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A PALEY-WIENER THEOREM FOR THE ASKEY-WILSON FUNCTION TRANSFORM

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ABSTRACT: We define an analogue of the Paley-Wiener space in the context of the Askey-Wilson function transform, compute explicitly its reproducing kernel and prove that the growth of functions in this space of entire functions is of order two and type $\ln q^{-1}$, providing a Paley-Wiener Theorem for the Askey-Wilson transform. Up to a change of scale, this growth is related to the refined concepts of exponential order and growth proposed by J. P. Ramis. The Paley-Wiener theorem is proved by combining a sampling theorem with a result on interpolation of entire functions due to M. E. H. Ismail and D. Stanton.

KEYWORDS: Askey-Wilson function, Paley-Wiener theorem, Reproducing Kernels, Sampling Theorem.

1. Introduction

Let $M(r;f) = \sup\{|f(z)| : |z| \leq r\}$ and consider the space $\mathcal{A}$, constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ satisfying

$$M(r;f) = O(e^{\pi r}). \tag{1}$$

Consider also the space PW constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ such that, for some $u \in L^2(-\pi,\pi)$,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{izt} u(t)\, dt. \tag{2}$$

A celebrated classical theorem of Paley and Wiener says that

$$\mathcal{A} = PW.$$

The growth condition (1) means that $f : \mathbb{C} \longrightarrow \mathbb{C}$ has *order one* and type π and the space PW is called the Paley-Wiener space of *band-limited* functions; it is the *reproducing kernel Hilbert space* mapped via the Fourier transform into L^2 functions supported on the interval $[-\pi,\pi]$. See [24] for more details.

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Another famous result, the Whittaker-Shannon-Kolte´nikov sampling theorem, asserts that every function in the space PW admits the following representation

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}. \tag{3}$$

As a result, research concerning extensions of the sampling theorem has been historically associated with the corresponding extensions of the Paley-Wiener theorem.

The sampling theorem is known to hold for more general transforms, including the Hankel and Jacobi functions transforms [13], [25] and the Paley-Wiener theorem is know to extend to such special functions transforms [9].

Many sampling theorems have been recently considered in the q-case [1], [2],[5], [16]. When thinking about these extensions, one should keep in mind that many of the classical q-functions are special cases of a very general basic hypergeometric function known as *the Askey-Wilson function.* This fact is known as "the Askey-Wilson transform scheme" [8].

Recently, one of us has found a sampling theorem for the Askey-Wilson function transform [6]. Thus, it is natural to ask for the associated Paley-Wiener theorem. It is the purpose of this paper to address this question, providing a Paley-Wiener theorem for the Askey-Wilson function transform. This will be done after rephrasing the results in [6] in the convenient reproducing kernel Hilbert space setting.

Recent research concerning q-difference equations [19], interpolation of entire functions [17] and moment problems [4], strongly suggests that in order to del with basic hypergeometric functions one should use the following concepts. A function f has *logarithmic order* ρ if

$$\lim_{r\to+\infty} \sup \frac{\ln \ln M(r;f)}{\ln \ln r} = \rho$$

and f with logarithmic order ρ has logarithmic type c if

$$\lim_{r\to+\infty} \sup \frac{\ln M(r;f)}{(\ln r)^{\rho}} = c.$$

This is because basic hypergeometric functions are of order zero and therefore require a refined concept of order to define their growth. However, we will approach the topic in a slightly different manner in this paper: Instead of considering a function in μ, we will considerer a function in $z = q^{\mu}$. Looking at objects from this point of view, our Askey-Wilson Paley-Wiener space turns out to be constituted by functions of order two with type $\ln(1/q)$. This is equivalent to say that, in the variable $z = q^{\mu}$, they have logarithmic order two and logarithmic type $\ln(1/q)$.

We have organized the paper in the following way. The next section reviews the definitions of the Askey-Wilson polynomials and functions an provides a short outline of the L^2 theory of the Askey-Wilson transform. Then, in the third section, we present a detailed study of the reproducing kernel Hilbert space which is naturally associated to the Askey-Wilson functions transform (in much the same way PW is associated to the Fourier transform). We compute a basis for this space as well as the explicit formula for the reproducing kernel and recover by this method the sampling theorem of [6]. Finally, in the last section we prove a Paley-Wiener theorem, by describing the growth of functions in the reproducing kernel Hilbert space in terms of their order and type.

2. The Askey-Wilson function transform

2.1. The Askey-Wilson polynomials. Choose a number q such that $0 < q < 1$. The notational conventions from [11]

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1 - aq^{k-1}),$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad (a_1, ..., a_m; q)_n = \prod_{l=1}^{m}(a_l;q)_n, \quad |q| < 1,$$

where $n = 1, 2, \ldots,$ will be used. The symbol ${}_{r+1}\phi_r$ stands for the function

$${}_{r+1}\phi_r\left(\begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array}\middle| q, z\right) = \sum_{n=0}^{\infty}\frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n.$$

The Askey-Wilson polynomials $p_n(x; a, b, c, d)$, with $x = \frac{z+z^{-1}}{2}$, are defined by

$$p_n(\frac{z+z^{-1}}{2}; a, b, c, d) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{n-1}abcd, az, a/z \\ ab, ac, ad \end{array}\middle| q; q\right). \tag{4}$$

If $a, b, c, d \in \mathbb{C}$ are four reals or two reals and one pair of conjugates, or two pairs of conjugates such that $|ab|, |ac|, |ad|, |bc|, |cd| < 1$, then the

Askey-Wilson polynomials are real valued and their orthogonality can be written as an integral over $x = \frac{z+z^{-1}}{2} \in [-1, 1]$ plus a finite sum over a discrete set with mass points outside $[-1, 1]$. This finite sum does no occur if $|a|, |b|, |c|, |d| < 1$. When $\max(|a|, |b|, |c|, |d|) < 1$, the Askey-Wilson polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} p_n(x; a, b, c, d) p_m(x; a, b, c, d) w(x) dx = h_n \delta_{m,n},$$

where

$$w(x) = \frac{(x^2, 1/x^2; q)_\infty \sin\theta}{(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q)_\infty},$$

and

$$h_n = \frac{2\pi(abcdq^{2n}; q)_\infty (abcdq^{n-1}; q)_n}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.$$

The Askey-Wilson function is defined as

$$\phi_\gamma(z) = \frac{1}{(bc, q/ad; q)_\infty} {}_4\phi_3\left(\begin{array}{c} \tilde{a}/\gamma, \tilde{a}\gamma, az, a/z \\ ab, ac, ad \end{array} \middle| q; q\right) +$$

$$\frac{(\tilde{a}/\gamma, \tilde{a}\gamma, qb/d, qc/d, az, a/z; q)_\infty}{(q\gamma/\tilde{d}, q/\gamma\tilde{d}, ab, ac, bc, ad/q, qz/d, q/zd; q)_\infty} {}_4\phi_3\left(\begin{array}{c} q\gamma/\tilde{d}, q/\gamma\tilde{d}, qz/d, q/zd \\ qb/d, qc/d, q^2/ad \end{array} \middle| q; q\right),$$

where

$$\begin{aligned} \tilde{a} &= \sqrt{q^{-1}abcd}, \\ \tilde{b} &= ab/\tilde{a} = q\tilde{a}/cd, \\ \tilde{c} &= ac/\tilde{a} = q\tilde{a}bd, \\ \tilde{d} &= ad/\tilde{a} = q\tilde{a}/bc. \end{aligned}$$

The function ϕ_γ is introduced in [15] and it can also be defined as a single ${}_8\phi_7$ with a very-well poised ${}_8W_7$ structure [23]. The function ϕ_γ is meromorphic in γ. Moreover, its poles are simple and can be removed multiplying it by the factor $(q\gamma/\tilde{d}, q/\gamma\tilde{d}; q)_\infty$.

Now we will define the Askey-Wilson function transform, following the construction in [7]. A new weight function is defined as

$$W(x) = \Delta(x)\Theta(x)$$

where, using the notation $\theta(x) = (x, q/x; q)_\infty$ for the renormalized Jacobi theta function, the function Θ is defined as

$$\Theta(x) = \frac{\theta(dx, d/x)}{\theta(dtx, dt/x)}.$$

For generic parameters a, b, c, d such that the weight function W has simple poles we define a measure v, depending on these parameters, by

$$\begin{aligned}\int f(x)\,dv(x) &= \frac{K}{4i\pi}\int_{\mathbb{T}} f(x)\phi_\gamma(x)\,W(x)\,\frac{dx}{x} \\ &\quad + \frac{K}{2}\sum_{x\in D}\left(f(x) + f\left(x^{-1}\right)\right)\ Res_{y=x}\left(\frac{W(y)}{y}\right),\end{aligned}$$

where K is a constant (the exact value will not be required), $S = S_- \cup S_+$ is the infinite, discrete set given by

$$\begin{aligned} S_- &= \{dtq^k;\ k \in \mathbb{Z},\ dtq^k < -1\}, \\ S_+ &= \{aq^k;\ k \in \mathbb{Z},\ aq^k > 1\}. \end{aligned}$$

In the next sections we will often refer to the measure defined above as being of the form $v = v_c + \nu_d$, where v_c is the continuous measure

$$dv_c(x) = \Theta(x)\Delta(x)\,dx/x$$

continuous and ν_d is the discrete part, supported in the set S.

Now, let $L^2_+(v)$ be the Hilbert space with respect to the measure v constituted by functions f satisfying $f(x) = f\left(x^{-1}\right)$, ν-almost everywhere. The *Askey-Wilson function transform* is defined by

$$(\mathcal{F}f)(\gamma) = \int f(x)\,\phi_\gamma(x)\,dv(x)$$

for compactly supported functions $f \in L^2_+(\nu)$. Let $L^2_+(\widetilde{v})$ be the same space with respect to the same measure, but replacing the parameters a, b, c, d by the dual parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.The main result in [7] states that $\mathcal{F}$ extends to an *isometric isomorphism*

$$\mathcal{F} : L^2_+(v) \to L^2_+(\widetilde{v}).$$

3. The Askey-Wilson Paley-Wiener space

3.1. Reproducing kernel Hilbert spaces. We will now introduce some concepts concerning reproducing kernel Hilbert spaces. This exposition is taken from [13], [10] and [20].

Let H_{rep} be a class of complex valued functions, defined in a set $X \subset \mathbb{C}$, such that H_{rep} is a Hilbert space. We say that $k(\gamma, x)$ is a *reproducing kernel* of H_{rep} if $k(\gamma, x) \in H_{rep}$ for every $\gamma \in X$ and every $f \in H_{rep}$ satisfies the reproducing equation

$$f(\gamma) = \langle f(.), k(., \gamma)\rangle_{H_{rep}}.$$

Now we will use the language in Saitoh [20], and we proceed to give a brief account of the required results.

Consider a second Hilbert space, H. For each t belonging to a domain X, let $K(., t)$ belong to H. Then,

$$k(\gamma, x) = \langle K(., \gamma), K(., x)\rangle_H$$

is defined on $X \times X$. Suppose that we have an isometric transformation

$$(Fg)(\gamma) = \left\langle g, \overline{K(, \gamma)} \right\rangle_H$$

and denote the set of images by $F(H)$. The following theorem can be found in [20]:

Theorem A *If F is a one to one isometric transformation, the kernel $k(\gamma, x)$ determines uniquely a reproducing kernel Hilbert space for which it is the reproducing kernel. This reproducing kernel Hilbert space is precisely $F(H)$ and it can have no other reproducing kernel. If $\{S_n\}$* is a basis of $F(H)$, then

$$k(\gamma, x) = \sum_n S_n(\gamma) S_n(x).$$

There is a general formulation of the sampling theorem in reproducing kernel Hilbert spaces [14]. We will use the following "orthogonal basis case".

Theorem B *With the notations established earlier, we have: If there exists $\{t_n\}_{n \in \mathbf{I} \subset \mathbb{Z}}$ such that $\{K(., t_n)\}_{n \in \mathbf{I}}$ is an orthogonal basis, we then have the sampling expansion*

$$f(t) = \sum_{n \in \mathbf{I}} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)}$$

in $F(H)$, pointwise over $\mathbf{I}$, *and uniformly over any compact subset of X for which $\|K_t\|$ is bounded.*

The chief example of a reproducing kernel Hilbert space is PW. In this situation the reproducing kernel is the function $\sin\pi(x-\gamma)/\pi(x-\gamma)$, the sampling points are $t_n = n$ and the uniformly convergent expansion is the Whittaker-Shannon-Kolte´nikov sampling formula.

3.2. The Askey-Wilson function reproducing kernel. Let us look at the reproducing kernel Hilbert space associated to the Askey-Wilson function transform.

The first task is to consider a proper analogue of bandlimited functions. This is done by defining a finite continuous Askey-wilson function in much the same way it was done in [6].

We start by removing the poles of the function $\phi_{\tilde{a}q^\mu}$: Consider a function u_μ, analytic in the variable μ, defined as

$$u_\mu(x,a,b,c,d\mid q) = (\tilde{a}q^\mu;\tilde{a}q^{-\mu};q)_\infty \phi_{\tilde{a}q^\mu}\left(e^{i\theta}\right),\ x=\cos\theta.$$

Then we consider what is going to be the analogue of the transform (2): if $\max(|a|,|b|,|c|,|d|)<1$, the *finite continuous Askey-Wilson transform* $\mathcal{J}$ is defined by

$$\mathcal{J}(f)(\mu) = \int_{-1}^{1} f(x)u_\mu(x;a,b,c,d\mid q)\,w(x,a,b,c,d\mid q)dx. \tag{5}$$

The continuous Askey-Wilson relates to the Askey-Wilson transform as follows: If $\breve{f}$ is the analytic function such that $f(\cos\theta) = \breve{f}\left(e^{i\theta}\right)$, then

$$\mathcal{J}(f)(\mu) = \frac{4i\pi}{K}(\tilde{a}q^\mu;\tilde{a}q^{-\mu};q)_\infty \mathcal{F}\left(\frac{\breve{f}}{\Theta}\right)(\tilde{a}q^\mu).$$

Definition 1. *The* Askey-Wilson Paley-Wiener space, *PW_{AW}, is the space constituted by the analytic extension to the complex plane of the functions $f\in L^2_+(v)$ such that, for some $u\in L^2(w(x,a,b,c,d\mid q),dx)$,*

$$f=\mathcal{J}(u).$$

Let us look at this particular setting from the point of view of Theorem A.

Theorem 1. *If* $\max(|a|, |b|, |c|, |d|) < 1$, *then the set* PW_{AW} *is a Hilbert space of entire functions with reproducing kernel* $k(\gamma, \lambda)$. *The functions*

$$S_n^{(\tilde{a})}(\mu; q) = \frac{(-1)^n q^{\frac{n(n+1)}{2}} \left(1 - \tilde{a}^2 q^{2n}\right) \left(\tilde{a}q^{\mu}, \tilde{a}q^{-\mu}; q\right)_\infty}{(q;q)_n \left(a, \tilde{a}^2 q^n; q\right)_\infty \left(1 - \tilde{a}q^{n+\mu}\right)\left(1 - \tilde{a}q^{n-\mu}\right)}.$$

constitute an orthogonal basis of PW_{AW} *and the reproducing kernel is given explicitly by*

$$k(\gamma, \lambda) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})}(\gamma; q)\, S_n^{(\tilde{a})}(\lambda; q).$$

Proof: To fulfill the conditions in Theorem A, we need to show that the finite continuous Askey-Wilson is a one to one isomorphism between A_{AW} and PW_{AW}. To see that it is one-to-one, observe that, since, if

$$\int_{-1}^{1} f(x) u_\mu(x; a, b, c, d \mid q)\, w(x, a, b, c, d \mid q) dx = 0, \text{for all } \mu \in \mathbb{C},$$

then we have, in particular, that

$$\int_{-1}^{1} f(x) u_n(x; a, b, c, d \mid q)\, w(x, a, b, c, d \mid q) dx, \quad \text{for } n = 0, 1, ...$$

Since for integer values of μ, u_μ is a multiple of the Askey-Wilson polynomials,

$$u_n(x; a, b, c; d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab, ac, bc; q)_n} d^{-n} p_n(x; a, b, c, d), \tag{6}$$

we can use the completeness of the system of the Askey-Wilson polynomials to get $f = 0$. Consequently, $\mathcal{J}(f)$ is one to one. From the definition,

$$PW_{AW} = \mathcal{J}\left[L^2\left(w(x, a, b, c, d \mid q) dx\right)\right].$$

Therefore, endowing PW_{AW} with the inner product

$$\langle \mathcal{J}(f), \mathcal{J}(g) \rangle_{PW_{AW}} = \int_{-1}^{1} f(x)\, \overline{g(x)} w(x, a, b, c, d \mid q) dx, \tag{7}$$

the finite Askey-Wilson transform $\mathcal{J}$ becomes a Hilbert space isometry between $L^2(w(x, a, b, c, d \mid q)dx)$ and PW_{AW}.

It remains to show that the functions $S_n^{(\tilde{a})}(\mu; q)$ provide an orthogonal basis for PW_{AW}. By the definition (5) and (6),

$$\mathcal{J}(u_n)(\mu) = \int_{-1}^{1} u_n(x) u_\mu(x)\, w(x, a, b, c, d \mid q) dx$$

$$= \frac{(-1)^n q^{-n(n-1)/2}}{d^n(ab, ac, bc; q)_n} \int_{-1}^{1} p_n(x) u_\mu(x)\, w(x, a, b, c, d \mid q) dx.$$

Now we can use Proposition 6 of [6] to conclude that

$$\mathcal{J}(u_n)(\mu) = S_n^{(\tilde{a})}(\mu; q).$$

By (6), $\{u_n\}$ is an orthogonal basis of $L^2(w(x, a, b, c, d \mid q)dx)$. Since $\mathcal{J}$ is isometric onto PW_{AW}, it follows that $S_n^{(\tilde{a})}(\mu; q)$ is a basis of PW_{AW}. ■

Remark 1. *The functions* $S_n^{(\tilde{a})}(\gamma; q)$ *play the same role in our setting as do the functions* $\sin \pi (x - n) / \pi (x - n)$ *in the Paley-Wiener space.*

Now, Theorem 1 and Theorem B give the following sampling theorem. This has been proved in [6], but the approach with reproducing kernels provides the uniform convergence that will be used in the next section.

Theorem 2. *For* $f \in PW_{AW}$ *we have*

$$f(\mu) = \sum_{n=0}^{\infty} f(n)\, S_n^{(\tilde{a})}(\mu; q). \tag{8}$$

where $S_n^{(\tilde{a})}(\mu; q)$ *is given by*

$$S_n^{(\tilde{a})}(\mu; q) = \frac{(-1)^n q^{\frac{n(n+1)}{2}} \left(1 - \tilde{a}^2 q^{2n}\right) \left(\tilde{a} q^{\mu}, \tilde{a} q^{-\mu}; q\right)_\infty}{(q; q)_n \left(a, \tilde{a}^2 q^n; q\right)_\infty \left(1 - \tilde{a} q^{n+\mu}\right)\left(1 - \tilde{a} q^{n-\mu}\right)}.$$

The convergence is uniform on every compact subset of the real line.

Proof: Observe that, from

$$S_n^{(\tilde{a})}(m; q) = \delta_{n,m},$$

we obtain:

$$g(\mu, m) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})}(\mu; q)\, S_n^{(\tilde{a})}(m; q) = S_m^{(\tilde{a})}(\mu; q).$$

Moreover,

$$g(m, m) = S_m^{(\tilde{a})}(m; q) = 1,$$

and the result follows from Theorem 1 and Theorem B. ■

4. The Askey-Wilson-Paley-Wiener theorem

Recall that the entire function f is of order ρ if

$$\lim_{r\longrightarrow\infty}\frac{\ln\ln(M(r;f))}{\ln r}=\rho.$$

A constant has order zero, by convention.

The entire function f of positive order ρ is of type τ if

$$\lim\frac{\ln(M(r;f))}{r^{\rho}}=\tau.$$

Definition 2. *The space $\mathcal{A}_{\mathcal{AW}}$, which will be the analogue of $\mathcal{A}$ in the Askey-Wilson setting, is the space constituted by the analytic continuation of the functions from $L^2(w(x,a,b,c,d\mid q),dx)$ such that*

$$M(r;f)=O(e^{\ln(1/q)r^2}),$$

that is, of order 2 and type $\ln(1/q)$.

It is easy to see that the functions in $\mathcal{A}_{\mathcal{AW}}$ satisfy the conditions in [17, Theorem 3.1]. We rewrite this statement as:

Theorem C *Every $f\in\mathcal{A}_{\mathcal{AW}}$ admits the expansion*

$$f\left(\mu\right)=\sum_{n=0}^{\infty}f\left(n\right)S_n^{(\tilde{a})}\left(\mu;q\right). \tag{9}$$

The next result is the Paley-Wiener theorem for the Askey-Wilson functions transform. The cornerstone of its proof is the fact that the entire function expansion (9) and the sampling expansion (8) are *exactly* the same.

Theorem 3. *If* $\max(|a|,|b|,|c|,|d|)<1$, *then* $\mathcal{A}_{\mathcal{AW}}=PW_{AW}$.

Proof: Take $f\in PW_{AW}$. By definition we have, for some $u\in L^2(w(x,a,b,c,d\mid q),dx)$,

$$f\left(\mu\right)=\ \mathcal{J}\left(u\right)\left(\mu\right)=\int_{-1}^{1}u(x)u_{\mu}\left(x\right)w(x,a,b,c,d\mid q)dx.$$

We need to study the growth of

$$M\left(r;u_{\mu}\right).$$

From formula (5.4) in [23] and for $0\leq\theta\leq\pi$, we have

$$u_r\left(x\right)=\frac{\left(ae^{i\theta},be^{i\theta},ce^{i\theta},qe^{i\theta}/d;q\right)_{\infty}}{\left(ab,ac,bc,e^{2i\theta};q\right)_{\infty}}\left(q^{1-r}/e^{i\theta}d;q\right)_{\infty}\left[1+o\left(1\right)\right],\ \text{ as }\ r\longrightarrow\infty.$$

Let $-1 < \delta < 0$, then

$$M(n+\delta; u_\mu) = O\left(\left(q^{1-\delta-n}/d; q\right)_n\right).$$

This implies

$$M(n+\delta; u_\mu) = O\left((q/d)^n q^{-n(n+1+2\delta)/2}\right).$$

Therefore,

$$\lim_{r\to\infty} \sup \frac{\ln\ln(M(r; u_\mu))}{\ln r} = 2,$$

and

$$\lim_{r\to\infty} \sup \frac{\ln(M(r; u_\mu))}{r^2} = \ln\left(1/q\right).$$

This condition implies that u_μ is of order 2 and type at most $\ln(1/q)$. Therefore,

$$u_\mu\left(x\right) \in \mathcal{A}_{\mathcal{AW}}.$$

This shows that $f \in \mathcal{A}_{\mathcal{AW}}$. Conversely, let $f \in \mathcal{A}_{\mathcal{AW}}$. By Theorem C,

$$f\left(\mu\right) = \sum_{n=0}^{\infty} f\left(n\right) S_n^{(\tilde{a})}\left(\mu; q\right),$$

In the end of the proof of Theorem 1 we have seen that

$$S_n^{(\tilde{a})}\left(\mu; q\right) = \mathcal{J}\left(u_n\right)\left(\mu\right).$$

Then, the sampling formula of Theorem 2 can be written as

$$f\left(\mu\right) = \sum_{n=0}^{\infty} f\left(n\right) \mathcal{J}\left(u_n\right)\left(\mu\right)$$

$$= \sum_{n=0}^{\infty} f\left(n\right) \int_{-1}^{1} u_n(x) u_\mu\left(x\right) w(x, a, b, c, d \mid q) dx.$$

The uniform convergence of the sampling series allows to interchange the integral with the sum in such a way that

$$f\left(\mu\right) = \int_{-1}^{1} \left(\sum_{n=0}^{\infty} f\left(n\right) u_n(x)\right) u_\mu\left(x\right) w(x, a, b, c, d \mid q) dx.$$

Then we have written f in the form

$$f\left(\mu\right) = \mathcal{J}(u)(\mu),$$

with

$$u(x) = \left(\sum_{n=0}^{\infty} f(n) u_n(x) \right) \in L^2(w(x, a, b, c, d \mid q), dx).$$

As a result, $f \in PW_{AW}$. ■

References

[1] L. D. Abreu, *A q-sampling theorem related to the q-Hankel transform*, Proc. Amer. Math. Soc. 133 (2005), 1197-1203.

[2] M. H. Annaby, *q-type sampling theorems,* Result. Math., 44, (3-4), 214-225 (2003).

[3] R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 54 (1985), no. 319.

[4] C. Berg, H. L. Pedersen and with an appendix by Walter Hayman, *Logarithmic order and type of indeterminate moment problems,* Proceedings of the International Conference : "Difference equations, special functions and orthogonal polynomials", Munich, July 25-30, 2005. World Scientific Publishing Company, Singapore 2007.

[5] F. Bouzeffour, *A q-sampling theorem and product formula for continuous q-Jacobi functions,* Proc. Amer. Math. Soc 135 (2007), 2131-2139.

[6] F. Bouzeffour, *A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Askey-Wilson functions,* J. Nonlinear Math. Phys. 14 (2007), no. 3, 367–380.

[7] E. Koelink, J.V. Stokman, *The Askey-Wilson functions transform*, Int. Math. Res. Not. IMRN , no. 22, 1203–1227 (2001).

[8] E. Koelink, J.V. Stokman, *The Askey-Wilson functions transform scheme,* Special functions 2000: current perspective and future directions (Tempe, AZ), 221–241, NATO Sci. Ser. II Math. Phys. Chem., 30, Kluwer Acad. Publ., Dordrecht, (2001).

[9] T. H. Koornwinder, *A new proof of a Paley-Wiener type theorem for the Jacobi transform.* Ark. Mat. 13 (1975), 145–159.

[10] A. G. Garcia, *Orthogonal sampling formulas: a unified approach.* SIAM Rev. 42 (2000), no. 3, 499–512.

[11] G. Gasper and M. Rahman, "*Basic Hypergeometric Series*," Cambridge University Press, Cambridge, UK, 1990.

[12] G. H. Hardy, *Notes on special sistems of orthogonal functions IV*, Proc. Cambridge Phil. Soc., 37 (1941), 331-348.

[13] J. R. Higgins, *An interpolation series associated with the Bessel-Hankel transform* , J. Lond. Math. Soc. **5**, (1972) 707-714.

[14] J. R. Higgins, *A sampling principle associated with Saitoh's fundamental theory of linear transformations.* Analytic extension formulas and their applications (Fukuoka, 1999/Kyoto, 2000), 73–86, Int. Soc. Anal. Appl. Comput., 9, Kluwer Acad. Publ., Dordrecht, 2001.

[15] M. E. H. Ismail, M. Rahman, *The Associated Askey-Wilson polynomials,* Trans. Amer. Math. Soc. 328 (1991), 201-239.

[16] M. E. H. Ismail, A. I. Zayed, *A q-analogue of the Whittaker-Shannon-Kotel ´nikov sampling theorem*, Proc. Amer. Math. Soc. 131 (2003), 3711-3719.

[17] M. E. H Ismail, D. Stanton, *q-Taylor theorems, polynomial expansions, and interpolation of entire functions.* J. Approx. Theory 123 (2003), no. 1, 125–146.

[18] M. E. H Ismail, D. Stanton, *Applications of q-Taylor theorems,* J. Comput. Appl. Math. 153 (2003), no. 1-2, 259–272.

[19] J. P. Ramis, *About the growth of entire functions solutions of linear algebraic q-difference equations.* Ann. Fac. Sci. Toulouse Math. (6) 1 (1992), no. 1, 53–94.

[20] S. Saitoh, *Integral Transforms, Reproducing Kernels And Their Applications* (Harlow: Longman), Pitman Research Notes in Mathematics Series, 369. Longman, Harlow, 1997.

[21] J. V. Stokman *An expansion formula for the Askey-Wilson function.* J. Approx. Theory 114 (2002), no. 2, 308–342.

[22] S. K. Suslov, *Some orthogonal very-well-poised* ${}_8\varphi_7$*-functions*, J. Phys. A 30 (1997), pp 5877-5885.

[23] S. K. Suslov, *Some orthogonal very-well-poised* ${}_8\varphi_7$*-functions that generalize the Askey-Wilson polynomials,* Ramanujan J., 5, 183-218, (2001).

[24] Young, R. M., "*An Introduction To Nonharmonic Fourier Series*" , revised first edition, Academic Press, New York, 2001.

[25] A. I. Zayed, *Advances In Shannon's Sampling Theory.* CRC Press, Boca Raton, FL, 1993.

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