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POSITIVE SEMIDEFINITE DIAGONAL MINUS TAIL FORMS ARE SUMS OF SQUARES

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ABSTRACT: By a *diagonal minus tail* form (of even degree) we understand a real homogeneous polynomial $F(x_1, ..., x_n) = F(\underline{x}) = D(\underline{x}) - T(\underline{x})$, where the *diagonal* part $D(\underline{x})$ is a sum of terms of the form $b_i x_i^{2d}$ with all $b_i \geq 0$ and the *tail* $T(\underline{x})$ a sum of terms $a_{i_1 i_2 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$ with $a_{i_1 i_2 \ldots i_n} > 0$ and at least two $i_\nu \geq 1$. We show that an arbitrary change of the signs of the tail terms of a positive semidefinite diagonal minus tail form will result in a sum of squares of polynomials.

1. Introduction

The problem of the representability of positive semidefinite (psd) polynomials as sums of squares of polynomials (sos) has been treated by many authors. The first significant publication prompted by a question of Minkowski seems to be Hilbert's article [Hi] where it is shown that not all psd forms are sos. The first explicit examples for this phenomenon were given by Motzkin [Mo] in 1967 and shortly after by Robinson [Ro].

In the nineteen eighties and nineties this area was very much advanced by Reznick, partly in collaboration with Choi and Lam. Note that all psd polynomials are necessarily of even degree. One of Reznick's results [Re1] is that the form $F(\underline{a}, \underline{x}) = a_1 x_1^{2d} + \ldots + a_n x_n^{2d} - 2d x_1^{a_1} \ldots x_n^{a_n}$ can be written as a sum of at most $3n - 4$ squares of polynomials. With more squares allowed, $F(\underline{a}, \underline{x})$ can be written even as a sum of binomial squares (sobs), i.e. as sum of expressions of the form $(\alpha x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} - \beta x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})^2$, see [Re2] (where always $\alpha = \beta$). If α or β is 0, these forms are squares of monomials.

Similar but much less economic representations than Reznick's had been found earlier by Hurwitz's [Hu]; see e.g. [BB, p8]. Reznick's paper [Re2] undertakes a profound investigation of forms obtained from the arithmetical geometrical inequality by substitution of its variables by monomial squares. He calls these forms agiforms.

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We use one of his results on 'Hurwitz agiforms' to show that all psd *diagonal minus tail* (dmt) forms are sums of binomial and monomial squares. The psd dmt forms have the aspect $F(x_1,\dots,x_n) = F(\underline{x}) = D(\underline{x}) - T(\underline{x})$, where the *diagonal* part $D(\underline{x})$, and *tail* $T(\underline{x})$ are defined via writing $a_{\underline{i}}\underline{x}^{\underline{i}} = a_{i_1\dots i_n}x_1^{i_1}\dots x_n^{i_n}$, by

$$D(\underline{x}) = \sum_{i=1}^{n} b_i x_i^{2d}, \quad \text{and} \quad T(\underline{x}) = \sum_{\underline{i}\in I} a_{\underline{i}}\underline{x}^{\underline{i}}, \qquad d \in \mathbb{Z}_{\geq 1},\ b_i, a_{\underline{i}} \geq 0,$$

and $I \subseteq \{\underline{i} = (i_1,\dots,i_n) \in \mathbb{Z}^n_{\geq 0} : 0 \leq i_1,\dots,i_n \leq 2d-1,\ i_1+\dots+i_n = 2d\}$. Call a dmt-form *elementary* if its tail consists of at most one term. Such forms are generalizations of Reznick's agiforms on Hurwitz trellises in that the coefficients in the diagonal part are not necessarily the powers occurring in the tail term and its coefficient not necessarily equals the sum of the diagonal coefficients.

The main contribution is to adapt Reznick's idea to prove in theorem 2.3 that elementary psd dmt forms are sobs and theorem 2.7 saying that a psd dmt form F can be written as a sum of elementary psd dmt forms. Consequently all psd dmt forms are sobs. As a byproduct we also obtain an easy proof that a psd dmt form restricted to the standard simplex has exactly one local minimum, see corollary 2.8. Other corollaries are corollary 2.9 saying that the sobs property persists even after changing the signs of the tail terms of a psd dmt form; and corollary 2.10 - a result due to Robinson - according to which adding $\beta(x_1^{2d}+\dots+x_n^{2d})$ to any form of degree $2d$ will for large enough β result in a sobs. We obtain this result in completely a different manner from Robinson's.

In section 3 the findings are put together in algorithm 3.1 for writing a psd dmt form as a sobs. We work an example for illustration. We also present examples due to Robinson, and Choi and Lam showing that in general it is false that a psd form even if having prominent diagonal part is sos.

For following the proofs of lemma 2.2 and theorem 2.3, and the algorithm and example in section 3, availability of [Re2] will be necessary; otherwise the essence of the paper can be understood without specialized knowledge.

2. Results

If context avoids confusion, we write an n-tuple $(x_1, \ldots, x_n)$ often as $\underline{x}$. For parts of $\underline{x}$, like $(x_1, \ldots, x_k)$, $(x_{k+1}, \ldots, x_n)$, we may write $\underline{x}_{1:k}, \underline{x}_{k+1:n}$, respectively. Special vectors we use are the standard vectors $\underline{e}_i = (0, ..., 0, 1, 0, ..., 0)$, with a single 1 at positions $i = 1, \ldots, n$, and their sum $\underline{1}_n = (1, 1, ..., 1) \in \mathbb{R}^n$. Consider now a polynomial $P(\underline{x}) = \sum_{\underline{i} \in I} c_{\underline{i}} \underline{x}^{\underline{i}} \in \mathbb{R}[\underline{x}]$. We say (quite naturally) that a *term* $c_{\underline{i}} \underline{x}^{\underline{i}}$ *occurs* in P if $c_{\underline{i}} \neq 0$, a variable x_j *occurs* in a such a term if $i_j \geq 1$, and the variable *occurs* in P is there exists a $c_{\underline{i}} \neq 0$ such that $i_j \geq 1$. Denoting by $\mathrm{var}(P)$ the set of variables occurring in P, the *dimension* of P is $\dim(P) = \#\mathrm{var}(P)$. A representation of a homogeneous polynomial (or *form*) of degree≥ 1 as a sum $P = P_1 + \ldots + P_k$ with nonzero forms P_i, so that for all $1 \leq i < j \leq k$, $\mathrm{var}(P_i) \cap \mathrm{var}(P_j) = \emptyset$ will be called a *decomposition* of P. If such a representation forces $k = 1$, P is *indecomposable.* The decomposition is *complete* if the P_i are indecomposable. A complete decomposition is easily seen to be unique up to the order of the summands P_i.

For example, $3x_1x_2 - x_2x_3 + 7x_4x_5 = P_1 + P_2$ is a complete decomposition with indecomposable $P_1 = 3x_1x_2 - x_2x_3$ and $P_2 = 7x_4x_5$. It is convenient to assign the zero polynomial the degree of the set of polynomials which we work with. This degree will be fixed and assumed *positive* throughout.

We begin with a simple lemma. It concerns variables and decomposability of forms, in particular psd forms and dmt forms. Its part b explains why much of the theory can be reduced to the investigation of indecomposable forms.

Lemma 2.1. a. If $F = D - T$ is psd dmt, then $\mathrm{var}(T) \subseteq \mathrm{var}(D)$.
$\mathrm{b}_{\mathrm{psd}}$ & $\mathrm{b}_{\mathrm{dmt}}$. If $F = F_1 + F_2 + \ldots + F_k$ is a decomposition of a psd form (dmt form), then $F_1, F_2, \ldots, F_k$ are psd forms (dmt forms).
c. Assume $F = \sum_{\underline{i} \in I} c_{\underline{i}} \underline{x}^{\underline{i}}$ is any form and $F = F_1 + F_2 + \ldots + F_k$ a decomposition. Let $\tilde{F}$ be obtained from F by altering the signs of some of its terms: $\tilde{F} = \sum_{\underline{i} \in I} \varepsilon_{\underline{i}} c_{\underline{i}} \underline{x}^{\underline{i}}$, with $\varepsilon_{\underline{i}} \in \{-1, +1\}$. Let $\tilde{F_i}$ be obtained from F_i by giving each term in F_i the sign it has in $\tilde{F}$. Then $\tilde{F} = \tilde{F}_1 + \tilde{F}_2 + \ldots + \tilde{F}_k$ is a decomposition of $\tilde{F}$.

Proof. a. Assume a variable, x_1 say, occurs in $\mathrm{var}(T)$ but not in $\mathrm{var}(D)$. Then $-T(x_1, 1, ..., 1) \to -\infty$ as $x_1 \to \infty$, while $D(x_1, 1, ..., 1) = \sum_{i=2}^{n} b_i$ remains constant. Hence $F(x_1, 1, ..., 1) < 0$ for large enough x_1, contradicting psdness of F. The other variables can be treated similarly.

$\mathrm{b}_{\mathrm{psd}}$. Consider a decomposition of a psd form F into two parts first: $F = F_1 + F_2$. Assume, say, $\mathrm{var}(F_1) = \{x_1, \ldots, x_k\}$, $\mathrm{var}(F_2) = \{x_{k+1}, \ldots, x_n\}$. Then $0 \leq F(\underline{0}_{1:k}, \underline{x}_{k+1:n}) = F_1(\underline{0}) + F_2(\underline{x}_{k+1:n}) = F_2(\underline{x}_{k+1:n})$. Hence F_2 is psd. Similarly F_1 is psd. The general case follows by induction. This result can also be seen as a special case of a result involving the Newton polytope of a psd polynomial: see e.g. theorem 3.6iv [Re2, p442].

$\mathrm{b}_{\mathrm{dmt}}$. Let $F = D - T$ be dmt and again consider a decomposition $F = F_1 + F_2$ into two parts first. Since the family of nonzero terms in a decomposition $F_1 + F_2$ of F is precisely the disjoint union of the terms in F_1 with the terms in F_2, we can write $F_1 = D_1 - T_1$ and $F_2 = D_2 - T_2$, where $T = T_1 + T_2$, $D = D_1 + D_2$. It follows that F_1, F_2 are dmt forms. The general case follows by induction.

c. Is immediate. □

For lemma 2.2, the reader will have to absorb especially the following pieces from Reznick's long paper [Re2]: framework and trellis [p433c7] (page 433, about 7 centimeters from the first textrow), $C(\mathcal{U}), E(\mathcal{U}), \bar{A}(\mathcal{B})$, definition of (simplicial) agiforms $f(\mathcal{U}, \underline{\lambda}, \underline{w})$ [p434c8], the Hurwitz trellis [p435c-5, p436c1], $\mathcal{U}$-mediated sets [p438c3], the definition of $\mathcal{U}^*$, theorem 2.2 [p438], theorem 2.8 [p440], H-trellis [p439c2], and theorem 4.4 [p443c-5]. The following is proved but enunciated slightly weaker as corollary 4.10 in [Re2, p444].

Lemma 2.2. Every simplicial agiform $f(\mathcal{U}, \underline{\lambda}, \underline{w})$ on a H-trellis $\mathcal{U}$ is a sum of $|\mathcal{U}^* \setminus \mathcal{U}|$ binomial squares.

Proof. Let $f(\mathcal{U}, \underline{\lambda}, \underline{w}) = \lambda_1 \underline{x}^{\underline{u}_1} + \ldots + \lambda_m \underline{x}^{\underline{u}_m} - x^{\underline{w}}$ be a simplicial agiform on a H-trellis. Then $\underline{w} \in C(\mathcal{U}) = \mathcal{U}^*$. By Reznick's theorem 2.2, $\mathcal{U}^*$ is $\mathcal{U}$-mediated. Then his theorem 4.4 says that f is a sum of $|\mathcal{U}^* \setminus \mathcal{U}|$ binomial squares. □

In [Re1] it is shown that for the special case of the Hurwitz trellis $\mathcal{U} = \mathcal{H}_{n,2d} = \{2d\underline{e}_i : i = 1, \ldots, n\}$ which is the one that interests us here most, one can obtain representations of less (but not necessarily binomial) squares. See our section 3 for some remarks on this.

Note that the form $E(\underline{x})$ figuring in the next theorem is elementary dmt if and only if $\mu \geq 0$. Also note that dimension $n = 1$ would not even allow us to speak of a_i or μ since the form would collaps to a form $b_i' x_i^{2d}$.

Theorem 2.3. Let $n \geq 2$, $b_1, \dots, b_n \in \mathbb{R}_{\geq 0}$, $d \in \mathbb{Z}_{\geq 1}$, and assume

$$E(\underline{x}) = b_1 x_1^{2d} + \dots + b_n x_n^{2d} - \mu x_1^{a_1} \cdots x_n^{a_n}$$

to be an indecomposable form of dimension n. Then all $a_i \in \mathbb{Z}_{\geq 1}$. Hence

$$\mu_0 = 2d \prod_{i=1}^{n} \left(\frac{b_i}{a_i}\right)^{a_i/2d}$$

is well defined and we have:

a. The following are equivalent:
 i. E is psd.
 ii. $\mu \leq \mu_0$ and all a_i are even or $|\mu| \leq \mu_0$ and some a_i are odd.
 iii. E is sobs and hence sos.

b. If $\mu = \mu_0$, then $E|\mathbb{R}^n_{\geq 0}$ has (projectively counted) precisely one zero, namely at the point

$$\left(\tfrac{a_1}{b_1}\right)^{1/2d} : \left(\tfrac{a_2}{b_2}\right)^{1/2d} : \dots : \left(\tfrac{a_n}{b_n}\right)^{1/2d}.$$

Proof. Since $\dim(E) = n$, every x_i, $i = 1, \dots, n$, occurs in E. Also $n \geq 2$ implies $\mu \neq 0$, for otherwise the form would be decomposable. If x_1, say would not occur in $\mu \underline{x}^{\underline{a}} = \mu x_1^{a_1} \cdots x_n^{a_n}$, then $a_1 = 0$. So $b_1 x_1^2 + (\sum_{i=2}^n b_i x_i^2 - \mu \underline{x}^{\underline{a}})$ would be a decomposition of E, contradicting the hypothesis. Similarly for any other x_i. We conclude that all $a_i \in \mathbb{Z}_{\geq 1}$. Since all $b_i \geq 0$, the expression for μ_0 is well defined.

i⇒ii. Let E be psd. Assume first that some b_i are 0, w.l.o.g. $b_1 = 0$, say. Then $\mu_0 = 0$. Putting $x_{2:n} = \underline{1}_{n-1}$, we find $E(\underline{x}) = \sum_{i=2}^n b_i - \mu x_1^{a_1}$. Going with x_1 towards $-\infty$ or $+\infty$ we see that psd-ness of E implies $\mu \leq 0$ and a_1 is even. If there would be an odd a_i then choosing $x_i = -1$ (maintaining the other $x_2, \dots, x_n$ equal to 1) yields $E(\underline{x}) = \sum_{i=2}^n b_i + \mu x_1^{a_1}$. Then $\mu < 0$ again yields a contradiction. So either $\mu < 0$ and all a_i are even or $\mu = 0$. This shows ii in the case that some $b_i = 0$.

Assume now that $b_i > 0$ for $i = 1, \dots, n$. Then the map

$$\mathbb{R}^n_{\geq 0} \ni (u_1, u_2, \dots, u_n) \overset{T}{\mapsto} \left(\left(\tfrac{a_1}{2db_1} u_1\right)^{1/2d}, \dots, \left(\tfrac{a_n}{2db_n} u_n\right)^{1/2d}\right) \in \mathbb{R}^n_{\geq 0}$$

is well defined. It is easy to see that T is a bijectve. Now

$$E(T(x)) = \frac{a_1}{2d}x_1 + ... + \frac{a_n}{2d}x_n - \underbrace{\left(\frac{\mu}{2d}\prod_{i=1}^{n}\left(\frac{a_i}{b_i}\right)^{a_i/2d}\right)}_{\mu_1} \cdot \prod_{i=1}^{n} x_i^{a_i/2d}.$$

By the arithmetic-geometric mean inequality, see e.g. [BB, p13c-0], and definition of μ_1, we thus find $E|\mathbb{R}^n_{\geq 0} \geq 0$ iff $E \circ T|\mathbb{R}^n_{\geq 0} \geq 0$ iff $\mu_1 \leq 1$ iff $\mu \leq \mu_0$. Evidently we also have

$E|\mathbb{R}^n \geq 0$ iff for all $\underline{\varepsilon} \in \{-1, +1\}^n$ and $\underline{x} \in \mathbb{R}^n_{\geq 0}$ $E_{\underline{\varepsilon}}(\underline{x}) = E(\underline{\varepsilon} \circ \underline{x}) \geq 0$,

where $\underline{\varepsilon} \circ \underline{x} = (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$.

Now $E_{\underline{\varepsilon}}(\underline{x}) = E(\underline{\varepsilon} \circ \underline{x}) = D(\underline{x}) - \mu \prod_{i=1}^{n} \varepsilon_i^{a_i} \underline{x}^{\underline{a}}$. So what we found above is applicable to $E_{\underline{\varepsilon}}$, that is, $E_{\underline{\varepsilon}}|\mathbb{R}^n_{\geq 0} \geq 0$, iff $\mu \prod_{i=1} \varepsilon_i^{a_i} \leq \mu_0$. This will hold for every $\underline{\varepsilon} \in \{-1, +1\}^n$ iff (ii) holds. This establishes (ii).

i$\Leftarrow$ii. Assume $b_i = 0$ for some i. Then $\mu_0 = 0$ and (ii) implies by virtue of $\mu \neq 0$ that $\mu < 0$ and all a_i are even. This implies obviously that E is sobs and hence psd. In the case that all $b_i > 0$, the equivalences we have used in the implication before yield (i).

ii$\Rightarrow$iii. Assume (ii). We have observed just before that existence of some $b_i = 0$ implies E is sos. So we assume now all $b_i > 0$. Then we have a well defined bijective map from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by

$$\mathbb{R}^n \ni (u_1, u_2, \ldots, u_n) \overset{T}{\mapsto} (\left(\frac{a_1}{2db_1}\right)^{1/2d} u_1, \ldots, \left(\frac{a_n}{2db_n}\right)^{1/2d} u_n) \in \mathbb{R}^n.$$

Then

$$G(\underline{x}) = 2dE(T(\underline{x})) = a_1 x_1^{2d} + \ldots + a_n x_n^{2d} - \mu_1 2d \cdot \prod_{i=1}^{n} x_i^{a_i}.$$

Since T is bijective the equivalence of (i) and (ii) already proved and applied to $G(\underline{x})$ yields $E(\underline{x})$ is psd iff $G(\underline{x})$ is psd iff $\mu_1 \leq 1$, and all a_i are even or $|\mu_1| \leq 1$ and some a_i are odd.

If $\mu_1 < -1$, then all the a_i are even; and since $-\mu_1 \cdot 2d > 0$, it is clear that $G(\underline{x})$ is sum of $n+1$ monomial squares.

Now assume $0 \leq \mu_1 \leq 1$. We can write

$$G(\underline{x}) = 2d\mu_1(\frac{a_1}{2d}x_1^{2d} + \ldots + \frac{a_n}{2d}x_n^{2d} - \prod_{i=1}^{n} x_i^{a_i}) + 2d(1-\mu_1)(\sum_{i=1}^{n} \frac{a_i}{2d}x_i^{2d})$$

Noting that $(a_1, \ldots, a_n) = \sum_{i=1}^{n} \frac{a_i}{2d} 2d\underline{e}_i$, we see that the expression in the first pair of parentheses is a simplicial agiform on a H-trellis, and therefore by lemma 2.2 a sum of binomial squares. Hence evidently $G(\underline{x})$ is a sum of binomial and monomial squares.

Finally assume $-1 \leq \mu_1 < 0$. If all a_i are even then $G(\underline{x})$ is a sum of monomial squares. So assume now some of the a_i, among others, w.l.o.g. a_1, say, are odd. Then since $|\mu_1| \leq 1$, we have by the result just proved that

$$\tilde{G}(\underline{x}) = G(-x_1, x_{2:n}) = a_1 x_1^{2d} + \ldots + a_n x_n^{2d} - |\mu_1| 2d \cdot \prod_{i=1}^{n} x_i^{a_i}$$

is a sum of binomial and monomial squares. Since the property of being a square of a binomial or monomial remains invariant under change of sign of x_1, it follows that $G(\underline{x}) = G(x_1, x_{2:n}) = \tilde{G}(-x_1, x_{2:n})$ is a sum of binomial and monomial squares. Finally, since $E(\underline{x}) = \frac{1}{2d} G(T^{-1}(\underline{x}))$, and since the linear diagonal operator T^{-1} transforms binomials into bionomials, (iii) follows.

iii$\Rightarrow$i. This implication is trivial.

b. In the implication i$\Rightarrow$ii above we have established using the arithmetic geometric inequality that $E(T(x)) \geq 0$. The sharper form of the inequality tells us also that in case $\mu_1 = 1$ we have for $\underline{x} \in \mathbb{R}^n_{\geq 0}$, $\underline{x} \neq \underline{0}$, that $E(T(\underline{x})) = 0$ iff $x_1 = \ldots = x_n$; that is iff $x_1 : \ldots : x_n = 1 : 1 : \ldots : 1$. Now $T(\underline{1}_n) = ((\frac{a_1}{2db_1})^{1/2d}, \ldots, (\frac{a_n}{2db_n})^{1/2d})$, and since T is bijective on $\mathbb{R}^n_{\geq 0}$, the claim follows. $\square$

To fix notation for the sequel, we consider a dmt form F written as

$$F(\underline{x}) = F(x_1, ..., x_n) = \sum_{i=1}^{n} b_i x_i^{2d} - \sum_{\underline{i} \in I} a_{\underline{i}} \underline{x}^{\underline{i}}, \tag{2.3'}$$

with

$$I = \{\underline{i} \in \mathbb{Z}^n_{\geq 0} : a_{\underline{i}} > 0, \text{ all } i_\nu \leq 2d-1\}.$$

The standard $n-1$-simplex in the hyperplane $H_{n-1} = \{\underline{x} \in \mathbb{R}^n : \sum_i x_i = 1\}$ is $\Delta_{n-1} = H_{n-1} \cap \mathbb{R}^n_{\geq 0}$. Its (relative) interior and its (relative) boundary are int $(\Delta_{n-1}) = \{\underline{x} \in \Delta_{n-1} : \forall i\, x_i > 0\}$, and bd $(\Delta_{n-1}) = \{\underline{x} \in \Delta_{n-1} : \exists i\, x_i = 0\}$, respectively.

The following lemma will be generalized in corollary 2.

Lemma 2.4. Let F be a indecomposable psd dmt form F of dimension $n \geq 2$. Then $F|\Delta_{n-1}$ has a local minimum in int (Δ_{n-1}).

Proof. The first case to consider is $\dim(F) = 2$. Here we have to consider a form

$$F(x, y) = b_1 x^{2d} + b_2 y^{2d} - \sum_{\substack{i \geq 1, j \geq 1 \\ i+j=2d}} a_{ij} x^i y^j$$

on $\Delta_1 = \{(x, y) : x + y = 1, x, y \geq 0\}$. By lemma 1a, $\mathrm{var}(D) = \{x, y\}$ since otherwise the dimension would be 1. So $b_1, b_2 > 0$. Note $\mathrm{bd}(\Delta_1) = \{(1, 0), (0, 1)\}$ and $F(1, 0) = b_1$ and $F(0, 1) = b_2$. The natural derivative D on the line $x + y = 1$ is given by

$$DF(x, y) := \frac{dF}{dt}(x - t, y + t)|_{t=0} = (\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x})(x, y).$$

Since $DF(1, 0) = -2db_1 - a_{2d-1,1} < 0$ and similarly $DF(0, 1) = 2db_2 + a_{1,2d-1} > 0$, we find that at points $(x, y) \in$ int (Δ_1) in the neighborhood of $(1, 0)$ and $(0, 1)$ we have $F(x, y) < F(1, 0)$ and $F(x, y) < F(0, 1)$, respectively. Consequently we have a local minimum in int (Δ_1). Note that this argument works hypothizing $b_1, b_2 > 0$, even without indecomposability.

We return to the general case. Let $\Delta = \Delta_{n-1}$. Assume there exists $\underline{z} \in \Delta$ so that $F(\underline{z}) = 0$. Assume $\{i : z_i = 0\} \neq \emptyset$; w.l.o.g. $\{i : z_i = 0\} = \{1, 2, ..., k\}$, say. Then $\underline{z} = (0, ..., 0, z_{k+1}, \ldots, z_n)$. By lemma 2.1a the variables $x_1, \ldots, x_n$ occur in D. So $D(\underline{z}) > 0$, consequently $T(\underline{z}) > 0$. So, there occur terms $a_{\underline{i}} \underline{z}^{\underline{i}} = a_{\underline{i}} 0^{i_1} ... 0^{i_k} z_{k+1}^{i_{k+1}} \cdots z_n^{i_n} \neq 0$ in $T(\underline{z})$, and we must conclude $i_1 = \ldots = i_k = 0$ for all such terms. Thus $x_1, \ldots, x_k \notin \mathrm{var}(T)$. But since $x_1, \ldots, x_k \in \mathrm{var}(D)$, we find that F decomposes as $F = (\sum_{i=1}^k b_i x_i^{2d}) + (\sum_{i=k+1}^n b_i x_i^{2d} - T(\underline{x}))$, a contradiction. This shows: a zero $\underline{z}$ of $F|\Delta_{n-1}$ if existing must lie in int (Δ_{n-1}). Hence psd-ness of F implies that for all $\underline{x} \in$ bd (Δ) we have $F(\underline{x}) > 0$.

Since $\mathrm{bd}(\Delta)$ is compact we infer that there exists a point $\hat{\underline{x}} \in \mathrm{bd}\ (\Delta)$ such that $0 < F(\hat{\underline{x}}) = \min F|\mathrm{bd}\ (\Delta)$. Assume w.l.o.g. that $\hat{\underline{x}} = (\hat{x}_1, \ldots, \hat{x}_{n-1}, 0)$.

For any pair x, y with $x + y = 1$, we compute

$$\tilde{F}(x,y) = F(x\hat{\underline{x}} + y\underline{e}_n) = \left(\sum_{i=1}^{n-1} b_i \hat{x}_i^{2d}\right) x^{2d} + b_n y^{2d} - \sum_{\underline{i} \in I} a_{\underline{i}} \hat{x}_1^{i_1} \ldots \hat{x}_{n-1}^{i_{n-1}} x^{i_1+\ldots+i_{n-1}} y^{i_n}.$$

Note $\tilde{F}(1,0) = F(\hat{\underline{x}}) > 0$. Also $\underline{e}_n \in \mathrm{bd}\ (\Delta)$ guarantees $\tilde{F}(0,1) = F(\underline{e}_n) \geq F(\hat{\underline{x}}) > 0$. This guarantees by the case $\dim(F) = 2$ considered at the beginning that there are $0 < \bar{x}, \bar{y} < 1$, $\bar{x} + \bar{y} = 1$ so that $\tilde{F}(\bar{x}, \bar{y}) < \tilde{F}(1,0) = F(\hat{\underline{x}})$. The point $\bar{x}\hat{\underline{x}} + \bar{y}\underline{e}_n$ is not a boundary point of Δ, for otherwise we would have a contradiction to the choice of $\hat{\underline{x}}$. Hence this point is in the relative interior of Δ and at it F assumes a smaller value than at any point of $\mathrm{bd}\ (\Delta)$. So $F|\Delta$ has a global, hence local minimum in $\mathrm{int}\ (\Delta)$. □

Deleting the word 'indecomposable' the previous lemma becomes false. The decomposable psd dmt form $x^4 + y^4 + z^4 - 2x^2y^2 = (x^2 - y^2)^2 + z^4$ on Δ_2 has its only local minimum at $(x, y, z) = (.5, .5, 0) \in \mathrm{bd}\ (\Delta_2)$

Next we show how to construct elementary positive semi definite dmt-forms of prescribed tail and entrywise positive zero.

Proposition 2.5. Let $\underline{u} \in \mathrm{int}\ (\Delta_{n-1})$, $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^n a_i = 2d \geq 2$, and $\mu > 0$. Define b_i, $i = 1, \ldots, n$ and a form E by

$$b_i = \frac{\mu}{2d} a_i \frac{u_1^{a_1} \cdots u_n^{a_n}}{u_i^{2d}} \text{ and } E(\underline{x}) = \sum_{i=1}^n b_i x_i^{2d} - \mu x_1^{a_1} \cdots x_n^{a_n}.$$

Then the b_i are well defined. If $E \neq 0$ then E is a indecomposable psd dmt form and has a zero is $\underline{u}$. If $\dim(E) = n$, $\underline{u}$ is the unique zero in Δ_{n-1}.

Proof. The hypotheses on $\underline{u}$ and d imply that b_i is well defined. Furthermore we have $a_i = 0$ iff $b_i = 0$ from where it follows that E is either indecomposable or identically 0 (which is the case iff $a_i = 2d$ for some i). Assume now $E \neq 0$. If $\dim(E) = n$, we can compute the threshold value μ_0 by means of theorem

2.3. With the current definitions of the b_i, we find

$$\begin{aligned}\mu_0 &= 2d\prod_{i=1}^{n}\left(\frac{b_i}{a_i}\right)^{a_i/2d}\\ &= 2d\prod_{i=1}^{n}\left(\frac{\mu}{2d}\frac{u_1^{a_1}\cdots u_n^{a_n}}{u_i^{2d}}\right)^{a_i/2d}\\ &= \mu\prod_{i=1}^{n}\frac{(u_1^{a_1}\cdots u_n^{a_n})^{a_i/2d}}{u_i^{a_i}}\\ &= \mu,\end{aligned}$$

since $\sum a_i = 2d$. So by theorem 2.3 E is psd and has in $\Delta_{n-1} \subseteq \mathbb{R}^n_{\geq 0}$ only one zero.

If $\dim(E) \leq n-1$, then some of the variables x_i do not occur; i.e. the set $I = \{i : a_i = 0\} = \{i : b_i = 0\}$ is nonempty. Let $\Delta' = \{\underline{x} \in \Delta : x_i = 0, \text{ for } i \in I\}$. Δ' is an $n - |I| - 1$-dimensional simplex and we can repeat above calculations and reasoning for $E(\underline{x})$ restricted to Δ'. The indices i above have to be restricted to the $i \in I^c$. We get that $\underline{u}'$ defined by $u'_i = u_i$ iff $i \in I^c$ and $u'_i = 0$ iff $i \in I$ is a zero of E. But since E does not depend of the $x_i, i \in I$, $\underline{u}$ is also a zero of E and we are done. □

Lemma 2.6. Let F be a psd homogeneous polynomial of positive degree and dimension n. Assume $\underline{u} \in$ int (Δ_{n-1}) is a local minimum of $F|\Delta_{n-1}$. Then there exists a $\lambda \leq 0$ such that $0 = \nabla F(\underline{u}) + \lambda \underline{1}_n$.

Proof. Consider the function $[0,\infty[\ni t \overset{\varphi}{\mapsto} F(t\underline{u})$. By our homogeneity assumptions, $\varphi(t) = t^d F(\underline{u})$ for some $d \geq 1$. Since $\underline{u} \in \Delta$, $I = \{i : u_i \neq 0\} = \{i : u_i > 0\} \neq \emptyset$. Hence $F(\underline{u}) \geq 0$ implies

$$0 \leq \frac{d\varphi(t)}{dt} = \frac{d}{dt}F(t\underline{u}) = \sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(t\underline{u})u_i = \sum_{i\in I}\frac{\partial F}{\partial x_i}(t\underline{u})u_i.$$

So there exists an $i \in I$ such that $\frac{\partial F}{\partial x_i}(\underline{u}) \geq 0$.

We have $\nabla(\sum_{i=1}^n x_i) = \underline{1}_n$. So the Lagrange Caratheodory multiplier rule (see [Po] for a geometric discussion not often found in textbooks) tells us that there is a λ so that $0 = \nabla F(\underline{u}) + \lambda \underline{1}_n$. In particular for the chosen $i \in I$, $\frac{\partial F}{\partial x_i}(\underline{u}) + \lambda = 0$, implying $\lambda \leq 0$. □

Theorem 2.7. Let F be a psd dmt form. Then F is a sum of elementary psd dmt forms and in particular sum of binomial squares.

Proof. Assume first that F is an indecomposable form of dimension n. Let $\Delta = \Delta_{n-1}$.

By lemma 2.4 we know that there exists a local minimum $\underline{u}$ of F in int (Δ). Using notation (2.3'), by lemma 2.6 we have for some $\lambda \leq 0$ the equations

$$2db_k u_k^{2d-1} - \sum_{\underline{i} \in I} a_{\underline{i}} \cdot i_k \cdot u_1^{i_1} \dots u_k^{i_k-1} \dots u_n^{i_n} + \lambda = 0, \quad k = 1, \dots, n.$$

Putting
$$b_{\underline{i}k} := \frac{a_{\underline{i}}}{2d} i_k \frac{\underline{u}^{\underline{i}}}{u_k^{2d}},$$

we find for $k = 1, \dots, n$, that

$$\begin{aligned} b_k &= \sum_{\underline{i} \in I} \frac{a_{\underline{i}}}{2d} i_k \frac{u_1^{i_1} \dots u_k^{i_k-1} \dots u_n^{i_n}}{u_k^{2d-1}} - \frac{\lambda}{2du_k^{2d-1}} \\ &= \sum_{\underline{i} \in I} b_{\underline{i}k} - \frac{\lambda}{2du_k^{2d-1}}. \end{aligned}$$

For each $\underline{i} \in I$, proposition 2.5 tells us that

$$E_{\underline{i}}(\underline{x}) = \sum_{k=1}^{n} b_{\underline{i}k} x_k^{2d} - a_{\underline{i}} \underline{x}^{\underline{i}}$$

is a elementary indecomposable psd dmt form that has a zero at $\underline{u}$. Now we compute

$$\begin{aligned} F(\underline{x}) &= \sum_{k=1}^{n} b_k x_k^{2d} - \sum_{\underline{i} \in I} a_{\underline{i}} \underline{x}^{\underline{i}} \\ &= \sum_{k=1}^{n} \left(\sum_{\underline{i} \in I} b_{\underline{i}k} - \frac{\lambda}{2du_k^{2d-1}} \right) x_k^{2d} - \sum_{\underline{i} \in I} a_{\underline{i}} \underline{x}^{\underline{i}} \\ &= \sum_{\underline{i} \in I} \left(\sum_{k=1}^{n} b_{\underline{i}k} x_k^{2d} - a_{\underline{i}} \underline{x}^{\underline{i}} \right) - \sum_{k=1}^{n} \frac{\lambda}{2du_k^{2d-1}} x_k^{2d} \\ &= \sum_{\underline{i} \in I} E_{\underline{i}}(\underline{x}) + \sum_{k=1}^{n} \frac{(-\lambda)}{2du_k^{2d-1}} x_k^{2d} \end{aligned}$$

Since $-\lambda \geq 0$, the last summand is a elementary psd dmt form without tail. We thus have shown that every indecomposable psd dmt form is a sum of elementary psd dmt forms. In the general case, by lemma 2.1b, one can write $F = F_1 + F_2 + \ldots + F_l$ with indecomposable psd dmt forms. Write each of these as a sum of elementary psd dmt forms. Evidently, each elementary psd dmt form $E = E' + D'$ with an indecomposable elementary psd dmt form E' and a psd diagonal form D'. By theorem 2.3, E' is sobs and obviously D' is sum of monomial squares. The theorem follows. $\square$

Originally we tried to establish the uniqueness result of the following corollary as a base for theorem 2.7. An attempt for a direct proof caused us considerable troubles even in the case $n = 2$. It is amusing that rather than a base, it is an easy consequence of theorem 2.7.

Corollary 2.8. (of proof of theorem 2.7) Let F be a indecomposable psd dmt form of dimension n. Then F has on Δ_{n-1} a unique local minimum; it is of course also the global minimum of $F|\Delta_{n-1}$.

Proof. In the proof of theorem 2.7 we established that if $\underline{u}$ is a by lemma 2.4 existing local minimum of $F|\Delta_{n-1}$, then we can write

$$F(\underline{x}) = \sum_{\underline{i} \in I} E_{\underline{i}}(\underline{x}) + D'(\underline{x}), \text{ with diagonal } D'(x) = \sum_{k=1}^{n} \frac{(-\lambda)}{2du_k^{2d-1}} x_k^{2d}$$

for certain elementary indecomposable psd dmt forms $E_{\underline{i}}(\underline{x})$, $\underline{i} \in I$ which all have $\underline{u}$ as a zero in Δ_{n-1}. The zero is unique for all forms $E_{\underline{i}}$ of dimension n. However in general only the components u_i for which $x_i \in \text{var}(E_{\underline{i}})$ are unique, the other components are arbitrary. For any $\underline{x} \in \Delta_{n-1}$ and any $\underline{i} \in I$ we have $*$: $E_{\underline{i}}(\underline{x}) \geq 0 = E_{\underline{i}}(\underline{u})$. Now since F is indecomposable of dimension n we find that $\text{var}(F) = \bigcup_{\underline{i} \in I} \text{var}(E_{\underline{i}})$. Thus for any $\underline{x}$ the inequality $*$ is strict for some $\underline{i}$.

Viewing the E_i, F, and D' as functions on the hyperplane H_{n-1} containing Δ_{n-1}, we now see that $\sum_{\underline{i} \in I} E_{\underline{i}}(\underline{x}) > \sum_{\underline{i} \in I} E_{\underline{i}}(\underline{u}) = 0$ for all $\underline{x} \in \Delta_{n-1} \setminus \{\underline{u}\}$. Further, by Lagrange multipliers applied to $D'|H_{n-1}$ we see that provided $D' \neq 0$, $D'|H_{n-1}$ has only one local (and hence global) minimum, namely in $\underline{u}$. Consequently $D'(\underline{x}) > D'(\underline{u})$ for all $\underline{x} \in H_{n-1} \setminus \{\underline{u}\}$. This proves: for all $\underline{x} \in \Delta_{n-1} \setminus \{\underline{u}\}$ we have $F(\underline{x}) > F(\underline{u})$. It follows that $F|\Delta_{n-1}$ has $\underline{u}$ as its unique global minimum. Since our reasoning began with an arbitrary local minimum and showed it is the unique global minimum, the claim is established. □

Sometimes one will be able to show that a given form is sobs by relating it to a psd dmt form via the following corollary.

Corollary 2.9. (of proof of theorem 2.7) Consider the forms P and F of degree $2d$ given by

$$P(\underline{x}) = \sum_{i=1}^{n} b_i x_i^{2d} + \sum_{\underline{i}\in I} a_{\underline{i}} \underline{x}^{\underline{i}} \quad \text{and} \quad F(\underline{x}) = \sum_{i=1}^{n} b_i x_i^{2d} - \sum_{\underline{i}\in I} |a_{\underline{i}}| \underline{x}^{\underline{i}},$$

where for all $\underline{i} \in I$, $\max \underline{i} \le 2d-1$. If $F(\underline{x})$ is psd, then $P(\underline{x})$ is sum of squares of binomials.

Proof. The conditions imposed on the $\underline{i}$ guarantee that $F(\underline{x})$ is a dmt form. Evidently there exist signs $\varepsilon_{\underline{i}} \in \{-1, +1\}$ such that

$$P(\underline{x}) = \textstyle\sum_{i=1}^{n} b_i x_i^{2d} - \sum_{\underline{i}\in I} \varepsilon_{\underline{i}} |a_{\underline{i}}| \underline{x}^{\underline{i}}.$$

Let us assume first and for simplicity of notation that $F = D - T$ is indecomposable, where D and T are diagonal and tail of F, respectively.

From the proof of theorem 2.7 we know that $F(\underline{x}) = \sum_{\underline{i}\in I} E_{\underline{i}}(\underline{x}) + D'(\underline{x})$ for certain elementary indecomposable psd dmt forms $E_{\underline{i}}(\underline{x}) = D_{\underline{i}}(\underline{x}) - |a_{\underline{i}}| \underline{x}^{\underline{i}}$, where $D_{\underline{i}}$, D' are psd diagonal and hence sobs. To each $E_{\underline{i}}$ associate $E'_{\underline{i}}(\underline{x}) = D_{\underline{i}}(\underline{x}) - \varepsilon_{\underline{i}} |a_{\underline{i}}| \underline{x}^{\underline{i}}$. By theorem 2.3a $E'_{\underline{i}}$ is psd and hence sobs. Evidently $P(\underline{x}) = \sum_{\underline{i}\in I} E'_{\underline{i}}(\underline{x}) + D'(\underline{x})$. Thus P is sobs.

In the general case of a decomposable form F, by lemma 1b we can write $F = F_1 + \ldots + F_k$ as a sum of indecomposable psd dmt forms $F_i = D_i - T_i$, $i = 1, \ldots, k$, where D_i and T_i are diagonal and tail of F_i respectively and can apply above reasoning to each of the indecomposable forms F_i. Invoking lemma 1c concludes the proof that P is sobs. □

Robinson [Ro, p269] has shown the following interesting fact by a quite different method.

Corollary 2.10. If $F(\underline{x})$ is any real form of degree 2d, then for large enough β, the polynomial $P(\underline{x}) = F(\underline{x}) + \beta(x_1^{2d} + \ldots x_n^{2d})$ is a sum of squares of binomials.

Proof. Change all positive coefficients in $F(\underline{x})$ not pertaining to a monomials x_i^{2d} to their negative counterpart. Let $\hat{F}(\underline{x})$ be the resulting polynomial. On the unit sphere it assumes its minimum. Hence for large enough β, the polynomial $\hat{F}(\underline{x}) + \beta(x_1^{2d} + \ldots x_n^{2d})$ will be a positive definite diagonal minus tail form and hence sobs by theorem 2.7. Since the operation 'adding $\beta \sum_i x_i^{2d}$' and 'changing signs of monomials $\neq x_i^{2d}$' commute, corollary 2.9 guarantees that $P(\underline{x})$ will be sobs. □

3. An Algorithm and Examples

The theorems of this paper are constructive. To decide whether a diagonal minus tail form $F(\underline{x})$ can be written as a sum of squares and if so to give the an explicit representation, one can use the following algorithm.

Algorithm 3.1.

1. Write the complete decomposition of F, $F = F_1 + \ldots + F_k$ into dmt forms F_i; see lemma 2.1.

2. Determine for each of the indecomposable F_i the by corollary 2.8 unique local minimum $\underline{u}_i$ in the relative interior of the standard simplex spanned by the $\underline{e}_\ell$ for which $x_\ell \in \mathrm{var}(F_i)$.

3. If there exists an i such that $F_i(\underline{u}_i) < 0$, then stop: F_i and hence by lemma 2.1b F cannot be sum of squares.

4. Otherwise write each of the F_i as a sum of elementary nondiagonal dmt forms plus one diagonal form as in the proof of theorem 2.7 (which recurs to proposition 2.5).

5. Write each of the nondiagonal elementary dmt forms obtained in step 4 according to the proof of theorem 2.3a, ii⇒iii as a sum of squares of binomials. This also requires applying the method given in the proof of [Re2], theorem 4.4 to $\mathcal{L} = C(\mathcal{H}_{n,2d})$.

6. Upon summing the sos representations of all the F_i in step 5 and the respective diagonal forms of step 4 we get a representation of F as a sum of binomial and monomial squares.

To illustrate the algorithm we consider the example

$$F(x, y, z) = x^4 + y^4 + z^4 + 2v^4 + w^4 - 0.7x^3y - 0.9x^2yz - yz^3 - v^2w^2.$$

This is a dmt form. As step 1 we note that the complete decomposition of F is $F = F_1 + F_2$ with

$$F_1 = x^4 + y^4 + z^4 - 0.7x^3y - 0.9x^2yz - yz^3, \quad \text{and} \quad F_2 = 2v^4 + w^4 - v^2w^2.$$

We next find that the local minimum of F_1 on Δ_2 is assumed in $\underline{u}_1 = (0.343, 0.313, 0.343)$. Here and below we cut off the digits following the third one; much more precise values are listed below to 16 decimal places. Since $F_1(\underline{u}_1) = 0.004 > 0$, F_1 is positive semidefinite. Similarly F_2 assumes on Δ_1 its local minimum in $\underline{u}_2 = (0.448, 0.551)$ and the value of $F_2(\underline{u}_2) = 0.112 > 0$. This completes steps 2,3 of the algorithm.

Next we have to write F_1 and F_2 as sums of elementary dmt forms. By the proof of theorem 2.7 we can find for F_1 a representation $F_1 = E_{11} + E_{12} + E_{13} + D_1$, where D_1 is psd diagonal, and the E_{1i} are elementary dmt forms having respective tails $-0.7x^3y$, $-0.9x^2yz, -1yz^3$, and the common zero $\underline{u}_1$; for F_2 we have similarly to find a representation $F_2 = E_{21} + D_2$ where E_{21} has tail $-v^2w^2$ and zero $\underline{u}_2$, and D_2 is diagonal.

Proposition 2.5 gives us formulae for finding the Es. The Ds are then found by the obvious subtractions. The result of these calculations are as follows. For F_1,

$$\begin{aligned} E_{11} &= 0.479x^4 + 0.230y^4 + 0z^4 - .7x^3y, \\ E_{12} &= 0.410x^4 + 0.295y^4 + 0.205z^4 - .9x^2yz, \\ E_{13} &= 0x^4 + 0.328y^4 + 0.684z^4 - 1.0yz^3, \\ D_1 &= 0.110x^4 + 0.144y^4 + 0.110z^4; \end{aligned}$$

while for F_2,

$$\begin{aligned} E_{21} &= 0.755v^4 + 0.330w^4 - v^2w^2, \\ D_2 &= 1.244v^4 + 0.669w^4. \end{aligned}$$

This was step 4. Next for step 5, glancing at the proof of theorem 2.3 we find that if E is a elementary dmt form with a zero, then there $\mu_1 = 1$. We shall apply the formula

$$G(\underline{x}) = 2dE(T(x)) = 2d(\frac{a_1}{2d}x_1^{2d} + \ldots + \frac{a_n}{2d}x_n^{2d} - \prod_{i=1}^{n} x_i^{a_i})$$

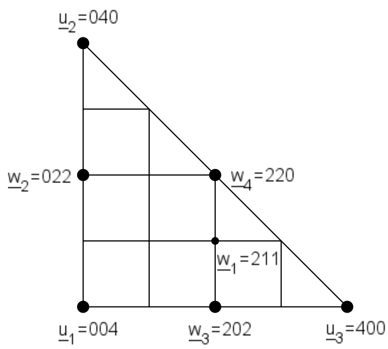
with $d = 2$ and $n = 3$ in case of $E = E_{12}$ and $n = 2$ for E_{11}, E_{13}, E_{21}, write $G(\underline{x})$ as sobs using Reznick's agiform theory, and obtain the desired sobs representations for E as $E(\underline{x}) = \frac{1}{4}G(T^{-1}(\underline{x}))$.

We do the case $E = E_{12}$ in detail, and let the remaining simpler cases for verification to the reader.

With $\underline{\lambda} = (\frac{a_1}{2d}, \dots, \frac{a_n}{2d})$, $\underline{w} = (a_1, \dots, a_n)$, the form in parenthesis is in Reznick's notation the simplicial agiform $f(\mathcal{H}_{n,2d}, \underline{w}) = f(\mathcal{H}_{n,2d}, \underline{\lambda}, \underline{w}) = \frac{G}{2d}$. For $E = E_{12}$, the relevant objects are $\underline{w} = (2, 1, 1)$, and

$$\mathcal{U}^* = H^*_{3,4} = C(\mathcal{H}_{3,4}) = cvx\{(0,0,4), (0,4,0), (4,0,0)\}.$$

Since $\mathcal{H}_{3,4}$ is by p438c-4 a trellis, $\mathcal{F} = E(\mathcal{H}_{3,4}) \cup \{\underline{w}\}$ is by theorem 2.8, p440 a $\mathcal{U}$-mediated set. Then the proof of theorem 4.4, p443 tells us how to write $f(\mathcal{H}_{3,4}, (2, 1, 1))$ as a sobs. Namely, with the notations in our figure, reminiscent of the figure on p435, we find that $\underline{w}_1, \dots, \underline{w}_4$ satisfies in terms of Reznick's case distinctions (4.5) the following cases, respectively.



$$\begin{array}{rcll} \underline{w}_1 & = & \frac{1}{2}(w_3 + w_4) & \text{case iii} , \\ \underline{w}_2 & = & \frac{1}{2}(u_1 + u_2) & \text{case i} , \\ \underline{w}_3 & = & \frac{1}{2}(u_1 + u_3) & \text{case i} , \\ \underline{w}_4 & = & \frac{1}{2}(u_2 + u_3) & \text{case i} . \end{array}$$

Hence we have to invert the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \text{ obtaining } 4A^{-1} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

So by the reasoning in theorem 4.4, $4f(\mathcal{H}_{3,4},\underline{w}) = 2(xz-xy)^2+1(z^2-x^2)^2+(x^2-y^2)^2$. Using that

$$T^{-1}(\underline{x}) = ((\frac{2db_i}{a_i})^{1/2d}x_i)_{i=1}^3 = ((\frac{2db_i}{a_i})^{1/2d}x_i)_{i=1}^3 = ((2b_1)^{1/4}x,(4b_2)^{1/4}y,(4b_3)^{1/4}z),$$

where b_1, b_2, b_3 are the coefficients of x^4, y^4, z^4 in E_{12}, we get for

$$E_{12}(\underline{x}) = E(x) = f(\mathcal{H}_{3,4},\underline{w})(T^{-1}(\underline{x})),$$

that

$$E_{12}(\underline{x}) = \frac{1}{4}(0.906x^2 - 1.088y^2)^2 + \frac{1}{2}(-0.993xy + 0.906xz)^2 + \frac{1}{4}(-0.906x^2 + 0.906z^2)^2.$$

With the obvious modifications, one similarly has for E_{11}, E_{13} and E_{21} the representations

$$\begin{aligned}
E_{11} &= \frac{1}{2}(-0.799x^2 + 0.875xy)^2 + \frac{1}{4}(-0.799x^2 + 0.959y^2)^2,\\
E_{13} &= \frac{1}{2}(-0.799z^2 + 0.875yz)^2 + \frac{1}{4}(-0.799z^2 + 0.959y^2)^2,\\
E_{21} &= \frac{1}{2}(1.229v^2 - 0.813w^2)^2.
\end{aligned}$$

This finishes step 5. The sum of all sobs representations found for the Es plus the monomial square representations for the Ds is a representation of the original polynomial F as a sum of monomial and binomial squares as desired. □

The following examples illustrate the necessity of some requirements like our hypothesis that F be psd in theorem 2.9. Robinson [Ro, p272] shows that the form by later authors denoted R, and given by

$$R(x,y,z) = x^6+y^6+z^6-(x^4y^2+x^4z^2+y^4x^2+y^4z^2+z^4x^2+z^4y^2)+3x^2y^2z^2$$

is psd but not sos. In [CL] one finds the result that each one of the ininitely many fully symmetric forms

$$H_\mu(x,y,z) = \sum^3 x^{2\mu+4} - \sum^6 x^{2\mu+2}y^2 + \sum^3 x^{2\mu+4}y^2z^2,\ \mu = 1,2,3,...,$$

is psd but not sos. Since $R = H_1$ this is a generalization of Robinson's result. So forms can have a significant diagonal part and be psd but need not be sos. Consistent with our findings, if we change some of the signs of positive nondiagonal terms into their negative counterparts the obtained forms are not anymore psd.

Our main theorem, that every psd dmt form is sum of binomial squares has the consequence, of course, that every such form is sum of squares. If one gives up the requirement that these squares be binomial squares, then the number of squares can be reduced. To this end one modifies theorem 2.3 iii to 'E is sos' and uses in the proof of ii$\Rightarrow$iii the method of Reznick's paper [Re1] instead of his theory of agiforms. There he uses that the binary psd forms to which his inductive reduction of the number of variables lead, although they are sobs, can more economically be written as sums of two squares. This follows by well known reasoning using the fundamental theorem of algebra and iteratively the identity $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$.

By the way, not every binary psd form is sobs: A representation of $f(x,y) = (x-y)^2(x-2y)^2$ as sobs would mean that each of the binomials involved must vanish when $x=y$ and also when $x=2y$, easily seen to be impossible.

For those wishing to verify the above calculations, we conclude with the list of the reals abbreviated above and increasingly ordered according to their modulus.

0.1101683087515428, 0.1122521354097604, 0.144926205793132, 0.2053457749034901,

0.2302121753633875, 0.2959870826100697, 0.3133395876912924, 0.3288745362334108,

0.3307276430512695, 0.3433302061543539, 0.4106915498069803, 0.4485189743648009,

0.4791401414414769, 0.5514810256351992, 0.6692723569487306, 0.684485916344967,

0.7559089941606274, 0.7992831300121186, 0.813299013956453, 0.875784779780586,

0.906301881060588, 0.959608618892906, 0.993046598277813, 1.088093897804908,

1.229560079183305, 1.244091005839373.

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