j, $j = 2 \leq j \leq N - 1$, and an upper obstacle problem for u_N. This fact has interesting consequences and, similarly to the theory of the obstacle problem that we recall briefly for completeness, allows us to characterize the N-membranes problem as a nonlinear parabolic system with known discontinuous nonlinearities on the right hand side as in (1.4).

3.1. Dual estimates for obstacle type problems. We consider the scalar two--obstacles problem for the nonlinear operator P defined in (1.1), with compatible Cauchy-Dirichlet data on $\partial_p\Omega_T$. Let

$$\varphi \in L^{p'}(\Omega_T), \qquad \eta \in W^{1,p}_0(\Omega)) \cap L^2(\Omega), \tag{3.32}$$

$$\psi_1,\, \psi_2 \in L^p(0, T; W^{1,p}(\Omega)), \quad \psi_1 \geq \psi_2 \ \text{ in } \Omega_T, \quad \psi_1 \geq 0 \geq \psi_2 \text{ on } \Sigma_T, \tag{3.33}$$

and, for $j = 1, 2$,

$$\partial_t \psi_j \in L^{p'}(0, T; W^{-1,p'}(\Omega)), \quad P\psi_j \in L^{p'}(\Omega_T), \quad \psi_1(0) \geq \eta \geq \psi_2(0) \text{ on } \Omega_0. \tag{3.34}$$

Using the Lipschitz continuous function θ_ε defined in subsection 2.2 for each $\varepsilon > 0$, we may easily show that the problem

$$Pw^\varepsilon + \zeta_2\theta_\varepsilon(w^\varepsilon - \psi_2) - \zeta_1\theta_\varepsilon(\psi_1 - w^\varepsilon) = \varphi \quad \text{ in } \Omega_T, \tag{3.35}$$

$$w^\varepsilon = 0 \quad \text{ on } \Sigma_T \qquad \text{and} \qquad w^\varepsilon = \eta \quad \text{ on } \Omega_0, \tag{3.36}$$

where $\zeta_1 = (P\psi_1 - \varphi)^-$ and $\zeta_2 = (P\psi_2 - \varphi)^+$, has a unique solution $w^\varepsilon \in L^p(0, T; W^{1,p}_0(\Omega)) \cap C([0, T]; L^2(\Omega))$, with $Pw^\varepsilon \in L^{p'}(\Omega_T)$, uniformly in $\varepsilon \leq 1$.

Similarly to Proposition 2.2, it is easy to show that

$$\psi_2 - \varepsilon \leq w^\varepsilon \leq \psi_1 + \varepsilon \qquad \text{a.e. in } \Omega_T,$$

and, when $\varepsilon \to 0$, as in Theorem 2.4, that

$$w^\varepsilon \to w \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)),$$

where w is the unique solution of the double obstacle problem

$$w \in \mathbb{K}_{\psi_2}^{\psi_1} = \{v \in L^p(0,T;W_0^{1,p}(\Omega)) : \psi_1 \geq v \geq \psi_2 \text{ in } \Omega_T\}, \tag{3.37}$$

$$\int_{\Omega_T} (Pw - \varphi)(v - w) \geq 0, \qquad \forall\, v \in \mathbb{K}_{\psi_2}^{\psi_1}, \qquad \text{a.e. } t \in (0,T), \tag{3.38}$$

such that $w(0) = \eta$ on Ω. The solution w satisfies also

$$w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega)) \qquad \text{and} \qquad Pw \in L^{p'}(\Omega_T)$$

and, arguing as in Proposition 4.1 of [16], we can state the following important property.

Proposition 3.1. *The solution* w *to* (3.37)-(3.38)*, under assumptions* (3.32)-(3.34)*, satisfies the parabolic nonlinear equation*

$$Pw = \varphi + (P\psi_2 - \varphi)^+ \chi_{\{w=\psi_2\}} - (P\psi_1 - \varphi)^- \chi_{\{w=\psi_1\}} \qquad \textit{a.e. in } \Omega_T. \tag{3.39}$$

In addition, we have the Lewy-Stampacchia inequalities

$$\varphi - (P\psi_1 - \varphi)^- = \varphi \wedge P\psi_1 \leq Pw \leq \varphi \vee P\psi_2 = \varphi + (P\psi_2 - \varphi)^+ \qquad \textit{a.e. in } \Omega_T \tag{3.40}$$

and the a.e. in Ω_T *necessary conditions for contact with the obstacles*

$$\{w = \psi_1\} \subset \{P\psi_1 \leq \varphi\} \qquad \textit{and} \qquad \{w = \psi_2\} \subset \{P\psi_2 \geq \varphi\} \tag{3.41}$$

being the inclusions valid up to subsets of Ω_T *with zero measure.*

Remark 3.2. *We note that for the case of only one-obstacle, we have similar properties. In fact, if we formally take* $\psi_1 \equiv +\infty$*, we have a lower obstacle problem*

$$w \geq \psi_2 \qquad \textit{and} \qquad \varphi \leq Pw \leq \varphi \vee P\psi_2 \qquad \textit{a.e. in} \qquad \Omega_T, \tag{3.42}$$

and, with $\psi_2 \equiv -\infty$*, an upper obstacle problem*

$$w \leq \psi_1 \qquad \textit{and} \qquad \varphi \wedge P\psi_1 \leq Pw \leq \varphi \qquad \textit{a.e. in} \qquad \Omega_T. \tag{3.43}$$

Analogously, the semilinear equation holds in each case with the corresponding characteristic function, respectively.

Remark 3.3. *As observed in* [16], *we have*

$$Pw = \varphi \qquad a.e.\ in\ \{\psi_2 < w < \psi_1\} \tag{3.44}$$

and due to the fact that both Pw *and* $P\psi_j$ *are integrable, we have*

$$Pw = P\psi_j \qquad a.e.\ in\ \{w = \psi_j\} \qquad for\ j = 1, 2. \tag{3.45}$$

Using the regularity of Theorem 2.4, we easily see that each component u_i of the N-membranes problem solves an obstacle type problem (3.37)-(3.38) with $\varphi = f_i$, $\psi_1 = u_{i-1}$ and $\psi_2 = u_{i+1}$ (with the conventions $u_0 \equiv +\infty$ and $u_{N+1} \equiv -\infty$ corresponding to the one-obstacle problems). Hence, we have from (3.42), (3.40) and (3.43), respectively, a.e. in Ω_T,

$$\begin{array}{ccccc}
f_1 & \leq & Pu_1 & \leq & f_1 \vee Pu_2 \\
\vdots & & & & \vdots \\
f_i \wedge Pu_{i-1} & \leq & Pu_i & \leq & f_i \vee Pu_{i+1} \qquad (i = 2, \ldots, N-1) \\
\vdots & & & & \vdots \\
f_N \wedge Pu_{N-1} & \leq & Pu_N & \leq & f_N \qquad \text{a.e. in } \Omega_T.
\end{array}$$

By simple iteration, we have shown the following Lewy-Stampacchia type inequalities, that extend Theorem 3.5 of [3] to the evolution N-membranes problem

Theorem 3.4. *The solution* $\boldsymbol{u}$ *of* (2.12)-(2.13) *satisfies*

$$\bigwedge_{j=1}^{i} f_j \leq Pu_i \leq \bigvee_{j=i}^{N} f_j \qquad a.e.\ in\ \Omega_T, \qquad i = 1, \ldots, N.$$

3.2. The nonlinear N-system. As a consequence of the equivalence of the N-membranes inequality with two one–obstacle problems and $N-2$ two–obstacles problems, we may prove the equivalence of this inequality with a N-system of equations, strongly coupled by the $\frac{N(N-1)}{2}$ coincidence sets $I_{j,k}$ defined in (1.5). Indeed, we can argue as in section 4 of [3], and since we know that $Pu_i \in L^{p'}(\Omega_T)$, for all $i = 1, \ldots, N$, we have on each coincidence set

$$Pu_j = \cdots = Pu_k \qquad \text{a.e. in} \qquad I_{j,k} = \{u_j = \cdots = u_k\}$$

and we conclude, for each $j \leq i \leq k$,

$$Pu_i = \langle \boldsymbol{f} \rangle_{j,k} \qquad \text{a.e. in} \qquad I_{j,k},$$

where we introduce the averages of $\boldsymbol{f}$ by

$$\langle \boldsymbol{f} \rangle_{j,k} = \frac{f_j + \cdots + f_k}{k - j + 1}, \qquad 1 \leq j \leq k \leq N.$$

On the other hand, in the complementary sets $\Omega_T \setminus I_{j,k}$, for each $i > k > j$ or $i < j < k$, we have

$$Pu_i = f_i \qquad \text{a.e. in} \qquad \Omega_T \setminus I_{j,k},$$

and we conclude, as in [3], the following explicit form for (1.4).

Theorem 3.5. *The variational solution of the N-membranes problem* (2.12)-(2.13) *satisfies the system* ($i = 1, \ldots, N$)

$$Pu_i = f_i + \sum_{1 \leq j < k \leq N,\, j \leq i \leq k} b_i^{j,k}[\boldsymbol{f}]\, \chi_{j,k} \qquad \textit{a.e. in } \Omega_T, \tag{3.46}$$

where $\chi_{j,k}$ denotes the characteristic function of each $I_{j,k}$ and

$$b_i^{j,k}[\boldsymbol{f}] \;=\; \begin{cases} \langle \boldsymbol{f} \rangle_{j,k} - \langle \boldsymbol{f} \rangle_{j,k-1} & \textit{if } \ i = j \\[2ex] \langle \boldsymbol{f} \rangle_{j,k} - \langle \boldsymbol{f} \rangle_{j+1,k} & \textit{if } \ i = k \\[2ex] \frac{2}{(k-j)(k-j+1)} \left(\langle \boldsymbol{f} \rangle_{j+1,k-1} - \frac{1}{2}(\boldsymbol{f}_j + \boldsymbol{f}_k) \right) & \textit{if } \ j < i < k. \end{cases}$$

For the particular case $N = 3$ (and $N = 2$), we can easily deduce (1.6) from (3.46).

3.3. Convergence of coincidence sets. From Theorem 2.6, we know that if for sequences

$$\boldsymbol{f}^\nu \underset{\nu}{\longrightarrow} \boldsymbol{f} \qquad \text{in } \boldsymbol{L}^q(\Omega_T), \qquad q = p' \wedge 2, \tag{3.47}$$

$$\boldsymbol{h}^\nu \underset{\nu}{\longrightarrow} \boldsymbol{h} \qquad \text{in } \boldsymbol{L}^2(\Omega_T), \tag{3.48}$$

then, the corresponding solutions of (2.12)-(2.13) also converge

$$\boldsymbol{u}^\nu \underset{\nu}{\longrightarrow} \boldsymbol{u} \qquad \text{in } C^0([0,T]; \boldsymbol{L}^2(\Omega)) \cap L^p(0,T; \boldsymbol{W}_0^{1,p}(\Omega)). \tag{3.49}$$

Consequently, we have

$$\Delta_p \boldsymbol{u}^\nu \underset{\nu}{\rightharpoonup} \Delta_p \boldsymbol{u} \qquad \text{in } L^{p'}(0,T; \boldsymbol{W}^{-1,p'}(\Omega)) - \text{weak}$$

and, by Theorem 3.4, also

$$P\boldsymbol{u}^\nu \underset{\nu}{\rightharpoonup} P\boldsymbol{u} \qquad \text{in } \boldsymbol{L}^q(\Omega_T) - \text{weak}.$$

Since the characteristic functions $\chi^\nu_{j,k} = \chi_{\{u^\nu_j=\cdots=u^\nu_k\}}$ satisfy $0 \le \chi^\nu_{j,k} \le 1$ a.e. in Ω_T, there are $\chi^*_{j,k} \in L^\infty(\Omega_T)$ such that

$$\chi^\nu_{j,k} \underset{\nu}{\longrightarrow} \chi^*_{j,k} \qquad \text{in } \boldsymbol{L}^\infty(\Omega_T) - \text{weak} * .$$

Passing to the limit in

$$Pu^\nu_i = f^\nu_i + \sum_{1\le j<k\le N,\, j\le i\le k} b^{j,k}_i[\boldsymbol{f}^\nu]\,\chi^\nu_{j,k},$$

we obtain, for each $i = 1,\dots,N$,

$$Pu_i = f_i + \sum_{1\le j<k\le N,\, j\le i\le k} b^{j,k}_i[\boldsymbol{f}]\,\chi^*_{j,k},$$

which compared with (3.46) yields

$$\sum_{1\le j<k\le N,\, j\le i\le k} b^{j,k}_i[\boldsymbol{f}](\chi_{j,k} - \chi^*_{j,k}) = 0 \qquad \text{a.e. in } \Omega_T.$$

Arguing exactly as in the proof of Theorem 4.6 of [3], we conclude, under the same nondegeneracy assumption for the limit data, namely

$$\langle \boldsymbol{f}\rangle_{i,j} \ne \langle \boldsymbol{f}\rangle_{j+1,k}, \text{ a.e. in } \Omega_T, \quad \text{for all } i,j,k \in \{1,\dots,N\}, \text{ with } i\le j\le k, \tag{3.50}$$

that $\chi_{j,k} = \chi^*_{j,k}$ and prove the following stability property for the respective coincidence sets $I^\nu_{j,k} = \{u^\nu_j = \cdots = u^\nu_k\}$.

Theorem 3.6. *Under the convergence assumptions* (3.47) *and* (3.48), *the characteristic functions associated with the convergent variational solutions* (3.49) *also converge*

$$\chi_{\{u^\nu_j=\cdots=u^\nu_k\}} \underset{\nu}{\longrightarrow} \chi_{\{u_j=\cdots=u_k\}} \qquad \text{in } L^s(\Omega_T),$$

for any $1 \le s < \infty$, *all* $1 \le j < k \le N$, *provided the nondegeneracy condition* (3.50) *holds.*

3.4. Asymptotic stabilization as $t \to \infty$. In this section we assume $p \ge 2$ and we consider the unique solution $\boldsymbol{u}^\infty$ to the stationary N-membranes problem for a given $\boldsymbol{f}^\infty \in \boldsymbol{L}^{p'}(\Omega)$:

$$\boldsymbol{u}^\infty \in \mathbb{K}: \qquad \int_\Omega \nabla_p \boldsymbol{u}^\infty : \nabla(\boldsymbol{v}-\boldsymbol{u}^\infty) \ge \int_\Omega \boldsymbol{f}^\infty\cdot(\boldsymbol{v}-\boldsymbol{u}^\infty), \qquad \forall \boldsymbol{v}\in\mathbb{K}. \tag{3.51}$$

Supposing that the problem (2.12)-(2.13) is solvable for all $T < \infty$ and that $\boldsymbol{f}(t) \longrightarrow \boldsymbol{f}^\infty$ in $\boldsymbol{L}^{p'}(\Omega)$ as $t \to \infty$ in the sense

$$\int_t^{t+1} \int_\Omega |\boldsymbol{f}(t) - \boldsymbol{f}^\infty|^{p'} \longrightarrow 0 \qquad \text{as } t \to \infty, \tag{3.52}$$

by the results of [19], the evolutive solution $\boldsymbol{u}(t)$ is such that

$$\boldsymbol{u}(t) \underset{t\to\infty}{\longrightarrow} \boldsymbol{u}^\infty \qquad \text{in } \boldsymbol{L}^2(\Omega), \tag{3.53}$$

$$\int_t^{t+1} \int_\Omega |\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^\infty|^p \longrightarrow 0 \qquad \text{as } t \to \infty. \tag{3.54}$$

By the results of [3], the stationary solution also solves the nonlinear N-system

$$-\Delta_p u_i^\infty = f_i^\infty + \sum_{1 \le j < k \le N,\ j \le i \le k} b_i^{j,k}[\boldsymbol{f}^\infty]\, \chi_{j,k}^\infty \qquad \text{a.e. in } \Omega, \tag{3.55}$$

where $\chi_{j,k}^\infty = \chi_{\{u_j^\infty = \cdots = u_k^\infty\}}$ denotes the characteristic function of the limit coincidence set $I_{j,k}^\infty = \{x \in \Omega : u_j^\infty(x) = \cdots = u_k^\infty(x)\}$.

Denoting by $\chi_{j,k}(t)$ the characteristic functions of $I_{j,k}(t) = \{u_j(t) = \cdots = u_k(t)\}$ at time t, we have the following asymptotic convergence result as $t \to \infty$.

Theorem 3.7. *Under assumption* (3.52)*, the variational solution of the evolution N-membranes problem converges to the corresponding stationary solution in the sense* (3.53) *and* (3.54)*. In addition, the characteristic functions satisfy*

$$\chi_{j,k}(t) \longrightarrow \chi_{j,k}^\infty \qquad \text{as } t \to \infty \qquad \text{in } L^s(\Omega), \tag{3.56}$$

for any $1 \le s < \infty$, for all $1 \le j < k \le N$, provided we assume

$$\langle \boldsymbol{f}^\infty \rangle_{i,j} \ne \langle \boldsymbol{f}^\infty \rangle_{j+1,k} \qquad \text{a.e. in } \Omega, \qquad \text{for all } 1 \le i \le j < k \le N. \tag{3.57}$$

Proof: We rewrite (3.54) for $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\infty$ as

$$\int_t^{t+1} \int_\Omega |\nabla \boldsymbol{w}(\tau)|^p d\tau = \int_0^1 \int_\Omega |\nabla \boldsymbol{w}(t+\sigma)|^p d\sigma \longrightarrow 0 \qquad \text{as } t \to \infty,$$

and this convergence can be interpreted as

$$\boldsymbol{w}_\sharp(t) \longrightarrow 0 \qquad \text{as } t \to \infty \text{ in } L^p(0,1; \boldsymbol{W}_0^{1,p}(\Omega)), \tag{3.58}$$

where we define $\boldsymbol{w}_\sharp \in L^\infty(0,\infty; L^p(0,1; \boldsymbol{W}_0^{1,p}(\Omega)))$ as

$$\boldsymbol{w}_\sharp(t)(\sigma, \cdot) = \boldsymbol{w}(t+\sigma, \cdot) \in \boldsymbol{W}_0^{1,p}(\Omega), \quad \sigma \in (0,1).$$

Consequently, from (3.58) we have

$$\Delta_p \boldsymbol{u}_\sharp(t) \rightharpoonup \Delta_p \boldsymbol{u}_\sharp^\infty \qquad \text{as } t \to \infty \text{ in } L^{p'}(0,1;\boldsymbol{W}^{-1,p'}(\Omega)) - \text{weak}$$

and, recalling the estimates of Theorem 3.4 and the assumption (3.52), we may conclude

$$(\partial_t \boldsymbol{u}_\sharp - \Delta_p \boldsymbol{u}_\sharp)(t) \rightharpoonup -\Delta_p \boldsymbol{u}_\sharp^\infty \qquad \text{as } t \to \infty \text{ in } L^{p'}(0,1;\boldsymbol{L}^{p'}(\Omega)) - \text{weak}.$$

Since $\boldsymbol{u}_\sharp(t)$ solves (3.46), a.e. in Ω and for a.e. $t > 0$, we can pass to the limit, as $t \to \infty$, in $L^{p'}(0,1;\boldsymbol{L}^{p'}(\Omega))$. As in the proof of Theorem 3.6 (and Theorem 4.6 of [3]), we conclude that assumption (3.57) implies the convergence $\chi_{j,k}(t) \longrightarrow \chi_{j,k}^\infty$ as $t \to \infty$, first as functions of $L^\infty(0,1;L^\infty(\Omega))$ with the weak-$*$ topology and, afterwards, also in the sense of (3.56). Indeed, since they are characteristic functions and any subsequence of $\chi_{j,k}(t)$ has the same limit $\chi_{j,k}^\infty$, their weak convergence implies the strong convergence in $L^s(\Omega)$ for all $1 \le s < \infty$. ■

References

[1] Akagi, G. & Ôtani, M., *Evolution inclusions governed by subdifferentials in reflexive Banach spaces*, J. Evol. Eq. **4** (2004), 519–541.

[2] Azevedo, A. & Rodrigues, J.F. & Santos, L., *Remarks on the two and three membranes problem*, Recent advances in elliptic and parabolic problems, 19–33, World Sci. Publ., Hackensack, NJ, 2005.

[3] Azevedo, A. & Rodrigues, J.F. & Santos, L., *The N-membranes problem for quasilinear degenerate systems*, Interfaces Free Bound. **7** (2005), nº 3, 319–337.

[4] Boccardo, L. & Dall'Aglio, A. & Gallouët, T. & Orsina, L., *Existence and regularity results for some nonlinear parabolic equations*, Adv. Math. Sci. Appl. **9** (1999), nº 2, 1017–1031.

[5] Boccardo, L. & Murat, F., *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. **19** (1992), nº 6, 581–597.

[6] Charrier, P. & Troianiello, G.M., *On strong solutions to parabolic unilateral problems with obstacle dependent on time*, J. Math. Anal. Appl. **65** (1978), 110–125.

[7] Chen, Y.Z. & DiBenedetto, E., *Boundary estimates for solutions of nonlinear degenerate parabolic systems*, J. Reine Angew. Math. **395** (1989), 102–131.

[8] Chipot, M. & Vergara-Cafarelli, G., *The N-membranes problem*, Appl. Math. Optim. **13** (1985), nº 3, 231–249.

[9] DiBenedetto, E., *Degenerate parabolic equations*, Springer-Verlag, New York, 1993.

[10] DiBenedetto, E. & Urbano, J.M. & Vespri, V., *Current issues on singular and degenerate evolution equations*, Handbook of Differential Equations, Evolutionary Equations, vol. **1**, pp. 169–286, Elsevier, 2004.

[11] Donati, F., *A penalty method approach to strong solutions of some nonlinear parabolic unilateral problems*, Nonlinear Anal. **6** (1982), nº 6, 585–597.

[12] Donati, F. & Mazteu, M., *On the strong solutions of some nonlinear evolution problems in ordered Banach spaces*, Boll. Un. Mat. Ital. B (5) **16–B** (1979), nº 1, 54–73.

[13] Ladyzhenskaja, O.A. & Solonnikov, V.A. & Ural'ceva, N.N., *Linear and quasilinear equations of parabolic type*, AMS Transl. Math. Mono., Vol. **23**, Providence, 1968.

[14] Lieberman, G., *Boundary and initial regularity for solutions of degenerate parabolic equations*, Nonlinear Anal. **20** (1993), nº 5, 551–569.

[15] Lions, J.L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier–Villars, Paris, 1969.

[16] Rodrigues, J.F. & Santos, L., *On a constrained reaction-diffusion system related to multiphase problems*, accepted for publication in DCDS, Issue in honour of Mayan Mimura, arXiv:0711.2814.

[17] Rodrigues, J.F. & Santos, L. & Urbano, J.M., *The quasilinear N-membranes problem with time-dependent Dirichlet data*, in preparation.

[18] Showalter, R.E., *Monotone operators in Banach spaces and nonlinear partial differential equations*, AMS, Providence, 1997.

[19] Simon, J., *Quelques propriétés de solutions d'équations et d'inéquations d'évolution paraboliques nonlinéaires*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), nº 4, 585–609.

[20] Vergara-Caffarelli, G., *Regolarità di un problema di disequazioni variazionali relativo a due membrane*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **50** (1971), 659–662.

[21] Vergara-Caffarelli, G., *Variational inequalities for two surfaces of constant mean curvature*, Arch. Rational Mech. Anal. **56** (1974/75), 334–347.

[22] Zheng, S., *Nonlinear evolution equations*, Monographs and Surveys in Pure and Applied Mathematics, Vol. **133**, Chapman & Hall/CRC, Boca Raton, FL, 2004.

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