Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 09–29

# BRANCHED COVERINGS OF LINKS WHOSE GEOMETRIC GRAPH CONTAINS ONE CYCLE

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ABSTRACT: Let $M$, $M'$ be compact, connected, oriented 3-manifolds and $L'$ a link in $M'$ with irreducible exterior $E'$. We prove that if the geometric graph of $E'$ has a single cycle then there exists at most one prime number $d$ for which $M$ is a cyclic covering of $M'$, of order $d$, branched over $L'$.

KEYWORDS: Group actions, cyclic branched coverings, orbifolds.
AMS SUBJECT CLASSIFICATION (2000): 57M12, 57M25, 57M50.

## 1. Introduction

Let $M, M'$ be compact connected oriented 3-manifolds. Thurston [3, Problem 3.16] asked for which manifolds $M'$ every covering of $M'$ by $M$ has the same degree. Wang, Wu and Yu [4, 5] proved that the only 3-manifolds $M'$ admitting a geometric decomposition that did not have this property are that those which are not covered by a product (surface)$\times\mathbb{S}^1$ or by a torus bundle over $\mathbb{S}^1$.

Now consider a link $L'$ in $M'$ and suppose that there are two finite groups $G_1, G_2 \subset \mathrm{Diff}^+(M)$, acting on $M$, whose quotient is an orbifold with underlying space $M'$, branched over $L'$. We can similarly ask if the the degrees of $G_1$ and $G_2$ must be the same. There are explicit examples that show that these degrees can be different [1]. The simplest example is given by the cyclic rotations on $\mathbb{S}^3$ around a trivial knot, whose quotients have the same topological type, and can have any degree. Similar examples occur for lens spaces. If we allow the degrees to be any integer we can exhibit several other examples for which these degrees are different. However, for prime degrees, we conjecture that this can only happen for lens spaces. In fact we have proved the following results [1, 2]:

**Proposition 1.1.** *Let $M$ and $M'$ be compact orientable 3-manifolds and $L'$ a link in $M'$. If the geometric graph of the exterior of $L'$ is a tree then there*

Received August 7, 2009.

*exists at most one prime number $d$ for which $M$ is a cyclic covering of $M'$, of degree $d$, branched over $L'$.*

**Proposition 1.2.** *Let $M$ and $M'$ be compact orientable 3-manifolds and $L'$ a link in $M'$. If the geometric graph of the exterior of $L'$ is complete then there exists at most one prime number $d$ for which $M$ is a cyclic covering of $M'$, of degree $d$, branched over $L'$.*

The trees and the complete graphs can be considered the extreme cases. In this paper we show that the degree is also determined in intermediate cases. More precisely, we consider the case where the the geometric graph of the exterior of $L'$ has one cycle. We prove the following result:

**Proposition 1.3.** *Let $M$ and $M'$ be compact orientable 3-manifolds and $L'$ a link in $M'$. If the geometric graph of the exterior of $L'$ has one cycle then there exists at most one prime number $d$ for which $M$ is a cyclic covering of $M'$, of degree $d$, branched over $L'$.*

The paper is organized as follows. In section 2 we review some results necessary to prove Proposition 1.3 and section 3 is devoted to the proof.

## 2. Preliminaries

Let $M, M'$ be compact, connected, oriented 3-manifolds and $L'$ a link in $M'$ with irreducible exterior $E'$. For $i = 1, 2$, let $G_i =< g_i >$ be two groups of orientation preserving diffeomorphisms of $M$, of prime order $d_i$, such that the quotient $M/G_i$ induces a branched cyclic covering $p_i : M \to O_i$ over an orbifold $O_i$ of topological type $(M', L')$. Let $\Gamma$, $\Gamma'$ be the geometric graphs of $M$ and $E'$. The actions of $G_1$, $G_2$ on $M$ induce actions of the same degrees on $\Gamma$ whose quotient is $\Gamma'$ [1, 5].

We recall some results that we will be used in the proof of Proposition 1.3.

**Proposition 2.1** ([1, Proposition 5.2])**.** *If the graphs $\Gamma$ and $\Gamma'$ are isomorphic, then there exists at most a number $d$ for which $M$ is a covering of $(M', L')$ of degree $d$.*

We say that a covering $p : M \to (M', L')$ is of *type* $(u, v)$ on a Seifert piece $X$ of the geometric decomposition of $M$ if each fiber of the quotient $X/p$

is covered by $u$ fibers of the Seifert pieces of $M$ and $v$ is the degree of the restriction on the action on each regular fiber of $X$. Since we are considering prime degrees, if $p$ has degree $d$, then on each Seifert piece, it is either of type $(1, d)$ or type $(d, 1)$.

**Proposition 2.2** ([1, Proposition 5.3])**.** *Let $p_i : M \to (M', L')$, $i = 1, 2$, be two cyclic branched coverings of prime degree. If $p_1$ and $p_2$ have the same type $(u, v)$ for every Seifert piece of the geometric decomposition of $M$, then $d_1 = d_2$.*

The following proposition on distances is essential to the proof of Proposition 1.3, since it allows to compare distances in $\Gamma$ and $\Gamma'$.

**Proposition 2.3** ([1, Lemma 5.6])**.** *If a vertex $v$ of $\Gamma$ is $G_i$-invariant, then $d(v, w) = d(p_i(v), p_i(w))$, for every vertex $w$ of $\Gamma$.*

## 3. Proof of Proposition 1.3.

We begin the proof with the simplest case, that is, when $\Gamma'$ is a cycle. This will the be the base case for an argument by induction that will prove the general case.

**Proposition 3.1.** *Let $\Gamma'$ be a cycle. Then $p_1$ and $p_2$ have the same degree.*

*Proof.* By Proposition 2.1, we can assume that $\Gamma$ has vertices of degree greater than 2. Let $S$ be the set of these vertices. Now suppose $d_1 < d_2$. Since $d_1$, $d_2$ are prime, then the degree of each vertex of $S$ is $2d_1 = d_2 + 1$ and $S$ has at least two such vertices. Since $d_1 < d_2$, then $S$ has exactly two such vertices, whose images by each $p_i$ are opposite vertices of $\Gamma'$, which has necessarily even length. Then both actions are of type $(d_i, 1)$ on every vertex which contradicts Proposition 2.2. Therefore we conclude that $d_1 = d_2$. $\square$

The induction step will consist in removing the terminal vertices. To do this we need the following lemma, which we will prove after showing how it implies the general case.

**Lemma 3.2.** *If $p_1$ and $p_2$ have different degrees, then for each terminal vertex $v'$ of $\Gamma'$, $p_i^{-1}(v')$ has only terminal vertices, $i = 1, 2$.*

**Proposition 3.3.** *If $\Gamma'$ contains only one cycle, then $p_1$ and $p_2$ have the same degree.*

*Proof.* Let $T'$ be the set of terminal vertices of $\Gamma'$. If $T' = \emptyset$, the result follows from Proposition 3.1. If $T'$ is not empty, and the degrees are different, then, by the Lemma 3.2, $p_1^{-1}(T') = p_2^{-1}(T')$ are formed by the terminal vertices of $\Gamma$. Removing from $T'$ from $\Gamma$ and $p_1^{-1}(T') = p_2^{-1}(T')$ from $\Gamma$ shows that there are two (eventually) branched coverings of distinct prime orders $p_1, p_2 : M_0 \to (M_0, L_0')$. By induction (and eventually recurring to the unbranched case [5]), it follows that necessarily $d_1 = d_2$. □

Now we prove Lemma 3.2, by considering several cases.

*Proof of Lemma 3.2.* Since the degree of the action is prime, $p_i^{-1}(v')$ cannot have more than one internal vertex. Suppose $p_i^{-1}(v')$ has a single internal vertex $v$, of degree $d_i$, and consider the $G_i$-invariant vertex $w$ closest to $v$. Consider also a path $\gamma'$ from $v'$ to $p_i(w)$ of minimal length. This path lifts to $d_i$ parallel paths $\gamma_k$ from $v$ to $w$. Let $C = \cup_k \gamma_k$.

Suppose $d_1 < d_2$. We will consider two cases according to whether $d_1 = 2$ or $d_1 > 2$.

**Case 1:** $2 < d_1 < d_2$

Let $p_j$ be the other projection. If $d_j > d_i$ then clearly each edge incident on $v$ is $G_j$-invariant. If $2 < d_j < d_i$, then, since $d_i$ is prime, the edges incident on $v$ project over at least three edges of $\Gamma'$. If $p_j(v) \neq p_j(w)$, then the $d_i$ parallel paths connecting $v$ and $w$ would project to more than one cycle of $\Gamma'$. Therefore $p_j(v) = p_j(w)$ and $p_1(w)$ is also a terminal vertex of $\Gamma'$. Since $2 < d_i < d_j$, and $\Gamma'$ has a single cycle, each $p_2(\gamma_k)$ is an arc of $\Gamma'$.

Let $S = p_2^{-1}(\cup_k p_2(\gamma_k))$. Suppose there is an arc $\omega$ connecting two vertices $v_1$, $v_2$ of $S$, outside of $S$. For one of the projections these two vertices have distinct images. Suppose without loss of generality that $p_1(v_1) \neq p_1(v_2)$. Then, the projection $p_1(S \cup_k g_2^k(\omega))$ of all the $G_2$-translates of $S \cup \omega$ contains distinct cycles. Therefore every connected component of $\Gamma - S$ is isomorphic to a connected component of $\Gamma' - p_i(S)$ and $G_i$ permutes cyclically these components. Since we are assuming that $d_1 \neq d_2$, this is absurd.

Therefore $p_i^{-1}(v')$ has only terminal vertices.

**Case 2:** $d_1 = 2$

Consider first the set $p_2^{-1}(v')$ and suppose it has an internal vertex.

If $p_1(v) = p_1(w)$ then $p_2(w)$ is also a terminal vertex of $\Gamma'$. If there was no arc $\omega$ connecting two vertices $v_1$, $v_2$ of distinct $\gamma_k$'s, then $G_2$ would permute cyclically every vertex of $\Gamma - \{v, w\}$. Since $p_1$ can only identify pairs of vertices, a simple computation shows that this is impossible. We may thus consider such an arc $\omega$. It follows that $p_1(S \cup_k g_2^k(\omega))$ contains distinct cycles, which is impossible.

Suppose now that $p_1(v) \neq p_1(w)$. If $d_2 > 3$, then, as in Case 1, the $d_2$ parallel paths connecting $v$ and $w$ would project to more than one cycle of $\Gamma'$. For $d_2 = 3$, it may happen that $v, w$ are both $G_1$-invariant and $G_2$-invariant and the $p_2$-preimage of every terminal vertex distinct from $v'$ has no internal vertices.

Consider a connected component $X$ of $\Gamma - C$ connected to $C - \{w\}$. Since $p_1(C)$ is the only cycle of $\Gamma'$, then $X$ is connected to a single vertex of $C - w$ and it is a tree, isomorphic to a subgraph of $\Gamma' - p_1(C)$. Then $p_2^{-1}(p_1(C))$ is connected to $w$. Let $D$ be the set of vertices of $\Gamma$ at maximal distance from $w$. Since $w$ is $G_i$-invariant, then, by Proposition 2.3, $p_i(D)$ is the set of vertices of $\Gamma'$ at maximal distance from $p_i(w)$. These are, for both projections, the terminal vertices of $\Gamma'$ and $p_1(v)$. We denote this set by $D'$. Removing $D$ and $D'$ from $\Gamma$ and $\Gamma'$ induces two (branched) coverings of distinct prime orders $p_1, p_2 : M_0 \to (M_0, L_0')$, such that the JSJ graph of the exterior of the quotient is a tree. By Proposition 1.1 (or the unbranched case [5]), we conclude that $d_1 = d_2$.

Therefore $p_2^{-1}(v')$ has only terminal vertices.

Now suppose that $p_1^{-1}(v')$ has an internal vertex.

If $p_2(v) \neq p_2(w)$ then $p_2(C)$ is the cycle of $\Gamma'$. As before, $p_1^{-1}(p_2(C))$ is connected to $w$. Since $p_2(p_1^{-1}(p_2(C)))$ is a tree, $p_2(p_1^{-1}(p_2(v)))$ is a terminal vertex and $p_1^{-1}(p_2(v))$ is an interior vertex, which is impossible, as we have shown above.

If $p_2(v) = p_2(w)$ then let $S = p_1^{-1}(\cup_k\, p_1(\gamma_k))$. Suppose there is an arc $\omega$ connecting two vertices $v_1$, $v_2$ of $S$, outside of $S$ such that $p_2(\omega \cup S)$ contains a cycle. Then $p_1(S \cup_k g_2^k(\omega))$ contains distinct cycles. Therefore the cycle $O$ of $\Gamma'$ is connected to a single vertex of $p_1(C)$. Let $z$ be the $G_1$-invariant vertex of $\Gamma$ closest from $S$ and $\omega_1, \omega_2$ the arcs connecting $z$ to $S$. Then $p_2(S \cup \omega_1 \cup \omega_2)$ contains the cycle of $\Gamma'$. If $z$ were not $G_2$-invariant, then $p_1^{-1}(O)$ would be permuted cyclically by $G_2$ and we would obtain a second cycle in $\Gamma'$. Therefore $z$ is both $G_1$-invariant and $G_2$-invariant. By a reasoning similar to the case $d_2 = 3$ we can remove the vertices at maximal distance from $z$, $p_1(z)$ and $p_2(z)$ to obtain two (branched) coverings of distinct prime orders $p_1, p_2 : M_0 \to (M_0, L_0')$, such that the JSJ graph of the exterior of the quotient is a tree, contradiction.

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