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MEMORY IN THE BLACK-SCHOLES MODEL

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ABSTRACT: The evolution in time of European options is usually studied using the Black-Scholes formula. This formula is obtained from the equivalence between the Black-Scholes equation and a heat equation. The solution of the last equation presents infinite speed of propagation which induces the same property for European options. In this paper we study integro-differential equations which can be used to describe the evolution of European options and which is established replacing the heat equation by a delayed heat equation.

Key words: Black-Scholes equation, Fick's flux, non-Fickian flux, integro-differential equation.

Mathematics Subject Classification (2000): 35B35; 35B40; 35C10, 35K65

1. Introduction

The evolution in time of European options value is usually studied using the so called Black-Scholes equation. The formula used to compute the value $C(S,t)$ of an option, at time t and when the underlying asset has the value S, is established by rewriting Black-Scholes equation in the form of an equivalent heat equation. However it is well known that the behaviour of heat propagation and many other diffusion phenomena can not be exactly described by the classical diffusion equation. The physical arguments that support this assertion lie essentially on the fact that whereas in real world the propagation speed of information is finite, diffusion equation models phenomena where information travels with an infinite speed. As the solution of heat equation presents an infinite propagation speed, the same behaviour is presented by the solution of Black-Scholes equation. This means that to compute the value of an option $C(S,t)$ all the values of S in $\mathbb{R}^+$ are considered that is all these values influence the option value. Underlying this fact is the idea of infinite speed for stock price motion. In a certain sense Black-Scholes formulas have a non local character, that is the option price at any time depends on all possible prices for the asset.

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The drawbacks of Black-Scholes equation, related to the infinite propagation speed, are due to its parabolic character. To overcome such drawbacks integro-differential models are studied in this paper. The new models are of hyperbolic type which implies that the propagation speed is bounded and consequently the option value $C(S,t)$ depends only on the values of asset prices belonging to the domain of dependence, which amplitude is proportional to the stock volatility and to the time to expiry. This property can also be viewed as a memory effect that causes a certain delay in the time evolution of option values.

The question now arises of how to introduce memory in Black-Scholes equation. Memory effects in derivative pricing have been considered by several authors. For instance, in [18] a telegraph equation was introduced to model the evolution in time of asset prices. Memory effect can also be introduced by using the so called delay equations.In [5], [17], the authors assume that the asset price satisfies certain stochastic delayed equations obtaining integral forms for option prices. In [1] the authors suggest the use of an hyperbolic form of Black-Scholes model, but they do not present a mathematical study of the model. In this last approach closed formulas for the solution were not established.

In the present paper memory effects are introduced by using a modified nonfickian flux closely related with the concept of portfolio.In Section 2 general ideas about the use of a nonfickian flux in Partial Differential equations are presented.These ideas are used in Section 3 to establish a memory model for pricing European options which we call Telegraph Black-Scholes. In Section 4 closed formulas for a subclass of Telegraph Black-Scholes - which we call Wave Black-Scholes equations- will be deduced. An alternative Telegraph Black-Scholes will be considered in Section 5. Numerical simulations that illustrate the influence of the delay in the Telegraph Black-Scholes model and in the Wave Black -Scholes model are included in the paper.

2. Memory in Partial Differential Equations

Let $u(x,\tau)$ represents some kind of information at (x,τ), as for example temperature or concentration resulting from the diffusion of $u_0(x)$ at an initial time $\tau = 0$. Following the classical diffusion equation in an infinite media,

$u(x,\tau)$ is the unique solution of Cauchy problem

$$\begin{cases} u_\tau = u_{xx}, x \in \mathbb{R}, \tau > 0, \\ \\ u(x,0) = u_0(x), x \in \mathbb{R}, \end{cases} \tag{1}$$

where the dependent variable u is nondimensional. It is well known that the solution of (1) is given by

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds \,, x \in \mathbb{R}, \; \tau > 0. \tag{2}$$

An inspection of (2) shows that a sudden change in the initial condition is immediately felt everywhere, which means that "everything depends on everything" or otherwise said that the propagation speed of information is infinite. Equation (1) is established from the mass conservation law

$$u_\tau = -J_x, \tag{3}$$

where J stands for the flux defined by

$$J = -u_x. \tag{4}$$

To correct such anomalous behavior of the solution of (1) the flux J is replaced by a new flux $\bar{J}$ satisfying

$$\bar{J}(x,\tau) = J(x, \tau - \theta), \tau > \theta, \tag{5}$$

and

$$\bar{J}(x,\tau) = J(x,0), \tau \leq \theta, \tag{6}$$

where θ stands for a positive constant that represents a delay in the diffusion process.

Considering that θ is small enough, we obtain from (5) a first order approximation to $\bar{J}(x,\tau)$ defined by

$$\bar{J}(x,\tau) = \frac{1}{\theta} \int_\theta^\tau e^{\frac{s-\tau}{\theta}} J(x,s) ds + e^{\frac{\theta-\tau}{\theta}} \bar{J}(x,\theta), \tau > \theta. \tag{7}$$

Replacing J by $\bar{J}$ in (3) we obtain

$$\begin{cases} u_\tau(x,\tau) = \dfrac{1}{\theta}\displaystyle\int_\theta^\tau e^{\frac{s-\tau}{\theta}} u_{xx}(x,s)ds + e^{\frac{\theta-\tau}{\theta}} u_0^{''}(x), \tau > \theta, \\ u_\tau(x,\tau) = u_0^{''}(x), \tau \le \theta, \\ u(x,0) = u_0(x), \end{cases} \tag{8}$$

which is equivalent to

$$\begin{cases} u_\tau(x,\tau) = \dfrac{1}{\theta}\displaystyle\int_0^\tau e^{\frac{s-\tau}{\theta}} u_{xx}(x,s)ds + e^{-\frac{\tau}{\theta}} u_0^{''}(x), \tau > \theta, \\ u_\tau(x,\tau) = u_0^{''}(x), \tau \le \theta, \\ u(x,0) = u_0(x). \end{cases} \tag{9}$$

In (8) and (9) the solution u does not depend on τ for $\tau \in [0,\theta]$. To introduce in this interval some dependence on τ we modify $\bar{J}$ and consider an alternative flux J^* defined by

$$\begin{cases} J^*(x,\tau) + \theta J^*_\tau(x,\tau) = J(x,\tau), \tau > 0, \\ J^*(x,0) = J(x,0). \end{cases} \tag{10}$$

Solving the linear equation in (10) we have

$$J^*(x,\tau) = \frac{1}{\theta}\int_0^\tau e^{\frac{s-\tau}{\theta}} J(x,s)ds + e^{-\frac{\tau}{\theta}} J(x,0), \tau > 0, \tag{11}$$

and replacing J by J^* in (3) we finally obtain

$$\begin{cases} u_\tau(x,\tau) = \dfrac{1}{\theta}\displaystyle\int_0^\tau e^{\frac{s-\tau}{\theta}} u_{xx}(x,s)ds + e^{-\frac{\tau}{\theta}} u_0^{''}(x), \tau > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{12}$$

We note that the difference between equations (7) and (11) lies on the fact that the first is satisfied for $\tau > \theta$ while the second holds for $\tau > 0$. The previous models, (9) and (12), introduce a certain delay θ in the evolution of u : in (9) nothing happens for $\tau \le \theta$ while in (12) the solution begins to evolve for $\tau = 0$.

We note that in both models integrating by parts and taking limits when $\theta \to 0$, (1) is obtained. Problem (12) has been first considered by Cattaneo [7] and lately modified by Jeffrey [16] to study heat propagation in some types of materials. Without being exhaustive we point out that integro-differential models of type (12) were studied in [3], [4], [6], [8]-[15], [19] and [20].

To establish a partial differential equation equivalent to the integro-differential equation in (12) we derive this equation in time obtaining

$$u_{\tau\tau}(x,\tau) = \frac{1}{\theta}u_{xx}(x,\tau) - \frac{1}{\theta}e^{\frac{\theta-\tau}{\theta}}u_0''(x) - \frac{1}{\theta^2}\int_0^\tau e^{\frac{s-\tau}{\theta}}u_{xx}(x,s)ds \qquad (13)$$

From (12) and (13) we can then conclude that

$$\theta u_{\tau\tau} = u_{xx} - u_\tau. \qquad (14)$$

It can be easily established that if u satisfies (14) along with initial conditions

$$u(x,0) = u_0(x) \qquad (15)$$

and

$$u_\tau(x,0) = u_0''(x), \qquad (16)$$

then u satisfies (13).

The information $u(x,\tau)$ given by the solution of (14)-(16) travels at a finite speed $\frac{1}{\theta}$ and consequently it is no more true that "everything depends on everything". Otherwise said the value of u at (x,τ) depends only on the values of $u_0(x)$ for $x \in \left[x - \frac{\tau}{\sqrt{\theta}}, x + \frac{\tau}{\sqrt{\theta}}\right]$. We note that if instead of the diffusion equation a convection-diffusion equation of type

$$u_\tau = u_{xx} + ku_x \qquad (17)$$

was considered an analogous procedure would lead to the integro-differential equation

$$u_\tau = \frac{1}{\theta}\int_0^\tau e^{\frac{s-\tau}{\theta}}(u_{xx}(x,s) + ku_x(x,s))ds + e^{\frac{\theta-\tau}{\theta}}(u_0''(x) + ku_0'(x)), \tau > 0. \quad (18)$$

3. A new version of Black-Scholes equation

3.1. Preliminaries. Let S represent the current price of an underlying asset and C the value of an European call option on the asset. We define a portfolio Π by

$$\Pi(S,t) = C(S,t) - \Delta S, \qquad (19)$$

where Δ represents the number of units of the asset. In the framework of Black-Scholes equation it is assumed that there is:

i) Hedging that is Δ is selected in such a way that the random contributions in $d\Pi$ are canceled;
ii) Absence of arbitrage which means that the portfolio Π satisfies
$$d\Pi = r\Pi, \tag{20}$$
where r stands for a free risk interest rate.

Assuming a constant volatility σ, Black-Scholes equation is derived from (19), (i) and (ii) leading to
$$C_t + \frac{\sigma^2}{2} S^2 C_{SS} + rSC_S - rC = 0. \tag{21}$$

To compute $C(S, t)$, equation (21) is completed with the boundary conditions
$$C(0, t) = 0, \lim_{S \to +\infty} (C(S, t) - S) = 0, \tag{22}$$
and a final condition at maturity time T
$$C(S, T) = \max(S - E, 0), \tag{23}$$
where E represents the strike price.

It is easily verified that if we consider in (21)-(23) a change of variables defined by
$$\begin{cases} S = Ee^x, \\ t = T - \dfrac{2}{\sigma^2}\tau, \\ C = Ee^{\alpha x + \beta\tau} u, \end{cases} \tag{24}$$
with
$$\begin{cases} \alpha = -\dfrac{1-k}{2}, \\ \beta = -\dfrac{(1+k)^2}{2}, \end{cases} \tag{25}$$
where
$$k = \frac{2}{\sigma^2} r, \tag{26}$$
then
$$u_\tau = u_{xx}, \ x \in , \tau \in (0, \frac{\sigma^2}{2} T], \tag{27}$$

with

$$\begin{cases} u(x,0) = e^{-\alpha x}\max(e^x - 1, 0),\ x \in \mathbb{R}, \\ \lim\limits_{x\to-\infty} u(x,\tau) = 0,\ \lim\limits_{x\to+\infty} e^x\Big(e^{(\alpha-1)x+\beta\tau}u(x,\tau) - 1\Big) = 0, \tau > 0, \end{cases} \tag{28}$$

is obtained. We remark that x and τ are non-dimensional variables.

Using the solution of the diffusion equation (2) and returning to the initial variables S, t, C we have the following solution for Black-Scholes equation

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),\ S > 0,\ t > 0, \tag{29}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx,$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\Big(ln(\frac{S}{E}) + (r + \frac{\sigma^2}{2})(T-t)\Big)$$

and

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\Big(ln(\frac{S}{E}) + (r - \frac{\sigma^2}{2}(T-t)\Big).$$

To compute the value of an option at (S,t) all prices $S \in \mathbb{R}^+$ are involved in formula (29). This fact is obviously du to the parabolic character of Black-Scholes equation. However arguing as before only a finite range of variation for the prices should influence the value $C(S,t)$.

In Section 2.2 we modify the parabolic character of (21) by introducing a new definition of the portfolio in such a way that the resulting equation is hyperbolic. As a consequence for every S and every t only a finite range of variation of the asset prices will influence the computation of $C(S,t)$. It is expected that this admissible range depends on σ and T.

3.2. A new approach to Black-Scholes equation. Let us consider equation (21) and transform it into a convection-diffusion equation by defining the following change of variables

$$\begin{cases} S = Ee^x, \\ t = T - \frac{2}{\sigma^2}\tau, \\ C = Ev. \end{cases} \tag{30}$$

The equation obtained is

$$v_\tau = (v_x - v)_x + k(v_x - v), \tag{31}$$

where k is defined by (26).

Transforming Black-Scholes equation into the convection-diffusion equation (31) allows us to associate with this equation a flux J defined by

$$J(x,\tau) = -(v_x - v) \tag{32}$$

which can be related to the portfolio Π, previously defined, by

$$EJ(x,\tau) = \Pi(S,t). \tag{33}$$

Following a procedure analogous to the one introduced in Section 1, we replace in (31) J by a new flux J^*, as defined by (11), obtaining

$$\begin{aligned} v_\tau = {} & \frac{1}{\theta}\int_0^\tau e^{\frac{s-\tau}{\theta}}(v_{xx}(x,s) - v_x(x,s))ds + \frac{k}{\theta}\int_0^\tau e^{\frac{s-\tau}{\theta}}(v_x(x,s) - v(x,s))ds \\ & + e^{-\frac{\tau}{\theta}}(v_0^{''}(x) - v_0^{'}(x)) + k e^{-\frac{\tau}{\theta}}(v_0^{'}(x) - v_0(x)), \end{aligned} \tag{34}$$

where k is defined by (26).

Deriving equation (34) in τ we obtain the equivalent hyperbolic differential equation

$$\theta v_{\tau\tau} + v_\tau = (v_x - v)_x + k(v_x - v), \tag{35}$$

which is called telegraph equation. This equation is completed with the initial conditions

$$\begin{cases} v_\tau(x,0) = v_0^{''}(x) + (k-1)v_0^{'}(x) - kv_0(x), x \in \mathbb{R}, \\ \\ v(x,0) = v_0(x) = \max(e^x - 1, 0),\ x \in \mathbb{R}, \end{cases} \tag{36}$$

and with the boundary conditions

$$\begin{cases} \displaystyle\lim_{x\to-\infty} v(x,\tau) = 0,\ \tau > 0, \\ \\ \displaystyle\lim_{x\to+\infty} \big(v(x,\tau) - e^x\big) = 0,\ \tau > 0. \end{cases} \tag{37}$$

We remark that the condition for the initial velocity for v is given by

$$v_\tau(x,0) = \begin{cases} k,\ x > 0, \\ \\ 0,\ x < 0. \end{cases}$$

Returning to the original variables C, S, t we obtain

$$-\frac{2\theta}{\sigma^2}C_{tt} + C_t + \frac{\sigma^2}{2}C_{ss}S^2 + rC_sS - rC = 0, S \in \mathbb{R}^+, t < T, \tag{38}$$

with

$$\begin{cases} C_t(S,T) = \begin{cases} -rE, \ S > E, \\ 0, \ S < E, \end{cases} \\ C(S,T) = \max(S-E,0), S \in \mathbb{R}^+, \end{cases} \tag{39}$$

which is called in what follows Telegraph Black-Scholes(TBS) equation. Initial value problem (IVP) (38)-(39) is complemented with the boundary conditions (22).

Due to the parabolic character of Black-Scholes equation the values of $C(S,T)$, for all $S \in \mathbb{R}^+$, are involved in the computation of an option $C(P,t)$ for all $P \in \mathbb{R}^+$ and for all $t < T$. The solution $C(P,t)$ provided by Telegraph Black-Scholes IVP (38)-(39) involves values of $C(S,T)$, where S belongs to a bounded domain, that is the domain of dependence D of $C(P,t)$ defined by

$$D = [Pe^{-\frac{\sigma^2}{2\sqrt{\theta}}(T-t)}, Pe^{\frac{\sigma^2}{2\sqrt{\theta}}(T-t)}]. \tag{40}$$

As expected the amplitude of the domain of dependence D increases with σ^2 for fixed T, and increases with T, for fixed σ^2. In fact the more volatile the market is, or the larger is the expiry time T, the larger should be the admissible range of values that influence the solution $C(S,t)$.

We remark that if we return to the original variables C, S, t in (34) the following equation is obtained

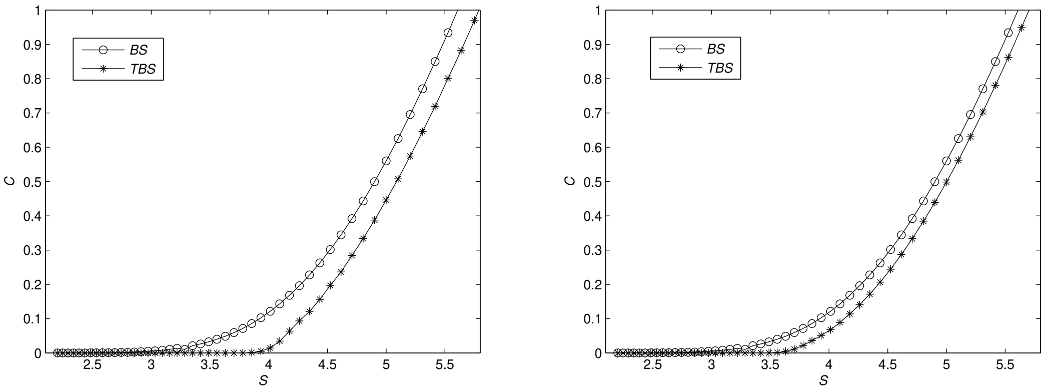
$$C_t = -\frac{\sigma^2}{2\theta}\int_t^T e^{\frac{\sigma^2}{2\theta}(-\xi+t)}(\frac{\sigma^2}{2}C_{SS}S^2 + rC_SS - rC)d\xi + e^{-(T-t)\frac{\sigma^2}{2\theta}}G(S), \tag{41}$$

where

$$G(S) = \begin{cases} -rE, \ S > E \\ 0, \ S < E. \end{cases}$$

In Figure1 we present plots of the solutions of Black-Scholes initial boundary value problem (IBVP) and Telegraph Black-Scholes IBVP computed for t=5,with $\sigma^2 = 10^{-2}, T = 10, E = 5, r = 10^{-2}$. In Figure 1 a) $\theta = 10^{-2}$, and in Figure 1b) $\theta = 5^{-3}$. The Black-Scholes solutions (BS) were compute using the Black-Scholes formula (29). In the computation of the solutions of Telegraph Black-Scholes IBVP(TBS) a standard finite difference discretization was used.

The delay effect of the parameter θ is illustrated in this figure. The convergence of the TBS solution to BS solution when $\theta \to 0$, is also suggested by the plots.



a) b)

FIGURE 1. The Black-Scholes solution (bullets) and the Telegraph Black-Scholes solution (stars) for $t = 5$ with $E = 5, T = 10, r = 10^{-2}, \sigma^2 = 10^{-2}$ for different values of θ: a) $\theta = 10^{-2}$, b) $\theta = 5^{-3}$.

Although many authors have criticized Black-Scholes model its big success is due to the fact that in most cases he furnishes good results. Another aspect that explains the intensive use of such model is the fact that very simple formulas can be obtained to evaluate $C(S, t)$. If Black-Scholes equation had included more complexity and many unobservable parameters it would have been useless. As far as Telegraph Black-Scholes equation is concerned the computation of $C(S, t)$ presented in Figure 1 result from the application of a numerical method. However it is possible to establish closed formulas for $C(S, t)$ using the characteristic curves of hyperbolic Telegraph Black-Scholes equation. This will be done in Section 4.

4. Closed formulas for a subclass of TBS

4.1. Weak solution. Let us consider equation (35). We define a change of the dependent variable such that

$$u = e^{-\alpha x - \beta \tau} v, \tag{42}$$

where α and β are free parameters. Equation (35) is then transformed into

$$\theta u_{\tau\tau}+(2\beta\theta+1)u_\tau-u_{xx}-(2\alpha+k-1)u_x+\Big(\theta\beta^2+\beta-\alpha^2-(k-1)\alpha+k\Big)u=0. \tag{43}$$

Equation (43) has three parameters. We select α,β,θ in order to eliminate the terms in u_τ, u_x and u, that is

$$\begin{cases} \alpha=\dfrac{1-k}{2}, \\ \beta=-\dfrac{1}{2\theta}, \\ \theta=\dfrac{\sigma^4}{(\sigma^2+2r)^2}. \end{cases} \tag{44}$$

As α and β are free parameters with non financial meaning any choice is admissible. As far as θ is concerned we must analyse if the value proposed in (44 represents a good choice. As the amplitude of the dependence domain D, defined by

$$l=P(e^{\frac{\sigma^2+2r}{2}(T-t)}-e^{-\frac{\sigma^2+2r}{2}(T-t)}), \tag{45}$$

is an increasing function of σ^2 for a fixed T and an increasing function of $T-t$ for a fixed volatility we conclude that the value of θ defined in (44) is admissible. This means that if the volatility increases then a larger interval of prices S will influence the value of $C(S,t)$ and that the larger is $T-t$ the larger should be the set of asset prices that can have influence in the option value which are sound properties. To obtain closed formulas for TBS we must solve the IVP

$$\theta u_{\tau\tau}=u_{xx}, x\in\mathbb{R}, \tau>0, \tag{46}$$

with

$$\begin{cases} u_\tau(x,0)=\begin{cases} -\beta e^{-\alpha x}(e^x-1)+ke^{-\alpha x},\ x\in\mathbb{R}^+, \\ 0,\ x\in\mathbb{R}^-, \end{cases} \\ u(x,0)=e^{-\alpha x}\max(e^x-1,0), x\in\mathbb{R}. \end{cases} \tag{47}$$

Let us represent $u(x,0)$ by $f(x)$ and $u_\tau(x,0)$ by $g(x)$. To solve exactly such problem we begin by regularizing initial conditions f and g by

$$f^\epsilon(x) = \int_{\mathbb{R}} j_\epsilon(x-y)f(y)\,dy,\ g^\epsilon(x) = \int_{\mathbb{R}} j_\epsilon(x-y)g(y)\,dy, \qquad (48)$$

where j_ϵ is given by

$$j_\epsilon(x) = \begin{cases} \dfrac{1}{\epsilon} e^{\frac{\epsilon^2}{\epsilon^2-x^2}},\ x \in (-\epsilon,\epsilon), \\ \\ 0,\ x \in \mathbb{R} - (-\epsilon,\epsilon). \end{cases}$$

In the following proposition we prove that the solution u^ϵ of the regularized problem converges to a weak solution of (46) (see [2] for some properties of the regularized functions).

Proposition 1. *Let u^ϵ be the solution of the IVP*

$$\theta u^\varepsilon_{\tau\tau} = u^\varepsilon_{xx},\ x \in \mathbb{R},\ \tau > 0,$$

with

$$\begin{cases} u^\varepsilon_\tau(x,0) = g^\varepsilon(x),\ x \in \mathbb{R}, \\ \\ u^\varepsilon(x,0) = f^\varepsilon(x), x \in \mathbb{R}, \end{cases}$$

where f^ε and g^ε are defined by (48). Then

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = u(x,t),$$

where

$$u(x,t) = \frac{1}{2}\Big(f(x+\frac{\tau}{\sqrt{\theta}}) + f(x-\frac{\tau}{\sqrt{\theta}})\Big) + \frac{\sqrt{\theta}}{2}\int_{x-\frac{\tau}{\sqrt{\theta}}}^{x+\frac{\tau}{\sqrt{\theta}}} g(s)\,ds \qquad (49)$$

satisfies

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(u_\tau(x,s)\varphi_\tau(x,s) - u_x(x,s)\varphi_x(x,s)\Big)dxds = 0, \qquad (50)$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$.

Proof: It is well known that $u^\epsilon(x,t)$ is given by

$$u^\epsilon(x,\tau) = \frac{1}{2}\Big(f^\epsilon(x+\frac{\tau}{\sqrt{\theta}}) + f^\epsilon(x-\frac{\tau}{\sqrt{\theta}})\Big) + \frac{\sqrt{\theta}}{2}\int_{x-\frac{\tau}{\sqrt{\theta}}}^{x+\frac{\tau}{\sqrt{\theta}}} g^\epsilon(s)\,ds. \qquad (51)$$

Taking limits in (51) when $\varepsilon \to 0$ we obtain the weak solution of (46) defined by (49). In fact, as f and g are continuous functions, $f^{\epsilon}(z) \to f(z), g^{\epsilon}(z) \to g(z)$ in all continuity points being the convergence uniform on any compact set of continuity points.

Moreover let φ be in $C_0^{\infty}(\mathbb{R} \times \mathbb{R}^+)$. As $f \in H^1(\mathbb{R})$ then $f^{\epsilon} \in H^1(\Omega_1)$ for every $\Omega_1 \subset \mathbb{R}$ and $f^{\epsilon} \to f$ in $H^1(\Omega_1)$. This convergence implies that

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} f^{\epsilon'}(x \pm \frac{s}{\sqrt{\theta}})\varphi_\tau(x,s)dxds = \int_{\Omega_\varphi} f^{\epsilon'}(x \pm \frac{s}{\sqrt{\theta}})\varphi_\tau(x,s)dxds$$
$$\to \int_{\Omega_\varphi} f'(x \pm \frac{s}{\sqrt{\theta}})\varphi_\tau(x,s)dxds = \int_0^{+\infty}\int_{-\infty}^{+\infty} f'(x \pm \frac{s}{\sqrt{\theta}})\varphi_\tau(x,s)dxds, \tag{52}$$

where Ω_φ is a compact containing the support of φ. Analogously we have

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} g^{\epsilon}(x \pm \frac{s}{\sqrt{\theta}})\varphi_\tau(x,s)dxds \to \int_0^{+\infty}\int_{-\infty}^{+\infty} g(x \pm \frac{s}{\sqrt{\theta}})\varphi_\tau(x,s)dxds, \tag{53}$$

for all φ be in $C_0^{\infty}(\mathbb{R} \times \mathbb{R}^+)$. From (52) and (53) we conclude the proof.

■

4.2. Closed formulas using characteristics. The characteristics curves of (46) are defined by

$$\frac{dx}{d\tau} = \pm\frac{1}{\sqrt{\theta}}. \tag{54}$$

Returning to the original variables C, S, t we obtain the characteristics curves of Wave Black-Scholes equation

$$t = \pm\frac{2\sqrt{\theta}}{\sigma^2}\ln\frac{S}{E} + Constant. \tag{55}$$

The domain of dependence D of $C(P,t)$ is given by (40) with $\theta = \dfrac{\sigma^4}{(\sigma^2+2r)^2}$.

Using the characteristic curves we can give a different form to the solution (49). In Figure 2 it is represented the two characteristic curves that contain (E,T). These two curves define three regions, R_1, R_2 and R_3, which we describe analytically by

$$
\begin{aligned}
R_1 &= \{(S,t) : S \geq 0, S \leq Ee^{-\frac{\sigma^2}{2\sqrt{\theta}}(T-t)},\ t \leq T\},\\
R_2 &= \{(S,t) : Ee^{-\frac{\sigma^2}{2\sqrt{\theta}}(T-t)} < S \leq Ee^{\frac{\sigma^2}{2\sqrt{\theta}}(T-t)}\},\\
R_3 &= \{(S,t) : S > Ee^{\frac{\sigma^2}{2\sqrt{\theta}}(T-t)},\ t \leq T\}.
\end{aligned}
\tag{56}
$$

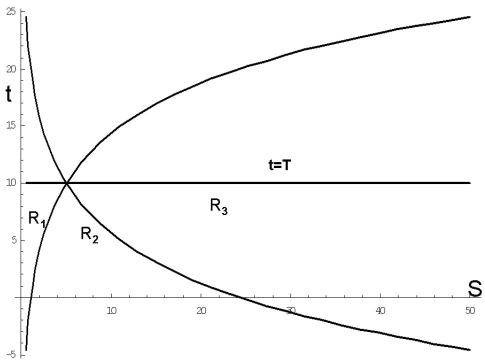


FIGURE 2. Regions of the Sot plane induced by the characteristic curves.

Recalling that the final condition in $t = T$ is $C(S,T) = \max(S-E, 0)$ we obtain the following formulas for the Wave Black- Scholes solution (WBS1)

$$
C(S,t) = \begin{cases}
0, & (S,t) \in R_1\\
\Big(\frac{S}{E}\Big)^{\alpha}\frac{1}{2}\Big(Se^{(T-t)\frac{\sigma^2}{2\sqrt{\theta}}} - E\Big) & \\
\quad +\frac{\sqrt{\theta}}{2}e^{(\beta-\frac{\alpha}{\sqrt{\theta}})(T-t)\frac{\sigma^2}{2}}\Big(Se^{(T-t)\frac{\sigma^2}{2\sqrt{\theta}}}\frac{\beta}{\alpha-1} - E\frac{\beta+k}{\alpha}\Big) & \\
\quad +\frac{\sqrt{\theta}}{2}e^{\beta(T-t)\frac{\sigma^2}{2}}\frac{S^{\alpha}}{E^{\alpha-1}}\frac{\alpha(k-1)-\beta}{\alpha(\alpha-1)}, & (S,t) \in R_2,\\
\frac{S^{(\alpha+1)}}{E^{\alpha}}e^{\beta(T-t)\frac{\sigma^2}{2}}cosh\Big(e^{(T-t)\frac{\sigma^2}{2\sqrt{\theta}}}\Big) & \\
\quad +\sqrt{\theta}e^{\beta(T-t)\frac{\sigma^2}{2}}\Big(\frac{S\beta}{\alpha-1}sinh\Big((1-\alpha)(T-t)\frac{\sigma^2}{2\sqrt{\theta}}\Big) & \\
\qquad +E\frac{\beta+k}{\alpha}sinh\Big(\alpha(T-t)\frac{\sigma^2}{2\sqrt{\theta}}\Big)\Big), & (S,t) \in R_3,
\end{cases}
\tag{57}
$$

where α, β and θ are defined by (44).

In Figure 3 it is represented a plot of the Black-Scholes solution (BS) and the Wave Black-Scholes solution (49) (WBS1) at $t = 5$ for $E = 5, T = 10, r = 10^{-2}, \sigma = 10^{-1}$. We observe that WBS1 solution is steeper than BS solution. For a certain (S, t) the option value given by BS is larger than the option value predicted by WBS1.

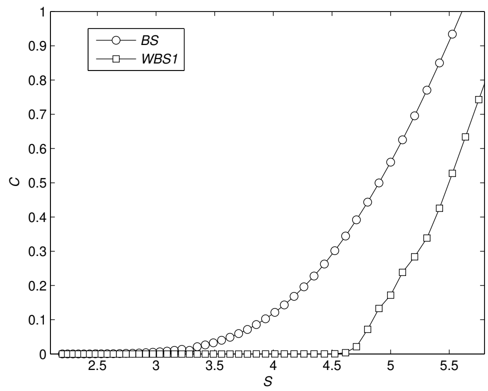


FIGURE 3. The Black-Scholes solution (BS) and the weak Black-Scholes solution (WBS) at $t = 5$ for $E = 5, T = 10, r = 10^{-2}, \sigma = 10^{-1}$.

5. An alternative Telegraph Black-Scholes equation

The main idea of our approach is to replace in (31) the flux J defined by (32) by a new flux J^* defined in (11). By (33) the flux J is closely related to the portfolio. If instead of (32) the flux

$$J = -v_x \tag{58}$$

is considered, the procedure followed in Section 3 leads to the modified flux

$$J^{**} = \frac{1}{\theta}\int_0^{\tau} e^{\frac{s-\tau}{\theta}} v_x(x, s)ds + e^{-\frac{\tau}{\theta}} v_0'(x). \tag{59}$$

Replacing then J in (31) by J^{**} a new Telegraph Black-Scholes equation (NTBS) is obtained

$$-\frac{2\theta}{\sigma^2}C_{tt} + (1 + \theta k)C_t + \frac{\sigma^2}{2}C_{ss}S^2 + rC_sS - rC = 0, \tag{60}$$

which in integro-differential form reads

$$C_t = -\frac{\sigma^2}{2\theta}\int_t^T e^{\frac{\sigma^2}{2\theta}(-\xi+t)}(\frac{\sigma^2}{2}C_{SS}S^2 + rC_SS)d\xi - rC + e^{-(T-t)\frac{\sigma^2}{2}}G(S), \tag{61}$$

with

$$G(S) = \begin{cases} -rS, \ S > E, \\ \\ 0, \ S < E. \end{cases}$$

We remark that (60) has been considered in [1]. However the authors in this last paper do not focuss on its deduction nor its meaning. To transform (60) in a wave equation, α, β and θ must satisfy

$$\begin{cases} \alpha = \dfrac{1-k}{2}, \\ \beta = -\dfrac{1}{2\theta}, \\ \theta = \dfrac{\sigma^4}{4r^2}. \end{cases} \tag{62}$$

In this case the domain of dependence of $C(P, t)$ is defined by

$$D = [Pe^{-(T-t)r}, Pe^{(T-t)r}] \tag{63}$$

whose amplitude is now an increasing function of r, but does not depend on the volatility. Let us call (60),(62),Wave Black-Scholes2 equation and denote its solution by WBS2. In Figure 4 we plot the Black- Scholes solution (BS), the Wave Black-Scholes solution 1 (WBS1) and the Wave Black-Scholes2 solution (WBS2) at $t = 5$ for $E = 5, T = 10, r = 10^{-2}, \sigma = 10^{-1}$. The delay effect in the models introduced in this section is well illustrated in this figure where WBS2 presents the larger delay.

6. Conclusions

The Black-Scholes models with memory presented in this paper have the property of involving only a set of asset prices, located around S, in the computation of $C(S, t)$. This set is the domain of dependence of the hyperbolic Black-Scholes equations with memory. As a consequence the solution of the memory models are less diffusive and exhibit a certain delay when compared with the Black-Scholes solutions. This implies that for every (S, t) the value $C(S, t)$ given by the Black-Scholes solution is larger than the corresponding value of the solution of the memory models.

From a mathematical point of view some questions still deserve attention as for example the existence of hedging and non arbitraging. These questions will be addressed in a future work.

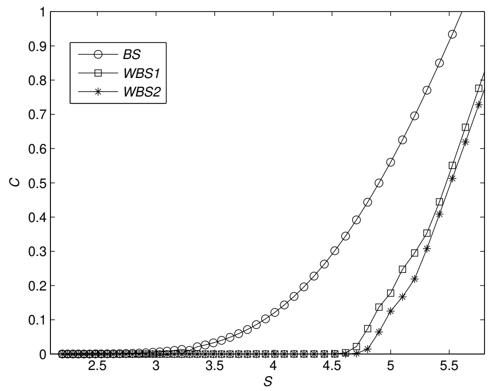


FIGURE 4. The Black-Scholes solution (BS), the weak Black-Scholes solution (WBS1) and the wave Black-Scholes solution (WBS2) at $t = 5$ for $E = 5, T = 10, r = 10^{-2}, \sigma = 10^{-1}$.

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