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INSERTION OF CONTINUOUS REAL FUNCTIONS ON SPACES, BISPACES, ORDERED SPACES AND POINT-FREE SPACES — A COMMON ROOT

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Abstract: We characterize normal and extremally disconnected biframes in terms of the insertion of a continuous real function in between given lower and upper semicontinuous real functions and show this to be the common root of several classical and new insertion results concerning topological spaces, bitopological spaces, ordered topological spaces and locales.

Keywords: Frame, locale, localic real function, insertion theorem, biframe, normal biframe, extremally disconnected biframe, bitopological space, ordered topological space.

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Introduction

Various kinds of spaces are characterized in terms of the existence of a continuous real function in between given lower and upper semicontinuous real functions, such as

- normal spaces (Katětov-Tong insertion theorem ([11, 25])),

and

- extremally disconnected spaces (Stone insertion theorem [26]).

These results rank among the fundamental results in point-set topology and provide as corollaries a separation-type theorem (e.g. in the normal case the celebrated Urysohn Lemma) and an extension-type theorem (e.g. in the normal case the well known Tietze Extension Theorem). The corresponding counterparts in point-free topology were obtained recently (see [18, 9, 6];

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in [7] and [4] perfectly normal, countably paracompact and completely normal point-free spaces are also characterized via insertion of continuous real functions and [8] presents a nice general point-free setting for arbitrary real functions in which those results can be nicely formalized). There are also in the literature versions of some of these results for bitopological spaces (as the Katětov-Tong version of Priestley for normal bitopological spaces [21] and the separation and the extension-type theorems of Sarma for extremally disconnected bitopological spaces [22]) and ordered topological spaces (Nachbin [17], Priestley [21], Edwards [3]).

The purpose of this paper is to exhibit a natural common root for these results, encompassing at once topological spaces, bitopological spaces, ordered topological spaces, point-free spaces (that is, frames and locales) and point-free bispaces (that is, biframes) in a single setting. The proper setting for this will be the category of *biframes* and, specifically, their (relative) lower and upper semicontinuous real functions.

After a brief survey of the required background and the introduction of a few new concepts in biframes, we begin with a new characterization of normal biframes from which it follows a Katětov-Tong-type insertion theorem for biframes (L_0, L_1, L_2) that generalizes all the above mentioned Katětov-Tong-type insertion results:

- The case $L_0 = L_1 = L_2$ provides the result for frames [18, 9]. Making further $L_0 = L_1 = L_2 = \mathfrak{T}$ for any topological space $(X, \mathfrak{T})$ provides the classical Katětov-Tong theorem.
- The case $(L_0, L_1, L_2) = (\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$ for a bitopological space $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ provides the Katětov-Tong-type result of Priestley [21] for bitopological spaces.
- The case $(L_0, L_1, L_2) = ({\downarrow}\mathfrak{T} \vee {\uparrow}\mathfrak{T}, {\downarrow}\mathfrak{T}, {\uparrow}\mathfrak{T})$ for an ordered topological space $(X, \mathfrak{T}, \leq)$, and some variants of it, provide the Katětov-Tong-type results of Priestley [21] and Nachbin [17] for ordered topological spaces.

Then we do a similar study for the extremally disconnected biframes and Stone insertion theorem. Here the application of our general theorem to bitopological spaces and ordered topological spaces provides new results in the literature.

1. Background on frames and biframes

We shall denote the category of *frames* and *frame homomorphisms* by Frm and the corresponding dual category of *locales* and *localic maps* by Loc. For general facts concerning frames, locales and various related ideas we refer to [10], [19] and [20]. Here we fix some terminology and notations.

A frame (or locale) L is *normal* if whenever $a \vee b = 1$ there exist $u, v \in L$ satisfying $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$ (equivalently, if there exists $u \in L$ such that $a \vee u = 1 = b \vee u^*$ for the pseudocomplement $u^* = \bigvee\{v \mid u \wedge v = 0\}$). Dually, L is *extremally disconnected* if whenever $a \wedge b = 0$ there exist $u, v \in L$ such that $u \vee v = 1$ and $a \wedge u = 0 = b \wedge v$. Equivalently, if $a^* \vee a^{**} = 1$ for every $a \in L$ (which is the same as asserting that the second De Morgan law $(a \wedge b)^* = a^* \vee b^*$ holds in L).

A subset S of L is a *sublocale* of L (i.e. a regular subobject of L in Loc) if, whenever $A \subseteq L$, $a \in L$ and $s \in S$, then $\bigwedge A \in S$ and $a \to s \in S$. Each sublocale S of L is a frame itself with $\wedge$ and $\to$ inherited from L (the top element of S is 1, while the bottom 0_S of S may differ from 0). It determines the surjection (frame quotient) $c_S : L \to S$ given by $c_S(x) = \bigwedge\{s \in S : x \leq s\}$.

The intersection of sublocales is again a sublocale, so that the set of all sublocales is a complete lattice under inclusion. In fact, it is a co-frame, in which $\{1\}$ is the bottom element and L is the top [19].

For notational reasons, we shall make the co-frame of all sublocales into a frame $\mathcal{S}(L)$ by considering the opposite ordering:

$$S_1 \leq S_2 \quad \Leftrightarrow \quad S_2 \subseteq S_1.$$

Thus, given $\{S_i \in \mathcal{S}(L) : i \in I\}$, we have

$$\bigvee_{i\in I} S_i = \bigcap_{i\in I} S_i \quad \text{and} \quad \bigwedge_{i\in I} S_i = \{\textstyle\bigwedge A : A \subseteq \bigcup_{i\in I} S_i\}.$$

Then $\{1\}$ is the top element and L is the bottom element in $\mathcal{S}(L)$ that we just denote by 1 and 0, respectively. The pseudocomplement of S in $\mathcal{S}(L)$ will standardly be denoted by S^*.

Among the important examples of sublocales are the *closed* and *open sublocales*

$$\mathfrak{c}_L(a) = \uparrow a = \{b \in L : a \leq b\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b : b \in L\}$$

for every $a \in L$. We denote them just by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ whenever there is no more than one frame L in question. The quotients $c_{\mathfrak{c}(a)}$ and $c_{\mathfrak{o}(a)}$ are given by

$$c_{\mathfrak{c}(a)}(x) = a \vee x \quad \text{and} \quad c_{\mathfrak{o}(a)}(x) = a \to x.$$

We shall freely use the following properties:

Properties 1.1. *For every $a \in L$ we have:*

(1) *The map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $\mathfrak{c}_L : L \to \mathcal{S}(L)$.*
(2) $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ *and* $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$. *In particular:*
- $\mathfrak{o}(a) \geq \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$,
- $\mathfrak{o}(a) \leq \mathfrak{c}(b)$ *if and only if* $a \vee b = 1$,

and, for any $S \in \mathcal{S}(L)$, $S \vee \mathfrak{c}(a) = 1$ if and only if $S \geq \mathfrak{o}(a)$.

For any frame L, we denote by $\mathfrak{c}L$ the subframe of $\mathcal{S}(L)$ consisting of all closed sublocales of L (by Properties 1.1 (1), L and $\mathfrak{c}L$ are isomorphic) and we denote by $\mathfrak{o}L$ the subframe of $\mathcal{S}(L)$ generated by all open sublocales of L.

Recall tat a subset M of a frame L is a *subframe* of L if it is closed under finite meets and arbitrary joins (in particular, $0, 1 \in M$). A *biframe* [2] is a triple (L_0, L_1, L_2) in which L_0 is a frame and L_1 and L_2 are subframes of L_0, which together generate it. This means that any element x of L_0 can be expressed as a join of finite meets of elements of $L_1 \cup L_2$, that is, $x = \bigvee_{i \in I}(a_i \wedge b_i)$ for some $a_i \in L_1$ and $b_i \in L_2$.

A *biframe homomorphism* [2] $f : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a frame homomorphism $f : L_0 \to M_0$ for which $f(L_i) \subseteq M_i$ $(i = 1, 2)$. Biframes and biframe homomorphisms form the category BiFrm of biframes [2]. For more information on BiFrm we refer to [23].

Given a biframe (L_0, L_1, L_2), we shall denote by $(\mathfrak{c}L_0)_i$ $(i = 1, 2)$ the system $\{\mathfrak{c}_{L_0}(a) \mid a \in L_i\}$ of all closed sublocales of L_0 determined by elements $a \in L_i$. Similarly, $(\mathfrak{o}L_0)_i$ $(i = 1, 2)$ denotes the set $\{\mathfrak{o}_{L_0}(a) \mid a \in L_i\}$. Note that $(\mathfrak{c}L_0, (\mathfrak{c}L_0)_1, (\mathfrak{c}L_0)_2)$ is a biframe isomorphic to (L_0, L_1, L_2).

The biframe

$$(\mathcal{S}_0(L_0), \mathcal{S}_1(L_1), \mathcal{S}_2(L_2))$$

is defined as follows (cf. Definition 1.12 in [23], written in terms of congruences):

$\mathcal{S}_i(L_i)$ $(i = 1, 2)$ is the subframe of $\mathcal{S}(L_0)$ generated by $(\mathfrak{c}L_0)_i \cup (\mathfrak{o}L_0)_j$ $(j = 1, 2, j \neq i)$ and $\mathcal{S}_0(L_0)$ is the subframe of $\mathcal{S}(L_0)$ generated by $\mathcal{S}_1(L_1) \cup \mathcal{S}_2(L_2)$.

The map $(L_0, L_1, L_2) \to (\mathcal{S}_0(L_0), \mathcal{S}_1(L_1), \mathcal{S}_2(L_2))$ given by $a \mapsto \mathfrak{c}_{L_0}(a)$ for all $a \in L_0$ is a biframe embedding.

Another important example of a biframe is the *biframe of reals*

$$(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}))$$

where $\mathfrak{L}(\mathbb{R})$ is the *frame of reals* ([1], [15]) defined by generators $(p, \text{—})$ and $(\text{—}, q)$ $(p, q \in \mathbb{Q})$ and relations

(R1) $(p, \text{—}) \wedge (\text{—}, q) = 0$ whenever $p \geq q$,
(R2) $(p, \text{—}) \vee (\text{—}, q) = 1$ whenever $p < q$,
(R3) $(p, \text{—}) = \bigvee_{r>p}(r, \text{—})$,
(R4) $(\text{—}, q) = \bigvee_{s<q}(\text{—}, s)$,
(R5) $\bigvee_{p\in\mathbb{Q}}(p, \text{—}) = 1$,
(R6) $\bigvee_{q\in\mathbb{Q}}(\text{—}, q) = 1$,

and

$$\begin{aligned}\mathfrak{L}_u(\mathbb{R}) &= \langle\{(p, \text{—}) : p \in \mathbb{Q},\ (p, \text{—}) \text{ satisfy (R3) and (R5) for all } p \in \mathbb{Q}\}\rangle\,, \\ \mathfrak{L}_l(\mathbb{R}) &= \langle\{(\text{—}, q) : q \in \mathbb{Q},\ (\text{—}, q) \text{ satisfy (R4) and (R6) for all } q \in \mathbb{Q}\}\rangle\,.\end{aligned}$$

A biframe (L_0, L_1, L_2) is called *normal* [23, 24] if, whenever $a \vee b = 1$ for some $a \in L_1$ and $b \in L_2$, there exist $u \in L_2$ and $v \in L_1$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

For $a \in L_i$ $(i = 1, 2)$ let $a^\bullet$ be the largest $b \in L_j$ $(j = 1, 2,\ j \neq i)$ for which $a \wedge b = 0$. Normality can then be equivalently expressed by the condition: whenever $a \vee b = 1$ for $a \in L_i$ and $b \in L_j$, there exists $u \in L_j$ such that $a \vee u = 1 = b \vee u^\bullet$.

We define *sub-bilocales* (S_0, S_1, S_2) of a biframe (L_0, L_1, L_2) as its subobjects in the dual category of BiFrm. Thus each (S_0, S_1, S_2) is given by a regular epimorphism $p_{(S_0,S_1,S_2)} : (L_0, L_1, L_2) \to (S_0, S_1, S_2)$ in BiFrm, that is, an onto frame homomorphism $p_{(S_0,S_1,S_2)} : L_0 \to S_0$ such that $p_{(S_0,S_1,S_2)}(L_i) \subseteq S_i$ $(i = 1, 2)$.

Remark 1.2. Given a subframe M and a sublocale S of L, the set $c_S(M)$ is a subframe of S. In particular, for any $x \in L$, the sets $x \vee M = \{x \vee m \mid m \in M\} = c_{\mathfrak{c}_L(x)}(M)$ and $x \to M = \{x \to m \mid m \in M\} = c_{\mathfrak{o}_L(x)}(M)$ are subframes of $\mathfrak{c}_L(x)$ and $\mathfrak{o}_L(x)$, respectively. Note that if $x \in M$ then $x \vee M$ is just $\mathfrak{c}_M(x)$.

Therefore, for each $x \in L_0$, $\mathfrak{c}^b(x) = (\mathfrak{c}_{L_0}(x), x \vee L_1, x \vee L_2)$ is a sub-bilocale of (L_0, L_1, L_2) with $p_{\mathfrak{c}^b(x)} : a \mapsto x \vee a$. We refer to these as the *closed sub-bilocales* of (L_0, L_1, L_2). In particular for any $x \in L_1$, $\mathfrak{c}^b(x) = (\mathfrak{c}_{L_0}(x), \mathfrak{c}_{L_1}(x), x \vee L_2)$ and for any $x \in L_2$, $\mathfrak{c}^b(x) = (\mathfrak{c}_{L_0}(x), x \vee L_1, \mathfrak{c}_{L_2}(x))$.

Further, for each $x \in L_0$, $\mathfrak{o}^b(x) = (\mathfrak{o}_{L_0}(x), x \to L_1, x \to L_2)$ is a sub-bilocale of (L_0, L_1, L_2) with $p_{\mathfrak{o}^b(x)} : y \mapsto (x \to y)$. We refer to these as the *open sub-bilocales* of (L_0, L_1, L_2).

2. Real functions on frames and biframes

In general topology one sometimes deals with arbitrary (not necessarily continuous) real-valued functions on a topological space X. This is also possible in the point-free setting with the approach recently introduced in [8] (which extends the approach to point-free continuous real functions of Banaschewski [1]):

Let L be a frame. A *real function* on L is an $f \in \mathsf{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$. It is

(1) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}L$,
(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}L$,
(3) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$.

$\mathsf{F}(L)$ is partially ordered by

$$\begin{aligned} f \le g \quad &\Leftrightarrow \quad f(p,\text{—}) \le g(p,\text{—}) \quad \text{for every } p \in \mathbb{Q} \\ &\Leftrightarrow \quad g(\text{—},q) \le f(\text{—},q) \quad \text{for every } q \in \mathbb{Q}. \end{aligned}$$

Now, given a biframe (L_0, L_1, L_2), a *real function on* (L_0, L_1, L_2) is a frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(L_0)$ such that

$$f : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}_0(L_0), \mathcal{S}_1(L_1), \mathcal{S}_2(L_2))$$

is a biframe homomorphism. Further:

(1) f is L_1-*usc* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq (\mathfrak{c}L_0)_1$,
(2) f is L_2-*lsc* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq (\mathfrak{c}L_0)_2$.

The following is obvious since $f(p,q) = f(p,-) \wedge f(-,q)$:

Proposition 2.1. *Let f be a real function on a biframe (L_0, L_1, L_2). Then f is L_1-usc and L_2-lsc if and only if f is continuous in L_0 and*

$$\mathfrak{c}_{L_0}{}^{-1} \circ f : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$$

is a biframe homomorphism. ∎

Note that, in particular, "L_1-usc+L_2-lsc" is stronger than "continuous in L_0".

Given a complemented $S \in \mathcal{S}_i(L_i)$ $(i = 1, 2)$ (i.e. such that $S^\bullet \vee S = 1$), the *characteristic map* $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(L_0)$ is defined by

$$\chi_S(p, -) = \begin{cases} 1 & \text{if } p < 0, \\ S^\bullet & \text{if } 0 \le p < 1, \\ 0 & \text{if } p \ge 1, \end{cases} \quad \text{and} \quad \chi_S(-, q) = \begin{cases} 0 & \text{if } q \le 0, \\ S & \text{if } 0 < q \le 1, \\ 1 & \text{if } q > 1, \end{cases}$$

for every $p, q \in \mathbb{Q}$. Then,

(1) χ_S is L_1-usc if and only if $S \in (\mathfrak{c}L_0)_1$,
(2) χ_S is L_2-lsc if and only if $S \in (\mathfrak{o}L_0)_2$.

3. Scales

A way of generating continuous real functions on frames by the so-called scales has been described in detail in [1] (cf. also [8]). In what follows we show how they can be adapted to real functions on biframes. We shall write $\mathfrak{c}_0$, $\mathfrak{c}_0(a)$ and $\mathfrak{o}_0(a)$ instead of $\mathfrak{c}_{L_0}$, $\mathfrak{c}_{L_0}(a)$ and $\mathfrak{o}_{L_0}(a)$, respectively.

Let (L_0, L_1, L_2) be a biframe. We say that a family $\mathcal{A} = \{a_p \mid p \in \mathbb{Q}\} \subseteq L_0$ is *ascending* (resp. *descending*) if $p < q$ implies $a_p \le a_q$ (resp. $a_p \ge a_q$).

An ascending family $\mathcal{A} = \{a_p \mid p \in \mathbb{Q}\} \subseteq L_1$ is an

- *upper L_1-scale* if $\bigvee_{p\in\mathbb{Q}} a_p = 1$ and $\bigvee_{p\in\mathbb{Q}} \mathfrak{o}_0(a_p) = 1$,
- *L_1-scale* if $\bigvee_{p\in\mathbb{Q}} a_p = 1 = \bigvee_{p\in\mathbb{Q}} a_p^\bullet$ and $a_p^\bullet \vee a_q = 1$ whenever $p < q$,

and, dually, a descending family $\mathcal{B} = \{b_p \mid p \in \mathbb{Q}\} \subseteq L_2$ is an

- *lower L_2-scale* if $\bigvee_{p\in\mathbb{Q}} b_p = 1$ and $\bigvee_{p\in\mathbb{Q}} \mathfrak{o}_0(b_p) = 1$,
- *L_2-scale* if $\bigvee_{p\in\mathbb{Q}} b_p = 1 = \bigvee_{p\in\mathbb{Q}} b_p^\bullet$ and $b_p \vee b_q^\bullet = 1$ whenever $p < q$.

Lemma 3.1. (1) *Let $\mathcal{B} = \{b_p \mid p \in \mathbb{Q}\}$ be a lower L_2-scale. The formulas*

$$f(p, -) = \bigvee_{r>p} \mathfrak{c}_0(b_r) \quad \textit{and} \quad f(-, q) = \bigvee_{s<q} \mathfrak{o}_0(b_s)$$

determine an L_2-lsc real function on (L_0, L_1, L_2).

(2) *Let $\mathcal{A} = \{a_p \mid p \in \mathbb{Q}\}$ be an upper L_1-scale. The formulas*

$$g(p, -) = \bigvee_{r>p} \mathfrak{o}_0(a_r) \quad \textit{and} \quad g(-, q) = \bigvee_{s<q} \mathfrak{c}_0(a_s)$$

determine an L_1-usc real function on (L_0, L_1, L_2).

Proof: (1) It suffices to show that f is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(L_0)$, since the rest is obvious by the definition of f. For that, we have to check that the map f turns the defining relations (R1)-(R6) of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathcal{S}_0(L_0)$:

(R1) If $p \geq q$ then $f(p,-) \wedge f(-,q) \leq \mathfrak{c}_0(b_p) \wedge \mathfrak{o}_0(b_q) \leq \mathfrak{c}_0(b_q) \wedge \mathfrak{o}_0(b_q) = 0$.

(R2) If $p < r < s < q$ then $f(p,-) \vee f(-,q) \geq \mathfrak{c}_0(b_r) \vee \mathfrak{o}_0(b_s) \geq \mathfrak{c}_0(b_s) \vee \mathfrak{o}_0(b_s) = 1$.

(R3) $\bigvee_{r>p} f(r,-) = \bigvee_{r>p} \bigvee_{s>r} \mathfrak{c}_0(b_s) = \bigvee_{s>p} \mathfrak{c}_0(b_s) = f(p,-)$.

(R4) $\bigvee_{s<q} f(-,s) = \bigvee_{s<q} \bigvee_{r<s} \mathfrak{o}_0(b_r) = \bigvee_{r<q} \mathfrak{o}_0(b_r) = f(-,q)$.

(R5) $\bigvee_{p\in\mathbb{Q}} f(p,-) = \bigvee_{p\in\mathbb{Q}} \bigvee_{r>p} \mathfrak{c}_0(b_r) = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}_0(b_r) = \mathfrak{c}_0(\bigvee_{r\in\mathbb{Q}} b_r) = \mathfrak{c}_0(1) = 1$.

(R6) $\bigvee_{q\in\mathbb{Q}} f(-,q) = \bigvee_{q\in\mathbb{Q}} \bigvee_{s<q} \mathfrak{o}(b_s) = \bigvee_{q\in\mathbb{Q}} \mathfrak{o}(b_q) = 1$.

(2) Similar to (1). ■

Lemma 3.2. *Let $\mathcal{B} = \{b_p \mid p \in \mathbb{Q}\}$ be an L_2-scale. The formulas*

$$h(p,-) = \bigvee_{r>p} \mathfrak{c}_0(b_r) \quad \text{and} \quad h(-,q) = \bigvee_{s<q} \mathfrak{c}_0(b_s^\bullet)$$

determine an L_1-usc and L_2-lsc real function on (L_0, L_1, L_2).

Proof: It suffices to show that h is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(L_0)$, since the rest is obvious by the definition of h. As before, it has to be checked that h turns the defining relations (R1)-(R6) of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathcal{S}_0(L_0)$:

(R1) If $p \geq q$ then $h(p,-) \wedge h(-,q) \leq \mathfrak{c}_0(b_p) \wedge \mathfrak{c}_0(b_q^\bullet) \leq \mathfrak{c}_0(b_q) \wedge \mathfrak{c}_0(b_q^\bullet) = 0$.

(R2) If $p < r < s < q$ then $h(p,-) \vee h(-,q) \geq \mathfrak{c}_0(b_r) \vee \mathfrak{c}_0(b_s^\bullet) = \mathfrak{c}_0(b_r \vee b_s^\bullet) = \mathfrak{c}_0(1) = 1$.

(R3) $\bigvee_{r>p} h(r,-) = \bigvee_{r>p} \bigvee_{s>r} \mathfrak{c}_0(b_s) = \bigvee_{s>p} \mathfrak{c}_0(b_s) = h(p,-)$.

(R5) $\bigvee_{p\in\mathbb{Q}} h(p,-) = \bigvee_{p\in\mathbb{Q}} \bigvee_{r>p} \mathfrak{c}_0(b_r) = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}_0(b_r) = \mathfrak{c}_0(\bigvee_{r\in\mathbb{Q}} b_r) = \mathfrak{c}_0(1) = 1$.

(R4) and (R6) may be proved analogously to (R3) and (R5), respectively. ■

4. Katětov-Tong type insertion

Lemma 4.1. *A biframe (L_0, L_1, L_2) is normal if and only if for any countable sets $\{a_k\}_{k\in\mathbb{N}} \subseteq L_1$ and $\{b_k\}_{k\in\mathbb{N}} \subseteq L_2$ satisfying $\bigwedge_{k\in\mathbb{N}} a_k \in L_1$, $\bigwedge_{k\in\mathbb{N}} b_k \in L_2$, $a_k \vee (\bigwedge_{\ell\in\mathbb{N}} b_\ell) = 1 = b_k \vee (\bigwedge_{\ell\in\mathbb{N}} a_\ell)$ for all $k \in \mathbb{N}$, there exists $u \in L_2$ such that $a_k \vee u = 1 = b_k \vee u^\bullet$ for every $k \in \mathbb{N}$.*

Proof: Let (L_0, L_1, L_2) be a normal biframe. Then, for each $k \in \mathbb{N}$, $a_k \vee (\bigwedge_{\ell\in\mathbb{N}} b_\ell) = 1$ implies, by normality, the existence of $v_k \in L_2$ satisfying $a_k \vee v_k = 1$ and $(\bigwedge_{\ell\in\mathbb{N}} b_\ell) \vee v_k^\bullet = 1$. Similarly, $b_k \vee (\bigwedge_{\ell\in\mathbb{N}} a_\ell) = 1$ implies the existence of $w_k \in L_1$ such that $b_k \vee w_k = 1$ and $(\bigwedge_{\ell\in\mathbb{N}} a_\ell) \vee w_k^\bullet = 1$. Then, for each $k \in \mathbb{N}$, we have

$$a_k \vee \Big(\bigwedge_{m=1}^{k} w_m^\bullet\Big) \geq \Big(\bigwedge_{\ell\in\mathbb{N}} a_\ell\Big) \vee \Big(\bigwedge_{m=1}^{k} w_m^\bullet\Big) = \bigwedge_{m=1}^{k}\Big(\big(\bigwedge_{\ell\in\mathbb{N}} a_\ell\big) \vee w_m^\bullet\Big) = 1$$

and, similarly,

$$b_k \vee \Big(\bigwedge_{m=1}^{k} v_m^\bullet\Big) \geq \Big(\bigwedge_{\ell\in\mathbb{N}} b_\ell\Big) \vee \Big(\bigwedge_{m=1}^{k} v_m^\bullet\Big) = \bigwedge_{m=1}^{k}\Big(\big(\bigwedge_{\ell\in\mathbb{N}} b_\ell\big) \vee v_m^\bullet\Big) = 1.$$

Now define, for each $k \in \mathbb{N}$, $u_k := v_k \wedge \bigwedge_{m=1}^{k} w_m^\bullet \in L_2$ and $z_k := w_k \wedge \bigwedge_{m=1}^{k} v_m^\bullet \in L_1$ and consider $u := \bigvee_{k\in\mathbb{N}} u_k \in L_2$ and $z := \bigvee_{k\in\mathbb{N}} z_k \in L_1$. Evidently,

$$a_k \vee u \geq a_k \vee u_k = (a_k \vee v_k) \wedge \Big(a_k \vee \big(\bigwedge_{m=1}^{k} w_m^\bullet\big)\Big) = 1$$

and

$$b_k \vee z \geq b_k \vee z_k = (b_k \vee w_k) \wedge \Big(b_k \vee \big(\bigwedge_{m=1}^{k} v_m^\bullet\big)\Big) = 1.$$

Furthermore,

$$u \wedge z = \bigvee_{k\in\mathbb{N}}\bigvee_{\ell\in\mathbb{N}} (u_k \wedge z_\ell) = \bigvee_{k\in\mathbb{N}}\bigvee_{\ell\in\mathbb{N}} \Big(v_k \wedge w_\ell \wedge \bigwedge_{m=1}^{k} w_m^\bullet \wedge \bigwedge_{m=1}^{l} v_m^\bullet\Big) = 0.$$

Thus $z \leq u^\bullet$ and, consequently, $b_k \vee u^\bullet \geq b_k \vee z = 1$.

The converse is easy: for each $a \in L_1$ and $b \in L_2$ with $a \vee b = 1$, it suffices to apply the hypothesis to the one-element families $\{a\}$ and $\{b\}$. ∎

Let $\{\alpha_k \mid k \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Then:

Lemma 4.2. *Let f and g be real functions on a normal biframe (L_0, L_1, L_2). If g is L_1-usc, f is L_2-lsc and $g \leq f$ then there exists $\{u_{\alpha_k}\}_{k\in\mathbb{N}} \subseteq L_2$ such that*

$$(q > \alpha_k) \;\Rightarrow\; (g(-, q) \vee \mathfrak{c}_0(u_{\alpha_k}) = 1), \tag{4.2.1}$$
$$(p < \alpha_k) \;\Rightarrow\; (f(p, -) \vee \mathfrak{c}_0(u_{\alpha_k}^\bullet) = 1), \tag{4.2.2}$$
$$(\alpha_{k_1} < \alpha_{k_2}) \;\Rightarrow\; (u_{\alpha_{k_1}} \vee u_{\alpha_{k_2}}^\bullet = 1). \tag{4.2.3}$$

Proof: We prove this by showing, using induction over $\mathbb{N}$, that, for every $i \in \mathbb{N}$, there exists $\{u_{\alpha_k}\}_{k\leq i} \subseteq L_2$ such that

$$\begin{aligned} (q > \alpha_k) \ &\Rightarrow \ (g(-,q) \vee \mathfrak{c}_0(u_{\alpha_k}) = 1), \\ (p < \alpha_k) \ &\Rightarrow \ (f(p,-) \vee \mathfrak{c}_0(u^\bullet_{\alpha_k}) = 1), \\ (\alpha_{k_1} < \alpha_{k_2}) \ &\Rightarrow \ (u_{\alpha_{k_1}} \vee u^\bullet_{\alpha_{k_2}} = 1), \ \text{for all } k_1, k_2 \leq i. \end{aligned}$$

Since $g \leq f$, we have

$$\begin{aligned} {\mathfrak{c}_0}^{-1}(g(-,q)) \vee \big(\textstyle\bigwedge_{p<\alpha_1} {\mathfrak{c}_0}^{-1}(f(p,-))\big) \ &\geq \ {\mathfrak{c}_0}^{-1}(g(-,q)) \vee {\mathfrak{c}_0}^{-1}(f(\alpha_1,-)) \\ &= \ {\mathfrak{c}_0}^{-1}(g(-,q) \vee f(\alpha_1,-)) \\ &\geq \ {\mathfrak{c}_0}^{-1}(g(-,q) \vee g(\alpha_1,-)) = 1 \end{aligned}$$

for every $q > \alpha_1$. Similarly, $(\bigwedge_{q>\alpha_1} {\mathfrak{c}_0}^{-1}(g(-,q))) \vee {\mathfrak{c}_0}^{-1}(f(p,-)) = 1$ for every $p < \alpha_1$. Then, by Lemma 4.1, there exists $u_{\alpha_1} \in L_2$ satisfying ${\mathfrak{c}_0}^{-1}(g(-,q)) \vee u_{\alpha_1} = 1$, for every $q > \alpha_1$, and ${\mathfrak{c}_0}^{-1}(f(p,-)) \vee u^\bullet_{\alpha_1} = 1$, for every $p < \alpha_1$, which shows the first step of the induction.

Now, consider $i \in \mathbb{N}$, and assume, by inductive hypothesis, that for any $k < i$ there is $u_{\alpha_k} \in L_2$ satisfying $g(-,q) \vee \mathfrak{c}_0(u_{\alpha_k}) = 1$, for every $q > \alpha_k$, $f(p,-) \vee \mathfrak{c}_0(u^\bullet_{\alpha_k}) = 1$, for every $p < \alpha_k$, and

$$(\alpha_{k_1} < \alpha_{k_2}) \Rightarrow (u_{\alpha_{k_1}} \vee u^\bullet_{\alpha_{k_2}} = 1), \ \text{for all } k_1, k_2 \leq i-1.$$

Further, define

$$\begin{aligned} \{a_\ell\}_{\ell\in\mathbb{N}} &:= \{{\mathfrak{c}_0}^{-1}(g(-,q)) \mid q > \alpha_i\} \cup \{u^\bullet_{\alpha_k} \mid k < i, \alpha_i < \alpha_k\}, \\ \{b_\ell\}_{\ell\in\mathbb{N}} &:= \{{\mathfrak{c}_0}^{-1}(f(p,-)) \mid p < \alpha_i\} \cup \{u_{\alpha_k} \mid k < i, \alpha_k < \alpha_i\}; \end{aligned}$$

Then $\{a_\ell\}_{\ell\in\mathbb{N}}$ and $\{b_\ell\}_{\ell\in\mathbb{N}}$ satisfy the conditions of Lemma 4.1:

(1) For each $q > \alpha_i$,

$${\mathfrak{c}_0}^{-1}(g(-,q)) \vee \Big(\bigwedge_{p<\alpha_i} {\mathfrak{c}_0}^{-1}(f(p,-)) \wedge \bigwedge_{k<i,\, \alpha_k<\alpha_i} u_{\alpha_k}\Big) = 1$$

since ${\mathfrak{c}_0}^{-1}(g(-,q)) \vee \big(\bigwedge_{p<\alpha_i} {\mathfrak{c}_0}^{-1}(f(p,-))\big) = 1$ (again because $g \leq f$) and, by inductive hypothesis,

$${\mathfrak{c}_0}^{-1}(g(-,q)) \vee \big(\bigwedge_{k<i,\, \alpha_k<\alpha_i} u_{\alpha_k}\big) = \bigwedge_{k<i,\, \alpha_k<\alpha_i} ({\mathfrak{c}_0}^{-1}(g(-,q)) \vee u_{\alpha_k}) = 1.$$

(2) For each $k < i$ such that $\alpha_i < \alpha_k$,

$$u^\bullet_{\alpha_k} \vee \Big(\bigwedge_{p<\alpha_i} {\mathfrak{c}_0}^{-1}(f(p,-)) \wedge \bigwedge_{m<i,\, \alpha_m<\alpha_i} u_{\alpha_m}\Big) = 1.$$

since $u^\bullet_{\alpha_k} \vee \left(\bigwedge_{p<\alpha_i} \mathfrak{c}_0{}^{-1}(f(p,-))\right) \geq u^\bullet_{\alpha_k} \vee \mathfrak{c}_0{}^{-1}(f(\alpha_i,-)) = 1$ and, by inductive hypothesis,

$$u^\bullet_{\alpha_k} \vee \Big(\bigwedge_{m<i,\,\alpha_m<\alpha_i} u_{\alpha_m}\Big) = \bigwedge_{m<i,\,\alpha_m<\alpha_i} (u^\bullet_{\alpha_k} \vee u_{\alpha_m}) = 1.$$

(3) Similarly to (1) and (2), respectively, one can prove that, for each $p < \alpha_i$,

$$\Big(\bigwedge_{q>\alpha_i} \mathfrak{c}_0{}^{-1}(g(-,q)) \wedge \bigwedge_{k<i,\,\alpha_i<\alpha_k} u^\bullet_{\alpha_k}\Big) \vee \mathfrak{c}_0{}^{-1}(f(p,-)) = 1,$$

and that, for each $k < i$ satisfying $\alpha_k < \alpha_i$,

$$\Big(\bigwedge_{q>\alpha_i} \mathfrak{c}_0{}^{-1}(g(-,q)) \wedge \bigwedge_{k<i,\,\alpha_i<\alpha_k} u^\bullet_{\alpha_k}\Big) \vee u_{\alpha_k} = 1.$$

So, it follows from Lemma 4.1 that there exists $u_{\alpha_i} \in L_2$ such that

$$\begin{aligned}
(q > \alpha_i) &\;\Rightarrow\; (\mathfrak{c}_0{}^{-1}(g(-,q)) \vee u_{\alpha_i} = 1),\\
(\alpha_i < \alpha_k) &\;\Rightarrow\; (u^\bullet_{\alpha_k} \vee u_{\alpha_i} = 1), \quad \text{for all } k < i,\\
(p < \alpha_i) &\;\Rightarrow\; (\mathfrak{c}_0{}^{-1}(f(p,-)) \vee u^\bullet_{\alpha_i} = 1),\\
(\alpha_k < \alpha_i) &\;\Rightarrow\; (u_{\alpha_k} \vee u^\bullet_{\alpha_i} = 1), \quad \text{for all } k < i.
\end{aligned}$$

This, together with the inductive hypothesis, gives us the required $\{u_{\alpha_k}\}_{k\leq i} \subseteq L_2$. ■

Theorem 4.3. *Let f and g be real functions on a biframe (L_0, L_1, L_2) such that g is L_1-usc, f is L_2-lsc and $g \leq f$. Then there exists an L_1-usc and L_2-lsc real function h on L satisfying $g \leq h \leq f$, for every such pair f, g, if and only if (L_0, L_1, L_2) is normal.*

Proof: $\Rightarrow$: Let $a \vee b = 1$ with $a \in L_1$ and $b \in L_2$. Then $\chi_{\mathfrak{c}_0(a)}$ is L_1-usc, $\chi_{\mathfrak{o}_0(b)}$ is L_2-lsc and $\chi_{\mathfrak{c}_0(a)} \leq \chi_{\mathfrak{o}_0(b)}$. Thus, by hypothesis, there exists an L_1-usc and L_2-lsc h such that $\chi_{\mathfrak{c}_0(a)} \leq h \leq \chi_{\mathfrak{o}_0(b)}$. By Proposition 2.1, $\mathfrak{c}_0{}^{-1} \circ h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ is a biframe homomorphism. Consider $(\mathfrak{c}_0{}^{-1} \circ h)(-, \frac{1}{2}) = v \in L_1$ and $(\mathfrak{c}_0{}^{-1} \circ h)(\frac{1}{2}, -) = u \in L_2$. Then $\mathfrak{c}_0(u) \wedge \mathfrak{c}_0(v) = h(0) = 0$, $\mathfrak{c}_0(a) \vee \mathfrak{c}_0(u) = \chi_{\mathfrak{c}_0(a)}(-, \frac{3}{4}) \vee h(\frac{1}{2}, -) = 1$ and $\mathfrak{c}_0(b) \vee \mathfrak{c}_0(v) = \chi_{\mathfrak{o}_0(b)}(\frac{1}{4}, -) \vee h(-, \frac{1}{2}) = 1$, which means that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$. This shows the normality of (L_0, L_1, L_2).

$\Leftarrow$: Let g be L_1-usc and let f be L_2-lsc such that $g \leq f$. By Lemma 4.2 there exists $\{u_{\alpha_k}\}_{k\in\mathbb{N}} \subseteq L_2$ satisfying conditions (4.2.1), (4.2.2) and (4.2.3).

Let $q \in \mathbb{Q}$. Then there exists $\alpha_q \in \mathbb{Q}$ such that $\alpha_q > q$ and it follows that from (4.2.1) that $g(q,-) \leq g(-,q)^* \leq \mathfrak{c}_0(u_{\alpha_k})$. Consequently,

$1 = \bigvee_{q\in\mathbb{Q}} g(q,-) \leq \bigvee_{\alpha_k\in\mathbb{Q}} \mathfrak{c}_0(u_{\alpha_k})$. Similarly, from (4.2.2) it follows that $1 = \bigvee_{q\in\mathbb{Q}} f(-,p) \leq \bigvee_{\alpha_k\in\mathbb{Q}} \mathfrak{c}_0(u_{\alpha_k^\bullet})$. Together with (4.2.3), this means that $\{u_{\alpha_k}\}_{k\in\mathbb{N}}$ is an L_2-scale thus, by Lemma 3.2, the formulas

$$h(p,-) := \bigvee_{\alpha_k>p} \mathfrak{c}_0(u_{\alpha_k}) \quad \text{and} \quad h(-,q) := \bigvee_{\alpha_k<q} \mathfrak{c}_0(u^\bullet_{\alpha_k})$$

establish an L_1-usc and L_2-lsc real function h on (L_0, L_1, L_2). Moreover, $g \leq h \leq f$:

(a) Condition (4.2.1) means that $g(-,q) \geq \mathfrak{o}_0(u_{\alpha_k}) \geq \mathfrak{c}_0(u^\bullet_{\alpha_k})$ for any $q \in \mathbb{Q}$ and each $\alpha_k < q$. Consequently,

$$g(-,q) \geq \bigvee_{\alpha_k<q} \mathfrak{c}_0(u^\bullet_{\alpha_k}) = h(-,q).$$

(b) Similarly, by (4.2.2), $f(p,-) \geq \mathfrak{o}_0(u^\bullet_{\alpha_k}) \geq \mathfrak{c}_0(u_{\alpha_k})$ whenever $p \in \mathbb{Q}$ and $\alpha_k > p$. Hence,

$$f(p,-) \geq \bigvee_{\alpha_k>p} \mathfrak{c}_0(u_{\alpha_k}) = h(p,-). \qquad \blacksquare$$

5. Consequences

5.1. Frames. The particular case $L_0 = L_1 = L_2$ in Theorem 4.3 gives us immediately the Katětov-Tong-type insertion theorem for frames ([9], [18]).

5.2. Bitopological spaces. Applied to

$$(\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$$

for a bitopological space $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ [12] one gets the following insertion theorem of Priestley ([21], Theorem 5):

Theorem. *Let f and g be real functions on a bitopological space $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ such that g is $\mathfrak{T}_1$-usc, f is $\mathfrak{T}_2$-lsc and $g \leq f$. Then there exists a $\mathfrak{T}_1$-usc and $\mathfrak{T}_2$-lsc real function h on X satisfying $g \leq h \leq f$, for every such pair f, g, if and only if $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ is normal.*

Proof: We verify the non-trivial implication. Let $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ be a normal bitopological space ([12]) and let $f : X \to \mathbb{R}$ be a $\mathfrak{T}_2$-lsc real function and $g : X \to \mathbb{R}$ a $\mathfrak{T}_1$-usc real function such that $g \leq f$. Note that

$$\{f^{-1}(]p, +\infty[) : p \in \mathbb{Q}\}$$

is a lower $\mathfrak{T}_2$-scale in the biframe $(\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$ (the only non-trivial fact to be proved is that $\bigvee_{p\in\mathbb{Q}} \mathfrak{o}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]p,+\infty[)) = 1$ which is an immediate consequence of the fact that, for every $p > q$, $\mathfrak{o}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]p,+\infty[)) \geq \mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]-\infty,q[)))$. Similarly,

$$\{g^{-1}(]-\infty,q[) : q \in \mathbb{Q}\}$$

is an upper $\mathfrak{T}_1$-scale. Thus, applying Lemma 3.1, we have a $\mathfrak{T}_2$-lsc function $\widetilde{f} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(\mathfrak{T}_1 \vee \mathfrak{T}_2)$ given by

$$\widetilde{f}(p,-) = \bigvee_{r>p} \mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]r,+\infty[)) \quad \text{and} \quad \widetilde{f}(-,q) = \bigvee_{s<q} \mathfrak{o}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]s,+\infty[))$$

and a $\mathfrak{T}_1$-usc function $\widetilde{g} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(\mathfrak{T}_1 \vee \mathfrak{T}_2)$ defined by

$$\widetilde{g}(p,-) = \bigvee_{r>p} \mathfrak{o}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(g^{-1}(]-\infty,r[)) \quad \text{and} \quad \widetilde{g}(-,q) = \bigvee_{s<q} \mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(g^{-1}(]-\infty,s[)).$$

Finally, we have $\widetilde{g} \leq \widetilde{f}$, that is,

$$\bigvee_{r>p} \mathfrak{o}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(g^{-1}(]-\infty,r[)) \leq \bigvee_{r>p} \mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]r,+\infty[)).$$

Indeed: let $r > p$ and take s such that $p < s < r$; then $g^{-1}(]-\infty,r[) \vee g^{-1}(]s,+\infty[) = X$ so $\mathfrak{o}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(g^{-1}(]-\infty,r[)) \leq \mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(g^{-1}(]s,+\infty[))$; finally, since $g \leq f$, then $g^{-1}(]s,+\infty[) \leq f^{-1}(]s,+\infty[)$ and therefore

$$\mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(g^{-1}(]s,+\infty[)) \leq \mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2}(f^{-1}(]s,+\infty[)).$$

Since $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ is normal, the corresponding biframe $(\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$ is normal and, by Theorem 4.3, there exists an L_1-usc and L_2-lsc $\widetilde{h}$ such that $\widetilde{g} \leq \widetilde{h} \leq \widetilde{f}$. By Proposition 2.1, $\widetilde{h} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}_0(\mathfrak{T}_1 \vee \mathfrak{T}_2)$ is continuous in $\mathfrak{T}_1 \vee \mathfrak{T}_2$ and

$$(\mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2})^{-1} \circ \widetilde{h} : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$$

is a biframe homomorphism. By Proposition 4 of [1] there is a continuous function $h : (X, \mathfrak{T}_1 \vee \mathfrak{T}_2) \to \mathbb{R}$ satisfying, for each $p, q \in \mathbb{Q}$,

$$p < h(x) < q \quad \Leftrightarrow \quad x \in ((\mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2})^{-1} \circ \widetilde{h})(p,q).$$

We have immediately that $g \leq h \leq f$, h is $\mathfrak{T}_1$-usc and $\mathfrak{T}_2$-lsc (since $h^{-1}(]-\infty,q[) \subseteq ((\mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2})^{-1} \circ \widetilde{h})(-,q) \in \mathfrak{T}_1$ and $h^{-1}(]p,+\infty[) \subseteq ((\mathfrak{c}_{\mathfrak{T}_1\vee\mathfrak{T}_2})^{-1} \circ \widetilde{h})(p,-) \in \mathfrak{T}_2$). ∎

Remark. Similarly, the monotone version of Theorem 4.3 (following and extending the lines of [5]) would give us Theorem 1 of [16].

5.3. Ordered topological spaces. An ordered topological space $(X, \mathfrak{T}, \leq)$ is:

(1) *normal* (normally ordered in [17, 21]) if, given disjoint closed subsets F_1, F_2 of X with F_1 decreasing and F_2 increasing, there exist disjoint open sets G_1, G_2 such that G_1 is decreasing, G_2 increasing and $F_j \subseteq G_j$, $j = 1, 2$;

(2) a N_i-space [21] if, given disjoint closed subsets F_1, F_2 of X with F_2 increasing, there exist disjoint open sets G_1, G_2 such that G_1 is decreasing, G_2 is increasing and $F_j \subseteq G_j$, $j = 1, 2$;

(3) a N_d-space [21] if, given disjoint closed subsets F_1, F_2 of X with F_1 decreasing, there exist disjoint open sets G_1, G_2 such that G_1 is decreasing, G_2 is increasing and $F_j \subseteq G_j$, $j = 1, 2$;

(4) a N-space [21] if, given closed sets F_1, F_2 satisfying $a_1 \not\geq a_2$ for every $a_1 \in F_1$ and $a_2 \in F_2$, there exist disjoint open sets G_1, G_2 such that G_1 is decreasing, G_2 is increasing and $F_j \subseteq G_j$, $j = 1, 2$.

The sets $\uparrow\mathfrak{T}$ (of open increasing sets) and $\downarrow\mathfrak{T}$ (of open decreasing sets) define two topologies on X and therefore the triples $(X, \uparrow\mathfrak{T}, \downarrow\mathfrak{T})$ and $(X, \downarrow\mathfrak{T}, \uparrow\mathfrak{T})$ are bitopological spaces. The following is obvious:

- $(X, \mathfrak{T}, \leq)$ is normal iff $(X, \uparrow\mathfrak{T}, \downarrow\mathfrak{T})$ is normal iff $(X, \downarrow\mathfrak{T}, \uparrow\mathfrak{T})$ is normal.
- If $(X, \mathfrak{T}, \leq)$ is N_i then $(X, \mathfrak{T}, \downarrow\mathfrak{T})$ and $(X, \downarrow\mathfrak{T}, \mathfrak{T})$ are normal.
- If $(X, \mathfrak{T}, \leq)$ is N_d then $(X, \uparrow\mathfrak{T}, \mathfrak{T})$ and $(X, \mathfrak{T}, \uparrow\mathfrak{T})$ are normal.
- $f : (X, \downarrow\mathfrak{T}) \to \mathbb{R}$ is usc iff $f : (X, \mathfrak{T}) \to \mathbb{R}$ is usc and f is monotone.
- $f : (X, \uparrow\mathfrak{T}) \to \mathbb{R}$ is lsc iff $f : (X, \mathfrak{T}) \to \mathbb{R}$ is lsc and f is monotone.

Furthermore:

($*$) An ordered topological space $(X, \mathfrak{T}, \leq)$ is an N-space iff it is simultaneously an N_i-space and an N_d-space and $\leq$ is a closed partial order [21, Lemma 3].

Using these results one can immediately obtain the following theorems as particular cases of our Theorem 4.3 above:

Theorem 1. *Let f and g be monotone real functions on an ordered topological space $(X, \mathfrak{T}, \leq)$ such that g is usc, f is lsc and $g \leq f$. Then there exists a continuous monotone real function h on X satisfying $g \leq h \leq f$, for every such pair f and g, if and only if $(X, \mathfrak{T}, \leq)$ is normal.*

Proof: Use the properties above and apply Theorem 4.3 to the biframe $({\downarrow}\mathfrak{T} \vee {\uparrow}\mathfrak{T}, {\downarrow}\mathfrak{T}, {\uparrow}\mathfrak{T})$. ■

Theorem 2. *Let f and g be real functions on an ordered topological space $(X, \mathfrak{T}, \leq)$ such that g is usc, f is lsc, $g \leq f$ and f is monotone. Then there exists a continuous monotone real function h on X satisfying $g \leq h \leq f$, for every such pair f and g, if and only if $(X, \mathfrak{T}, \leq)$ is N_i.*

Proof: Use the properties above and apply Theorem 4.3 to the biframe $(\mathfrak{T} \vee {\uparrow}\mathfrak{T}, \mathfrak{T}, {\uparrow}\mathfrak{T})$. ■

Theorem 3. *Let f and g be real functions on an ordered topological space $(X, \mathfrak{T}, \leq)$ such that g is usc, f is lsc, $g \leq f$ and g is monotone. Then there exists a continuous monotone real function h on X satisfying $g \leq h \leq f$, for every such pair f and g, if and only if $(X, \mathfrak{T}, \leq)$ is N_d.*

Proof: Use the properties above and apply Theorem 4.3 to the biframe $({\downarrow}\mathfrak{T} \vee \mathfrak{T}, {\downarrow}\mathfrak{T}, \mathfrak{T})$. ■

Then, immediately, using property $(*)$, Theorems 2 and 3 produce:

Corollary 4. *Let f and g be monotone real functions on an ordered topological space $(X, \mathfrak{T}, \leq)$ such that g is usc, f is lsc and $g \leq f$. If $(X, \mathfrak{T}, \leq)$ is an N-space then there exists a continuous monotone real function h on X satisfying $g \leq h \leq f$. The converse is true if $\leq$ is closed.* ■

Remark. Theorems 1 and 2 are, respectively, Theorems 1 and 2 of Priestley [21]. Note that, as in the bitopological case (subsection 5.2 above), the former provides as a corollary Nachbin's extension to normal ordered topological spaces of Urysohn's Lemma, with g, f characteristic functions of appropriate sets ([17], p. 30).

5.4. Urysohn-type lemma. Let (L_0, L_1, L_2) be a normal biframe, $a \vee b = 1$ with $a \in L_1$ and $b \in L_2$. Then $\chi_{\mathfrak{c}_0(a)}$ is L_1-usc, $\chi_{\mathfrak{o}_0(b)}$ is L_2-lsc and $\chi_{\mathfrak{c}_0(a)} \leq \chi_{\mathfrak{o}_0(b)}$. Therefore there exists an L_1-usc and L_2-lsc function h such that $\chi_{\mathfrak{c}_0(a)} \leq h \leq \chi_{\mathfrak{o}_0(b)}$. By Proposition 2.1,

$$\mathfrak{c}_0{}^{-1} \circ h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$$

is a biframe homomorphism. Further, $\chi_{\mathfrak{c}_0(a)} \leq h$ means that $\chi_{\mathfrak{c}_0(a)}(-, q) \leq h(-, q)$ for every $0 < q \leq 1$, that is, $h(-, 1) \leq \mathfrak{c}_0(a)$. Similarly, $h \leq \chi_{\mathfrak{o}_0(b)}$ implies $h(0, -) \leq \mathfrak{c}_0(b)$. Hence $(\mathfrak{c}_0{}^{-1} \circ h)(-, 1) \leq a$ and $(\mathfrak{c}_0{}^{-1} \circ h)(0, -) \leq b$. We have just obtained the Urysohn's Lemma for normal biframes of Schauerte [23, Proposition 6.2]:

Proposition. *A biframe* (L_0, L_1, L_2) *is normal if and only if, whenever* $a \vee b = 1$ *for* $a \in L_1$ *and* $b \in L_2$*, there exists a biframe homomorphism* $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ *such that* $h(-,1) \leq a$ *and* $h(0,-) \leq b$. ■

5.5. Tietze-type extension theorem. Given a sub-bilocale (S_0, S_1, S_2) of (L_0, L_1, L_2), we say that an S_1-usc and S_2-lsc real function f on (S_0, S_1, S_2) has a *continuous extension* to (L_0, L_1, L_2) if there exists an L_1-usc and L_2-lsc real function $\tilde{f}$ on (L_0, L_1, L_2) such that $\tilde{f}(p,q) \vee S_0 = f(p,q)$ for every $p, q \in \mathbb{Q}$. This means precisely that the diagram

$$\begin{tikzcd}
 & (\mathcal{S}_0(L_0), \mathcal{S}_1(L_1), \mathcal{S}_2(L_2)) \arrow[d, "p_S"] \\
(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (\mathcal{S}_0(S_0), \mathcal{S}_1(S_1), \mathcal{S}_2(S_2))
\end{tikzcd}$$

(with p_S defined by $p_S(T) = T \vee S_0$) is commutative.

Then we get what appears to be a new result for biframes (the Tietze-type extension theorem for normal biframes) that generalizes at once the results for frames [8], bitopological spaces [12] and ordered topological spaces [17, 21] (the details of the proof are rather technical and will appear elsewhere):

Theorem. *A biframe* (L_0, L_1, L_2) *is normal if and only if for every closed sub-bilocale* (S_0, S_1, S_2)*, each* S_1*-usc and* S_2*-lsc real function on* (S_0, S_1, S_2) *has a continuous extension to* (L_0, L_1, L_2).

6. Stone-type insertion

We say that a biframe (L_0, L_1, L_2) is *extremally disconnected* (cf. [13] for the corresponding bitopological notion) if whenever $a \wedge b = 0$ for some $a \in L_1$ and $b \in L_2$ there exist $u \in L_2$ and $v \in L_1$ such that $u \vee v = 1$ and $a \wedge u = 0 = b \wedge v$.

Remark 6.1. Note that a biframe is extremally disconnected if and only if $a^{\bullet\bullet} \vee a^{\bullet} = 1$ for every $a \in L_1$ (or, equivalently, if $b^{\bullet} \vee b^{\bullet\bullet} = 1$ for every $b \in L_2$). Indeed:

If (L_0, L_1, L_2) is extremally disconnected then, since $a^{\bullet\bullet} \wedge a^{\bullet} = 0$, there exist $u \in L_2$ and $v \in L_1$ such that $u \vee v = 1$ and $a^{\bullet\bullet} \wedge u = 0 = a^{\bullet} \wedge v$. Evidently, $1 = u \vee v \leq a^{\bullet} \vee a^{\bullet\bullet}$. Conversely, let $a \wedge b = 0$ with $a \in L_1$ and $b \in L_2$. Then $b \leq a^{\bullet}$. Take $u = a^{\bullet} \in L_2$ and $v = a^{\bullet\bullet} \in L_1$. Clearly, $u \vee v = 1$, $a \wedge u = 0$ and $b \wedge v = b \wedge a^{\bullet\bullet} \leq a^{\bullet} \wedge a^{\bullet\bullet} = 0$.

Given a biframe (L_0, L_1, L_2) and $S \in \mathcal{S}_0(L_0)$, we denote by $\mathrm{cl}_i(S)$ $(i = 1, 2)$ the *closure* of S in $\mathcal{S}_i(L_i)$, that is,

$$\mathrm{cl}_i(S) = \bigvee \{\mathfrak{c}_0(a) \mid a \in L_i, \mathfrak{c}_0(a) \leq S\}.$$

Note that $\mathrm{cl}_i(\mathfrak{o}_0(a)) = \mathfrak{c}_0(a^{\bullet})$ for each $a \in L_i$ $(i = 1, 2)$.

We need now to extend the lower and upper regularizations [6, 8] of frame real functions to biframe real functions. Let

$$f : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}_0(L_0), \mathcal{S}_1(L_1), \mathcal{S}_2(L_2)).$$

The *lower regularization* f° of f is defined by:

$$f^{\circ}(p,\text{—}) = \bigvee_{r>p} \mathrm{cl}_2(f(r,\text{—})) \quad \text{and} \quad f^{\circ}(\text{—}, q) = \bigvee_{s<q} \big(\mathrm{cl}_2(f(s,\text{—}))\big)^{*}.$$

Dually, the *upper regularization* f^{-} of f is defined by

$$f^{-}(p,\text{—}) = \bigvee_{r>p} \big(\mathrm{cl}_1(f(\text{—}, r))\big)^{*} \quad \text{and} \quad f^{-}(\text{—}, q) = \bigvee_{s<q} \mathrm{cl}_1(f(\text{—}, s)).$$

Properties 6.2. *The following hold for all real functions f, g in (L_0, L_1, L_2):*

(1) *$f^{\circ} \leq f$.*
(2) *f° is L_2-lsc.*
(3) *If g is L_2-lsc and $g \leq f$ then $g \leq f^{\circ}$.*
(4) *$f \leq f^{-}$.*
(5) *f^{-} is L_1-usc.*
(6) *If g is L_1-usc and $f \leq g$ then $f^{-} \leq g$.*

Proof: (1) For each $p \in \mathbb{Q}$, $f^{\circ}(p,\text{—}) = \bigvee_{r>p} \mathrm{cl}_2(f(r,\text{—})) \leq \bigvee_{r>p} f(r,\text{—}) = f(p, -)$.

(2) It is obvious.

(3) The fact that g is L_2-lsc means that $g(p, -) = \mathfrak{c}_0(b_p)$ for some $b_p \in L_2$. Then $\mathfrak{c}_0(b_p) = g(p, -) \leq f(p, -)$, which implies that $g(p, -) \leq \mathrm{cl}_2(f(p, -))$. Therefore

$$f^{\circ}(p, -) = \bigvee_{r>p} \mathrm{cl}_2(f(r,\text{—})) \geq \bigvee_{r>p} g(r, -) = g(p, -).$$

Properties (4)-(6) can be proved similarly. ■

Lemma 6.3. *Let g be an L_1-usc real function on (L_0, L_1, L_2) and let $g(-, q) = \mathfrak{c}_0(a_q)$ with $a_q \in L_1$. Then:*

(1) $g^\circ(p, -) \geq \bigvee_{r>p} \mathfrak{c}_0(a_r^\bullet)$.
(2) $g^{\circ-}(p, -) \leq \bigvee_{r>p} \mathfrak{o}_0(\bigvee_{s<r} a_s^{\bullet\bullet})$.
(3) *If $a_q^\bullet \vee a_q^{\bullet\bullet} = 1$ for every $q \in \mathbb{Q}$, then $g^{\circ-} = g^\circ$.*

Proof: (1) We need to check that $\bigvee_{r>p} \mathfrak{c}_0(a_r^\bullet) \leq \bigvee_{r>p} \mathrm{cl}_2(g(r, -))$. For each $r > p$ consider t such that $p < t < r$. Then $g(t, -) \vee \mathfrak{c}_0(a_r) = 1$, which implies $g(t, -) \geq \mathfrak{o}_0(a_r) \geq \mathrm{cl}_2(\mathfrak{o}_0(a_r)) = \mathfrak{c}_0(a_r^\bullet)$. Hence $\mathfrak{c}_0(a_r^\bullet) \leq g(t, -)$ and, consequently, $\mathfrak{c}_0(a_r^\bullet) \leq \mathrm{cl}_2(g(t, -))$.
(2) Since $g^{\circ-}(p, -) = \bigvee_{r>p}(\mathrm{cl}_1(g^\circ(-, r)))^*$, we have to show that

$$\bigvee_{r>p} (\mathrm{cl}_1(g^\circ(-, r)))^* \leq \bigvee_{r>p} \mathfrak{o}_0(\bigvee_{s<r} a_s^{\bullet\bullet}).$$

For that it suffices to check that, for any $r > p$, $\mathfrak{c}_0(\bigvee_{s<r} a_s^{\bullet\bullet}) \leq \mathrm{cl}_1(g^\circ(-, r))$. Since $g(s, -) \leq g(-, s)^* = \mathfrak{o}_0(a_s)$, we have $\mathrm{cl}_2(g(s, -)) \leq \mathrm{cl}_2(\mathfrak{o}_0(a_s)) = \mathfrak{c}_0(a_s^\bullet)$, from which it follows that $\mathfrak{c}_0(a_s^{\bullet\bullet}) = \mathrm{cl}_1(\mathfrak{o}(a_s^\bullet)) \leq \mathfrak{o}(a_s^\bullet) \leq \mathrm{cl}_2(g(s, -))^*$. Hence

$$\mathfrak{c}_0(\bigvee_{s<r} a_s^{\bullet\bullet}) = \bigvee_{s<r} \mathfrak{c}_0(a_s^{\bullet\bullet}) \leq \bigvee_{s<r} \mathrm{cl}_2(g(s, -))^* = g^\circ(-, r),$$

which ensures that $\mathfrak{c}_0(\bigvee_{s<r} a_s^{\bullet\bullet}) \leq \mathrm{cl}_1(g^\circ(-, r))$.
(3) By Properties 6.2 (4), it suffices to check that $g^{\circ-} \leq g^\circ$. Using (1) and (2), we only need to check that

$$\bigvee_{r>p} \mathfrak{o}_0(\bigvee_{s<r} a_s^{\bullet\bullet}) \leq \bigvee_{r>p} \mathfrak{c}_0(a_r^\bullet).$$

So, let $r > p$ and take t such that $p < t < r$. Since $a_t^{\bullet\bullet} \vee a_t^\bullet = 1$ we have $\mathfrak{o}_0(a_t^{\bullet\bullet}) \leq \mathfrak{c}_0(a_t^\bullet)$. This means that $\mathfrak{o}_0(\bigvee_{s<r} a_s^{\bullet\bullet}) \leq \mathfrak{o}_0(a_t^{\bullet\bullet}) \leq \mathfrak{c}_0(a_t^\bullet)$ and the required inequality follows. ■

Theorem 6.4. *Let f and g be real functions on a biframe (L_0, L_1, L_2) such that g is L_1-usc, f is L_2-lsc and $f \leq g$. Then there exists an L_1-usc and L_2-lsc real function h on (L_0, L_1, L_2) satisfying $f \leq h \leq g$, for every such pair f, g, if and only if L is extremally disconnected.*

Proof: $\Rightarrow$: Let $a \wedge b = 0$ with $a \in L_1$ and $b \in L_2$. Then $\chi_{\mathfrak{c}_0(a)}$ is L_1-usc, $\chi_{\mathfrak{o}_0(b)}$ is L_2-lsc and $\chi_{\mathfrak{o}_0(b)} \leq \chi_{\mathfrak{c}_0(a)}$. Thus, by hypothesis, there exists an L_1-usc and

L_2-lsc h such that $\chi_{\mathfrak{o}_0(b)} \leq h \leq \chi_{\mathfrak{c}_0(a)}$. By Proposition 2.1,

$$\mathfrak{c}_0{}^{-1} \circ h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$$

is a biframe homomorphism. Consider $(\mathfrak{c}_0{}^{-1} \circ h)(-, \frac{1}{2}) = v \in L_1$ and $(\mathfrak{c}_0{}^{-1} \circ h)(\frac{1}{2}, -) = u \in L_2$. Then $\mathfrak{c}_0(u) \vee \mathfrak{c}_0(v) = h(1) = 1$, $\mathfrak{c}_0(a) \wedge \mathfrak{c}_0(u) = \chi_{\mathfrak{c}_0(a)}(-, \frac{1}{2}) \wedge h(\frac{1}{2}, -) \leq \chi_{\mathfrak{c}_0(a)}(-, \frac{1}{2}) \wedge \chi_{\mathfrak{c}_0(a)}(\frac{1}{2}, -) = 0$ and $\mathfrak{c}_0(b) \wedge \mathfrak{c}_0(v) = \chi_{\mathfrak{o}_0(b)}(\frac{1}{2}, -) \wedge h(-, \frac{1}{2}) \leq h(\frac{1}{2}, -) \wedge h(-, \frac{1}{2}) = 0$, which means that $u \vee v = 1$ and $a \wedge u = 0 = b \wedge v$. This shows the extremal disconnectedness of (L_0, L_1, L_2).

$\Leftarrow$: Let f and g be real functions on a biframe (L_0, L_1, L_2) such that g is L_1-usc, f is L_2-lsc and $f \leq g$. By Properties 6.2, $f \leq g^\circ \leq g$. Since g is L_1-usc, we have $g(-, q) = \mathfrak{c}_0(a_q)$ for some $a_q \in L_1$. The extremal disconnectedness of (L_0, L_1, L_2) implies, by Lemma 6.3(3), that $g^\circ = g^{\circ -}$. We have here the required h (since g° is L_2-lsc and $g^{\circ -}$ is L_1-usc). ■

7. Consequences

7.1. Frames. The particular case $L_0 = L_1 = L_2$ in Theorem 6.4 gives us immediately the Stone-type insertion theorem for frames [14, 6].

7.2. Bitopological spaces. Applied to

$$(\mathfrak{T}_1 \vee \mathfrak{T}_2, \mathfrak{T}_1, \mathfrak{T}_2)$$

for a bitopological space $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ one gets the following insertion theorem for bitopological spaces that appears to be new:

Theorem. *Let f and g be real functions on a bitopological space $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ such that g is $\mathfrak{T}_1$-usc, f is $\mathfrak{T}_2$-lsc and $f \leq g$. Then there exists a $\mathfrak{T}_1$-usc and $\mathfrak{T}_2$-lsc real function h on X satisfying $f \leq h \leq g$, for every such pair f, g, if and only if $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ is extremally disconnected.*

Proof: Similar to the proof of Theorem 5.2. ■

This provides immediately as a corollary (with f, g characteristic functions of appropriate sets) the Urysohn-type lemma for extremally disconnected bitopological spaces of Sarma [22, Theorem 2.4].

7.3. Ordered topological spaces. Given an ordered topological space $(X, \mathfrak{T}, \leq)$ and a subset E of X, let $\mathrm{cl}{\uparrow}(E)$ denote the smallest closed increasing subset of X containing E and let $\mathrm{cl}{\downarrow}(E)$ denote the smallest closed decreasing subset of X containing E. The ordered space $(X, \mathfrak{T}, \leq)$ is *extremally disconnected* (extremally order-disconnected in [3]) if for each increasing open subset G_1 of X the set $\mathrm{cl}{\uparrow}(G_1)$ is open and for each decreasing open subset G_2 of X the set $\mathrm{cl}{\downarrow}(G_2)$ is open. After realizing that

> $(X, \mathfrak{T}, \leq)$ *is extremally disconnected iff* $(X, {\uparrow}\mathfrak{T}, {\downarrow}\mathfrak{T})$ *is extremally disconnected iff* $(X, {\downarrow}\mathfrak{T}, {\uparrow}\mathfrak{T})$ *is extremally disconnected,*

one can obtain immediately the following theorem as a particular case of our Theorem 6.4 above (cf. [3]):

Theorem. *Let f and g be monotone real functions on a ordered topological space $(X, \mathfrak{T}, \leq)$ such that g is usc, f is lsc and $f \leq g$. Then there exists a continuous monotone function h on X satisfying $f \leq h \leq h$, for every such pair f, g, if and only if $(X, \mathfrak{T}, \leq)$ is extremally disconnected.* ■

7.4. Urysohn-type lemma. Let (L_0, L_1, L_2) be a normal biframe, $a \wedge b = 0$ with $a \in L_1$ and $b \in L_2$. Then $\chi_{\mathfrak{c}_0(a)}$ is L_1-usc, $\chi_{\mathfrak{o}_0(b)}$ is L_2-lsc and $\chi_{\mathfrak{o}_0(b)} \leq \chi_{\mathfrak{c}_0(a)}$. Therefore there exists an L_1-usc and L_2-lsc function h such that $\chi_{\mathfrak{o}_0(b)} \leq h \leq \chi_{\mathfrak{c}_0(a)}$. By Proposition 2.1,

$$\mathfrak{c}_0{}^{-1} \circ h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$$

is a biframe homomorphism. Further, $\chi_{\mathfrak{o}_0(b)} \leq h$ means that $\chi_{\mathfrak{o}_0(b)}(-, q) \leq h(-, q)$ for every $0 < q \leq 1$, that is, $\mathfrak{o}_0(b) \leq h(-, 1)$. Similarly, $h \leq \chi_{\mathfrak{c}_0(a)}$ implies $h(0, -) \leq \mathfrak{o}_0(a)$. Since $(\mathfrak{c}_0{}^{-1} \circ h)(-, 1) = u$ for some $u \in L_1$ and $(\mathfrak{c}_0{}^{-1} \circ h)(0, -) = v$ for some $v \in L_2$, we have $\mathfrak{o}_0(b) \leq \mathfrak{c}_0(u)$ and $\mathfrak{c}_0(v) \leq \mathfrak{o}_0(a)$, that is, $b \vee u = 1$ and $a \wedge v = 0$. We have just obtained the following separation-type lemma for extremally disconnected biframes, which appears to be new:

Proposition. *A biframe (L_0, L_1, L_2) is extremally disconnected if and only if whenever $a \wedge b = 0$ for $a \in L_1$ and $b \in L_2$, there exists a biframe homomorphism $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (L_0, L_1, L_2)$ such that $a \wedge h(0, -) = 0$ and $b \vee h(-, 1) = 1$.* ■

7.5. Tietze-type extension theorem. Similarly to 5.5, we can also produce an extension-type result for extremally disconnected biframes:

Theorem. *A biframe* (L_0, L_1, L_2) *is extremally disconnected if and only if for every open sub-bilocale* (S_0, S_1, S_2), *each* S_1*-usc and* S_2*-lsc real function on* (S_0, S_1, S_2) has a continuous extension to (L_0, L_1, L_2).

This is new for biframes and extends at once the results for frames [14, 6], bitopological spaces [22] and ordered topological spaces.

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