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LOCAL ESTIMATES FOR PARABOLIC EQUATIONS WITH NONLINEAR GRADIENT TERMS

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Abstract: In this paper we deal with local estimates for parabolic problems in $\mathbb{R}^N$ with absorption first order terms, whose model is

$$\begin{cases} u_t - \Delta u + u|\nabla u|^q = f(t,x) & \text{in } (0,T) \times \mathbb{R}^N\,, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases}$$

where $T > 0$, $N \geq 3$, $1 < q \leq 2$, $f(t,x) \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Keywords: Local estimates, parabolic equations, lower order terms, large solutions.

1. Introduction

In this paper we deal with local estimates for parabolic problems in $\mathbb{R}^N$ with absorption first order lower order terms. In particular, our main goal concerns the proof the existence of a solution for a Cauchy problem whose model is

$$\begin{cases} u_t - \Delta_p u + u|\nabla u|^q = f(t,x) & \text{in } (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases} \tag{1}$$

where $p-1 < q \leq p$, $f(t,x) \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, without any prescribed condition at infinity.

Such a problem is obviously strictly related to the possibility of proving estimates for the solutions that are independent from the behavior at infinity. This is a peculiarity of nonlinear equations with *strong absorption* lower order terms. If such term does not depend on the gradient, i.e. for problems of the type

$$\begin{cases} u_t - \Delta_p u + b(u) = f(t,x) & \text{in } (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases} \tag{2}$$

with $f(t,x) \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, the existence (and regularity) of distributional solutions has been recently studied in [18] (see also [6]).

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Since a strong regularizing effect is needed in order prove local estimates, a sign condition on the lower order term is assumed, namely $b(s)s \geq 0$, $\forall s \in \mathbb{R}$.

Moreover, in order to prove the existence of a solution, a condition on the growth of the function $b(\cdot)$ is also necessary, namely

$$\int^{\infty} \frac{ds}{(b(s)s)^{\frac{1}{p}}} < \infty \,. \tag{3}$$

In order to prove such kind of estimates the presence of the regularizing effect of the absorption lower order term is crucial. For instance, if we consider the heat equation in $\mathbb{R}^N$ it is well known that the solution is represented by the convolution of the data with the heat kernel. Thus roughly speaking, $\forall (t,x) \in (0,T) \times \mathbb{R}^N$ the solution depends even on what happens *far away* from (t,x).

For $p = 2$ and if b is increasing (at least at infinity), (3) is equivalent to the well-known Keller-Osserman condition. This condition has been introduced in the papers [15] and [21] in order to prove a local (uniform) bound for any subsolution of the nonlinear elliptic equation

$$-\Delta u + b(u) = f(x) \quad \text{in } \Omega \,, \tag{4}$$

where Ω is a bounded open set and f is bounded. This tool is strictly related to the posibility of constructing solutions that blow-up at the boundary (such solutions are known in literature as *large solutions*). A huge number of papers has been devoted to the study of such topic: we mention, among the others, [2], [27] in which the existence of such kind of solutions has been proved. We want to stress that (3), (as well as Keller-Osserman condition) is the necessary and sufficient condition for having a solution for the ordinary differential equation associated to the equation.
As already noticed, the key tools in order to face problem (4) are *local estimates*. Because of this fact, the study of such problems turns out to be strictly related to the study of the same problem in the whole $\mathbb{R}^N$ without conditions at infinity (we refer to the papers [8], [5] and [17]).

On the other hand, equations with nonlinear gradient terms have been studied since a lot of years. Recently the problem of both existence of solutions in the whole space (entire solutions) and existence of large solutions related to such kind of equations has been considered (see [24] and [19]). More precisely, in [19] the existence of both a large solution and an entire solution without any condition at infinity (or at ∂D if D is bounded open subset of $\mathbb{R}^N$) is proved for equations of the type

$$-\Delta_p u + u + u|\nabla u|^q = f(x) \quad \text{in } D\,,$$

where $D \subseteq \mathbb{R}^N$, $N \geq 3$, $p > 1$, $p-1 < q \leq p$ and $f(x)$ is a possibly singular datum, say $L^1_{\mathrm{loc}}(D)$.

The purpose of this paper is twofold: from one side we want to extend the results of [18] to problems with lower order terms that depend on the gradient. On the other hand, since we deal with local estimates, our aim is show that we can construct solutions that assume, in a suitable way, the value "$+\infty$"at $(0,T) \times \partial\Omega$.

First of all we have to define what we mean by a solution for such a problem. In this framework a lot of notions of solutions have been introduced. Following the outlines of [12] (see also the more recent papers [11] and [22]) we use a renormalized formulation. In fact, the notion of renormalized solution turns out to be stronger than the distributional one, and it is very useful in order to face problems involving solution which have no local finite energy. The main idea relies on the the fact that, roughly speaking, we look at a family of problems solved by suitable *truncations* of the solution.
As a matter of fact, local estimates are the key tool to prove existence of large solutions in bounded domains. We shall deal with this problem in Section 4. In order to deal with the explosive condition $u = +\infty$ at the boundary, we have to define the meaning of how this value is attained in a suitable way, since, due to the possible lack of continuity of these solutions, this condition can not be satisfied (for instance) in a pointwise sense. For this purpose we extend the definition given in [19] to the parabolic framework; roughly, we say that a solution matches $+\infty$ on the boundary if any of its truncation at level k has constant trace k on the boundary.

The plan of the paper is the following: in Section 2 we give the main hypothesis and we state our main results, while in Section 3 we collect some useful technical tools. The existence results are proved in Section 4, while Section 5 will be devoted to de description of some further regularity results.

2. Assumptions and statement of the main results

Let D be an open subset of $\mathbb{R}^N$, $N \geq 3$, possibly $\mathbb{R}^N$ itself. Throughout the paper we will denote, for any $r > 0$ and $\forall s \in (0, T]$, by $Q_r^s = (0, s) \times B_r$, and by $Q_D^s = (0, s) \times D$.

We consider the following equation

$$u_t - \operatorname{div} a(t, x, u, \nabla u) + g(t, x, u, \nabla u) = f(t, x) \quad \text{in } Q_D^T \,, \tag{1}$$

where $f(t, x) \in L^1(0, T; L^1_{\mathrm{loc}}(D))$ while $a(t, x, s, \varsigma) : \mathbb{R} \times D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that:

$$\exists \alpha > 0 : \quad a(t, x, s, \varsigma) \cdot \varsigma \geq \alpha |\varsigma|^p \,, \tag{2}$$

$$\exists \beta > 0 : \quad |a(t, x, s, \varsigma)| \leq \beta |\varsigma|^{p-1} \,, \tag{3}$$

and

$$(a(t, x, s, \varsigma) - a(t, x, s, \eta)) \cdot (\varsigma - \eta) > 0 \,, \tag{4}$$

for a.e. $x \in D$ $\forall t > 0$, $\forall s \in \mathbb{R}$, and $\forall \varsigma\,, \eta \in \mathbb{R}^N$ such that $\eta \neq \varsigma$.

Under the above hypothesis div $a(t, x, u, \nabla u)$ turns out to be a Leray-Lions type operator, acting from $L^p(0, T; W_0^{1,p}(D))$ into its dual.

As far as the lower order term is concerned, we suppose that $g(t, x, s, \varsigma)$: $\mathbb{R} \times D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that:

$$g(t, x, s, \varsigma) s \geq 0 \,, \tag{5}$$

$$\begin{gathered} \forall k > 0 \quad \sup_{|s| \leq k} |g(t, x, s, \varsigma)| \leq |g_k(t, x)| + \gamma_k |\varsigma|^p \,, \\ \gamma_k > 0, \quad g_k(t, x) \in L^1(0, T; L^1_{\mathrm{loc}}(\mathbb{R}^N)) \,, \end{gathered} \tag{6}$$

for a.e. $(t, x) \in (0, T) \times D$, $\forall s \in \mathbb{R}$ and $\forall \varsigma \in \mathbb{R}^N$, and

$$\exists L > 0 : \ |g(t, x, s, \varsigma)| \geq h(|\varsigma|^{p-1}) \,, \ \forall |s| \geq L, \, \forall \varsigma \in \mathbb{R}^N \,, \tag{7}$$

for a.e. $(t, x) \in (0, T) \times D$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^2(\mathbb{R}^+)$ convex function such that $h(0) = 0$ and satisfying the following growth condition,

$$\exists c_1 > 0 \text{ such that } \ h(\tau) \leq c_1 (\tau^{\frac{p}{p-1}} + 1) \,, \forall \tau \in \mathbb{R}^+ \,, \tag{8}$$

and the following conditions at infinity

$$\int^{\infty} \frac{d\tau}{h(\tau)} < \infty \,, \tag{9}$$

and

$$\limsup_{\tau\to\infty} \frac{\tau^2 h''(\tau)}{h'(\tau)\tau - h(\tau)} < \infty\,. \tag{10}$$

Some comments on these assumptions are in order to be given. Note that the absorption nature of the nonlinear lower order term depends on the sign condition (5), while (6) (and (8)) is known, in literature, as *natural growth condition*. We observe that condition (7) is a growth bound *from below* for the lower order term with respect to ς at infinity. This assumption is crucial, as it can be noticed in the proof of our main results, and in particular in the possibility of constructing suitable cut-off functions, in order to prove the local estimates we are interested in.

We remark that the hypotheses (9) referred to equation (1) corresponds to the already mentioned Keller-Osserman assumption for equation (4). In fact, in the same spirit of the stationary case, condition (9) it is what we have to impose in order to prove the existence of a solution for the ordinary differential equation associated to (1). For instance, a solution for the problem

$$\begin{cases} \left(|v'(s)|^{p-2}v'(s)\right)' = h\left(|v'(s)|^{p-1}\right) & \text{in } (0,+\infty)\,, \\ \lim\limits_{s\to 0^+} v(s) = +\infty\,, \end{cases}$$

is well defined if and only if (9) holds true. Finally, assumption (10) is technical and we suspect that it could be removed.

Let us introduce the truncations as $T_k(s) = \max\{-k \min\{k, s\}\}$. We will, in general, handle with measurable functions whose truncations (locally) belong to the energy space. To get rid of this fact we will made use of the notion of *generalized gradient* whose main feature is contained in the following lemma.

Lemma 2.1. *Let $D \subseteq \mathbb{R}^N$, $N \geq 3$, and let $w(t,x)$ be a measurable function such that its truncation belongs to $L^1(0,T;W^{1,1}_{\text{loc}}(D))$. Then there exists a measurable function $v : (0,T)\times D \to \mathbb{R}^N$ such that $\forall k >$ and for a.e. $(t,x) \in (0,T)\times D$,*

$$\nabla T_k(w) = v\chi_{\{|w|\leq k\}}\,. \tag{11}$$

Obviously, if $w \in L^1(0,T;W^{1,1}_{\text{loc}}(D))$, then the generalized gradient turns out to coincide with the classical distributional one. We will made use several times of this notion throughout the paper.

Let us introduce the following definition of *renormalized solution* which is the natural extension of the classical one (see for instance [12]).

Definition 2.2. Let D be an open subset of $\mathbb{R}^N$, $N \geq 3$. We say that a measurable function $u(t,x) \in L^\infty(0,T; L^1_{\rm loc}(D))$ such that $\forall k>0$, $T_k(u)$ belongs to $L^p(0,T; W^{1,p}_{\rm loc}(D))$ is a *renormalized solution* for equation (1), if $a(t,x,u,\nabla u) \in (L^1(0,T; L^1_{\rm loc}(D)))^N$ $f(t,x)$, $g(t,x,u,\nabla u) \in L^1(0,T; L^1_{\rm loc}(D))$, and the following identity holds true:

$$\begin{aligned} &- \int_D S(u_0)\psi(x,0) - \int_0^T < S(u)\,,\, \psi_t > \\ &+ \int_{Q_D^T} a(t,x,u,\nabla u)\cdot \nabla u S''(u)\psi + \int_{Q_D^T} a(t,x,u,\nabla u)\cdot \nabla\psi S'(u) \\ &+ \int_{Q_D^T} g(t,x,u,\nabla u) S'(u)\psi = \int_{Q_D^T} f(t,x)S'(u)\psi\,, \end{aligned} \tag{12}$$

$\forall \psi \in C^1_0([0,T)\times D)$, and for any $S(\tau) \in W^{2,\infty}(\mathbb{R})$ such that $S'(\tau)$ is compactly supported on $\mathbb{R}$. Moreover,

$$\lim_{l\to+\infty} \int_{Q_D^T \cap \{l \leq |u| \leq l+1\}} a(t,x,u,\nabla u)\cdot \nabla u\, \psi = 0\,, \quad \forall \psi \in C^\infty_0([0,T)\times D)\,. \tag{13}$$

Note that the regularity required to the solution is such that any term in the identity (12) makes sense. In fact the above definition is nothing that equation (1) formally multiplied by $S'(u)\psi$ and integrated on the cylinder . The fact that S' is compactly supported ensures that all but the first two terms in (12) involve only a truncation of u. Condition (13) is due to recover a uniform information on u on the set where it is *large*.

Let us also emphasize that, a priori, we are not in the position to apply Theorem 1.1 in [23] in order to deduce that $u \in C^0([0,T]; L^1_{\rm loc}(D))$, since we do not imposed any regularity on u_t. Anyway, this result can be applied to $S(u)$, for any S as above, since, by (12), the distributional time-derivative $S(u)_t$ turns out to belongs to $L^1(0,T;L^1(\omega)) + L^{p'}(0,T;W^{-1,p'}(\omega))$, for any $\omega \subset\subset D$. Moreover, it is easy to deduce, using that $g(t,x,u,\nabla u) \in L^1(0,T; L^1_{\rm loc}(D))$ and (7)–(9), that $|\nabla u|^{p-1} \in L^1(0,T; L^1_{\rm loc}(\mathbb{R}^N))$.

Here we state our existence result concerning entire solutions.

Theorem 2.3. *Assume that $a(t,x,s,\varsigma)$ and $g(t,x,s,\varsigma)$ satisfy hypotheses (2)–(4) and (5)–(10), respectively. Then for any $f \in L^1(0,T; L^1_{\rm loc}(\mathbb{R}^N))$ and* *for any* $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *there exists a renormalized solution of the Cauchy problem*

$$\begin{cases} u_t - \ \mathrm{div}\ a(t,x,u,\nabla u) + g(t,x,u,\nabla u) = f(t,x) & \textit{in}\, (0,T)\times\mathbb{R}^N \\ u = u_0(x) & \textit{in}\, \mathbb{R}^N\,. \end{cases} \tag{14}$$

As a consequence of the local estimates proved in the previous result, we are able to show the existence of a large solution for the boundary value problem associated to equation (1). More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 3$. We consider the following problem:

$$\begin{cases} u_t - \ \mathrm{div}\ a(t,x,u,\nabla u) + g(t,x,u,\nabla u) = f(t,x) & \text{in}\, (0,T)\times\Omega \\ u(t,x) = +\infty & \text{on}\, \partial_{\mathcal{P}} Q^T_\Omega \\ u = u_0(x) & \text{in}\, \Omega\,. \end{cases} \tag{15}$$

where $\partial_{\mathcal{P}} Q^T_\Omega$ indicates the parabolic lateral boundary $(0,T)\times\partial\Omega$

In the sequel we will need a localized version of (6), namely

$$\begin{gathered} \forall k>0 \quad \sup_{|s|\leq k} |g(t,x,s,\varsigma)| \leq |g_k(t,x)| + \gamma_k|\varsigma|^p\,, \\ \gamma_k > 0, \quad g_k(t,x)\in L^1(Q^T_\Omega)\,. \end{gathered} \tag{16}$$

Let us also specialize the definition of renormalized solution to this particular boundary value problem. We recall that large solutions for parabolic equations have been introduced in [1], for a different class of equations. Actually in such a case, since the solutions are continuous, the blow-up condition is assumed in a pointwise sense. For our purpose, we need to reformulate this condition in a suitable weak sense adapted to our framework. More precisely, the value "$u = +\infty$"at $\partial\Omega$ is assumed trough a condition on the trace of $T_k(u)$.

Definition 2.4. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. For any $f(t,x) \in L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$, we define a *renormalized large solution* for problem (15) to be a measurable function $u(t,x)$ such that $T_k(u) \in L^p(0,T;W^{1,p}(\Omega))$, $a(t,x,u,\nabla u) \in (L^1(0,T;L^1_{\mathrm{loc}}(D)))^N$, $g(t,x,u_n,\nabla u_n) \in L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$ and it satisfies both (12) and (13) in Ω. Moreover the boundary condition is assumed in the following sense:

$$k - T_k(u) \in L^p(0,T;W^{1,p}_0(\Omega)) \qquad \forall k>0\,. \tag{17}$$

Our result concerning the existence of a large solution is the following.

Theorem 2.5. *Assume that $a(t,x,s,\varsigma)$ and $g(t,x,s,\varsigma)$ satisfy hypotheses (2)–(4) and (5), (7)–(10) and (16). Thus for any $f \in L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$ such that $f^- \in L^1(Q^T_\Omega)$ and for any $u_0 \in L^1_{\mathrm{loc}}(\Omega)$ such that $u_0^- \in L^1(\Omega)$ there exists a (renormalized) large solution of the problem (15).*

Let us introduce, for any $0 < q < \infty$, the Marcinkiewicz space $M^q(Q^T_D)$ as the space of all measurable functions f such that there exists $c > 0$, with

$$\mathrm{meas}\{(t,x) \in (0,T) \times D : |f(t,x)| \geq k\} \leq \frac{c}{k^q},$$

for every positive k endowed with the seminorm

$$\|f\|_{M^q(Q^T_D)} = \inf\left\{c > 0 : \mathrm{meas}\{(t,x) : |f(t,x)| \geq k\} \leq \left(\frac{c}{k}\right)^q\right\}.$$

Let us recall that, since Q^T_D is bounded, then for $q > 1$ we have the following continuous embeddings

$$L^q(Q^T_D) \hookrightarrow M^q(Q^T_D) \hookrightarrow L^{q-\varepsilon}(Q^T_D),$$

for every $\varepsilon \in (0, q-1]$.

We stress that from the definition of renormalized solution we can not, a priori, deduce neither any summability properties nor is u (and not $S(u)$) assumes the initial value in some sense. Next proposition gets rid of this tools.

Proposition 2.6. *Any renormalized solution of (14) satisfies the following estimates:*

$$\|u\|_{M^{p-1+\frac{p}{N}}(Q^T_D)} \leq c_1 \quad \text{and} \quad \|\nabla u\|_{M^{p-\frac{N}{N+1}}(Q^T_D)} \leq c_2,$$

where c_1 and c_2 are positive constants only depending on N, R, T, f and p. Moreover, $u \in C^0([0,T];L^1_{\mathrm{loc}}(D))$.

We finally want to investigate how the local summability of the datum $f(t,x)$ influences the local summability of the renormalized solutions. In particular we will see that that the regularity of the solutions is, locally, the same of the solution of the equation in (14) without the lower order term and with homogeneous Dirichlet boundary conditions. For simplicity we deal with bounded initial datum.

The technique we will use are nowadays classic and follow, for instance [14] and [7]. However, since a localization is needed, the role of the lower order term (and in particular the growth condition (9)) is crucial. Actually we will only sckech the proof of such result, underlying the main difference with the cases treated both in [14] and in [7].

Theorem 2.7. *Suppose $1 < p < N$, $q > 1$, $m > 1$ and suppose that $f(x)$ belongs to $L^m(0,T;L^q_{\mathrm{loc}}(D))$. Thus for any renormalized solution of (14) there exists $C_0 > 0$ such that: $\forall u_0 \in L^\infty_{\mathrm{loc}}(D)$*

(1) *if*

$$1 < \frac{1}{m} + \frac{N}{pq} \leq 1 + \frac{N}{pm}, \tag{18}$$

then

$$\|u\|_{L^s(0,T;L^s_{\mathrm{loc}}(D))} \leq C_0\,, \quad \textit{where } s = \frac{mq(N+p) + N(p-2)(q(m-1)+m)}{mN - pq(m-1)};$$

moreover

$$\|u\|_{L^{s_1}(0,T;L^{s_2}_{\mathrm{loc}}(D))} \leq C_0\,, \quad \textit{where } s_1 = m's_0\,,\ s_2 = q's_0$$

$$\textit{and } s_0 = \frac{mq(q-1) + q(m-1)[p(N+1) - 2N]}{mN - pq(m-1)}.$$

(2) *If*

$$\frac{1}{m} + \frac{N}{pq} > 1 + \frac{N}{pm}. \tag{19}$$

then

$$\|u\|_{L^s(0,T;L^s_{\mathrm{loc}}(D))} \leq C_0\,,$$

where

$$s = \frac{[N(p-1)(q-1) + N - pq](N+p)}{N(N-pq)} + p - 2;$$

moreover

$$\|u\|_{L^{s_1}(0,T;L^{s_2}_{\mathrm{loc}}(D))} \leq C_0$$

where

$$s_1 = m's_0\,,\ s_2 = q's_0 \quad \textit{and}\, s_0 = \frac{N(q-1)(p-1)}{N-pq}.$$

(3) *If*

$$\frac{1}{m} + \frac{N}{pq} < 1\,, \tag{20}$$

then

$$\|u\|_{L^\infty(0,T;L^\infty_{\mathrm{loc}}(D))} \leq C_0.$$

Notation. Let us define $\varphi_\lambda(s) = se^{\lambda s^2}$; we recall that $\varphi_\lambda(s)$ enjoy the following useful property:

$$\forall a > 0\,,\, b > 0\,,\, \forall \lambda > \frac{b^2}{8a^2} \qquad a\varphi'_\lambda(s) - b|\varphi_\lambda(s)| \geq 1\,, \quad \forall s \in \mathbb{R}\,. \tag{21}$$

We will also make use of the following functions related with the trucations:

$$S(\tau) = S_j(\tau) \qquad S_j(\tau) = \int_0^\tau [1 - T_1(G_j(s))]ds\,, \tag{22}$$

and $G_k(s) = s - T_k(s)$.

By $< \cdot, \cdot >$ we mean the duality between suitable spaces in which function are involved. In particular we will consider both the duality between $W_0^{1,p}(D)$ and $W^{-1,p'}(D)$ and $W^{-1,p'}(D) + L^1(D)$ and $W_0^{1,p}(D) \cap L^\infty(D)$.

Finally, we use the following notation for sequences: $\forall a, b > 0$, by $\varepsilon(a, b)$ we denote a sequence such that

$$\lim_b \lim_a \varepsilon(a, b) = 0\,.$$

3. Technical results

In this section we collect some technical results that will be useful in the rest of the paper. The first one concerns the construction of suitable cut-off functions.

Proposition 3.1. *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^2, convex function, such that $h(0) = 0$, and such that (9) and (10) hold. Then, for any δ there exists a constant $C_0 = C_0(\delta) > 0$ and a function $\sigma : [0, 1] \to [0, 1]$, $\sigma \in C^0[0, 1] \cap C^1(0, 1)$ with $\sigma(0) = \sigma'(0) = 0$, $\sigma(1) = 1$, such that*

$$\forall v > 0, \qquad v\sigma'(s) \leq \delta h(v)\sigma(s) + C_\delta\,, \quad \forall s \in [0, 1]\,. \tag{1}$$

To construct the function σ, we are lead to use suitable *spatial* cut-off functions. We denote by $\xi = \xi_R^\rho(|x|)$ a $C_0^1(\mathbb{R}^N)$ function such that $\forall \rho > 0$

$$\begin{cases} \xi \equiv 1 & \text{if } |x| \leq R \\ 0 < \xi < 1 & \text{if } |x| < R + \rho \\ \xi \equiv 0 & \text{if } |x| \geq R + \rho\,. \end{cases} \tag{2}$$

Another fundamental tool in our arguments relies on the so called generalized Young's inequality with the function h which appears in (7)-(9), so that we have to introduce the Legendre transform for h together with its first properties that we will use in the sequel.

We recall that h is a C^2 increasing and convex function such that $h(0) = 0$. Moreover by the convexity and since hypothesis (9) is in force (i.e. roughly speaking h "a bit more" than superlinear at infinity) it follows that

$$\lim_{s\to\infty} h'(s) = +\infty\,.$$

Let us consider the so called *Legendre tranform* of h defined through

$$h^*(q) = \sup_{r\in\mathbb{R}} \left[qr - h(r)\right]\,.$$

Here we recall the so called generalized Young inequality namely: for any positive z, w,

$$wz \le h(z) + h^*(w)\,. \tag{3}$$

It is clear that h^* is continuous, increasing and, since (3) holds, superlinear at infinity. Consequently h^{*-1} is well defined and moreover

$$\lim_{q\to\infty} h^{*-1}(q) = +\infty\,.$$

Moreover, since h is smooth, $\forall q > 0$, we have

$$h^*(q) = q[(h')^{-1}(q)] - h((h')^{-1}(q))\,,$$

so that

$$h^*(h'(y)) = yh'(y) - h(y)\,, \quad \forall\ y > 0\,.$$

Proposition 3.1 is based on the possibility of construct a solution of a suitable Cauchy problem, as stated in the following Lemma.

Lemma 3.2. *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^2, convex function, such that $h(0) = 0$, and such that (9) and (10) hold. Then, for any $\delta > 0$ there exists $C_0 = C_0(\delta)$ and a function $\sigma = \sigma_\delta : [0, 1] \to [0, 1]$, $\sigma \in C^0[0, 1] \cap C^1(0, 1)$ solution of the problem*

$$\begin{cases} \sigma'(s) = \delta\sigma(s)h^{*-1}\left(\dfrac{C_0}{\delta\sigma(s)}\right) & \text{in } (0,1)\,, \\ \sigma(0) = 0\,, \quad \sigma(s) > 0\,. \end{cases} \tag{4}$$

Moreover

$$\sigma(1) = 1\,, \quad \textit{and} \quad \lim_{s\to 0^+} \sigma'(s) = 0\,. \tag{5}$$

Proof: Let us define $\sigma(s)$ trough the implicit formula

$$\int_0^{\sigma(s)} \frac{dt}{(h^*)^{-1}\left(\frac{C}{t}\right)t} = s\delta\,. \tag{6}$$

Step 1: Near 0. We first prove that $\forall C_0 > 0$, $\sigma(s)$ is well defined in a neighborhood of $s = 0$. Indeed, through the change of variable defined by the relationship $h'(z) = (h^*)^{-1}\left(\frac{C}{\sigma(s)}\right)$, and by the properties of h and h^* we have stated at the end of the previous section, it follows that:

$$\int_0 \frac{dt}{(h^*)^{-1}\left(\frac{C}{t}\right)t} \quad \Leftrightarrow \int^{+\infty} \frac{zh''(z)}{h'(z)[h'(z)z - h(z)]}dz\,.$$

Recalling hypothesis (10) and since $h'(z)z - h(z) > 0$, for any $z > 0$, there exists a constant c_1 such that

$$\int^{+\infty} \frac{1}{zh'(z)}\frac{z^2h''(z)}{h'(z)z - h(z)}dz \le c_1 \int^{+\infty} \frac{dz}{h'(z)z}dz \le c_1 \int^{+\infty} \frac{dz}{h(z)}\,,$$

where the last inequality holds since h is convex; by (9) last integral is finite and so σ is well defined near 0.

Step 2: The choice of C_0. It follows by Step 1, through the change $\rho = \frac{\tau}{\delta t}$, $\forall \delta > 0$, that

$$\lim_{\tau\to+\infty} \frac{1}{\delta}\int_{\frac{\tau}{\delta}}^{\infty} \frac{d\rho}{\rho(h^*)^{-1}(\rho)} = 0\,;$$

on the other hand, since $(h^*)^{-1}(0) = 0$,

$$\lim_{\tau\to 0} \frac{1}{\delta}\int_{\frac{\tau}{\delta}}^{\infty} \frac{d\rho}{\rho(h^*)^{-1}(\rho)} = +\infty\,.$$

Thus there exists C_0 such that

$$\frac{1}{\delta}\int_{\frac{C_0}{\delta}}^{\infty} \frac{d\rho}{\rho(h^*)^{-1}(\rho)} = \int_0^1 \frac{dt}{(h^*)^{-1}\left(\frac{C}{t}\right)t} = 1\,.$$

Step 3: The limit of σ'. Recalling the definition of σ' from (4), we want to prove

$$\lim_{s\to 0^+} \delta\sigma(s)h^{*-1}\left(\frac{C_0}{\delta\sigma(s)}\right) = 0\,. \tag{7}$$

This is equivalent to prove that

$$\lim_{\tau\to+\infty}\frac{h'(\tau)}{h^*(h'(\tau))}=\lim_{\tau\to+\infty}\frac{h'(\tau)}{h'(\tau)\tau-h(\tau)}=0$$

since τ is such that $(h^*)^{-1}(\frac{C_0}{\delta\sigma(s)})=h'(\tau)$. Using that $h^*(h'(\tau))\to+\infty$ as τ diverges and by De l'Hopital rule we deduce that (7) holds, and so the Lemma is proved. ■

Proof of Proposition 3.1: Let $\sigma(s)$ be the function defined in Lemma 3.2. Thus it is clear that inequality (1) is verified at $s=0$, and we can multiply and divide the left hand side by $\sigma(s)$; using (3) we get

$$\delta\sigma(s)v\frac{\sigma'(s)}{\delta\sigma(s)}\le\delta\sigma(s)h(v)+\delta\sigma(s)h^*\left(\frac{\sigma'(s)}{\delta\sigma(s)}\right).$$

Recalling that σ is the solution of the Cauchy problem defined in (4), (1) holds true. ■

In the sequel we will also handle with dualities involving the time derivatives of suitable functions; to this aim we will use the following Landes-type (see [16]) regularization result.

Lemma 3.3. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and let w be a function in $L^p(0,T;W_0^{1,p}(\Omega))$ and $w_0\in L^1(\Omega)$. Then, for any $\nu>0$, there exists a function $\eta_\nu=\eta_\nu(w,w_0)\in L^p(0,T;W_0^{1,p}(\Omega))$, such that*

$$\frac{d}{dt}\eta_\nu=\nu(w-\eta_\nu),$$

and $\eta_\nu(w,w_0)(0,x)=\eta_{0,\nu}\in L^2(\Omega)$, with

$$\eta_{0,\nu}\overset{\nu\to\infty}{\longrightarrow}w_0\quad in\quad L^1(\Omega).$$

If furthermore $w\in L^\infty(Q_\Omega^T)$, then

$$\|\eta_\nu\|_{L^\infty(Q)}\le\|w\|_{L^\infty(Q_\Omega^T)}\,. \tag{8}$$

Moreover, if $w_t=w^{(1)}+w^{(2)}\in L^1(Q_\Omega^T)+L^{p'}(0,T;W^{-1,p'}(\Omega))$, then it is possible to choose η_ν such that $\frac{d}{dt}\eta_\nu=\rho_\nu^{(1)}+\rho_\nu^{(2)}$, with both

$$\rho_\nu^{(1)}\overset{\nu\to\infty}{\longrightarrow}w^{(1)}\quad in\quad L^1(Q_\Omega^T)$$

and

$$\rho_\nu^{(2)}\overset{\nu\to\infty}{\longrightarrow}w^{(2)}\quad in\quad L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Proof: See [13], Lemma 2.1. ■

Here we state a useful result which allows us to handle with functions such that do not have the time derivative belonging to the dual of the energy space. In fact it consists in a generalized integration by parts formula, whose proof can be found in [11] (see also [9]).

Lemma 3.4. *Let D be any domain in $\mathbb{R}^N$, $N \geq 3$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is zero away from a compact set of $\mathbb{R}$; let us denote $F(s) = \int_0^s f(r)dr$. If $v \in L^p(0,T,W_0^{1,p}(D))$ is such that $v_t \in L^{p'}(0,T;W^{-1,p'}(D)) + L^1(Q)$ and if $\psi \in C^\infty([0,T] \times \overline{D})$, then we have*

$$\int_0^T \langle v_t, f(v)\psi\rangle \ = \int_D F(v(T))\psi(T) \ - \int_D F(v(0))\psi(0) \ - \int_{Q_T} \psi_t \ F(v) \ . \quad (9)$$

We observe that $v_t \in L^{p'}(0,T;W^{-1,p'}(D)) + L^1(Q_D^T)$ implies that there exist $\eta_1 \in L^{p'}(0,T;W^{-1,p'}(D))$ and $\eta_2 \in L^1(Q_D^T)$ such that $u_t = \eta_1 + \eta_2$. Even if η_1 and η_2 are not uniquely determined, the integration by parts formula turn out to be independent on the representation of v_t; moreover, according with the notation introduced before, $\langle \cdot, \cdot \rangle$ will indicate the duality between $L^{p'}(0,T;W^{-1,p'}(D)) + L^1(Q_D^T)$ and $L^p(0,T;W_0^{1,p}(D)) \cap L^\infty(Q_D^T)$.

We also recall the following classical result due to Gagliardo and Nirenberg.

Theorem 3.5 (Gagliardo-Nirenberg)**.** *Let $\Omega \subset \mathbb{R}^N$, opend and bounded, and let v be a function in $W^{1,\mu}(\Omega) \cap L^\lambda(\Omega)$ with $\mu \geq 1$, $\lambda \geq 1$. Then there exists a positive constant C, depending on N, q and λ, such that*

$$\|v\|_{L^\eta(\Omega)} \leq C \|\nabla v\|^\theta_{(L^\mu(\Omega))^N} \|v\|^{1-\theta}_{L^\lambda(\Omega)} \,,$$

for every θ and η satisfying

$$0 \leq \theta \leq 1, \quad 1 \leq \eta \leq +\infty, \quad \frac{1}{\eta} = \theta \left(\frac{1}{\mu} - \frac{1}{N}\right) + \frac{1-\theta}{\lambda}.$$

Proof: See [20], Lecture II. ■

The following embedding results are consequences of the previous theorem. We will use them in the last section but we give their statement here for completeness.

Corollary 3.6. *Let $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega))$, with $q \geq 1$, $\gamma \geq 1$. Then $v \in L^\sigma(Q_\Omega^T)$ with $\sigma = q\frac{N+\gamma}{N}$ and*

$$\int_{Q_\Omega^T} |v|^\sigma \, dxdt \leq C\|v\|_{L^\infty(0,T;L^\gamma(\Omega))}^{\frac{\gamma q}{N}} \int_{Q_\Omega^T} |\nabla v|^q \, dxdt \,. \tag{10}$$

Corollary 3.7. *Let $\Omega \subset \mathbb{R}^N$, opend and bounded, $\tau > 0, 1 < p < N$ and let further $w \in L^\infty(0,\tau;L^p(\Omega)) \cap L^p(0,\tau;W_0^{1,p}(\Omega))$. Then there exists a positive constant K depending only on N and p such that*

$$\left[\int_0^\tau \left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \leq K\left(\sup_{t\in[0,\tau]}\int_\Omega |w|^p + \int_0^\tau\int_\Omega |\nabla w|^p\right)$$

for all μ and σ satisfying

$$p \leq \sigma \leq p^*, \quad p \leq \mu \leq \infty, \quad \frac{N}{p\sigma} + \frac{1}{\mu} = \frac{N}{p^2}. \tag{11}$$

We want also recall the interpolation inequality, we will make use in the proof of Theorem 2.7. Assume that $z \in L^\infty(0,T;L^q(D)) \cap L^p(0,T;L^r(D))$, $p,q,r \geq 1$. Thus z in $L^\eta(Q_D^T)$ and

$$\begin{gathered} \|z\|_{L^\eta(Q)} \leq C\|z\|_{L^\infty(0,T;L^q(D))}^{1-\theta}\|z\|_{L^p(0,T;L^r(D))}^{\theta} \\ \text{with} \quad \tfrac{1}{\eta} = \tfrac{\theta}{r} + \tfrac{1-\theta}{q}, \; p \geq \theta\eta. \end{gathered} \tag{12}$$

A useful application of Corollary 3.6 is the following.

Proposition 3.8. *Let $R > 0$ and $B_R \in \mathbb{R}^N$ a ball of radius R and let $p > 1$. Let $w \in L^\infty(0,T;L^1(B_R))$ such that $T_k(w) \in L^p(0,T;W_0^{1,p}(B_R))$, for any $k > 0$. If $|\nabla w|^{p-1} \in L^1(Q_R^T)$, then $w^{p-1} \in L^1(Q_R^T)$.*

Proof: We deal only with the case $p > 2$, since for $p \leq 2$ it is trivial. Since both $w \in L^\infty(0,T;L^1(B_R))$ and $|\nabla w|^{p-1} \in L^1(Q_R^T)$, we have that $w \in L^1(0,T;W_0^{1,1}(B_R))$. Then, we can apply Corollary 3.6 with $q = \gamma = 1$ to obtain that $w \in L^{\frac{N+1}{N}}(Q_R^T)$. Now, if $p \leq 1 + \frac{N+1}{N}$ we are finished, otherwise $w \in L^1(0,T;W_0^{1,\frac{N+1}{N}}(B_R))$ and we apply again Corollary 3.6 with $\gamma = 1$ and $q = \frac{N+1}{N}$ to conclude that $w \in L^{\left(\frac{N+1}{N}\right)^2}(Q_R^T)$. It is clear that, iterating this procedure, we get the result in a finite number of steps. ■

The estimates contained in the following lemma are standard and turn out to coincide, for instance, with the one proved in [4] (see also Lemma 3.7 in [18]).

Lemma 3.9. *Let w be a measurable function finite almost everywhere on Q_R^T. If there exists $C > 0$ is such that*

$$\int_{Q_R^T} |\nabla T_k(w)|^p \le C(k+1),$$

for any $k > 0$, then, both

$$\|w\|_{M^{p-1+\frac{p}{N}}(Q_R^T)} \le c_1 \quad \text{and} \quad \|\nabla w\|_{M^{p-\frac{N}{N+1}}(Q_R^T)} \le c_2,$$

where c_1 and c_2 are positive constants only depending on C, N, R, T and p.

Finally let us state the following classical result due to Stampacchia.

Lemma 3.10 ([26])**.** *Let $\zeta(j,\rho)\ : [0,+\infty) \times [0,R)$ be a function such that $\zeta(\cdot,\rho)$ is nonincreasing and $\zeta(j,\cdot)$ nondecreasing. Moreover, suppose that $\exists K_0 > 0$, $\mu \ge 1$, and $C,\nu,\gamma > 0$ such that*

$$\zeta(j,\rho) \le C\frac{\sigma(k,R)^\mu}{(j-k)^\nu (R-\rho)^\gamma} \quad \forall j > k > K_0,\ \forall \rho \in (0,R].$$

Then for every $\delta \in (0,1)$, there exists $d > 0$ such that:

$$\zeta(K_0 + d, (1-\delta)R) = 0,$$

where

$$d^\nu = C' 2^{\frac{\mu(\nu+\gamma)}{\mu-1}} \frac{\zeta(K_0,1)^{\mu-1}}{(1-\delta)}, \qquad C' > 0.$$

Proof: See [26]. ■

4. proofs of Theorem 2.3 and Theorem 2.5

Proof of Theorem 2.3: Let us consider u_n as the weak solutions of the following problem

$$\begin{cases} (u_n)_t - \operatorname{div} a(t,x,u_n,\nabla u_n) + g(t,x,u_n,\nabla u_n) = f_n(t,x) & \text{in } Q_T^n, \\ u_n(t,x) = 0 & \text{on } \partial_{\mathcal{P}} Q_n^T, \\ u_n(0,x) = u_n^0(x) & \text{in } B_n(0), \end{cases} \tag{1}$$

where $f_n(t,x) = T_n(f(t,x))$ and $u_n^0(x) = T_n(u_0(x))$. Note that, thanks to the result of [10] (see also [23]), there exists (at least) a weak solution for (1), i.e.

a function $u_n \in L^p(0,T;W_0^{1,p}(B_n))$ such that $(u_n)_t \in L^{p'}(0,T;W^{-1,p'}(B_n))$, $g(t,x,u_n,\nabla u_n) \in L^1((0,T)\times B_n)$, and the following identity holds true

$$\begin{aligned} \int_0^T < (u_n)_t\,,\,\psi > + \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\psi \\ + \int_{Q_n^T} g(t,x,u_n,\nabla u_n)\psi = \int_{Q_n^T} f_n\psi\,, \end{aligned} \tag{2}$$

$\forall\psi \in L^p(0,T;W_0^{1,p}(B_n)) \cap L^\infty(Q_n^T)$.

We will prove Theorem 2.3 by showing that the terms in (1) are compact in suitable spaces. In order to do it, here and throughout the whole proof, we fix a ball B_R, centered at the origin, and we will prove suitable estimates for u_n in $(0,T)\times B_R$. Moreover, a weak solution on Q_n^T turns out to be obviously a weak solution in Q_n^t for any $0<t<T$. So that, with an abuse of notation we will often invoque (2) by tacitly understanding its counterpart on Q_n^t.

Local estimate on truncations. For any $n \geq R+\rho$ (for any fixed $\rho>0$), let us choose in (2) $\psi = \varphi_\lambda(T_k(u_n))\xi$, where $\varphi_\lambda(s) = se^{\lambda s^2}$ enjoys property (21), $\lambda>0$ will be fixed later, $k>L$, and $\xi(x)$ is a cut-off function such that (2) holds true (we will often omit the dependence on x). Thus we have

$$\begin{aligned} \int_0^t < (u_n)_t, \varphi_\lambda(T_k(u_n))\xi > \\ + \int_{Q_n^t} a(t,x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\varphi_\lambda'(T_k(u_n))\xi \\ + \int_{Q_n^t} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\varphi_\lambda(T_k(u_n)) \\ + \int_{Q_n^t} g(t,x,u_n,\nabla u_n)\varphi_\lambda(T_k(u_n))\xi = \int_{Q_n^t} f_n(t,x)\varphi_\lambda(T_k(u_n))\xi\,. \end{aligned} \tag{3}$$

Since ξ does not depend on time, using the Lemma 3.4,

$$\int_0^t < (u_n)_t, \varphi_\lambda(T_k(u_n))\xi >= \int_{B_n} \Phi_{\lambda,k}(u_n(t,x))\xi - \int_{B_n} \Phi_{\lambda,k}(u_n(x,0))\xi\,,$$

where

$$\Phi_{\lambda,k}(s) = \int_0^s \varphi_\lambda(T_k(\tau))d\tau = \begin{cases} \frac{1}{2\lambda}(e^{\lambda s^2} - 1) & \text{if } |s| \leq k\,, \\ \varphi_\lambda(k)(|s| - k) + \frac{1}{2\lambda}(e^{\lambda k^2} - 1) & \text{if } |s| > k\,. \end{cases}$$

Note that

$$\varphi_\lambda(k)|s| - e^{\lambda k^2}\left(k^2 - \frac{1}{2\lambda}\right) - \frac{1}{2\lambda} \leq \Phi_{\lambda,k}(s) \leq \varphi_\lambda(k)|s|\,,$$

so that we deduce

$$\begin{aligned} &\int_0^t < (u_n)_t, \varphi_\lambda(T_k(u_n))\xi > \\ &\geq \varphi_\lambda(k)\int_{B_n} |u_n(t,x)|\xi - \left[e^{\lambda k^2}\left(k^2 - \frac{1}{2\lambda}\right) + \frac{1}{2\lambda}\right]\text{meas}\{B_{R+\rho}\} \\ &\quad -\varphi_\lambda(k)\int_{B_n} |u_0(x)|\xi\,. \end{aligned} \tag{4}$$

Moreover, since $k > L$, by (5)–(9), we have

$$\begin{aligned} \int_{Q_n^t} g(t,x,u_n,\nabla u_n)\varphi_\lambda(T_k(u_n))\xi \geq \int_{Q_n^t} h(|\nabla u_n|^{p-1})|\varphi_\lambda(T_k(u_n))|\xi \\ - \int_{Q_n^t}\Big(\widetilde{\gamma}_k|\nabla T_k(u_n)|^p + |\widetilde{g}_k(t,x)|\Big)|\varphi_\lambda(T_k(u_n))|\xi\,, \end{aligned}$$

where $\widetilde{\gamma}_k = \gamma_k + c_1$ and $\widetilde{g}_k = g_k + c_1$. On the other hand by (3) and choosing $\xi = \sigma(\eta)$, where η satisfies (2) and sigma is is the function defined in Lemma 3.2, we can apply Proposition 3.1 with $\delta = \frac{1}{2\beta}$. Thus there exists a constant $C = C(\lambda, k, \beta, T)$ such that

$$\begin{aligned} &\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\varphi_\lambda(T_k(u_n)) \\ &\geq -\beta\int_{Q_n^T} |\nabla u_n|^{p-1}|\nabla\xi||\varphi_\lambda(T_k(u_n))| \\ &\geq -\int_{Q_n^t}\frac{1}{2}h\left(|\nabla u_n|^{p-1}\right)|\varphi_\lambda(T_k(u_n))|\xi - C\text{meas}\{B_{R+\rho}\}\,. \end{aligned}$$

By substituting the above inequalities into (3), we deduce

$$\begin{aligned}&\varphi_\lambda(k)\int_{B_n}|u_n(t,x)|\xi+\int_{Q_n^t}|\nabla T_k(u_n)|^p\Big[\alpha\varphi_\lambda'(T_k(u_n))-\widetilde{\gamma}_k|\varphi_\lambda(T_k(u_n))|\Big]\xi\\&\le C\mathrm{meas}\{B_{R+\rho}\}+\varphi_\lambda(k)\left[\int_{B_n}|u_0(x)|\xi+\int_{Q_n^t}\Big(|f_n(t,x)|+|\widetilde{g}_k(t,x)|\Big)\xi\right].\end{aligned}\tag{5}$$

Note that both $f_n(t,x)\xi$ and $\widetilde{g}_k(t,x)\xi$ are bounded in $L^1(Q_T)$, so that, by (21) applied with $a=\alpha$ and $b=\widetilde{\gamma}_k$ we deduce that there exists a constant (depending on k) such that

$$\sup_{t\in(0,T)}\int_{B_R}|u_n(t,x)|+\int_{Q_R^T}|\nabla T_k(u_n)|^p\le C(k)\,,\qquad\forall k>0\,.\tag{6}$$

This implies, since obviously $\|T_k(u_n)\|_{L^p((0,T)\times B_R)}\le C(R,T)k$, that $T_k(u_n)$ is bounded in $L^p(0,T;W^{1,p}(B_R))$, $\forall R>0$. Thus, up to subsequences (not relabeled) $T_k(u_n)$ weakly converges toward a function v_k in $L^p(0,T;W^{1,p}(B_R))$. Moreover the sequence $\{u_n\}$ is bounded in $L^\infty(0,T;L^1(B_R))$.

Hence, from (6) we deduce (integrating between 0 and T), $\forall j>0$,

$$\begin{aligned}&j\mathrm{meas}\{(t,x)\in(0,T)\times B_R\,:\,|u_n|\ge j\}\\&\le\int_{\{(0,T)\times B_{R+\rho}\}\cap\{(t,x):|u_n|\ge j\}}|u_n(t,x)|\xi\le\int_0^T\int_{B_n}|u_n(t,x)|\xi\le CT\,,\end{aligned}$$

so that

$$\mathrm{meas}\{(t,x)\in(0,T)\times B_R\,:\,|u_n|\ge j\}\le\frac{CT}{j}\,.\tag{7}$$

Moreover, choosing $S_k'(u_n)\xi$ as test function in (2), we deduce that $\big(S_k(u_n)\xi\big)_t$ is bounded in $L^1(Q_T)+L^{p'}(0,T;W^{-1,p'}(B_{R+\rho}))$ and so, using the compactness result of [25], we have that $S_k(u_n)\xi$ is strongly compact in $L^1((0,T)\times B_{R+\rho})$. Hence, up to subsequences (not relabeled), it converges a.e. as n diverges. Using a diagonal argument, it follows that $u_n\to u$ for a.e. $(t,x)\in(0,T)\times B_R$, $\forall R>0$, and consequently there exists a measurable function $u(t,x)$ such that $u_n\to u$ a.e. in $(0,T)\times\mathbb{R}^N$. Finally, we note that (7) implies that u_n is compact in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and consequently that

$$T_k(u_n)\rightharpoonup T_k(u)\quad\text{in }L^p(0,T;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))\,.\tag{8}$$

Estimate on the lower order term. Let us choose $\psi = \frac{T_\varepsilon(u_n)}{\varepsilon}\xi$ as test function in (2), so that we have:

$$\int_{Q_n^T} \frac{d}{dt}\left(\frac{\Theta_\varepsilon(u_n)}{\varepsilon}\right)\xi + \frac{1}{\varepsilon}\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot \nabla T_\varepsilon(u_n)\xi$$
$$+\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\frac{T_\varepsilon(u_n)}{\varepsilon}$$
$$+\int_{Q_n^T} g(t,x,u_n,\nabla u_n)\frac{T_\varepsilon(u_n)}{\varepsilon}\xi \le \int_{Q_n^T}|f_n(t,x)|\xi\,,$$

where $\Theta_k(s) = \int_0^s T_k(\tau)d\tau$, $\forall k>0$. We first note that

$$0 \le \frac{\Theta_\varepsilon(s)}{\varepsilon} \le |s|, \quad \forall s\in\mathbb{R},$$

and by (2), we deduce

$$\frac{\alpha}{\varepsilon}\int_{Q_n^T}|\nabla T_\varepsilon(u_n)|^p\xi + \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\frac{T_\varepsilon}{\varepsilon}(u_n)$$
$$+\int_{Q_n^T} g(t,x,u_n,\nabla u_n)\frac{T_\varepsilon(u_n)}{\varepsilon}\xi \le \|f_n(t,x)\|_{L^1(Q_{R+\rho}^T)} + \int_{B_n}|u_n^0(x)|\xi\,.$$

Moreover, using (5)–(9), we have (as above $\widetilde{\gamma}_L = \gamma_L + c_1$ and $\widetilde{g}_L = g_L + c_1$)

$$\int_{Q_n^T} g(t,x,u_n,\nabla u_n)\frac{T_\varepsilon}{\varepsilon}(u_n)\xi$$
$$\ge \frac{1}{2}\int_{Q_n^T} h(|\nabla u_n|^{p-1})\xi + \frac{1}{2}\int_{Q_n^T}|g(t,x,u_n,\nabla u_n)|\left|\frac{T_\varepsilon}{\varepsilon}(u_n)\right|\xi$$
$$-\widetilde{\gamma}_L\int_{Q_T^n\cap\{|u_n|\le L\}}|\nabla T_L(u_n)|^p\xi - \int_{Q_T^n\cap\{|u_n|\le L\}}|\widetilde{g}_L(t,x)|\xi\,,$$

Now we choose $\xi=\sigma(\eta)$ where η is chosen as in (2); thus by using Proposition 3.1 with $\delta = \frac{1}{4\beta}$, we have

$$\left|\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\frac{T_\varepsilon}{\varepsilon}(u_n)\right|$$
$$\le \frac{1}{4}\int_{Q_n^T} h\left(|\nabla u_n|^{p-1}\right)\xi + C\ \mathrm{meas}(B_{R+\rho})\,.$$

Thus, dropping positive terms, we have

$$\frac{1}{4}\int_{Q_n^T} h(|\nabla u_n|^{p-1})\xi + \frac{1}{2}\int_{Q_n^T} |g(t,x,u_n,\nabla u_n)|\left|\frac{T_\varepsilon}{\varepsilon}(u_n)\right|\xi$$
$$\leq \gamma_L \int_{B_n\cap\{|u_n|\leq L\}} |\nabla T_L(u_n)|^p\xi + \int_{B_n\cap\{|u_n|\leq L\}} |g_L(t,x)|\xi$$
$$+\int_{B_n} |u_n^0(x)|\xi + C\text{meas}(B_{R+\rho})\,,$$

and, by (6) and (6), the right hand side of the previous inequality is uniformly bounded (with respect to n). Thus, letting $\varepsilon \to 0$, Fatou's Lemma yields,

$$\int_{Q_R^T} h(|\nabla u_n|^{p-1}) + \int_{Q_R^T} |g(t,x,u_n,\nabla u_n)| \leq C_R\,. \tag{9}$$

Equiintegrability of the lower order term and uniform estimate on the "stripes". Let us choose $\psi = \gamma_j(u_n)\xi$, $\forall j > L$, in (2) where $\gamma_j(s) = T_1(G_j(s))$, and moreover we denote by $\Gamma_j(s) = \int_0^s \gamma_j(t)dt$; we note that

$$|G_{j+1}(s)| \leq \Gamma_j(s) \leq |G_j(s)|\,. \tag{10}$$

Thus we have:

$$\int_0^T < (u_n)_t\,,\,\gamma_j(u_n)\xi > + \int_{Q_n^t} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\gamma_j(u_n)$$
$$+\int_{Q_n^t} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\gamma_j'(u_n)\xi$$
$$+\int_{Q_n^t} g(t,x,u_n,\nabla u_n)\gamma_j(u_n)\xi = \int_{Q_n^t} f_n(t,x)\gamma_j(u_n)\xi\,.$$

Thus, since $j > L$, using that $|\gamma(s)| \leq 1$ and (2) we get

$$\int_{B_n} \Gamma_j(|u_n(t,x)|)\xi + \int_{Q_n^t} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\gamma_j(u_n)$$
$$+\int_{Q_n^t\cap\{j\leq|u_n|\leq j+1\}} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\xi$$
$$+\frac{1}{2}\int_{Q_n^t\cap\{|u_n|\geq j\}} h(|\nabla u_n|^{p-1})|\gamma_j(u_n)|\xi + \frac{1}{2}\int_{Q_n^t\cap\{|u_n|\geq j\}} |g(t,x,u_n,\nabla u_n)||\gamma_j(u_n)|\xi$$
$$\leq \int_{Q_n^t\cap\{|u_n|\geq j\}} |f_n(t,x)|\xi + \int_{B_n} \Gamma_j(|u_n^0(x)|)\xi\,.$$

On the other hand by (3), and choosing ξ as above, we deduce by Proposition 3.1 applied with $\delta = \frac{1}{2\beta}$,

$$\int_{Q_n^t\cap\{|u_n|\geq j\}} |a(t,x,u_n,\nabla u_n)\cdot\nabla\xi||\gamma_j(u_n)|$$

$$\leq \frac{1}{2}\int_{Q_n^t\cap\{|u_n|\geq j\}} h\left(|\nabla u_n|^{p-1}\right)|\gamma_j(u_n)|\xi$$

$$+C\text{meas}\{(t,x)\in(0,T)\times B_{R+\rho}\,:\,|u_n|\geq j\}\,,$$

and the last term tends to 0 (uniformly with respect to n) as j diverges by (7). Moreover, by using (10) we deduce, dropping the positive term,

$$\int_{B_n} G_{j+1}(|u_n(t,x)|)\xi + \int_{Q_n^t\cap\{j\leq|u_n|\leq j+1\}} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\xi$$

$$+\frac{1}{2}\int_{Q_n^t\cap\{|u_n|\geq j\}} |g(t,x,u_n,\nabla u_n)||\gamma_j(u_n)|\xi$$

$$\leq \int_{Q_n^t\cap\{|u_n|\geq j\}} |f_n(t,x)|\xi + \int_{B_n} G_j(|u_n^0(x)|)\xi + \varepsilon(j)$$

Since both $u_0^n(x)\xi$ and $f_n(t,x)\xi$ are strongly compact in $L^1(B_{R+\rho})$ and in $L^1(Q^T_{R+\rho})$ respectively, we obtain, dropping positive terms,

$$\begin{aligned}\liminf_{j\to\infty}\sup_{n\in\mathbb{N}}\Bigg[&\int_{Q_n^T\cap\{j\leq|u_n|\leq j+1\}} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\xi\\ &+\int_{Q_n^T\cap\{|u_n|\geq j+1\}}|g(t,x,u_n,\nabla u_n)|\xi\Bigg]=0\,.\end{aligned}\tag{11}$$

Note that the above estimate, in fact, allows us to say, using (7) and since $h(s)$ is superlinear at infinity, that

$$\sup_{n\in\mathbb{N}}\int_{Q_R^T\cap\{|u_n|\geq j\}} |\nabla u_n|^{p-1} = \varepsilon(j)\,.\tag{12}$$

Strong convergence of truncations. Let $\varphi_\lambda(s)$ be the function introduced in (21), where $\lambda>0$ will be fixed in the sequel. We set $T_k(u)_\nu = \eta_\nu(T_k(u),T_k(u_0))$, where $\eta_\nu(\cdot)$ has been defined in Lemma 3.3.

Now we choose $\psi = \varphi_\lambda(z_{n,\nu})S_j'(u_n)\xi$ as test function in (2), where $z_{n,\nu} = T_k(u_n) - T_k(u)_\nu$, $k \geq L$, and $S_j(s)$ is defined in (22). Thus we have

$$
\begin{gathered}
\int_0^T < S_j(u_n)_t\,,\ \varphi_\lambda(T_k(u_n) - T_k(u)_\nu)\xi > \\
+ \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\,\varphi_\lambda(z_{n,\nu})S_j'(u_n) \\
+ \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\,\varphi_\lambda'(z_{n,\nu})S_j'(u_n)\xi \\
+ \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\,\varphi_\lambda(z_{n,\nu})S_j''(u_n)\xi \\
+ \int_{Q_n^T} g(t,x,u_n,\nabla u_n)\varphi_\lambda(z_{n,\nu})S_j'(u_n)\xi \\
= \int_{Q_n^T} f_n(t,x)\varphi_\lambda(z_{n,\nu})S_j'(u_n)\xi\,.
\end{gathered}
\tag{13}
$$

We first note that

$$
\begin{aligned}
&\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\,\varphi_\lambda'(z_{n,\nu})S_j'(u_n)\xi \\
&= \int_{Q_n^T\cap\{|u_n|\leq k\}} a(t,x,u_n,\nabla u_n)\cdot\nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\,\varphi_\lambda'(z_{n,\nu})S_j'(u_n)\xi \\
&\quad - \int_{Q_n^T\cap\{|u_n|\geq k\}} a(t,x,u_n,\nabla u_n)\cdot\nabla T_k(u)_\nu\,\varphi_\lambda'(z_{n,\nu})S_j'(u_n)\xi\,.
\end{aligned}
$$

Using (6) and recalling that $\mathrm{Supp}(S_j') \subset [-j-1, j+1]$, there exists $\varsigma_{k,j} \in (L^{p'}(Q_R^T))^{N+1}$ such that, using Egorov Theorem,

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{Q_n^T\cap\{|u_n|\geq k\}} a(t,x,u_n,\nabla u_n)\cdot\nabla T_k(u)_\nu\,\varphi_\lambda'(z_{n,\nu})S_j'(u_n)\xi \\
&\qquad = \int_{Q_{R+\rho}^T\cap\{|u|\geq k\}} \varsigma_{k,j}\cdot\nabla T_k(u)_\nu\,\varphi_\lambda'(z_\nu)S_j'(u)\xi\,,
\end{aligned}
$$

and last integral tends to 0 as ν diverges, since $T_k(u)_\nu \to T_k(u)$ strongly in $L^p(0,T;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))$, and consequently $|\nabla T_k(u)_\nu|\chi_{\{|u|\geq k\}}$ tends to zero strongly in $L^p(0,T\,;\,L^p_{\mathrm{loc}}(\mathbb{R}^N))$. Thus

$$
\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\,\varphi_\lambda'(z_{n,\nu})S_j'(u_n)\xi
$$

$$= \int_{Q_n^T} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\, \varphi'_\lambda(z_{n,\nu}) S'_j(u_n)\xi + \varepsilon(n,\nu)\,.$$

On the other hand, since $k \geq L$,

$$\int_{Q_n^T} g(t,x,u_n,\nabla u_n)\varphi_\lambda(z_{n,\nu})S'_j(u_n)\xi$$

$$\geq \int_{Q_n^T} h(|\nabla u_n|^{p-1})|\varphi_\lambda(z_{n,\nu})|S'_j(u_n)\xi - \int_{Q_n^T\cap\{|u_n|\leq k\}} |\widetilde{g}_k(t,x)||\varphi_\lambda(z_{n,\nu})|\xi$$

$$-\frac{\widetilde{\gamma}_k}{\alpha}\int_{Q_n^T\cap\{|u_n|\leq k\}} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u)_\nu)S'_j(u_n)|\varphi_\lambda(z_{n,\nu})|\xi$$

$$+\frac{\gamma_k}{\alpha}\int_{Q_n^T} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla T_k(u)_\nu S'_j(u_n)\varphi_\lambda(z_{n,\nu})\xi\,,$$

where $\widetilde{\gamma}_k = \gamma_k + c_1$ and $\widetilde{g}_k = g_k + c_1$. Reasoning as before,

$$\frac{\gamma_k}{\alpha}\int_{Q_n^T} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla T_k(u)_\nu \varphi_\lambda(z_{n,\nu})S'_j(u_n)\xi = \varepsilon(n,\nu)\,,$$

and

$$\int_{Q_n^T} \Big[|f_n(t,x)| + |\widetilde{g}_k(t,x)|\Big]\varphi_\lambda(z_{n,\nu})S'_j(u_n)\xi = \varepsilon(n,\nu)\,.$$

Gathering the above informations together, we deduce:

$$\begin{gathered}
\int_0^T < S_j(u_n)_t\,,\ \varphi_\lambda(T_k(u_n) - T_k(u)_\nu)\xi > \\
+\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\xi\,\varphi_\lambda(z_{n,\nu})S'_j(u_n) \\
+\int_{Q_n^T} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\,\varphi'_\lambda(z_{n,\nu})S'_j(u_n)\xi \\
+\int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\,\varphi_\lambda(z_{n,\nu})S''_j(u_n)\xi \\
-\tfrac{\widetilde{\gamma}_k}{\alpha}\int_{Q_n^T} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla\Big(T_k(u_n) - T_k(u)_\nu\Big)\varphi_\lambda(z_{n,\nu})S'_j(u_n)\xi \\
+\int_{Q_n^T} h(|\nabla u_n|^{p-1})|\varphi_\lambda(z_{n,\nu})|S'_j(u_n)\xi \leq \varepsilon(n,\nu)\,.
\end{gathered} \tag{14}$$

Moreover the second integral in (14) is estimated, using that $\xi = \sigma(\eta)$, η chosen as in (2), and by Proposition 3.1,

$$\left| \int_{Q_n^T} a(t,x,u_n,\nabla u_n) \cdot \nabla \xi \, \varphi_\lambda(T_k(u_n) - T_k(u)_\nu) \right|$$

$$\leq \frac{1}{2} \int_{Q_n^T} h\Big(|\nabla u_n|^{p-1}\Big) |\varphi_\lambda(T_k(u_n) - T_k(u)_\nu)| S_j'(u_n)\xi + \varepsilon(n,\nu)$$

Thus by (14) we have, dropping positive terms,

$$\begin{aligned} &\int_0^T < S_j(u_n)_t \,,\, \varphi_\lambda(T_k(u_n) - T_k(u)_\nu)\xi > \\ &+ \int_{Q_n^T} a(t,x,u_n,\nabla T_k(u_n)) \cdot \nabla z_{n,\nu} \Big[\varphi_\lambda'(z_{n,\nu}) - \frac{\widetilde{\gamma}_k}{\alpha}\varphi_\lambda(z_{n,\nu})\Big] S_j'(u_n)\xi \\ &+ \int_{Q_n^T} a(t,x,u_n,\nabla u_n) \cdot \nabla u_n \, \varphi_\lambda(z_{n,\nu}) S_j''(u_n)\xi \leq \varepsilon(n,\nu)\,. \end{aligned} \tag{15}$$

Noticing that, by definition of $T_k(u)_\nu$,

$$- \int_{Q_n^T} a(t,x,u_n,\nabla T_k(u)_\nu) \cdot \nabla z_{n,\nu} \Big[\varphi_\lambda'(z_{n,\nu}) - \frac{\widetilde{\gamma}_k}{\alpha} |\varphi_\lambda(z_{n,\nu})|\Big] S_j'(u_n)\xi = \varepsilon(n,\nu)\,,$$

we can add this quantity in both sides of (15). Moreover using (11) we also have:

$$\left| \int_{Q_n^T} a(t,x,u_n,\nabla u_n) \cdot \nabla u_n \, \varphi_\lambda(z_{n,\nu}) S_j''(u_n)\xi \right| \leq \varepsilon(j)\,.$$

Finally, in order to get rid of the integral involving the time derivative of $S_j(u_n)$, we apply the following inequality, whose proof is postponed at the end of this Section.

Claim. $\forall j \geq j_0$:

$$\int_0^t < S_j(u_n)_t \,,\, \varphi_\lambda(T_k(u_n) - T_k(u)_\nu)\xi > \geq \varepsilon(n,\nu)\,. \tag{16}$$

Using (16) in (15) we deduce that, for j large enough,

$$\int_{Q_n^T} (a(t,x,u_n,\nabla T_k(u_n)) - a(t,x,u_n,\nabla T_k(u)_\nu)) \cdot \nabla z_{n,\nu} \\ \times \left[\varphi'_\lambda(z_{n,\nu}) - \tfrac{\widetilde{\gamma}_k}{\alpha}\varphi'_\lambda(z_{n,\nu})\right] S'_j(u_n)\xi \le \varepsilon(n,\nu) + \varepsilon(j)\,.$$

By a suitable choice of λ (according with (21) applied with $a = 1$ and $b = \frac{\widetilde{\gamma}_k}{\alpha}$) we deduce that

$$\int_{Q_n^T} (a(t,x,u_n,\nabla T_k(u_n)) - a(t,x,u_n,\nabla T_k(u)_\nu)) \cdot \nabla z_{n,\nu} \\ \times S'_j(u_n)\xi \le \varepsilon(n,\nu) + \varepsilon(j)\,.$$

Lemma 5 in [BMP] yields

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(0,T;W^{1,p}(B_R))\,. \tag{17}$$

Note that the above convergence implies that, up to subsequences, $\nabla T_k(u_n)$ a.e. converges to $\nabla T_k(u)$, and by a diagonal argument we conclude that, again up to subsequences not relabeled,

$$\nabla u_n \to \nabla u \quad \text{a.e.}\,. \tag{18}$$

Moreover combining (17) with (12) and (18) we deduce, by using Vitali's Theorem, that

$$|\nabla u_n|^{p-1} \to |\nabla u|^{p-1} \quad \text{strongly in } L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N))\,, \tag{19}$$

and by (11), (17) and (18) we have

$$g(x,u_n,\nabla u_n)\xi \to g(x,u,\nabla u)\xi \quad \text{strongly in } L^1((0,T)\times\mathbb{R}^N)\,. \tag{20}$$

Passing to the limit. Let us choose $\psi = \phi(t,x)S'(u_n)$ in (2) where ϕ is a $C^1_0([0,T)\times\mathbb{R}^N)$ and $S(s)$ chosen as in the Definition 2.2. We have:

$$\int_0^T < (u_n)_t\,,\, \phi S'(u_n) > + \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla\phi S'(u_n) \\ + \int_{Q_n^T} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n S''(u_n)\phi \\ + \int_{Q_n^T} g(t,x,u_n,\nabla u_n)\phi = \int_{Q_n^T} f_n(t,x)\phi\,. \tag{21}$$

We first note that there exists $R > 0$ such that supp $\phi(x,t) \subset (0,T) \times B_R$, so that, integrating by parts, and recalling that $\phi(T,x) = 0$ we get:

$$\lim_{n\to+\infty} \int_0^T < (u_n)_t \,,\, \phi S'(u_n) >$$

$$= \lim_{n\to+\infty} \int_{\mathbb{R}^N} S(u_n)\phi(T,x) - \int_{\mathbb{R}^N} S(u_n^0)\phi(0,x) - \int_{Q^T} S(u_n)\,\phi_t(t,x)$$

$$= -\int_{\mathbb{R}^N} S(u_0)\phi(0,x) - \int_{Q^T} S(u)\,\phi_t(t,x)\,,$$

where $Q^T = (0,T) \times \mathbb{R}^N$. Moreover, by (3) and (19), we have

$$\lim_{n\to+\infty} \int_{Q_n^T} a(t,x,u_n,\nabla u_n) \cdot \nabla\phi S'(u_n) = \int_{Q_n^T} a(t,x,u,\nabla u) \cdot \nabla\phi S'(u)\,,$$

while by (17) and (20), (11) we deduce that both

$$\lim_{n\to+\infty} \int_{Q_n^T} a(t,x,u_n,\nabla u_n) \cdot \nabla u_n\, S''(u_n)\phi = \int_{Q_n^T} a(t,x,u,\nabla u) \cdot \nabla u\, S''(u)\phi\,,$$

and

$$\lim_{n\to+\infty} \int_{Q_n^T} g(t,x,u_n,\nabla u_n) S'(u_n)\phi = \int_{Q_n^T} g(t,x,u,\nabla u) S'(u)\phi\,.$$

Finally, since $f_n\xi \to f\xi$ in $L^1(Q_T)$ it allows us to pass to the limit in the last integral in (21). Consequently $u(t,x)$ is a solution for (14) in the sense of Definition 2.2.

To complete the proof of Theorem 2.3 we need to prove that inequality (16) holds.

Proof of (16): We recall that, by previous estimates, $T_k(u_n)$ converges to $T_k(u)$ weakly in $L^p(0,T;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))$. Here we use an approximation argument by using Lemma 3.3. We set, for every $\sigma > 0$, $u_{n,\sigma} = \eta_\sigma(u_n,u_n^0)$; we know that $u_{n,\sigma} \in L^p(0,T;W_0^{1,p}(B_n))$, $(u_{n,\sigma})_t \in L^p(0,T;W_0^{1,p}(B_n))$, and moreover, both

$$u_{n,\sigma} \longrightarrow u_n \ \text{ in } \ L^p(0,T;W_0^{1,p}(B_n))\,,$$

and

$$(u_{n,\sigma})_t \longrightarrow (u_n)_t \ \text{ in } \ L^{p'}(0,T;W^{-1,p'}(B_n)) + L^1(Q_n^T)\,,$$

with $u_{n,\sigma}(0,x) = u_n^0$.

This approximation argument will allow us to consider derivatives with respect to t of the composition between Lipschitz functions and $u_{n,\sigma}$. Thanks to these properties we have that

$$\begin{aligned}&\int_0^T < S_j(u_n)_t\,,\, \varphi_\lambda(T_k(u_n)-T_k(u)_\nu)\xi > \\ &= \lim_{\sigma\to 0}\int_0^T < S_j(u_{n,\sigma})_t\,,\, \varphi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)\xi >\end{aligned} \tag{22}$$

Our aim is to prove that

$$\int_0^T < S_j(u_{n,\sigma})_t\,,\, \varphi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)\xi > \geq \varepsilon(n,\nu)\,.$$

So, note that, for any $j>k$, we can write

$$S_j(u_{n,\sigma}) = T_k(u_{n,\sigma}) + G_k(S_j(u_{n,\sigma}))$$

thus, if we define $\phi_\lambda(s)=\int_0^s\varphi_\lambda$, we have

$$\begin{aligned}&\int_0^T < S_j(u_{n,\sigma})_t\,,\, \varphi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)\xi > \\ &= \int_{B_R}\phi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)(T)\xi - \int_{B_R}\phi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)(0)\xi \\ &\quad + \int_0^T < G_k(S_j(u_{n,\sigma}))_t\,,\, \varphi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)\xi > \\ &\quad + \int_{Q_n^T}\nu(T_k(u)-T_k(u)_\nu)\varphi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)\xi\,,\end{aligned}$$

where we used that $(T_k(u)_\nu)_t = \nu(T_k(u)-T_k(u)_\nu)$. Using that $\phi_\lambda(s)>0$, $\forall s\in\mathbb{R}$ and that $T_k(u_n^0)\xi \to T_k(u_0)\xi$ weakly-$*$ in $L^\infty(B_R)$, we deduce that

$$\int_{B_R}\phi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)(T)\xi - \int_{B_R}\phi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)(0)\xi \geq \varepsilon(n,\nu)\,.$$

Moreover

$$\begin{aligned}&\int_{Q_n^T}\nu(T_k(u)-T_k(u)_\nu)\varphi_\lambda(T_k(u_{n,\sigma})-T_k(u)_\nu)\xi \\ &= \int_{Q_n^T}\nu(T_k(u)-T_k(u)_\nu)\varphi_\lambda(T_k(u)-T_k(u)_\nu)\xi + \varepsilon(\sigma,n) \geq \varepsilon(\sigma,n)\,,\end{aligned}$$

since $s \cdot se^{\lambda s^2} \geq 0$. Finally, we deal with the term

$$\int_0^T < G_k(S_j(u_{n,\sigma}))_t \, , \, \varphi_\lambda(T_k(u_{n,\sigma}) - T_k(u)_\nu)\xi > \; .$$

Integrating by parts we deduce that

$$\begin{aligned}
&\int_0^T < G_k(S_j(u_{n,\sigma}))_t \, , \, \varphi_\lambda(T_k(u_{n,\sigma}) - T_k(u)_\nu)\xi > \\
&= \int_{B_R} G_k(S_j(u_{n,\sigma}))(T)\varphi_\lambda(T_k(u_{n,\sigma}) - T_k(u)_\nu)(T)\xi \\
&- \int_{B_R} G_k(S_j(u_{n,\sigma}))(0)\varphi_\lambda(T_k(u_{n,\sigma}) - T_k(u)_\nu)(0)\xi \\
&- \int_0^T < G_k(S_j(u_{n,\sigma}))\varphi_\lambda'(T_k(u_{n,\sigma}) - T_k(u)_\nu)\xi \, , \, (T_k(u_{n,\sigma}) - T_k(u)_\nu)_t > \; .
\end{aligned}$$

Thus, the first term is positive since

$$\begin{aligned}
&\int_{B_R} G_k(S_j(u_{n,\sigma}))(T)\varphi_\lambda(T_k(u_{n,\sigma}) - T_k(u)_\nu)(T)\xi \\
&= \int_{B_R \cap \{u_{n,\sigma} > k\}} G_k(S_j(u_{n,\sigma}))(T)\varphi_\lambda(k - T_k(u)_\nu)(T)\xi \\
&+ \int_{B_R \cap \{u_{n,\sigma} < -k\}} G_k(S_j(u_{n,\sigma}))(T)\varphi_\lambda(-k - T_k(u)_\nu)(T)\xi \geq \varepsilon(\sigma),
\end{aligned}$$

while in the second term vanishes passing to the limit with respect to σ and then ν Concerning the last one, we note that since $G_k(S_j(u_{n,\sigma}))$ is 0 if $|u_{n,\sigma}| \leq k$, thus

$$\begin{aligned}
&- \int_0^T < G_k(S_j(u_{n,\sigma}))\varphi_\lambda'(T_k(u_{n,\sigma}) - T_k(u)_\nu)\xi \, , \, (T_k(u_{n,\sigma}) - T_k(u)_\nu)_t > \\
&= \nu \int_{Q_n^T} G_k(S_j(u_{n,\sigma}))\varphi_\lambda'(T_k(u_{n,\sigma}) - T_k(u)_\nu)(T_k(u) - T_k(u)_\nu)\xi \, .
\end{aligned}$$

Finally, taking the limit respectively in σ and n, we have

$$\begin{aligned}
&- \int_0^T < G_k(S_j(u_{n,\sigma}))\varphi_\lambda'(T_k(u_{n,\sigma}) - T_k(u)_\nu)\xi \, , \, (T_k(u_{n,\sigma}) - T_k(u)_\nu)_t > \\
&= \nu \int_{Q_n^T} G_k(S_j(u))\varphi_\lambda'(T_k(u) - T_k(u)_\nu)(T_k(u) - T_k(u)_\nu)\xi + \varepsilon(\sigma)
\end{aligned}$$

$$= \nu \int_{Q\cap\{u>k\}} G_k(S_j(u))\varphi'_\lambda(k - T_k(u)_\nu)(k - T_k(u)_\nu)\xi$$

$$+\nu \int_{Q\cap\{u<-k\}} G_k(S_j(u))\varphi'_\lambda(-k - T_k(u)_\nu)(-k - T_k(u)_\nu)\xi + \varepsilon(\sigma, n) \geq \varepsilon(\sigma, n)$$

since $\varphi'_\lambda(s) > 0$, $\forall s \in \mathbb{R}$ and the lemma is proved because of (22).

This conclude the proof of Theorem 2.3. ■

Proof of Theorem 2.5. Let us consider u_n as the weak solutions of the following problem

$$\begin{cases} (u_n)_t - \operatorname{div} a(t, x, u_n, \nabla u_n) + g(t, x, u_n, \nabla u_n) = f_n(t, x) & \text{in } Q^T_\Omega, \\ u_n(t, x) = n & \text{on } \partial_{\mathcal{P}} Q^T_\Omega\,, \\ u_n(x, 0) = u^0_n(x) & \text{in } \Omega, \end{cases} \tag{23}$$

where $f_n(t, x) = T_n(f(t, x))$ and $u^0_n(x) = T_n(u_0(x))$. The existence of a weak solution for (1) is still a consequence of the result of [10]. It means that there exists a function u_n such that $u_n - n \in L^p(0, T; W^{1,p}_0(\Omega))$, $(u_n)_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $g(t, x, u_n, \nabla u_n) \in L^1((0, T) \times \Omega)$, and the following identity holds true

$$\begin{gathered} \int_0^T < (u_n)_t\,,\, \psi > + \int_{Q^T_\Omega} a(t, x, u_n, \nabla u_n) \cdot \nabla\psi \\ + \int_{Q^T_\Omega} g(t, x, u_n, \nabla u_n)\psi = \int_{Q^T_\Omega} f_n\psi\,, \\ \forall\psi \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(Q^T_\Omega)\,. \end{gathered} \tag{24}$$

The idea of the proof is similar to the one of Theorem 2.3. The main difference relies on the fact that now we need to have an information about u_n (and consequently u) at the boundary, and so we need first to prove a "global "(i.e. on the whole Ω) estimate on the truncations in the energy space. On the other hand, for the second part of the proof, we follow exactly the same outline of the one of Theorem 2.3. Indeed, the estimates proved there are localized in $(0, T) \times B_R$, $\forall R > 0$. Since, in order to pass to the limit in the equation, we need to use such estimates on any compact subset $\varpi \subset\subset$

$(0,T)\times\Omega$, we observe that there exists $\omega\subset\subset\Omega$ such that $\varpi\subset\subset(0,T)\times\omega$. Thus

$$\exists M\in\mathbb{N} \text{ such that } x_i\in\Omega,\ r_i>0,\ i=1,...,M, \text{ and } \omega\subset\bigcup_{i=1}^{M}B_{r_i}(x_i).$$

It is clear that is enough to prove all the estimates on a ball and without loss of generality we can suppose it centered at the origin.

Global estimate on truncations. We choose $\forall n\geq k$, $\psi=\varphi_\lambda(T_k(u_n)-k)$, where $\lambda>0$ to be fixed, as test function in (24). Thus we have

$$\begin{aligned}\int_\Omega \Upsilon_{\lambda,k}(u_n(t,x))-\int_\Omega \Upsilon_{\lambda,k}(u_n^0(x))\\ +\int_{Q_\Omega^T} a(t,x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\varphi'_\lambda(T_k(u_n)-k)\\ +\int_{Q_\Omega^T} g(t,x,u_n,\nabla u_n)\varphi_\lambda(T_k(u_n)-k)\\ =\int_{Q_\Omega^T} f_n^+\varphi_\lambda(T_k(u_n)-k)-\int_{Q_\Omega^T} f_n^-\varphi_\lambda(T_k(u_n)-k)\,,\end{aligned}$$

where

$$\Upsilon_{\lambda,k}(s)=\begin{cases}-2ke^{4\lambda k^2}(s+k)+\frac{1}{2\lambda}\left[e^{4\lambda k^2}-e^{\lambda k^2}\right] & \text{if}\, s<-k\,,\\ \frac{1}{2\lambda}\left[e^{\lambda(s-k)^2}-e^{\lambda k^2}\right] & \text{if } -k\leq s<k\,,\\ \frac{1}{2\lambda}\left[1-e^{\lambda k^2}\right] & \text{if}\, s\geq k\,,\end{cases}$$

is a primitive of $\varphi_\lambda(T_k(s)-k)$. Note that, since $\Upsilon_{\lambda,k}(s)$ is decreasing and $\Upsilon(0)=0$, thus

$$\begin{aligned}&\int_\Omega \Upsilon_{\lambda,k}(u_n(t,x))-\int_\Omega \Upsilon_{\lambda,k}(u_n^0(x))\\ &\geq\int_{\Omega\cap\{0\leq u_n\leq k\}}\Upsilon_{\lambda,k}(u_n(t,x))+\int_{\Omega\cap\{u_n>k\}}\Upsilon_{\lambda,k}(u_n(t,x))\\ &\quad-\int_{\Omega\cap\{u_n^0\leq -k\}}\Upsilon_{\lambda,k}(u_n^0(x))-\int_{\Omega\cap\{-k\leq u_n^0\leq 0\}}\Upsilon_{\lambda,k}(u_n^0(x))\\ &\geq-\Big(\frac{1}{\lambda}\left[e^{\lambda k^2}-1\right]+\frac{1}{\lambda}\left[e^{4\lambda k^2}-e^{\lambda k^2}\right]\Big)|\Omega|-2ke^{4\lambda k^2}\|(u_n^0)^-\|_{L^1(\Omega)}\,.\end{aligned}$$

Thus, by (2), (16) and the assumptions on f we deduce, since the function $\varphi_\lambda(T_k(s)-k)\le 0$, $\forall s\in\mathbb{R}$,

$$\alpha\int_{Q_\Omega^T}|\nabla T_k(u_n)|^p\varphi_\lambda'(T_k(u_n)-k)-\int_{Q_\Omega^T}\gamma_k|\nabla T_k(u_n)|^p|\varphi_\lambda(T_k(u_n)-k)|$$
$$\le\varphi_\lambda(2k)\int_{Q_\Omega^T}\Big[|f_n^-|+|g_k(t,x)|\Big]+2ke^{4\lambda k^2}\|(u_n^0)^-\|_{L^1(\Omega)}$$
$$+\Big(\tfrac{1}{2\lambda}\Big[e^{\lambda k^2}-1\Big]+\tfrac{1}{\lambda}\Big[e^{4\lambda k^2}-e^{\lambda k^2}\Big]+2k^2e^{4\lambda k^2}\Big)|\Omega|\,.$$

By fixing a suitable $\lambda>0$, according with property (21) of the function $\varphi_\lambda(s)$, we deduce that $k-T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and so, up to subsequences (not relabeled), it converges weakly in $L^p(0,T;W_0^{1,p}(\Omega))$.

As already pointed out, the conclusion of the Theorem follows exactly using the same steps of Theorem 2.3. ■

5. Further Regularity

In this section we are going to describe some local regularity properties for a renormalized solution of problem

$$\begin{cases} u_t-\ \operatorname{div} a(t,x,u,\nabla u)+g(t,x,u,\nabla u)=f(t,x) & \text{in}\,(0,T)\times D\\ u(0,x)=u_0(x) & \text{in}\,D\,. \end{cases}\tag{1}$$

Let us first emphasize that, in this section we would like to be able to choose test functions of the type $S'(u)\psi$ with S' not compactly supported on $\mathbb{R}$ and such that $\psi(T,x)\neq 0$. In principle, according to Definition 2.2, we are not allowed to do that. Anyway, up to suitably modify our definition, this fact can be made rigourous by an easy density argument. In fact, we can choose $S'(u)=S_j'(u)M(u)$ where M is a Lipschitz function and S_j is defined in (22), in the renormalized formulation. Then, we take the limit as j diverges and we observe that $S_j'(u)$ converges to 1 both a.e. and $*-$weak in $L^\infty(Q_D^T)$, and the term involving S'' vanishes tanks to (13). On the other hand, to deal with cut-off functions such that $\psi(T,x)\neq 0$ (we will actually handle with functions which do not depend on time) we can choose a family $\psi_\delta(t,x)$ of functions in $C_0^1([0,T)\times D)$ such that they converge in $C^1([0,T]\times D)$ to a function $\psi(x)$. Thus, according with Lemma 3.4, Proposition 2.6, and considering that $\psi_t(x)=0$, the formulation change into the following one, that is the useful one in order to get rid of our regularity estimates.

$$
\begin{aligned}
\int_D \mathcal{M}(u(t,x))\psi(x) + \int_{Q_D^t} a(t,x,u,\nabla u)\cdot\nabla u M'(u)\psi \\
+ \int_{Q_D^t} a(t,x,u,\nabla u)\cdot\nabla\psi M(u) \\
+ \int_{Q_D^t} g(t,x,u,\nabla u)M(u)\psi = \int_{Q_D^t} f(t,x)M(u)\psi + \int_D \mathcal{M}(u_0)\psi(x)\,,
\end{aligned}
\tag{2}
$$

for any $0 < t \leq T$ and where $\mathcal{M}'(s) = M(s)$, $\mathcal{M}(0) = 0$.

Finally, we observe that, since the estimates we are going to prove in this section are localized, we will proceed as follows. We fix a ball (without loss of generality, centered at 0) of radius R contained in D. Thus there exists $\rho > 0$ such that $B_{R+\rho} \subset\subset D$ and we will prove the estimate in $(0,T)\times B_R$, in dependence of quantities computed on $(0,T)\times B_{R+\rho}$. By covering any compact $\omega \subset\subset D$ with balls we get the results.

We start proving Proposition 2.6.

Proof of Proposition 2.6. According to the formulation above we are allowed to choose $\psi(t,x) = \xi(x)$, where ξ is chosen as in (2) and such that Proposition 3.1 holds true, and $M(s) = T_k(s)$, $\forall k \geq L$. Thus we have:

$$
\begin{aligned}
\int_{B_{R+\rho}} \Theta_k(u(x,t))\xi^p + \frac{\alpha}{2^{p-1}}\int_{Q_{R+\rho}^T} |\nabla(T_k(u)\xi)|^p + \frac{1}{2}\int_{Q_{R+\rho}^T} h(|\nabla u|^{p-1})T_k(u)\xi^p \\
\leq k\|f\|_{L^1(Q_{R+\rho}^T)} + C_0 + \alpha\int_{Q_{R+\rho}^T} |T_k(u)|^p|\nabla\xi|^p + k\|u_0\|_{L^1(B_{R+\rho})}\,.
\end{aligned}
$$

Note that, since Proposition 3.8 holds true, then u^{p-1} belongs to $L^1(Q_R^T)$ and so the last integral can be estimated with Ck, for suitable $C > 0$. Thus we deduce, by dropping positive terms,

$$
\int_{Q_R^T} |\nabla T_k(u)|^p \leq C(k+1)\,,
$$

and so we conclude applying Lemma 3.9. As we have already noticed, the continuity with values in L^1_{loc} is an easy consequence of Theorem 1.1 of [23].

■

Sketch of the Proof of Theorem 2.7: We first give an idea of the proof of parts (1) and (2). Let us fix any $0 < R < R+\rho$ and consider $B_R \subset$

$B_{R+\rho} \subset\subset D$. Let us choose $M(s) = v_{\varepsilon,j}(s)$, and $\psi = \xi^{\lambda}$ in (2), where $\lambda = \max\{p, \frac{q'p}{q'p-1}, \frac{q'p'}{q'p'-1}\}$, $\xi(x)$ is as in (2) and

$$v_{\varepsilon,j}(s) = [(|T_j(s)| + \varepsilon)^{\gamma} - \varepsilon^{\gamma}] \operatorname{sign} s,$$

for any $0 < \gamma \leq \overline{\gamma}$, with

$$\overline{\gamma} = \begin{cases} \frac{Nm(q-1)+q(m-1)[p(N+1)-2N]}{Nm-pq(m-1)} & \text{if (18) holds} \\ \frac{N(p-1)}{(N-p)q'-N} & \text{if (19) holds}. \end{cases} \tag{3}$$

We follow the same ideas of previous estimates, we use the ellipticity condition, assumption (7) and Proposition 3.1, and we finally let ε vanishing. Thus, we deduce that there exists a constant $C = C(\alpha, \beta, N, R, \rho, u_0, L) > 0$, but independent on j, such that,

$$\begin{aligned} &\|\xi^{\frac{\lambda(\gamma+p-1)}{p(\gamma+1)}} T_j(u)\|^{\gamma+1}_{L^{\infty}(0,T;L^{\gamma+1}(B_{R+\rho}))} + \int_0^T \|\xi^{\frac{\lambda}{p}} |T_j(u)|^{\frac{\gamma+p-1}{p}}\|^p_{L^{p^*}(B_{R+\rho})} \\ &\leq C \left[\|f\|_{L^m(0,T;L^q(B_{R+\rho}))} \left(\int_0^T \left(\int_{B_{R+\rho}} \xi^{\frac{\lambda}{p}} |T_j(u)|^{\gamma q'} \right)^{\frac{m'}{q'}} \right)^{\frac{1}{m'}} \right. \\ &\qquad \left. + \int_{Q^T_{R+\rho}} \xi^{\lambda-p} |T_j(u)|^{p+\gamma-1} + 1 \right], \end{aligned}$$

where we have applied a space-time Hölder inequality on the term involving the datum $f(t,x)$. Using an interpolation inequality, we deduce both that

$$\|T_j(u)\|_{L^{\infty}(0,T;L^{\gamma+1}(B_R))} \leq C_1 \left[1 + \|T_j(u)\|^{p+\gamma-1}_{L^{p+\gamma-1}(Q^T_{R+\rho})} \right]^{\frac{1}{\gamma+1}}, \tag{4}$$

and

$$\|T_j(u)\|_{L^{\gamma+p-1}(0,T;L^{\frac{\gamma+p-1}{p}p^*}(B_R))} \leq C_1 \left[1 + \|T_j(u)\|^{p+\gamma-1}_{L^{p+\gamma-1}(Q^T_{R+\rho})} \right]^{\frac{1}{\gamma+p-1}}, \tag{5}$$

where C_1, now, depends on f, too. By (4), (5) and by applying the interpolation inequality (to the function $|T_j(u)|^{\frac{\gamma+p-1}{p}}$), we have:

$$\begin{aligned} &\|T_j(u)\|^{\frac{p+\gamma-1}{p}}_{L^{\frac{p+\gamma-1}{p}\eta}(Q^T_R)} \\ &\leq C \|T_j(u)\|^{1-\theta}_{L^{\infty}(0,T;L^{\gamma+1}(B_R))} \|T_j(u)\|^{\theta}_{L^{\gamma+p-1}(0,T;L^{\frac{\gamma+p-1}{p}p^*}(B_R))}, \end{aligned} \tag{6}$$

where η and θ satisfy

$$\frac{1}{\eta} = \frac{\theta}{p^*} + \frac{(1-\theta)(\gamma+p-1)}{p(\gamma+1)}, \quad \frac{p}{\theta} \geq \eta.$$

Consequently,

$$\eta = p\left[1 + \frac{p(\gamma+1)}{N(\gamma+p-1)}\right]$$

turns out to optimize the above constraints. Thus gathering together (6), (4) and (5) we deduce that there exists $C > 0$ such that

$$\|T_j(u)\|_{L^{\frac{\eta(\gamma+p-1)}{p}}(Q_R^T)} \leq C\left[1 + \|T_j(u)\|^{\gamma+p-1}_{L^{\gamma+p-1}(Q^T_{R+\rho})}\right]^{\frac{p}{\gamma+p-1}\max\{\frac{1}{\gamma+p-1},\frac{1}{\gamma+1}\}}. \tag{7}$$

Actually, roughly speaking, we control the norm of $T_j(u)$ in $L^{\frac{\eta(\gamma+p-1)}{p}}$ in a cylinder with the norm in $L^{\gamma+p-1}$ in a slightly larger cylinder. Moreover the such estimate is uniform with respect to j. Noticing that that $\eta > p$, in order to conclude it is enough to perform an iteration method. We can construct both $\overline{k}+1$ radii $0 = \rho_0 < \rho_1, ..., \rho_{\overline{k}-1} < \rho_{\overline{k}} = \rho$ and $\overline{k}+1$ exponents $\gamma_0 < s_1, ..., \gamma_{\overline{k}-1} < \gamma_{\overline{k}} = \overline{\gamma}$, such that

$$s_0 + p - 1 < p - 1 + \frac{p}{N},$$

and $\frac{\eta(s_{\overline{k}}+p-1)}{p}$ is our desired summability. Thus the application of (7) $\overline{k}+1$ times and by Proposition 2.6, we get the result. To get rid of the different summability in space and time stated in Theorem 2.7 we can argue in a similar way by applying Hölder inequality.

Now we deal with part 3) of the Theorem. Let us denote by

$$A_{k,r} = \{x \in B_\rho(x_0) : \ |u(t,x)| > k\}, \ \forall r > 0, \tag{8}$$

and let us set for any fixed $\delta \in (0,1)$,

$$\begin{gathered} t_1 = \left[\frac{1-\delta}{|B_{R+\rho}|^{\lambda_1}}\right]^{\lambda_2} \\ \lambda_1 = \left(1 - \frac{\sigma}{q(\sigma-p)}\right)\left(1 - \frac{p}{\sigma}\right) \quad \text{and} \quad \lambda_2 = \frac{m\mu}{m(\mu-p)-\mu}, \end{gathered} \tag{9}$$

and moreover

$$\sigma = p\frac{Nm' + pq'}{Nm'}, \qquad \mu = p\frac{Nm'+pq'}{Nq'}. \tag{10}$$

Let us choose $M(u) = T_j(G_k(u))$ in (2), $j > k > L$ and $\psi = \xi^p$ (ξ chosen as in (2)) in the cylinder of *height* t_1, where we will fix δ (and so t_1) later.

We also recall that for ξ chosen as in (2), we have $|\nabla\xi| \le \frac{c}{\rho}$. Thus, by nowadays standard computations, we have

$$\begin{aligned}\int_0^{t_1}\int_{A_{k,R+\rho}} |\nabla T_j(G_k(u))|^p\xi^p &\le \frac{c_1}{\rho}\int_{Q_R^{t_1}} |T_j(G_k(u))|\xi^{p-1} \\ &+ \int_0^{t_1}\int_{A_{k,R+\rho}} |f||T_j(G_k(u))|^p\xi^p + \int_0^{t_1}\int_{A_{k,R+\rho}} |f|\xi^p\,.\end{aligned} \tag{11}$$

Moreover, we choose $M_\varepsilon(u) = [(|T_j(G_k(u))| + \varepsilon)^{p-1} - \varepsilon^{p-1}]\operatorname{sign} u$, $j > k > L$ and $\psi = \xi^p$ in (2). Thus, dropping positive terms, as ε goes to zero, we get

$$\begin{aligned}\sup_{t\in(0,t_1)}\int_{A_{k,R+\rho}} \Theta_j(|G_k(u)|^p)(t)\xi^p &\le \frac{c_2}{\rho}\int_0^{t_1}\int_{A_{k,R+\rho}} |T_j(G_k(u))|^{p-1}\xi^{p-1} \\ &+c_3\int_0^{t_1}\int_{A_{k,R+\rho}} |f|\xi^p + c_4\int_0^{t_1}\int_{A_{k,R+\rho}} |f||T_j(G_k(u))|^p\xi^p\,,\end{aligned} \tag{12}$$

where $\Theta_j(s)$ denotes the primitive of $T_j(s)$, such that $\Theta_j(0) = 0$. Now we apply Corollary 3.7 with $w = T_j(|G_k(u)|)\xi$, $\Omega = B_{R+\rho}$ and $T = t_1$. Thus, for all μ and σ satisfying (11), we deduce, by adding (11) and (12), and by applying Corollary 3.7

$$\begin{aligned}\left[\int_0^{t_1}\left(\int_{A_{k,R+\rho}} (|T_j(G_k(u))|\xi)^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} &\le \frac{c_5}{\rho^p}\int_0^{t_1}\int_{A_{k,R+\rho}} |T_j(G_k(u))|^{p_0} \\ &+c_6\int_0^{t_1}\int_{A_{k,R+\rho}} |f||T_j(G_k(u))|^p\xi^p + c_7\int_0^{t_1}\int_{A_{k,R+\rho}} |f|\xi^p\,,\end{aligned} \tag{13}$$

with $p_0 = \max\{1, p-1\}$. Recalling the definitions of μ and σ (see (10)) and noticing that both of them are greater than p, we apply Hölder inequality to estimate the right hand side of (13), so that

$$\begin{aligned}&\int_0^{t_1}\int_{A_{k,R+\rho}} |f||T_j(G_k(u))|^p\xi^p \\ &\le |B_{R+\rho}|^{\lambda_1} t_1^{\frac{1}{\lambda_2}} \|\xi T_j(u)\|^p_{L^\sigma(0,t_1;L^\mu(A_{k,R+\rho}))}\|f\|_{L^m(0,t_1;L^q(B_{R+\rho}))}\,,\end{aligned}$$

where λ_1 and λ_2 have been defined in (9). We fix now δ (and consequently we fix t_1) such that $c_6\|f\|(1-\delta) < \frac{1}{2}$; note that this quantity depends on the data of the problem but not on u. Thus from (13) we deduce

$$\begin{aligned}&\left[\int_0^{t_1}\left(\int_{A_{k,R+\rho}}|T_j(G_k(u))|^\sigma\xi^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}}\\&\le\int_0^{t_1}\int_{A_{k,R+\rho}}|f|\xi^p+\frac{c}{\rho^p}\int_0^{t_1}\int_{A_{k,R+\rho}}|T_j(G_k(u))|^{p_0}\,.\end{aligned}\tag{14}$$

Moreover, by Hölder inequality it follows that, for every $h>k$,

$$\left[\int_0^{t_1}\left(\int_{A_{k,R+\rho}}|T_j(G_k(u))|^\sigma\xi^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}}\ge(h-k)^p\left(\int_0^{t_1}|A_{h,R}|^{\frac{m'}{q'}}\right)^{\frac{p}{\mu}}.$$

On the other hand we have to estimate the right hand side of (14): we first note that

$$\int_0^{t_1}\int_{A_{k,R+\rho}}|f|\xi^p\le\|f\|_{L^m(0,t_1;L^q(B_{R+\rho}))}\left(\int_0^{t_1}|A_{k,R+\rho}|^{\frac{m'}{q'}}\right)^{\frac{1}{m'}},$$

and moreover

$$\int_0^{t_1}\int_{A_{k,R}}|T_j(G_k(u))|^{p_0}\le c\|T_j(G_k(u))\|^{p_0}_{L^{rp_0}(Q^{t_1}_{R+\rho})}\left(\int_0^{t_1}|A_{k,R+\rho}|^{\frac{m'}{q'}}\right)^{\frac{1}{m'}},$$

where $d=\max\{q,m\}$. Now, we observe that $f(t,x)\in L^{m_0}(0,t_1;L^{q_0}(B_{R+\rho}))$, $\forall m_0,q_0$ such that $1<m_0\le m$, $1<q_0\le q$. In particular we can choose m_0,q_0 such that

$$m_0q=mq_0\quad\text{and}\quad\tfrac{1}{m_0}+\tfrac{N}{pq_0}=1+\varepsilon\,,$$

$$\forall\varepsilon<\min\left\{\frac{m_0q_0(N+p)+N(p-2)(q_0(m_0-1)+m_0)}{p_0d},\frac{N}{pm_0}\right\}.$$

Using the first part of the Theorem we deduce that $u\in L^{\hat s}(Q^{t_1}_{R+\rho})$, where $\hat s=\frac{m_0q_0(N+p)+N(p-2)(q_0(m_0-1)+m_0)}{\varepsilon}$. Since $\hat s\ge p_0d$ we deduce

$$\frac{c}{\rho}\int_0^{t_1}\int_{A_{k,R}}|T_j(G_k(u))|^{p_0}\le\frac{c_1\|u\|_{L^{\hat s}(Q^{t_1}_{R+\rho})}}{\rho}\left(\int_0^{t_1}|A_{k,R+\rho}|^{\frac{m'}{q'}}\right)^{\frac{1}{m'}}.$$

Gathering together the above informations, we finally deduce, using also that $\frac{\mu}{\sigma}=\frac{m'}{pq'}$ and passing to the limit with respect to j, that there exists $C>0$

such that

$$\int_0^{t_1} |A_{h,R}|^{\frac{m'}{q'}} \leq \frac{c}{(h-k)^{\mu}\rho^{\frac{\mu}{p}}} \left(\int_0^{t_1} |A_{k,R+\rho}|^{\frac{m'}{q'}}\right)^{\frac{1}{m'}} .$$

Since (20) is in force, we have

$$\frac{\mu}{m'p} = \frac{1}{q'} + \frac{p}{m'N} = 1 + \frac{p}{N} - \frac{p}{N}\left(\frac{1}{m} + \frac{N}{pq}\right) > 1\,,$$

and so we can apply Lemma 3.10 to the function

$$\zeta(h,d) = \int_0^{t_1} |A_{k,d}|^{\frac{m'}{q'}}(t)\; dt\,.$$

Thus the proof is complete for $0 \leq t_1 < T$. As already remarked, it is clear that the the choice of t_1 only depends on the data of the problem and thus we can iterate and conclude the same estimate in the whole cylinder in a finite number of steps. ■

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