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SEMI-PERFECT CATEGORY-GRADED ALGEBRAS

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Abstract: We introduce the notion of algebras graded over a small category and give a criterion for such algebras to be semi-perfect.

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1. Introduction

There is a deep connection between algebras and the categories of their modules. The interplay between their properties contributes to both the structure theory of algebras and the theory of abelian categories.

It is often the case that the study of the category of modules over a particular algebra can lead to the use of other abelian categories, which are non-equivalent to module categories over any algebra. One of such examples is the category of modules over a C-graded algebra, where C is a small category. Such categories appear in the joint work of the author and A. P. Santana [18] on homological properties of Schur algebras.

Algebras graded over a small category generalise the widely known group graded algebras (see [16, 12, 8, 15, 14]), the recently introduced groupoid graded algebras (see [10, 11, 9]), and $\mathbb{Z}$-algebras, used in the theory of operads (see [17]).

One of the important properties of some abelian categories, that considerably simplifies the study of their homological properties, is the existence of projective covers for finitely generated objects. Such categories were called *semi-perfect* in [7]. We give in this article the characterisation of C-graded algebras whose categories of modules are semi-perfect. This result will be of particular importance in the above mentioned work on Schur algebras.

We now introduce the definitions used in the paper and explain the main result in more detail. A C-*graded algebra* is a collection of vector spaces A_α parametrised by the arrows α of C with preferred elements $e_s \in A_{1_s}$ for every object s of C and a collection of maps $\mu_{\alpha,\beta}\colon A_\alpha \otimes A_\beta \to A_{\alpha\beta}$ for every

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composable pair of morphisms α, β of C. For every composable triple α, β, and γ of arrows in C we require associativity

$$\begin{tikzcd} A_\alpha \otimes A_\beta \otimes A_\gamma \arrow[r, "\mu_{\alpha,\beta}\otimes A_\gamma"] \arrow[d, "A_\alpha\otimes\mu_{\beta,\gamma}"'] & A_{\alpha\beta} \otimes A_\gamma \arrow[d, "\mu_{\alpha\beta,\gamma}"] \\ A_\alpha \otimes A_{\beta\gamma} \arrow[r, "\mu_{\alpha,\beta\gamma}"] & A_{\alpha\beta\gamma}. \end{tikzcd}$$

Also for every $\gamma\colon s \to t$ the unitary axiom

$$\mu_{1_t,\gamma}(e_t, a) = a = \mu_{\gamma,1_s}(a, e_s)$$

for all $a \in A_\gamma$, holds.

A C-*graded module* M over a C-graded algebra A is a collection of vector spaces M_γ parametrised by the arrows γ of C with maps $r_{\alpha,\beta}\colon A_\alpha \otimes M_\beta \to M_{\alpha\beta}$ for every composable pair of morphisms in C. These are subject to the associativity

$$\begin{tikzcd} A_\alpha \otimes A_\beta \otimes M_\gamma \arrow[r, "\mu_{\alpha,\beta}\otimes M_\gamma"] \arrow[d, "A_\alpha\otimes r_{\beta,\gamma}"'] & A_{\alpha\beta} \otimes M_\gamma \arrow[d, "r_{\alpha\beta,\gamma}"] \\ A_\alpha \otimes M_{\beta\gamma} \arrow[r, "r_{\alpha,\beta\gamma}"] & A_{\alpha\beta\gamma} \end{tikzcd}$$

and unitarity

$$r_{1_t,\gamma}(e_t, m) = m, \ \forall m \in M_\gamma$$

axioms. An A-*homomorphism* between two C-graded A-modules M and N is a collection of linear homomorphisms $f_\gamma\colon M_\gamma \to N_\gamma$ such that for every composable pair of morphisms α, $\beta \in$ C the diagram

$$\begin{tikzcd} A_\alpha \otimes M_\beta \arrow[r, "A_\alpha\otimes f_\beta"] \arrow[d, "r_{1,\alpha,\beta}"'] & A_\alpha \otimes N_\beta \arrow[d, "r_{2,\alpha,\beta}"] \\ M_{\alpha\beta} \arrow[r, "f_{\alpha\beta}"] & N_{\alpha\beta} \end{tikzcd}$$

is commutative.

We denote the category of all C-graded A-modules by A-mod.

Given a morphism $\gamma\colon s \to t$ of C define *the left stabiliser* St^l_γ of γ by

$$St^l_\gamma = \{\alpha \in \mathrm{C}(t,t) | \alpha\gamma = \gamma\}.$$

Then St^l_γ is a submonoid of $\mathrm{C}(t,t)$ since for every α_1, $\alpha_2 \in St^l_\gamma$ we have

$$\alpha_1\alpha_2\gamma = \alpha_1\gamma = \gamma.$$

For every C-graded algebra A the multiplication maps $\mu_{\alpha,\beta}$ induce an algebra structure on the vector space

$$A^l(\gamma) := \bigoplus_{\alpha \in St^l_\gamma} A_\alpha$$

with unity e_t.

The main result of this paper is

Theorem 1.1. *Let* C *be a small category. The category of* C*-graded modules over a* C*-graded algebra A is semi-perfect if and only if for all arrows γ of* C *the algebras $A^l(\gamma)$ are semi-perfect.*

Note that if C is a group G considered as a category with one object, then we recover a criterion of Dăscălescu [4] for a G-graded algebra to be semi-perfect.

This paper is written with the reader unfamiliar with category theory in mind. Thus the author tried to give all relevant definitions and prove all the properties in full detail. The names of the sections are self-explaining.

Throughout this paper C denotes a small category.

2. Free objects

A C-graded vector space is just a collection V_γ of vector spaces parametrised by the arrows $\gamma \in$ C. A map between two C-graded vector spaces is a collection of linear homomorphisms $f_\gamma \colon V_\gamma \to W_\gamma$. We denote the category of C-graded vector spaces by $\mathcal{V}_{\mathrm{C}}$.

Given a C-graded algebra A and a C-graded vector space V, we define a *free* C-graded A-module $F_A(V)$ by the formula

$$F_A(V)_\gamma = \bigoplus_{\gamma = \alpha\beta} A_\alpha \otimes V_\beta,$$

with the structure map $r_{\delta,\gamma} \colon A_\delta \otimes F_A(V)_\gamma \to F_A(V)_{\delta\gamma}$ defined by the requirement that its restriction to the component $A_\delta \otimes A_\alpha \otimes V_\beta$ is $\mu_{\delta,\alpha} \otimes V_\beta$. We say that a C-graded A-module M is *free* if $M \cong F_A(V)$ for some $V \in \mathcal{V}_{\mathrm{C}}$. Let $f \colon V \to W$ be a map of C-graded vector spaces. Define $F_A(f)$ by the formulas

$$F_A(f)_\gamma|_{A_\alpha \otimes V_\beta} = f_\alpha \otimes V_\beta, \text{for all } \alpha, \beta \in \mathrm{C}^1 \text{ such that } \alpha\beta = \gamma.$$

With these definitions, we get a functor $F_A \colon \mathcal{V}_{\mathrm{C}} \to A\text{-mod}$.

Proposition 2.1. *The functor F_A is a left adjoint to the forgetful functor $U\colon A$-mod $\to \mathcal{V}_{\mathrm{C}}$.*

Proof: We will denote by Id the identity functor. By Theorem 3.1.5. in [3], to prove the proposition it is enough to show the existence of natural transformations $\eta\colon \mathrm{Id} \to UF_A$ and $\epsilon\colon F_AU \to \mathrm{Id}$, such that $(\epsilon F_A)(F_A\eta) = 1_{F_A}$ and $(U\epsilon)(\eta U) = 1_U$.

For every C-graded vector space V define $\eta_V\colon V \to A \otimes V$ by

$$(\eta_V)_\gamma(v) = e_t \otimes v,$$

where t is the target of γ. For every $f\colon V \to W$ we have

$$((A \otimes f) \circ (\eta_V))_\gamma(v) = (A \otimes f)_\gamma(e_t \otimes v) = e_t \otimes f(v) = (\eta_W \circ f)_\gamma(v),$$

which shows that η is a natural transformation of functors.

For every A-module M define $\epsilon_M\colon A \otimes M \to M$ to be the action of A on M. That ϵ is a natural transformation of functors follows just from the definition of homomorphism of A modules.

It is left to check the above stated equalities. Let M be an A-module and $m \in M_\gamma$. Then

$$(U\epsilon)(\eta U)(m) = (U\epsilon)(e_t \otimes m) = m.$$

Let V be a C-graded vector space. Then $F_A(V) = A \otimes V$. The vector space $F_A(V)_\gamma$ is a direct sum of vector spaces $A_\alpha \otimes V_\beta$ with $\alpha\beta = \gamma$. Let $a \in A_\alpha$ and $v \in V_\beta$. Denote by t the target of β. Then

$$(\epsilon F_A)(F_A\eta)(a \otimes v) = (\epsilon F_A)(a \otimes e_t \otimes v) = a \otimes v.$$

■

Denote by $\mathcal{M}$ the composition UF_A. Then $\mathcal{M}$ has a structure of a monad $(\mathcal{M}, \mu, \eta)$, where $\mu\colon \mathcal{M}^2 \to \mathcal{M}$ is given by $\mu_V = U(\varepsilon_{F_A(V)})$. For information on monads and there connection with adjoint functors the reader is referred to [2].

Let us compute the explicit formula for μ. For every $\alpha\beta\gamma = \delta$, $a_1 \in A_\alpha$, $a_2 \in A_\beta$, $w \in V_\gamma$

$$(\mu_V)_\delta(a_1 \otimes a_2 \otimes w) = U(\varepsilon_{F_A(V)})_\delta(a_1 \otimes a_2 \otimes w) = \mu_{\alpha,\beta}(a_1 \otimes a_2) \otimes w.$$

Recall that an $\mathcal{M}$-*algebra* is a C-graded vector space V together with a map $r\colon \mathcal{M}(V) \to V$ such that the diagrams

$$\begin{tikzcd} \mathcal{M}^2(V) \arrow[r, "\mu_V"] \arrow[d, "\mathcal{M}(r)"'] & \mathcal{M}(V) \arrow[d, "r"] \\ \mathcal{M}(V) \arrow[r, "r"] & V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[dr, "1_V"'] & \mathcal{M}(V) \arrow[d, "r"] \\ & V \end{tikzcd}$$

are commutative. The map between $\mathcal{M}$-algebras V and W is a map of C-graded vector spaces $f\colon V \to W$ such that the diagram

$$\begin{tikzcd} F(V) \arrow[r, "F(f)"] \arrow[d, "r"'] & F(W) \arrow[d, "r"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

is commutative. From the above computation it immediately follows that the category of $\mathcal{M}$-algebras is the same as the category of A-modules. We will use this fact to prove some basic properties of the category of A-modules.

3. Abelian category

For the definition and examples of abelian categories the reader is referred to [13].

Recall that a discrete category is a category without non-trivial maps. We may consider the category $\mathcal{V}_{\mathrm{C}}$ as a category of functors from the discrete category whose set of objects coincide with the set of morphisms of C to the category of vector spaces $\mathcal{V}$. Since $\mathcal{V}$ is an abelian category it follows from the second paragraph of p.65 in [13] that the category $\mathcal{V}_{\mathrm{C}}$ is abelian as well.

Proposition 3.1. *Let A be a* C*-graded algebra. Then the category A-mod is abelian.*

Proof: We prove the theorem by applying Proposition 5.3 of [5]. This proposition states that if $\mathcal{C}$ is an abelian category and $\mathcal{M}$ is an additive monad on $\mathcal{C}$ that preserves cokernels, then the category of $\mathcal{M}$-algebras is abelian. We know that the category of A-modules is equivalent to the category of $\mathcal{M}$-algebras, where $\mathcal{M}$ is the monad defined in the previous section.

Thus it is enough to check that $\mathcal{M}$ is additive and preserves cokernels. Note that direct sums and cokernels in $\mathcal{V}_{\mathrm{C}}$ are defined componentwise. Now for any two C-graded vector spaces V and W and any map $f\colon V \to W$ we have

$$\mathcal{M}(V\oplus W)_\gamma = F_A(V\oplus W)_\gamma = \bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes (V\oplus W)_\beta = \bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes (V_\beta \oplus W_\beta)$$

$$= \bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes V_\beta \oplus A_\alpha \otimes W_\beta = \bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes V_\beta \oplus \left(\bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes W_\beta \right)$$

$$= F_A(V)_\gamma \oplus F_A(W)_\gamma = \mathcal{M}(V)_\gamma \oplus \mathcal{M}(W)_\gamma.$$

Also

$$\operatorname{Coker}(\mathcal{M}(f))_\gamma = \operatorname{Coker}(F_A(f))_\gamma = \operatorname{Coker}(F_A(f)_\gamma) = \operatorname{Coker}\left(\bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes f_\beta \right)$$

$$= \bigoplus_{\alpha\beta=\gamma} \operatorname{Coker}(A_\alpha \otimes f_\beta) = \bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes \operatorname{Coker}(f_\beta)$$

$$= \bigoplus_{\alpha\beta=\gamma} A_\alpha \otimes \operatorname{Coker}(f)_\beta = F_A(\operatorname{Coker}(f))_\gamma = \mathcal{M}(\operatorname{Coker}(f))_\gamma.$$

■

4. Grothendieck category

Given an abelian category C, the set of objects $\{M_\alpha | \alpha \in I\}$ is called a *generating set* if every object of $\mathcal{C}$ is a quotient of direct sums from $\{M_\alpha | \alpha \in I\}$.

Denote by $\mathbb{K}[\gamma]$ the C-graded vector space defined by

$$\mathbb{K}[\gamma]_\alpha = \begin{cases} \mathbb{K} & \text{if } \alpha = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\left\{ \mathbb{K}\left[\gamma\right] \middle| \gamma \in \mathrm{C}^1 \right\}$ is a generating set of the category $\mathcal{V}_{\mathrm{C}}$.

Proposition 4.1. *The set* $\left\{ F_A\left(\mathbb{K}\left[\gamma\right]\right) \middle| \gamma \in \mathrm{C}^1 \right\}$ *is a generating set of the category A-mod.*

Proof: The functor F_A commutes with direct sums, and so $F_A(V)$ is a direct sum of modules $F_A\left(\mathbb{K}\left[\gamma\right]\right)$ for every C-graded vector space V. Let M be a C-graded A-module. Denote by $\psi\colon \mathcal{V}_{\mathrm{C}}(M,M) \to A\text{-mod}(F_A(M),M)$ the adjunction isomorphism. Then ψ is given by the formula

$$\psi(f)_\gamma(e_t \otimes m) = f_\gamma(m)$$

for all $m \in M_\gamma$ and $t = t(\gamma)$. In particular,

$$\psi(1_M)(e_t \otimes m) = m.$$

Therefore M is a quotient of a direct sum of objects $F_A(\mathbb{K}[\gamma])$. ∎

A complete and cocomplete abelian category $\mathcal{C}$ is called *Grothendieck* if for every object X and every ascending family of subobjects U_i and subobject V of X we have

$$V \cap \cup_i U_i = \cup_i (V \cap U_i).$$

Here $\cup_i U_i$ is the image of the direct limit $\varinjlim_{i \in I} U_i$ under the structure map from the direct limit to X. Analogously for $\cup_i(V \cap U_i)$. The *intersection* $V \cap U$ is defined as a pull-back of the diagram

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ U & \longrightarrow & X. \end{array}$$

Note that in the modern literature the completness condition is usually omited in the definition of Grothendieck category. But the reader should be aware that Harada uses it in his article [7]. Since we quote the result from this paper we stick to his definition.

Proposition 4.2. *The category A-mod is complete and cocomplete.*

Proof: The category $\mathcal{V}_{\mathrm{C}}$ is complete as a category of functors from C^1 to a complete category $\mathcal{V}$. Now the category $\mathcal{M}$-alg is complete by Corollary 4.3 in [2].

To show that an abelian category is cocomplete it is enough to check that it contains arbitrary direct sums. This is straightforward for A-modules since we can take direct sums componentwise. ∎

Proposition 4.3. *The category A-mod is Grothendieck.*

Proof: By Propositions 4.1 and 4.2 the category A-mod is a complete and cocomplete abelian category. Thus it is enough to check that for every C-graded module M, every A-submodule N of M and every ascending family of submodules $U_i \subset M$, $i \in I$, we have

$$N \cap (\cup_{i \in I} U_i) = \cup_{i \in I} N \cap U_i.$$

Since limits and colimits are given componentwise in the category A-mod we have for all $\gamma \in \mathrm{C}$

$$(N \cap (\cup_{i\in I} U_i))_\gamma = N_\gamma \cap (\cup_{i\in I} U_i)_\gamma.$$

Further,

$$(\varinjlim_{i\in I} U_i)_\gamma = \varinjlim_{i\in I} (U_i)_\gamma = \cup_{i\in I} (U_i)_\gamma \subset M_\gamma.$$

Thus

$$(\cup_{i\in I} U_i)_\gamma = \cup_{i\in I} (U_i)_\gamma$$

and

$$(N \cap (\cup_{i\in I} U_i))_\gamma = N_\gamma \cap (\cup_{i\in I} (U_i)_\gamma) = \cup_{i\in I} N_\gamma \cap (U_i)_\gamma = (\cup_{i\in I} N \cap U_i)_\gamma.$$

■

5. Projective cover

Let $\mathcal{C}$ be an abelian category. An object $P \in \mathcal{C}$ is called *projective* if for any epimorphism $\phi\colon X \twoheadrightarrow Y$ the map

$$\mathcal{C}(P, X) \to \mathcal{C}(P, Y)$$
$$f \mapsto \phi \circ f$$

is an epimorphism. It is straightforward that every object in $\mathcal{V}_\mathrm{C}$ is projective.

Proposition 5.1. *Let V be a* C*-graded vector space. Then $F_A(V)$ is a projective* C*-graded A-module.*

Proof: Let $h\colon M \twoheadrightarrow N$ be a surjective morphism of C-graded A-modules. We have to show that the map

$$A\text{-mod}(F_A(V), M) \to A\text{-mod}(F_A(V), N)$$
$$f \mapsto h \circ f$$

is surjective. Now the diagram

$$\begin{tikzcd}
A\text{-mod}\,(F_A(V), M) \arrow[r, "\phi"] \arrow[d, "h\circ-"'] & \mathcal{V}_\mathrm{C}(V, M) \arrow[d, "h\circ-"] \\
A\text{-mod}\,(F_A(V), N) \arrow[r, "\phi"] & \mathcal{V}_\mathrm{C}(V, N)
\end{tikzcd}$$

is commutative. Since the horizontal arrows are isomorphisms and the right arrow is surjective because V is projective in $\mathcal{V}_\mathrm{C}$, it follows that the left arrow is an epimorphism as well. ■

A family of maps $\{\phi_i \colon T_i \to X | i \in I\}$ is called *epimorphic* in $\mathcal{C}$ if for any $W \in \mathcal{C}$ the map

$$\mathcal{C}(X, W) \to \prod_{i \in I} \mathcal{C}(T_i, W)$$
$$f \mapsto (f \circ \phi_i)_{i \in I}$$

is injective.

We say that the map $\phi \colon X \to Y$ has a *small image* if for any map $\psi \colon T \to Y$ the family (ϕ, ψ) is epimorphic if and only if ψ is an epimorphism. A subobject T of Y is *small* if the natural inclusion $T \to Y$ has a small image.

A *projective cover* of an object Y is a projective object P together with an epimorphism $\psi \colon P \twoheadrightarrow Y$ such that the kernel of ψ is a small subobject of P.

The following theorem insures that a projective cover of an object Y, if it exists, is unique up to isomorphism.

Theorem 5.1. *Let P and $\tilde{P}$ be projective objects in $\mathcal{C}$ and $\pi \colon P \twoheadrightarrow Y$, $\tilde{\pi} \colon \tilde{P} \twoheadrightarrow Y$ epimorphisms. Suppose π is a projective cover. Then P is a direct summand of $\tilde{P}$, that is, there are $i \colon P \to \tilde{P}$ and $p \colon \tilde{P} \to P$ such that $p \circ i = 1_P$. Moreover, $\pi = \tilde{\pi} \circ i$ and $\tilde{\pi} = \pi \circ p$.*

Proof: Since both P and $\tilde{P}$ are projective and both π and $\tilde{\pi}$ are epimorphisms there are $\phi \colon \tilde{P} \to P$ and $\psi \colon P \to \tilde{P}$ such that $\tilde{\pi} = \pi \circ \phi$ and $\pi = \tilde{\pi} \circ \psi$. We will show that ϕ is surjective. Since P is projective, there is $\tau \colon P \to \tilde{P}$ such that $\phi \circ \tau = 1_P$. In particular P is a direct summand of $\tilde{P}$ and $\tilde{\pi} \circ \tau = \pi \circ \phi \circ \tau = \pi \circ 1_P = \pi$.

Let X be a kernel of π. Denote by $\theta \colon X \to P$ the inclusion of X in P. We will show that the family $\{\phi, \theta\}$ is epimorphic. Then since the image of θ is small this will imply that ϕ is an epimorphism.

Suppose there is $W \in \mathcal{C}$ such that the map

$$\mathcal{C}(P, W) \to \mathcal{C}(X, W) \oplus \mathcal{C}(\tilde{P}, W)$$
$$f \mapsto (f \circ \theta, f \circ \phi)$$

is not injective. Let f be a non-zero element in its kernel. Then we get a commutative diagram:

$$\begin{tikzcd}
 & & \tilde{P} \arrow[dl, "\phi"'] \arrow[d, "\tilde{\pi}"] \arrow[ddl, dotted] & \\
X \arrow[r, "\theta"] \arrow[dr, dotted] & P \arrow[r, "\pi"] \arrow[d, "f"'] & Y \arrow[r] \arrow[dl, "g"] & 0 \\
 & W & &
\end{tikzcd}$$

where doted arrows denote zero maps and g exists since Y is a cokernel of θ. Now

$$g \circ \tilde{\pi} = g \circ \pi \circ \phi = f \circ \phi = 0$$

and so

$$f = g \circ \pi = g \circ \tilde{\pi} \circ \psi = 0,$$

which is a contradiction. ∎

6. Small and finitely generated objects

An object X of an abelian category $\mathcal{C}$ is said to be small if for any direct sum $\bigoplus_{i \in I} M_i$ and arbitrary map $f\colon X \to \bigoplus_{i \in I} M_i$ there is a finite subset J of I such that $\mathrm{Im}(f) \subset \bigoplus_{j \in J} M_j$. It is straightforward that a C-graded vector space V is small if and only if all V_γ are finite dimensional and there are only finitely many γ such that $V_\gamma \neq 0$.

Proposition 6.1. *Let A be a* C*-graded algebra. A* C*-graded A-module M is small if and only there is a small* C*-graded vector space V and an inclusion of* C*-graded vector spaces $i\colon V \to M$ such that $\psi(i)\colon F_A(V) \to M$ is an epimorphism.*

Proof: Suppose there is an inclusion $i\colon V \to M$ satisfying the conditions of the proposition. Let $f\colon M \to \bigoplus_{j \in I} M_j$ be a map of C-graded A-modules. Then $f \circ i\colon V \to \bigoplus_{j \in I} M_j$ is a map of C-graded vector spaces, and since V is small there is a finite subset J of I such that $\mathrm{Im}(f \circ i) \subset \bigoplus_{j \in J} M_j$.

Let $m \in M_\gamma$. Then there are α_k, $\beta_k \in \mathrm{C}$ and $a_k \in A_{\alpha_k}$, $v_k \in V_{\beta_k}$ such that $\alpha_k \beta_k = \gamma$ and

$$\psi(i) \left(\sum_k a_k \otimes v_k \right) = m.$$

Hence

$$
\begin{aligned}
f(m) &= f \circ \psi(i) \left(\sum_k a_k \otimes v_k \right) \\
&= f \left(\sum_k a_k \otimes i(v_k) \right) = \sum_k a_k \otimes f \circ i(v_k) \in \bigoplus_{j \in J} M_j,
\end{aligned}
$$

since f is a homomorphism of C-graded A-modules and $\bigoplus_{j\in J} M_j$ is a C-graded A-module.

Conversely, let M be a small C-graded A-module. Since $F_A(M)$ is a projective object the natural epimorphism $\psi(1_M)\colon F_A(M) \to M$ has a splitting $\tau\colon M \to F_A(M)$. Now $F_A(M)$ is a direct sum of the C-graded A-modules $F_A(\mathbb{K}[\gamma])$ with the multiplicity of $F_A(\mathbb{K}[\gamma])$ equal to $\dim(M_\gamma)$. Since M is small there is a finite family J of morphisms in C such that $\operatorname{Im}(\tau) \subset \bigoplus_{\gamma\in J} F_A(\mathbb{K}[\gamma])$. Denote by V the direct sum $\bigoplus_{\gamma\in J} \mathbb{K}[\gamma] \subset \bigoplus_{\gamma\in J} F_A(\mathbb{K}[\gamma])$. Then V is small C-graded vector space, since the direct sum is finite. The restriction i of $\psi(1_M)$ to V is an inclusion and $\psi(i)$ is an epimorphism, since

$$
\psi(i) \circ \tau = \psi(1_M) \circ \tau = 1_M.
$$

∎

An object X of an abelian category C is said to be *finitely generated* if for every family of subobjects X_α with $\alpha \in I$ satisfying

$$
X = \bigcup_{\alpha \in I} X_\alpha
$$

there is a finite subset $J \subset I$ such that

$$
X = \bigcup_{\alpha \in J} X_\alpha.
$$

It is straightforward that a quotient of a finitely generated object in an abelian category is again finitely generated. Moreover, a projective object is small if and only if it is finitely generated (see [6, pg.105]). Thus we get

Corollary 6.1. *Every small object in A-mod is finitely generated.*

7. Harada's criterion

We say that a ring R is *semi-perfect* if the category of left R-modules is semi-perfect. An object M of an abelian category $\mathcal{C}$ is called *semi-perfect* if the ring $\mathcal{C}(M, M)$ is semi-perfect. An object M is called *completely indecomposable* if the ring $\mathcal{C}(M, M)$ is local. Since every local ring is semi-perfect (see [1, pg. 303]) every completely indecomposable object is a semi-perfect.

Theorem 7.1. *Let $\mathcal{C}$ be a Grothendieck category with a generating set $\{P_\alpha | \alpha \in I\}$ of semi-perfect projective objects. Then $\mathcal{C}$ is a semi-perfect category.*

Proof: In Corollary 1 to Theorem 4 of [7] it is proved that under the conditions of the theorem $\mathcal{C}$ is semi-perfect if and only if it has a generating set $\{P_\alpha | \alpha \in I\}$ of completely indecomposable projective objects.

Now, suppose that the set $\{P_\alpha | \alpha \in I\}$ is a generating set of semi-perfect projective objects. Then each ring $\mathcal{C}(P_\alpha, P_\alpha)$ is semi-perfect. By Theorem 27.6 of [1] for each α the ring $\mathcal{C}(P_\alpha, P_\alpha)$ has a complete orthogonal set $e_{\alpha,1}, e_{\alpha,2}, \ldots, e_{\alpha,n_\alpha}$ with each $e_{i,\alpha}\mathcal{C}(P_\alpha, P_\alpha)e_{i,\alpha}$ a local ring. Denote by $P_{\alpha,i}$ the direct summand of P_α that corresponds to the idempotent $e_{\alpha,i}$. Then $P_{\alpha,i}$ is projective as a direct summand of a projective object and $\mathcal{C}(P_{\alpha,i}, P_{\alpha,i}) \cong e_{\alpha,i}\mathcal{C}(P_\alpha, P_\alpha)e_{\alpha,i}$ is a local ring. It is clear that $\{P_{\alpha,i} | \alpha \in I, 1 \leq i \leq n_\alpha\}$ is a generating set for the category $\mathcal{C}$. ■

Theorem 7.2. *Let $\mathcal{C}$ be a semi-perfect Grothendieck category with a generating set of finitely generated projective objects. Then every finitely generated projective object P in $\mathcal{C}$ is semi-perfect.*

Proof: By Corollary 1 in [7] the category $\mathcal{C}$ has a generating set $\{P_\alpha | \alpha \in I\}$ of completely indecomposable projective objects. Now P is a quotient of a direct sum $\bigoplus_{\alpha \in J} P_\alpha$, where J is a family of elements in I. So if we denote the quotient map by ψ, we have that P is the union of the images of restrictions of ψ on P_α, $\alpha \in J$. Since P is finitely generated there is a finite subfamily S of J such that the restriction of ψ on $\tilde{P} = \bigoplus_{\alpha \in S} P_\alpha$ is surjective. As P is projective it is a direct summand of $\tilde{P}$ and therefore $P \cong \bigoplus_{\alpha \in S'} P_\alpha$ for some subfamily S' of S. From Lemma 2 [7, p.331] it follows that the ring $\mathcal{C}(P, P)$ is semi-perfect. ■

8. The main result

Recall from the introduction that given an arrow $\gamma\colon s \to t$ of a small category C we denoted by $A^l(\gamma)$ the algebra

$$\bigoplus_{\alpha \in St^l_\gamma} A_\alpha,$$

where $St^l(\gamma)$ is the left stabiliser of γ.

Theorem 8.1. *The category A-mod is semi-perfect if and only if for all $\gamma \in \mathrm{C}^1$ the algebras $A^l(\gamma)$ are semi-perfect.*

Proof: By Theorems 7.1 and 7.2 the category A-$\mathcal{V}_\mathrm{C}$ is semi-perfect if and only if all objects $F_A(\mathbb{K}[\gamma])$ are semi-perfect. Now, we have isomorphisms of vector spaces

$$\begin{aligned} A\text{-mod}\,(F_A(\mathbb{K}[\gamma]), F_A(\mathbb{K}[\gamma])) &\cong \mathcal{V}_\mathrm{C}(\mathbb{K}[\gamma], F_A(\mathbb{K}[\gamma])) \\ &\cong (F_A(\mathbb{K}[\gamma]))_\gamma \cong \bigoplus_{\alpha \in St^l_\gamma} A_\alpha = A^l(\gamma). \end{aligned}$$

We claim that this is an anti-isomorphism of algebras. Denote by v the element in $\mathbb{K}[\gamma]_\gamma$ that corresponds to the unit of $\mathbb{K}$. Let $f\colon F_A(\mathbb{K}[\gamma]) \to F_A(\mathbb{K}[\gamma])$ be a map of A-modules. Then the image of f under the first two isomorphisms is the evaluation of f on $e_t \otimes v$, where t is the target of γ. In particular, this is an element of $F_A(\mathbb{K}[\gamma])_\gamma \cong \bigoplus_{\alpha\gamma=\gamma} A_\alpha \otimes \mathbb{K}[\gamma]_\gamma$. Thus it has the form $a \otimes v$ for some $a \in A^l(\gamma)$. Now, let g be another endomorphism of the A-module $F_A(\mathbb{K}[\gamma])$. Suppose that $g(e_t \otimes v) = b \otimes v$. Then

$$g(f(e_t \otimes v)) = g(a \otimes v) = ag(e_t \otimes v) = ab \otimes v.$$

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