Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 09–07

SAMPLING AND INTERPOLATION IN THE BARGMANN-FOCK SPACE OF POLYANALYTIC FUNCTIONS

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ABSTRACT: We give a complete characterization of all lattice sampling and interpolating sequences in the Fock space of polyanalytic functions (poly-Fock spaces), displaying a "Nyquist rate" which increases with n, the degree of polyanaliticity of the space: A sequence of lattice points is sampling if and only if its density is strictly larger than n, and it is interpolating if and only if its density is strictly smaller than n. In our method of proof we introduce a unitary mapping between vector valued Hilbert spaces and poly-Fock spaces which allows the extension of Bargmann´s theory to polyanalytic spaces. Then we connect this mapping to Gabor transforms with Hermite windows and apply duality principles from time-frequency analysis in order to reduce the problem to a "purely holomorphic" situation.

KEYWORDS: time-frequency analysis, polyanalytic functions, Gabor frames and super frames, Bargmann transform, poly-Fock spaces, sampling, density conditions .

1. Introduction

1.1. The Bargmann-Fock space of polyanalytic functions. The Bargmann-Fock space of polyanalytic functions, $\mathbf{F}^n(\mathbb{C}^d)$, consists on all functions satisfying the equation

$$\left(\frac{d}{d\overline{z}}\right)^n F(z) = 0. \tag{1}$$

and such that

$$\int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} dz < \infty. \tag{2}$$

Functions satisfying (1) are known as *polyanalytic functions* of order n. Since (1) generalizes the Cauchy-Riemann equation

$$\frac{d}{d\overline{z}} F(z) = 0,$$

then the space $\mathbf{F}^n(\mathbb{C}^d)$ is a generalization of the Bargmann-Fock space of analytic functions, where, in $d = 1$, a complete description of the sampling and interpolation sets is known [26],[30],[31].

Received February 3, 2009.
Partial financial assistance by FCT grant SFRH/BPD/26078/2005 and CMUC.

Polyanalytic functions inherit some of the properties of analytic functions, often in a nontrivial form [3]. An obvious difference lies on the structure of the zeros. For instance, while nonzero entire functions don´t have sets of zeros with an accumulation point, polyanalytic functions can vanish along closed curves. Just take $F(z) = \overline{z}z - 1 = |z|^2 - 1$, a polyanalytic function of order 2.

In this paper we will study the spaces $\mathbf{F}^n(\mathbb{C}^d)$ using time-frequency analysis, offering a completely new point of view over this spaces. The basic theory of $\mathbf{F}^n(\mathbb{C}^d)$ is derived in such a way it leaves intact, with some variations, most of the structure of the classical analytic situation. Moreover, by means of the connection to time-frequency analysis, it is enriched by the intrinsic structure of Gabor spaces, providing us with tools that were unavailable using complex variables. Thanks to this approach, we will discover a duality between sampling and interpolation in $\mathbf{F}^n(\mathbb{C})$ and multi-sampling and interpolation in $\mathcal{F}(\mathbb{C})$, a problem studied in [6]. This results in the following complete characterization of all lattice sampling and interpolating sequences in $\mathbf{F}^n(\mathbb{C})$, in terms of Beurling density for lattices, $D(\Lambda) = |\det A|^{-1}$, where $\Lambda = A\mathbb{Z}^2$:

Theorem 5. The lattice Γ is a sampling sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if

$$D(\Gamma) > n.$$

Theorem 7. The lattice Γ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if

$$D(\Gamma) < n.$$

For convenience, we will also consider the spaces constituted by the functions satisfying (2), which are polyanalytic of order n, but are not polyanalytic of any lower order (in particular they have no analytic functions). These are the *true* poly-Fock spaces $\mathcal{F}^n(\mathbb{C}^d)$. The poly-Fock and true poly-Fock spaces are related by the following orthogonal decomposition (see Corollary 1 in section 3):

$$\mathbf{F}^n(\mathbb{C}^d) = \mathcal{F}^0(\mathbb{C}^d) \oplus ... \oplus \mathcal{F}^{k-1}(\mathbb{C}^d).$$

Before making a technical summary of our arguments, we would like to say that the "Nyquist rate" phenomenon has been studied in other spaces of analytic functions, first in the Paley-Wiener space [5],[24],[25] and then in Bargmann-Fock [26],[30],[31] and Bergman [32] spaces of analytic functions. The proofs use analytic function arguments and it is unclear how to extend them the polyanalytic situation. Therefore, we introduce new methods. They are based on the extension of Bargmann´s work [4] to the setting of polyanalytic function spaces. This will allow the application of ideas from signal analysis, by relating the problem to vector valued Gabor systems. It is also worth of notice that the density theorem in Gabor analysis has itself a very rich story, beginning with fundamental but imprecise statements by John Von Neumann and Dennis Gabor, which catched the attention of mathematicians after conjectures by Daubechies and Grossman [8]. See the survey article [19].

1.2. Technical summary. To give a context to our approach, we start from the classical connection between the Bargmann-Fock space and time-frequency analysis.

It is well known that, up to a weight, the Gabor transform with a Gaussian window belongs to the Fock-space of analytic functions. Moreover, it has been shown that this is the only choice leading to spaces of analytic functions [1].

However, a nice picture shows up when we take Hermite functions as windows. Then, the analytic situation generated by the gaussian window, becomes the tip of the iceberg of a larger structure involving spaces of polyanalytic functions. Indeed, the Gabor transform with the *nth Hermite function*, is, up to a weight, a polyanalytic function of order $n+1$. Each space $\mathcal{F}^n(\mathbb{C}^d)$ is associated with Gabor transforms with the *nth* Hermite window, with $\mathcal{F}^0(\mathbb{C}^d) = \mathcal{F}(\mathbb{C}^d)$, the Fock space of analytic functions. Such occurrence, which seems to have been hitherto unnoticed, will be fundamental our discussion. This observation is related to some recent developments in Gabor analysis with Hermite functions [16],[17],[13], to Janssen´s approach to the density theorem [22],[23] and also to the techniques used in [20],[21],[36], which suggest that wavelet spaces and polyanalytic functions share intriguing patterns.

We will follow Vasilevskii [35] and call *poly-Fock spaces* to the Fock spaces of polyanalytic functions. They are briefly mentioned in Balk´s monograph [3] and they are implicit in quantum mechanics, in connection to the Landau levels of the Schrödinger operator with magnetic field [29],[14] and displaced Fock states [34]. However, we were not able to find any reference to polyanalytic functions in the mathematical physics literature, apart from [35], where creation and annihilation operators are used.

To extend Bargmann´s theory [4] to the polyanalytic setting, we first introduce what we call the *true-poly-Bargmann transform*:

$$(\mathcal{B}^n f)(z) = n!^{-\frac{1}{2}}\pi^{\frac{n}{2}}e^{\pi|z|^2}\frac{d^n}{dz^n}\left[e^{-\pi|z|^2}F(z)\right].$$

Here F stands for the Bargmann transform of f. As we will see this is a unitary mapping from $L^2(\mathbb{R}^d)$ to $\mathcal{F}^n(\mathbb{C}^d)$. This mapping relates to Gabor transforms with Hermite windows Φ_n in the following way:

$$V_{\Phi_n}f(x,\omega) = e^{i\pi x\omega - \pi\frac{|z|^2}{2}}(\mathcal{B}^n f)(z).$$

Then we define, for vector-valued functions $\mathbf{f} = (f_0, ..., f_{n-1})$, the *poly-Bargmann* transform,

$$(\mathbf{B}^n\mathbf{f}) = \sum_{k=0}^{n-1}(\mathcal{B}^k f_k),$$

which will be unitary between $L^2(\mathbb{R}^d, \mathbb{C}^n)$ and $\mathbf{F}^n(\mathbb{C}^d)$.

With the tools described above at hand, our main argument will depend on two profound results. More specifically, we will combine two variations of the Janssen-Ron-Shen duality principle [28] with the characterization of multi sampling and interpolation sequences in the Fock space [6]. The first result and its variations reflect all the rich inner structure of Gabor frames. The second uses a deep elaboration on Beurling´s balayage technique [5] developed by Seip in [32]. We will proceed as follows. First, using an orthogonal basis for the poly-Fock spaces, we prove the unitary of $\mathbf{B}^n$ and $\mathcal{B}^n$. Then we study sampling in $\mathbf{F}^n(\mathbb{C})$. Using the unitary mapping $\mathbf{B}^n$ we show that the problem is equivalent to the study of vector valued frames with Hermite windows, also known as superframes [2],[17]. This problem has been recently studied in [17], but we provide an alternative proof, which is more natural in the context of sampling and interpolation: applying a vector valued version of Janssen-Ron-Shen duality we translate the statement into a problem concerning unions of Riesz sequences. After noticing that the latter is equivalent to a multi-interpolation problem in Fock spaces of analytic functions, we apply the interpolation result in [6]. Then we study interpolation in $\mathbf{F}^n(\mathbb{C})$. In order to do this, we "dualize" the arguments that we have used in the sampling part, once again using Ron-Shen duality, this time between vector-valued Riesz sequences and multi-frames with Hermite functions. This translates our interpolation problem into one of multi-sampling.

Noticing that this problem is equivalent to multi-sampling in Fock spaces, we apply the sampling result from [6].

It may be possible to find a proof of theorem 7 by using the methods of [17].

1.3. Outline. The paper is organized as follows. The next section contains the classical tools we are going to use. We list the basic properties of the Gabor transform, the Bargmann transform and the Hermite functions. In the third section, we introduce the true-poly-Bargmann and the poly-Bargmann transform. By making a connection to the Gabor transform, we study their basic properties, find an orthogonal basis for the poly-Fock spaces and prove the unitarity properties. Then we study the poly-Bargmann transform. Our main results are in the fourth and fifth sections, where we study sampling and interpolation for $\mathbf{F}^n(\mathbb{C})$.

2. Tools

2.1. The Gabor transform. Fix a function $g \neq 0$. Then the Gabor (short-time) Fourier transform of a function f with respect to the "window" g is defined, for every $x, \omega \in \mathbb{R}^d$ as

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}dt. \tag{3}$$

There is a very important property enjoyed by inner products of this transforms. The following relations are usually called *the orthogonal relations for the short-time Fourier transform.* Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then $V_{g_1}f_1, V_{g_2}f_2 \in L^2(\mathbb{R}^{2d})$ and

$$\langle V_{g_1}f_1, V_{g_2}f_2\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2\rangle_{L^2(\mathbb{R}^d)}}. \tag{4}$$

The Gabor transform provides an isometry

$$V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}),$$

that is, if $f, g \in L^2(\mathbb{R}^d)$, then

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{5}$$

For every $x, \omega \in \mathbb{R}^d$ define the operators translation by x and modulation by ω as

$$\begin{aligned} T_x f(t) &= f(t-x), \\ M_\omega f(t) &= e^{2\pi i\omega t}f(t). \end{aligned}$$

Using these operators we can write (3) as

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle_{L^2(\mathbb{R}^d)}\,.$$

2.2. The Bargmann transform. Here we will use multi-index notation, $z=(z_1,...z_d)$, $n=(n_{1,...}n_d)$ and $|n|=n_1+...+n_d$. The Bargmann transform is defined by

$$(\mathcal{B}f)(z)=\int_{\mathbb{R}^d} f(t)e^{2\pi tz-\pi z^2-\frac{\pi}{2}t^2}dt.$$

It is an isomorphism

$$\mathcal{B}: L^2(\mathbb{R}^d)\to \mathcal{F}(\mathbb{C}^d),$$

where $\mathcal{F}(\mathbb{C}^d)$ stands for the Bargmann-Fock space of analytic functions in $\mathbb{C}^d$ with the norm

$$\|F\|^2_{\mathcal{F}(\mathbb{C}^d)}=\int_{\mathbb{C}^d}|F(z)|^2 e^{-\pi|z|^2}dz. \tag{6}$$

The collection of the monomials of the form

$$e_n(z)=\left(\frac{\pi^{|n|}}{n!}\right)^{\frac{1}{2}} z^n=\prod_{j=1}^{d}\frac{\pi^{n_j}}{\sqrt{n_j!}}z^{n_j},\ n=(n_{1,...}n_d), \tag{7}$$

with $n_i\geq 0$, constitutes an orthonormal basis of $\mathcal{F}(\mathbb{C}^d)$. The reproducing kernel of $\mathcal{F}(\mathbb{C}^d)$ is the function

$$K(z,w)=e^{\pi\overline{w}z}. \tag{8}$$

Diferentiating k times the corresponding reproducing equation we obtain

$$\left\langle F(w), w^{n-k}e^{\pi\overline{w}z}\right\rangle_{\mathcal{F}(\mathbb{C}^d)}=\pi^{k-n}F^{(n-k)}(z). \tag{9}$$

A simple calculation shows that the Bargmann transform is related to the Gabor transform with the Gaussian window $\varphi(x)=2^{\frac{d}{4}}e^{-\pi x^2}$ by the formula

$$V_\varphi f(x,-\omega)=e^{i\pi x\omega}e^{-\pi\frac{|z|^2}{2}}(\mathcal{B}f)(z), \tag{10}$$

where $z=x+i\omega$.

We will need one more operator. Define a "translation operator" β_ζ on $\mathcal{F}(\mathbb{C}^d)$ by

$$\beta_\zeta F(z)=e^{i\pi x\omega-\pi\frac{|\zeta|^2}{2}}e^{\pi\overline{\zeta}z}F(z-\zeta). \tag{11}$$

The operator β_ζ satisfies the intertwining property

$$\beta_\zeta\mathcal{B}=\mathcal{B}M_\omega T_x,\qquad z=x+i\omega. \tag{12}$$

2.3. The Hermite functions. The *Hermite functions* can be defined via the so called Rodrigues Formula

$$h_n(t) = c_n e^{\pi t^2} \left(\frac{d}{dt}\right)^n \left(e^{-2\pi t^2}\right).$$

where c_n is chosen in such a way they can provide an orthonormal basis of $L^2(-\infty, \infty)$. Now let $n = (n_{1,\dots} n_d)$ and $x \in \mathbb{R}^d$. The *d-dimensional Hermite functions* are

$$\Phi_n(x) = \prod_{j=1}^{n} h_{n_j}(x).$$

They form a complete orthonormal system of $L^2(\mathbb{R}^d)$.

A very important property of the Hermite functions is that they are mapped into a basis of the Bargmann-Fock space via the Bargmann transform:

$$(\mathcal{B}\Phi_n)(z) = e_n(z). \tag{13}$$

3. Poly-Fock spaces and poly-Bargmann transforms

3.1. Definitions. We will use multi-index notation in such a way that there will be little difference between the one and the d-dimensional case. We thus write

$$\left(\frac{d}{dz}\right)^n = \frac{d^{|n|}}{dz_1^{n_1} \dots dz_n^{n_n}}$$

It is well known [3] that every polyanalytic function of order n can be uniquely expressed in the form

$$F(z) = \varphi_0(z) + \overline{z}\varphi_1(z) + \dots + \overline{z}^{n-1}\varphi_{n-1}(z), \tag{14}$$

where $\{\varphi_p(z)\}_{p=0}^{n-1}$ are analytic functions, each of them with a power series expansion

$$\varphi_p(z) = \sum_{j=0}^{\infty} c_{j,p} z^j,$$

As a result, the polyanalytic F has a power series expansion

$$F(z) = \sum_{p=0}^{n-1} \overline{z}^p \sum_{j=0}^{\infty} c_{j,p} z^j. \tag{15}$$

Definition 1. *The poly-Fock space, $\mathbf{F}^n(\mathbb{C})$, is the space of polyanalytic functions of order n satisfying*

$$\int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} dz < \infty.$$

The inner product is given by

$$\langle F, G \rangle_{\mathbf{F}^n(\mathbb{C})} = \int_{\mathbb{C}^d} F(z)\overline{G(z)} e^{-\pi|z|^2} dz.$$

3.2. The true poly-Bargmann transform.

Definition 2. *The* true poly-Bargmann transform of order n*, of a function on $\mathbb{R}^d$, is defined by the formula*

$$(\mathcal{B}^n f)(z) = n!^{-\frac{1}{2}} \pi^{\frac{n}{2}} e^{\pi|z|^2} \frac{d^n}{dz^n} \left[e^{-\pi|z|^2} F(z) \right], \tag{16}$$

where $F(z) = (\mathcal{B}f)(z)$.

Clearly $\mathcal{B}^0 f = \mathcal{B}f$ and $\mathcal{B}^n$ is a generalization of the Bargmann transform. We now provide the fundamental properties of $\mathcal{B}^n$. We try to stay as close as possible to the presentation of section 3.4 in [15]. The next proposition is the departing point of our study.

Proposition 1. *If f is a function on $\mathbb{R}^d$ with polynomial growth, then its true poly-Bargmann transform $\mathcal{B}^n f$ is a polyanalytic function of order $n+1$ on $\mathbb{C}^d$. If we write $z = x + i\omega$ then this transform is related to the Gabor transform with Hermite windows in the following way:*

$$V_{\Phi_n} f(x,\omega) = e^{i\pi x\omega - \pi\frac{|z|^2}{2}} (\mathcal{B}^n f)(z). \tag{17}$$

Moreover, if $f \in L^2(\mathbb{R})$ then

$$\|\mathcal{B}^n f\|_{L^2(\mathbb{C}^d, e^{-\pi|z|^2})} = \|f\|_{L^2(\mathbb{R}^d)}. \tag{18}$$

Proof: Let $F = \mathcal{B}f$. The following calculation is from Proposition 3.2 in [16], where (9) is used:

$$\begin{aligned} V_{\Phi_n} f(x,\omega) &= \langle f, M_\eta T_u \Phi_n \rangle_{L^2(\mathbb{R}^d)} = \langle F, \beta_w \mathcal{B}\Phi_n \rangle_{\mathcal{F}(\mathbb{C}^d)} \\ &= \frac{\pi^{|n|} e^{i\pi x\omega - \frac{\pi}{2}|z|^2}}{n!} \left\langle F(w), e^{\pi\overline{z}w}(w-z)^n \right\rangle_{\mathcal{F}(\mathbb{C}^d)} \\ &= \frac{e^{i\pi x\omega - \frac{\pi}{2}|z|^2}}{\sqrt{\pi^{|n|} n!}} \sum_{|k|=0}^{n} \binom{n}{k} (-\pi\overline{z})^k F^{(n-k)}(z). \end{aligned}$$

Now, since the Bargmann transform is an entire function [15, Proposition 3.4.1], the functions $F^{(n-k)}(z)$ are also entire, and from (14) we recognize the sum as a polyanalytic function of order $n+1$. To prove (17) observe that the last expression can be written as

$$e^{i\pi x\omega-\pi\frac{|z|^2}{2}}\frac{e^{\pi|z|^2}}{\sqrt{\pi^n n!}}\frac{d^n}{dz^n}\left[e^{-\pi|z|^2}F(z)\right]=e^{i\pi x\omega-\pi\frac{|z|^2}{2}}(\mathcal{B}^{(n)}f)(z).$$

The isometric property (18) is an immediate consequence of (17) and (5). ■

3.3. Orthogonal decomposition.

Definition 3. *For $k,n\in\mathbb{N}_0$, the functions $e_{k,m}$ are the polynomials defined as*

$$e_{k,m}(z)=\frac{e^{\pi|z|^2}}{\sqrt{\pi^k k!}}\left(\frac{d}{dz}\right)^k\left[e^{-\pi|z|^2}e_m(z)\right]. \tag{19}$$

Obviously,

$$e_{k,m}(z)=\frac{e^{\pi|z|^2}}{\sqrt{\pi^k k!}}\left(\frac{d}{dz}\right)^k\left[e^{-\pi|z|^2}(\mathcal{B}\Phi_m)(z)\right]=(\mathcal{B}^k\Phi_m)(z), \tag{20}$$

Theorem 1. *The set $\{e_{k,m}\}_{k=0,1,...n-1;m=0,1,...}$ is an orthogonal basis of $\mathbf{F}^n(\mathbb{C}^d)$.*

Proof: The orthogonality follows from (20) and (17), since

$$\begin{aligned}\langle e_{k,m},e_{l,j}\rangle_{L^2(\mathbb{R}^{2d})} &= \left\langle \mathcal{B}^k\Phi_m,\mathcal{B}^{(l)}\Phi_j\right\rangle_{\mathcal{F}(\mathbb{C}^d)}\\ &= \left\langle e^{\pi\frac{|z|^2}{2}-i\pi x\omega}V_{\Phi_k}\Phi_m, e^{\pi\frac{|z|^2}{2}-i\pi x\omega}V_{\Phi_l}\Phi_j\right\rangle_{\mathcal{F}(\mathbb{C}^d)}\\ &= \langle V_{\Phi_k}\Phi_m,V_{\Phi_l}\Phi_j\rangle_{L^2(\mathbb{R}^{2d})}\\ &= \langle \Phi_m,\Phi_j\rangle_{L^2(\mathbb{R}^d)}\overline{\langle\Phi_k,\Phi_l\rangle_{L^2(\mathbb{R}^d)}}=\delta_{m,j}\delta_{k,l}\end{aligned}$$

To prove completeness of $\{e_{k,m}\}$ in $\mathbf{F}^n(\mathbb{C}^d)$, suppose that $F\in\mathbf{F}^n(\mathbb{C}^d)$ such that

$$\langle F,e_{k,m}\rangle_{\mathcal{F}(\mathbb{C}^d)}=0,\qquad k=0,1,...n-1;m=0,1,.....$$

For $k=0$, we can use the representation of F in power series (15). Interchanging the sums with the integrals and using the orthogonality of the functions (7), the result is

$$\langle F, e_{0,m}\rangle_{\mathcal{F}(\mathbb{C}^d)} = \sum_{p=0}^{n-1} c_{p+m,p}\frac{(p+m)!}{\sqrt{m!}\pi^{|2p+m|}} = 0, \quad m \geq 0 \tag{21}$$

For $k \geq 1$, a calculation using integration by parts gives:

$$\begin{aligned}\langle F, e_{k,m}\rangle_{\mathcal{F}(\mathbb{C}^d)} &= \int_{\mathbb{C}^d} e^{-\pi|z|^2}\overline{e_m(z)}p...(p-k+1)\sum_{p=k}^{n-1}\overline{z}^{p-k}\sum_{j\geq 0} c_{j,p}z^j dz \\ &= \sum_{p=k}^{n-1}\sum_{j\geq 0} c_{j,p}\frac{p...(p-k+1)\pi^{|m|}}{\sqrt{m!}}\int_{\mathbb{C}^d} z^j\overline{z}^{m+p-k}e^{-\pi|z|^2}dz.\end{aligned}$$

As a result,

$$\sum_{p=k}^{n-1}\frac{p...(p-k+1)(p+m-k)!}{\pi^{|m+2p-2k|}\sqrt{m!}}c_{m+p-k,p} = 0, \quad m \geq 0,\ k \leq n-1,$$

resulting in a triangular system for each m. Solving this system we obtain $c_{j,p} = 0$ for $|p| = 0, ..., |n|$ and $|j| = 0, 1, ...$ Therefore, $F = 0$. ■

Remark 1. *It is clear that these functions are reminiscent of the so-called special Hermite functions, which are the Wigner transforms of two Hermite functions* [33]*. They also appear in the study of Landau levels in* [14].

Definition 4. *The* true *poly-Fock space of order n are defined as*

$$\mathcal{F}^n(\mathbb{C}^d) = Span\left[\{e_{n,m}(z)\}_{m=0,1,...}\right]. \tag{22}$$

Remark 2. *Observe that*

$$\left(\frac{d}{dz}\right)^k\left[e^{-\pi|z|^2}z^m\right] = \frac{d^{m+n}}{dz^k d\overline{z}^m}\left[e^{-\pi|z|^2}\right].$$

Therefore, our functions $e_{n,m}$ are essentially the complex Hermitian functions introduced in [29, pag. 126] *and, as a result, according to theorem 7.1 in* [29] *, the true poly-Fock spaces are the eigenspaces of the Schrödinger operator with magnetic field in $\mathbb{R}^2$, associated to the eigenvalue $n + \frac{1}{2}$. Also, observe that the basis used in* [27] *approaches this one by a formal limit procedure.*

The orthogonal basis property has the following consequence.

Corollary 1. *The poly-Fock space, $\mathbf{F}^n(\mathbb{C}^d)$, admits the following decomposition in terms of true poly-Fock spaces $\mathcal{F}^k(\mathbb{C}^d)$.*

$$\mathbf{F}^n(\mathbb{C}^d) = \mathcal{F}^0(\mathbb{C}^d) \oplus ... \oplus \mathcal{F}^{k-1}(\mathbb{C}^d). \tag{23}$$

This results in a definition equivalent to the one in [36], where the spaces were defined using the decomposition. Observe that $\mathbf{F}^1(\mathbb{C}^d) = \mathcal{F}^0(\mathbb{C}^d) = \mathcal{F}(\mathbb{C}^d)$ and that functions in $\mathcal{F}^n(\mathbb{C}^d)$ are polyanalytic of order $n+1$.

3.4. Unitarity of $\mathcal{B}^n$.

Theorem 2. *The true poly-Bargmann transform is an isometric isomorphism*

$$\mathcal{B}^n : L^2(\mathbb{R}^d) \to \mathcal{F}^n(\mathbb{C}^d).$$

Proof: Since we know from (18) that $\mathcal{B}^n$ is isometric, we only need to show that $\mathcal{B}^n[L^2(\mathbb{R}^d)]$ is dense in $\mathcal{F}^n(\mathbb{C}^d)$. This is now easy, since the Hermite functions constitute a basis of $L^2(\mathbb{R}^d)$ and, by (20), they are mapped into the basis $\{e_{n,m}(z)\}$ of $\mathcal{F}^n(\mathbb{C})$. Since $\mathcal{B}^n[L^2(\mathbb{R}^d)]$ contains a basis of $\mathcal{F}^n(\mathbb{C}^d)$ it must be dense. ■

3.5. The poly-Bargmann transform. Now, consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, ..., f_{n-1})$ with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}} = \sum_{k=0}^{n-1} \langle f_k, g_k \rangle_{L^2(\mathbb{R}^d)} . \tag{24}$$

The *poly-Bargmann transform* of a function $\mathbf{f} = (f_0, ..., f_{n-1})$ is defined as

$$(\mathbf{B}^n \mathbf{f})(z) = \sum_{k=0}^{n-1} (\mathcal{B}^k f_k)(z). \tag{25}$$

This transform is also unitary.

Theorem 3. *The poly-Bargmann transform is an isometric isomorphism*

$$\mathbf{B}^n : \mathcal{H} \to \mathbf{F}^n(\mathbb{C}^d).$$

Proof: It is isometric, since, using the isometric property of $\mathcal{B}^n$, we have

$$\begin{aligned}\|\mathbf{B}^n\mathbf{f}\|^2_{\mathbf{F}^n(\mathbb{C}^d)} &= \sum_{k=0}^{n-1}\left\|\mathcal{B}^k f_k\right\|^2_{\mathcal{F}^n(\mathbb{C}^d)} \\ &= \sum_{k=0}^{n-1}\|f_k\|^2_{L^2(\mathbb{R}^d)} = \|f\|^2_{\mathcal{H}}\,.\end{aligned}$$

Moreover, $\mathbf{B}^n[L^2(\mathbb{R}^d)]$ is dense in $\mathbf{F}^n(\mathbb{C}^d)$, since, by the decomposition (23), every element $F \in \mathbf{F}^n(\mathbb{C}^d)$ can be written as $F = F_0 + ...F_{n-1}$, with $F_k \in \mathcal{F}^k(\mathbb{C}^d)$, $k = 0, ..., n-1$. Since $\mathcal{B}^k$ is unitary, there exists $f_k \in L^2(\mathbb{R}^d)$ such that $F_k = \mathcal{B}^k f_k$, for every $k = 0, ..., n-1$. It follows that $F = \mathbf{B}^n\mathbf{f}$, with $\mathbf{f}$ $=(f_0, ..., f_{n-1})$. ■

4. Sampling in $\mathbf{F}^n(\mathbb{C})$

From now on, we restrict to $d = 1$.

4.1. Definitions. We will work with lattices. A lattice is a discrete subgroup in $\mathbb{R}^2$ of the form $\Lambda = A\mathbb{Z}^2$, where A is an invertible 2×2 matrix. We will define the *density* of the lattice by

$$D(\Lambda) = \frac{1}{|\det A|}. \tag{26}$$

The adjoint lattice Λ^0 is defined as

$$\Lambda^0 = D(\Lambda)\Lambda.$$

Therefore,

$$D(\Lambda^0) = \frac{1}{D(\Lambda)}.$$

We will use the notation $\Gamma = \{z = x + i\omega\}$ to indicate the complex sequence associated to the sequence $\Lambda = (x, \omega)$. The density of Γ will be the density of the associated lattice, that is $D(\Gamma) = D(\Lambda)$.

Definition 5. Γ *is a sampling sequence for the space* $\mathbf{F}^n(\mathbb{C})$ *if there exist positive constants* A *and* B *such that, for every* $F \in \mathbf{F}^n(\mathbb{C})$,

$$A\,\|F\|^2_{\mathbf{F}^n(\mathbb{C})} \leq \sum_{z\in\Gamma}|F(z)|^2\,e^{-\pi|z|^2} \leq B\,\|F\|^2_{\mathbf{F}^n(\mathbb{C})}\,. \tag{27}$$

The definition of sampling in the spaces $\mathcal{F}^k(\mathbb{C})$ is exactly the same.

Now, we take the following definition, obtained from [6, page 114], by making a small simplification, (in the notation of [6, page 114] we set $\nu(z) = n$) and writing in our context (observe that the weight $e^{i\pi x\omega}$ makes no diference).

Definition 6. *A sequence Γ_n, consisting of n copies of Γ is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if, for every sequence $\{\alpha_{i,j}^{(k)}\}_{k=0,...n-1}$ such that $\{\alpha_{i,j}^{(k)}\}_{k=0,...n-1} \in l^2$, there exists $F \in \mathcal{F}(\mathbb{C})$ such that $\langle F, \beta_z e_k\rangle = \alpha_{i,j}^{(k)}$, for all $k = 0, ...n-1$ and every $z \in \Gamma$.*

Consider again the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, ..., f_{n-1})$ with the inner product (24). The time-frequency shifts act coordinate-wise in a obvious way.

Definition 7. *The vector valued system $\mathcal{G}(\mathbf{g}, \Lambda) = \{M_\omega T_x \mathbf{g}\}_{(x,w)\in\Lambda}$ is a* Gabor superframe *for $\mathcal{H}$ if there exist constants A and B such that, for every $\mathbf{f} \in \mathcal{H}$,*

$$A\left\Vert \mathbf{f}\right\Vert_{\mathcal{H}}^2 \leq \sum_{(x,w)\in\Lambda} \left\vert\langle \mathbf{f}, M_\omega T_x \mathbf{g}\rangle_{\mathcal{H}}\right\vert^2 \leq B\left\Vert \mathbf{f}\right\Vert_{\mathcal{H}}^2. \qquad (28)$$

This kind of frames were introduced by Balan in the context of "multiplexing" [2]. We will need a fundamental structure theorem from time-frequency analysis, namely the following version of the Janssen-Ron-Shen duality principle [17, Theorem 2.7].

Theorem A. Let $\mathbf{g} = (g_0, ..., g_{n-1})$ The vector valued system $\mathcal{G}(\mathbf{g}, \Lambda)$ is a *Gabor superframe* for $\mathcal{H}$ if and only if the union of Gabor systems $\cup_{k=0}^{n-1}\mathcal{G}(g_k, \Lambda^0)$ is a Riesz sequence for $L^2(\mathbb{R})$.

4.2. Duality principle. In this section we will obtain the following duality principle.

Theorem 4. Γ *is a sampling sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if the adjoint sequence Γ_n^0 is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$.*

We first prove two lemmas. Combining them with theorem A, gives theorem 4.

Lemma 1. *The union of Gabor systems $\cup_{k=0}^{n-1}\mathcal{G}(g_k, \Lambda)$ is a Riesz sequence for $L^2(\mathbb{R})$ if and only if Γ_n is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$.*

Proof: The union of Gabor systems $\cup_{k=0}^{n-1}\mathcal{G}(g_k,\Lambda)$ is a Riesz sequence for $L^2(\mathbb{R})$ if for every sequence $\{\alpha_{i,j}^{(k)}\}_{k=0,...n-1}\in l^2$ there exists a $f\in L^2(\mathbb{R})$ such that $\langle f, M_\omega T_x g_k\rangle=\alpha_{i,j}^{(k)}$, for all $k=0,...n-1$ and every $(x,\omega)\in\Lambda$. Using the unitarity of $\mathcal{B}$ and the intertwining property (12) gives

$$\langle f, M_\omega T_x g_k\rangle = \langle \mathcal{B}f, \beta_z e_k\rangle \,,$$

and setting $F=\mathcal{B}f$ shows that Γ_n, is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$. ■

The next lemma is a key step in our argument and it is at this point that the unitarity of the poly-Bargmann transform is essential.

Lemma 2. *Let* $\mathbf{h}_n=(h_0,...,h_{n-1})$. *Then the set* $\mathcal{G}(\mathbf{h}_n,\Lambda)$ *is a Gabor super frame for* $\mathcal{H}=L^2(\mathbb{R},\mathbb{C}^n)$ *if and only if the associated complex sequence* Γ *is a sampling sequence for* $\mathbf{F}^n(\mathbb{C})$.

Proof: Using the definition of the inner product (24), the identity (17) and the definition of the poly-Bargmann transform, it is clear that

$$\begin{aligned}
\langle \mathbf{f}, M_\omega T_x \mathbf{g}\rangle_{\mathcal{H}} &= \sum_{k=0}^{n-1}\langle f_k, M_\omega T_x g_k\rangle_{L^2(\mathbb{R})} && (29)\\
&= \sum_{k=0}^{n-1} e^{i\pi x\omega-\frac{\pi}{2}|z|^2}(\mathcal{B}^k f_k)(z)\\
&= e^{i\pi x\omega-\frac{\pi}{2}|z|^2}(\mathbf{B}^n\mathbf{f})(z). && (30)
\end{aligned}$$

Therefore, setting $F=\mathbf{B}^n\mathbf{f}$, the unitarity of $\mathbf{B}^n$ shows that the inequalities (28) are equivalent to (27). ■

4.3. Main result. We will need the concept of *Beurling density* of a sequence.

Let $n^-(r)$ denote the smallest (and $n^+(r)$ the biggest) number of points from Γ to be found in a translate of a compact set of measure 1 in the complex plane. We define the *lower* and the *upper* Beurling density of Γ to be

$$D^-(\Gamma)=\lim_{r\to\infty}\sup\frac{n^-(r)}{r^2} \text{ and } D^+(\Gamma)=\lim_{r\to\infty}\sup\frac{n^+(r)}{r^2},$$

respectively. When Γ is associated to the lattice Λ, $D^-(\Gamma)=D^+(\Gamma)=D(\Gamma)=D(\Lambda)$.

We will now use the following result, which is theorem 2.2 in [6]. Observe that we can remove the uniformly discrete condition from the statement in [6] since we are dealing only with lattices.

Theorem B. The sequence Γ_n is a multi-interpolating lattice sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if and only if $D(\Gamma_n) < 1$.

From this we obtain the characterization of sampling lattices in $\mathbf{F}^n(\mathbb{C})$.

Theorem 5. *The lattice* Γ *is a sampling sequence for* $\mathbf{F}^n(\mathbb{C})$ *if and only if* $D(\Gamma) > n$.

Proof: We know by the duality principle that Γ is a sampling sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if the adjoint sequence Γ_n^0 is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$. By definition of Beurling density, it is obvious that

$$D(\Gamma_n^0) = nD(\Gamma^0).$$

Therefore, theorem B says that Γ^0 is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if and only if

$$D(\Gamma^0) < \frac{1}{n}.$$

As a result, Γ is a sampling sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if

$$D(\Gamma) = \frac{1}{D(\Gamma^0)} > n.$$

■

Using lemma 1, we recover theorem 1.1 of [17].

Corollary 2. *Let* $\mathbf{h}_n = (h_0, ..., h_{n-1})$. *Then the set* $\mathcal{G}(\mathbf{h}_n, \Lambda)$ *is a Gabor super frame for* $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ *if and only if* $D(\Gamma) > n$.

5. Interpolation in $\mathbf{F}^n(\mathbb{C})$

5.1. Definitions.

Definition 8. *The sequence* Γ *is an interpolating sequence for* $\mathbf{F}^n(\mathbb{C})$ *if, for every sequence* $\{\alpha_{i,j}\} \in l^2$, *there exists* $F \in \mathbf{F}^n(\mathbb{C})$ *such that* $e^{i\pi x\omega - \frac{\pi}{2}|z|^2}F(z) = \alpha_{i,j}$, *for every* $z \in \Gamma$.

Definition 9. *The sequence* Γ_n, *consisting of* n *copies of* Γ *is is said to be a multi-sampling sequence for* $\mathcal{F}(\mathbb{C})$ *if there exist numbers* A *and* B *such that*

$$A\,\|F\|^2_{\mathcal{F}(\mathbb{C})} \leq \sum_{z\in\Gamma}\sum_{k=0}^{n-1} |\langle F, \beta_z e_k\rangle|^2 \leq B\,\|F\|^2_{\mathcal{F}(\mathbb{C})}\,. \tag{31}$$

Definition 10. *The set* $\cup_{k=0}^{n-1}\mathcal{G}(g_k,\Lambda)$ *is said to generate a* Gabor multi-frame *in* $L^2(\mathbb{R})$ *if there exist constants* A *and* $\mathbf{B}$ *such that, for every* $f \in L^2(\mathbb{R})$,

$$A\,\|f\|^2_{L^2(\mathbb{R})} \leq \sum_{(x,\omega)\in\Lambda}\sum_{k=0}^{n-1} \left|\langle f, M_\omega T_x g_k\rangle_{L^2(\mathbb{R})}\right|^2 \leq B\,\|f\|^2_{L^2(\mathbb{R})}\,. \tag{32}$$

Now we will need the dual of the duality principle stated in Theorem A. It is stated at the end of [18] in the following form.

Theorem C. The set $\mathcal{G}(\mathbf{g},\Lambda)$ is a Riesz sequence for $L^2(\mathbb{R})$ if and only if $\cup_{k=0}^{n-1}\mathcal{G}(g_k,\Lambda^0)$ is a Gabor multi-frame in $L^2(\mathbb{R})$.

5.2. Duality principle.

Now we prove the following duality.

Theorem 6. *The sequence* Γ *is an interpolating sequence for* $\mathbf{F}^n(\mathbb{C})$ *if and only if* Γ_n^0 *is a multi-sampling sequence for* $\mathcal{F}(\mathbb{C})$.

As in the sampling section, we prove first two lemmas which, combined with theorem C give the result. The next lemma requires only the unitarity of the Bargmann transform.

Lemma 3. *The set* $\cup_{k=0}^{n-1}\mathcal{G}(g_k,\Lambda)$ *is a Gabor multi-frame in* $L^2(\mathbb{R})$ *if and only if* Γ_n *is a multi-sampling sequence for* $\mathcal{F}(\mathbb{C})$.

Proof: Similar to lemma 1: using the unitarity of $\mathcal{B}$ and the intertwining property (12) gives $\langle f, M_\omega T_x g_k\rangle = \langle \mathcal{B}f, \beta_z e_k\rangle$; setting $F = \mathcal{B}f$ it follows from the unitarity of the Bargmann transform that (31) and (32) are equivalent. ■

Again, we make the key connection in the next step, where the unitarity of the poly-Bargmann transform is required.

Lemma 4. *The sequence* Γ *is an interpolating sequence for* $\mathbf{F}^n(\mathbb{C})$ *if and only if* $\mathcal{G}(\mathbf{h}_n,\Lambda)$ *is a Riesz sequence for* $\mathcal{H}$.

Proof: The sequence Γ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if, for every sequence $\{\alpha_{i,j}\} \in l^2$, there exists $F \in \mathbf{F}^n(\mathbb{C})$ such that $e^{i\pi x\omega - \frac{\pi}{2}|z|^2} F(z) = \alpha_{i,j}$, for every $z \in \Gamma$. Using the unitarity of $\mathbf{B}^n$, we find, for every $F \in \mathbf{F}^n(\mathbb{C})$, a vector valued function $\mathbf{f} \in \mathcal{H}$ such that $\mathbf{B}^n\mathbf{f} = F$ or, by (29)-(30), $\langle \mathbf{f}, M_\omega T_x \mathbf{h}_n \rangle_{\mathcal{H}} = F$. Therefore, the first assertion is equivalent to say that, for every sequence $\{\alpha_{i,j}\} \in l^2$, there exists a $\mathbf{f} \in \mathcal{H}$ such that $e^{i\pi x\omega - \frac{\pi}{2}|z|^2} \langle \mathbf{f}, M_\omega T_x \mathbf{h}_n \rangle_{\mathcal{H}} = \alpha_{i,j}$, for every $z \in \Gamma$. This says that $\mathcal{G}(\mathbf{h}_n, \Lambda^0)$ is a Riesz sequence for $\mathcal{H}$. ■

5.3. Main result. We will need the following result, which is contained in theorem 2.1 in [6]:

Theorem D. The sequence Γ_n is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if and only if $D(\Gamma_n) > 1$.

As before, we can obtain our main result from this one.

Theorem 7. *The lattice Γ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if $D(\Gamma) < n$.*

Proof: We know by the duality principle that Γ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if Γ_n, is a multi-sampling sequence for $\mathcal{F}(\mathbb{C})$. Once again we have $D(\Gamma_n^0) = nD(\Gamma^0)$. Therefore, theorem D says that Γ^0 is a multi-interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if and only if $D(\Gamma^0) > \frac{1}{n}$ As in theorem 5 it follows that Γ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if $D(\Gamma) < n$. ■

From this we obtain a new result characterizing all the lattices which generate vector valued Riesz sequences in $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$. This reveals, at least for lattices, the existence of a "Nyquist density" for vector-valued Gabor systems with Hermite functions.

Corollary 3. *$\mathcal{G}(\mathbf{h}_n, \Lambda)$ is a Riesz sequence for $\mathcal{H}$ if and only if $D(\Gamma) < n$.*

Remark 3. *We should remark that the reason we didn´t care about the Bessel condition in the equivalence of the Riesz sequence and interpolating property, used several times in the previous section is that the Hermite functions belong to Feichtinger´s algebra S_0 (see* [10],[9]*):*

$$\|\gamma\|_{S_0} = \int_{\mathbb{R}} |\langle \gamma, M_\omega T_x \varphi \rangle| \, dz < \infty,$$

and as a result they satisfy the Bessel condition [19, theorem 12].

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