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BILINEAR BIORTHOGONAL EXPANSIONS AND THE SPECTRUM OF AN INTEGRAL OPERATOR

LUÍS DANIEL ABREU, ÓSCAR CIAURRI AND JUAN LUIS VARONA

Abstract: We study an extension of the classical Paley-Wiener space structure, which is based on bilinear expansions of integral kernels into biorthogonal sequences of functions. The structure includes both sampling expansions and Fourier-Neumann type series as special cases. Concerning applications, several new results are obtained. From the Dunkl analogue of Gegenbauer's expansion of the plane wave, we derive sampling and Fourier-Neumann type expansions and an explicit closed formula for the spectrum of a right inverse of the Dunkl operator. This is done by stating the problem in such a way it is possible to use the technique due to Ismail and Zhang. Moreover, we provide a q-analogue of the Fourier-Neumann expansions in q-Bessel functions of the third type. In particular, we obtain a q-linear analogue of Gegenbauer's expansion of the plane wave by using q-Gegenbauer polynomials defined in terms of little q-Jacobi polynomials.

Keywords: Bilinear expansion, biorthogonal expansion, plane wave expansion, sampling theorem, Fourier-Neumann expansion, Dunkl transform, special functions, q-special functions.
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1. Introduction

1.1. Motivation. The function $K(x,t) = e^{ixt}$ has several well known bilinear expansion formulas. For example, the Fourier series expansion,

$$e^{ixt} = \sum_{n=-\infty}^{\infty} \frac{\sin(x-\pi n)}{x-\pi n}\, e^{i\pi n t}, \tag{1}$$

the expansion in terms of the prolate spheroidal wave functions φ_n,

$$e^{-ixt} = \sqrt{2\pi} \sum_{n=0}^{\infty} i^n \lambda_n \varphi_n(x) \varphi_n(t), \tag{2}$$

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where λ_n are the square roots of the eigenvalues arising from the time-band limiting integral equation (see the recent paper [31]), and Gegenbauer's expansion of the plane wave in ultraspherical polynomials and Bessel functions (see [18, § 4.8, formula (4.8.3), p. 116])

$$e^{ixt} = \Gamma(\beta)\left(\frac{x}{2}\right)^{-\beta}\sum_{n=0}^{\infty} i^n(\beta+n)J_{\beta+n}(x)C_n^{\beta}(t) \tag{3}$$

(in the particular case $\beta = 0$, this formula is the so-called Jacobi-Anger identity).

Each of the above expansions is associated to important developments in mathematical analysis. The first one is equivalent to the Whittaker-Shannon-Kotel'nikov sampling theorem, the second one is the prototype of a Mercer kernel, and the third one has been the main tool in the diagonalization of certain integral operators. Moreover, (3) provided the model for the introduction of a q-analogue of the exponential function with nice properties [20].

Our main interest are expansions of the type (3). To see why biorthogonality is required, recall the definition of the Paley-Weiner space PW,

$$PW = \left\{ f \in L^2(\mathbb{R}) : f(z) = (2\pi)^{-1/2}\int_{-1}^{1} u(t)e^{izt}\,dt,\ u \in L^2(-1,1)\right\}.$$

It is immediate to obtain orthogonal expansions for $f \in PW$ by simply integrating equations (1) and (2), by using the orthogonality of the exponentials and the prolate spheroidal functions. However, if we try to do the same thing in (3), we must restrict ourself to the case $\beta = 1/2$, when the weight function of the Gegenbauer's polynomials (actually Legendre) is 1. Therefore, even in this simple case it is not clear how to expand Paley-Weiner functions into Gegenbauer polynomials or Bessel functions with general parameter β.

Hence biorthogonality is required. Moreover, we have realized that the biorthogonality setting works for several structures involving special functions and their q-analogues. Therefore, to organize the presentation of our ideas, we first construct a structure involving biorthogonal expansions, from which the results are obtained, after explicit evaluation of some integrals.

1.2. Problems and results. We have identified three open problems connected to our set up. For each one of them we offer a solution. The first is the solution of the expansion problem mentioned in the previous subsection and its extension to the Dunkl kernel. The second and the third have been folk questions for a while, in connection with the Ismail and Zhang paper [20]. The problems are as follows.

(A) Expand functions in PW and its generalization studied in [8] in terms of Fourier-Neumann series.
(B) Provide an explicit solution of a right inverse spectral problem for the Dunkl operator in the real line in spaces weighted by the appropriated weight functions, as it has been done in [20] for the differential and the Askey-Wilson operator, in [19] for the q-difference operator and in [6] for the q^{-1}-Askey-Wilson operator.
(C) Find a q-linear version of Ismail and Zhang q-quadratic analogue of the plane wave expansion found in [20].

To have a glimpse of our results, we select a few of them in the context of problems (A)–(C) above. Regarding (A), from the following extension of (3) to the Dunkl kernel

$$E_\alpha(ixt) = \Gamma(\alpha+\beta+1)\left(\frac{x}{2}\right)^{-\alpha-\beta-1}\sum_{n=0}^{\infty} i^n(\alpha+\beta+n+1)J_{\alpha+\beta+n+1}(x)C_n^{(\beta+1/2,\alpha+1/2)}(t), \tag{4}$$

where $C_n^{(\beta+1/2,\alpha+1/2)}$ are the so-called generalized Gegenbauer polynomials and J_ν denotes the Bessel function of order ν, we obtain uniformly convergent Fourier-Neumann type expansions

$$f(x) = \sum_{n=0}^{\infty} a_n(f)(\alpha+\beta+n+1)x^{-\alpha-\beta-1}J_{\alpha+\beta+n+1}(x),$$

valid for $f \in PW_\alpha$ (the natural generalization of the Paley-Weiner space as in [8]), and where

$$a_n(f) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)\int_{\mathbb{R}} f(t)\frac{J_{\alpha+\beta+n+1}(t)}{t^{\alpha+\beta+1}}\,d\mu_{\alpha+\beta}(t).$$

Moreover, in some cases, the coefficients $a_n(f)$ are identified as Fourier coefficients.

The identity (4) is also the main tool to solve problem (B) as follows. We define the Dunkl operator of rank one in spaces weighted by the weight function of the generalized Gegenbauer polynomials and use this definition to find explicit formulas for the spectrum of its right inverse. It turns out that the eigenvalues of the resulting integral operator $T_{\beta,\alpha}$ are defined in terms of zeros of Bessel functions $\{\pm i/j_{\alpha+\beta+1,k}\}_{k\geq 1}$ and the eigenfunctions are

$$g_{\pm i/j_{\alpha+\beta+1,k}}(t) = \mp i \left(\frac{j_{\alpha+\beta+1,k}}{2}\right)^{\alpha+\beta+1} \frac{E_\alpha(\mp t j_{\alpha+\beta+1,k})}{\Gamma(\alpha+\beta+1)(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})}.$$

Our setting applies also to many situations involving q-special functions and we provide some q-analogues of the Fourier-Neumann expansions as illustration. To this end, we use some results from Koorwinder and Swarttouw [23]. As a byproduct of our approach we solve problem (C) and obtain the following q-linear analogue of the plane wave expansion, which somehow reminds Ismail and Zhang q-quadratic version:

$$e(itx;q^2) = \frac{(q^2;q^2)_\infty}{(q^{2\beta};q^2)_\infty} x^{-\beta} \sum_{n=0}^{\infty} i^n (1-q^{2n+2\beta}) J_{\beta+n}(xq^{[\frac{n}{2}]};q^2) C_n^\beta(t;q^2),$$

where $\left[\frac{n}{2}\right]$ denotes the biggest integer less or equal than $\frac{n}{2}$, $J_\nu(z;q)$ is the third Jackson q-Bessel function, and $C_n^\beta(t;q^2)$ are q-analogues of the Gegenbauer polynomials, defined in terms of the little q-Jacobi polynomials, perhaps it is natural to call them little q-Gegenbauer. This is obtained as a special case of a more general formula.

1.3. Outline. The paper is organized as follows. In the second section we describe our problem in abstract terms. First we build the general set up for bilinear orthogonal expansions and then we modify it in a way that some biorthogonal sequences are allowed in the expansions. In the third section we describe the results which are obtained in the case of Fourier kernel. The fourth section studies the expansions associated to the Dunkl kernel and its consequences for the Hankel transform. The fifth section is devoted to the diagonalization of the right inverse of the Dunkl operator. Then we present the applications to q-special functions in the sixth section. In the last section we collected the evaluation of some integrals involving special and q-special functions which were essential for the paper but could not be found in the literature.

2. Structure

2.1. Orthogonal expansions. We begin with $K(x,t)$, a function of two variables defined on $\Omega \times \Omega \subset \mathbb{R} \times \mathbb{R}$, and an interval $I \subset \Omega$. Using this function as a kernel, define on $L^2(\Omega, d\mu)$, with $d\mu$ a real measure, an integral transformation by

$$(\mathcal{K}f)(t) = \int_\Omega f(x)\overline{K(x,t)}\, d\mu(x), \tag{5}$$

with inverse

$$(\widetilde{\mathcal{K}}g)(x) = \int_\Omega g(t)K(x,t)\, d\mu(t), \tag{6}$$

and satisfying the multiplication formula

$$\int_\Omega (\mathcal{K}f)g\, d\mu = \int_\Omega (\mathcal{K}g)f\, d\mu. \tag{7}$$

Note that the previous identity implies the following one:

$$\int_\Omega (\widetilde{\mathcal{K}}f)g\, d\mu = \int_\Omega (\widetilde{\mathcal{K}}g)f\, d\mu.$$

Moreover, if in the multiplication formula we take $g = \overline{\mathcal{K}(f)}$ and use $\mathcal{K}(\overline{h}) = \overline{\widetilde{\mathcal{K}}(h)}$, we get $\|\mathcal{K}f\|_{L^2(\Omega,d\mu)} = \|f\|_{L^2(\Omega,d\mu)}$. That is, we guarantee that $\mathcal{K}$ is an isometry. The most common situation is to assume that the kernel satisfies $K(x,t) = K(t,x)$. In this way, the multiplication formula follows from Fubini's theorem.

As usually, it is enough to suppose that the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$, defined by (5) and (6), are valid for a suitable dense subset of $L^2(\Omega, d\mu)$, and later extended to the whole $L^2(\Omega, d\mu)$ in the standard way. Moreover, let us also assume that, as a function of t, $K(x,\cdot)\chi_I(\cdot)$ belongs to $L^2(\Omega, d\mu)$ (or, with other words, $K(x,\cdot) \in L^2(I, d\mu)$).

Now, let N be a subset of the entire numbers, $\{\phi_n\}_{n\in N}$ be an orthonormal basis of the space $L^2(I, d\mu)$ and $\{S_n\}_{n\in N}$ a sequence of functions in $L^2(\Omega, d\mu)$ such that, for every $n \in N$,

$$(\mathcal{K}S_n)(t) = \chi_I(t)\overline{\phi_n(t)}, \tag{8}$$

where χ_I stands for the characteristic function of I (note of the small abuse of notation given by the use of $\chi_I\overline{\phi_n}$; here, $\overline{\phi_n}$ is a function that is defined only on I, and with $\chi_I\overline{\phi_n}$ we mean that we extend this function to Ω by being the null function on $\Omega \setminus I$; we will use this kind of notation often along this paper). Consider the subspace $\mathcal{P}$ of $L^2(\Omega, d\mu)$ constituted by those functions f such that $\mathcal{K}f$ vanishes outside of I. This can also be written as

$$\mathcal{P} = \Big\{ f \in L^2(\Omega) : f(t) = \int_I u(t)K(x,t)\, d\mu(t),\ u \in L^2(I, d\mu) \Big\}.$$

On the one hand, by the isometrical property of $\mathcal{K}$, it follows that S_n is a *complete orthonormal* sequence in $\mathcal{P}$. This implies that every function f in $\mathcal{P}$ has a expansion of the form

$$f(x) = \sum_{n\in N} c_n S_n(x). \tag{9}$$

On the other hand, from (8) we have

$$\widetilde{\mathcal{K}}(\chi_I \overline{\phi_n}) = \widetilde{\mathcal{K}}(\mathcal{K} S_n) = S_n.$$

Consequently, the Fourier coefficients of $K(x,\cdot)$, as a function of t, in the basis $\{\phi_n\}_{n\in N}$ on $L^2(I, d\mu)$ are $S_n(x)$. As a result, $K(x,t)$ has the following bilinear expansion formula

$$K(x,t) = \sum_{n\in N} S_n(x)\phi_n(t). \tag{10}$$

Remark 1. The reader familiar with sampling theory, in particular with the generalization due to Kramer, has probably noticed strong ressemblances. Indeed, Kramer's lemma corresponds to a particular case of the above situation when there exists a sequence of points x_k such that $S_n(x_k) = \delta_{n,k}$. This implies that $\{K(x_n,\cdot)\} = \{\phi_n\}$ is an orthogonal basis of $L^2(I, d\mu)$ and that $\mathcal{P}$ has an orthogonal basis given by $\{\widetilde{\mathcal{K}}(\chi_I \overline{K(x_n,\cdot)})\} = \{\widetilde{\mathcal{K}}(\chi_I \overline{\phi_n(\cdot)})\} = \{S_n\}$. The orthogonal expansion in the basis $\{S_n\}_{n\in N}$ is the sampling theorem. In [15] it is given a detailed exposition of a similar structure, which, although restricted to sampling theory, is in its essence equivalent to the one we have described.

The objects that we are interested in here are mainly those expansions that fit in the above set up, but that are not sampling expansions. As we will see, there exists quite a few of these. We will see in this work a wealth of situations where explicit computation of certain integrals yield new expansion formulas of the type (10), but in general they are special cases of the more general setting we will provide in the next section.

Remark 2. Perhaps the most remarkable feature this set up inherits from sampling theory is the fact that, in many situations, uniform convergence can be granted, once we know that the expansion converges in norm. This happens because $\mathcal{P}$ is a Hilbert space with a reproducing kernel given by

$$k(x,y) = \sum_{n\in N} S_n(x)\overline{S_n(y)} = \int_I K(x,t)\overline{K(y,t)}\, d\mu(t).$$

This fact can be proved using Saitoh's theory of linear transformation in Hilbert space [27, 28] in a way similar to what has been done in [3] and also by the same arguments in [15]. The uniform convergence of the expansions (9) is now a consequence of the well known fact that if the sequence f_n converges to f in the norm of a Hilbert space with reproducing kernel $k(\cdot,\cdot)$, then the convergence is pointwise to f and uniform in every set where $\|K(x,\cdot)\|_{L^2(I,d\mu)}$ is bounded.

2.2. Biorthogonal expansions. We now consider the same set up as in subsection 2.1 (in particular, the notation for the operators $\mathcal{K}f$ and its inverse $\widetilde{\mathcal{K}}g$ in terms of a kernel $K(x,t)$ that satisfy the multiplication formula (7), but instead of the orthonormal basis $\{\phi_n\}_{n\in N}$ of the space $L^2(I,d\mu)$, we have a pair of complete biorthonormal sequences of functions in $L^2(I,d\mu)$, $\{P_n\}_{n\in N}$ and $\{Q_n\}_{n\in N}$, that is,

$$\int_I P_n(x)\overline{Q_m(x)}\,d\mu(x) = \delta_{n,m}$$

and every $g \in L^2(I,d\mu)$ can be written as

$$g(t) = \sum_{n\in N} c_n(g)P_n(t), \qquad c_n(g) = \int_I g(t)\overline{Q_n(t)}\,d\mu(t).$$

Let us also define, in $L^2(\Omega,d\mu)$, the sequences of functions $\{S_n\}_{n\in N}$ and $\{T_n\}_{n\in N}$ given by

$$S_n(x) = \widetilde{\mathcal{K}}(\chi_I\overline{Q_n})(x), \quad x\in\Omega, \tag{11}$$

and

$$T_n(x) = \overline{\mathcal{K}(\chi_I P_n)(x)}, \quad x\in\Omega$$

(note that, if $P_n = Q_n$ then $S_n = T_n$).

Our purpose is to prove the following theorem, which says that it is still possible to find a bilinear expansion in this context.

Theorem 1. *For each $x\in\Omega$, the following expansion* holds in $L^2(I,d\mu)$:*

$$K(x,t) = \sum_{n\in N} P_n(t)S_n(x), \quad t\in I. \tag{12}$$

*The condition $t\in I$ in the identity (12) is *not* a mistake. Although $K(x,t)$ is defined on $\Omega\times\Omega$, the functions $P_n(t)$ are defined, in general, only on I.

Moreover, $\{S_n\}_{n\in N}$ and $\{T_n\}_{n\in N}$ are a pair of complete biorthogonal sequences in $\mathcal{P}$, in such a way that every $f \in \mathcal{P}$ can be written as

$$f(x) = \sum_{n\in N} c_n(f) S_n(x), \quad x \in \Omega, \tag{13}$$

with

$$c_n(f) = \int_\Omega f(t)\overline{T_n(t)}\, d\mu(t).$$

The convergence is uniform in every set where $\|K(x,\cdot)\|_{L^2(I,d\mu)}$ is bounded.

Proof: Let us start proving (12). By being $K(x,\cdot) \in L^2(I,d\mu)$ for every $x \in \Omega$, as $\{P_n\}_{n\in N}$ and $\{Q_n\}_{n\in N}$ are a complete biorthogonal system on $L^2(I,d\mu)$, we can write

$$K(x,t) = \sum_{n\in N} b_n(x) P_n(t)$$

with (by (11))

$$b_n(x) = \int_I K(x,t)\overline{Q_n(t)}\, d\mu(t) = \widetilde{\mathcal{K}}(\chi_I \overline{Q_n})(x) = S_n(x).$$

Now, let us prove the biortogonality of $\{S_n\}_{n\in N}$ and $\{T_n\}_{n\in N}$. By definition and the multiplicative formula, we have

$$\begin{aligned}\int_\Omega S_n \overline{T_m}\, d\mu &= \int_\Omega S_n \mathcal{K}(\chi_I P_m)\, d\mu \\ &= \int_\Omega \mathcal{K}(S_n)\chi_I P_m\, d\mu = \int_I \overline{Q_n} P_m\, d\mu = \delta_{n,m}.\end{aligned}$$

Finaly, for $f \in \mathcal{P}$, by applying (12), interchanging the sum and the integral, and using the multiplicative formula, we have

$$\begin{aligned} f(x) &= \int_I u(t)K(x,t)\, d\mu(t) = \sum_{n\in N} \Big(\int_I u(t)P_n(t)\, d\mu(t) \Big) S_n(x) \\ &= \sum_{n\in N} \Big(\int_\Omega (\mathcal{K}f)(t)\chi_I(t)P_n(t)\, d\mu(t) \Big) S_n(x) \\ &= \sum_{n\in N} \Big(\int_\Omega f(t)\mathcal{K}(\chi_I P_n)(t)\, d\mu(t) \Big) S_n(x) = \sum_{n\in N} \Big(\int_\Omega f(t)\overline{T_n(t)}\, d\mu(t) \Big) S_n(x) \end{aligned}$$

and the proof is finished. ■

3. The Fourier kernel

As an example to clarify the technique, and to show how useful is the use of the *biorthogonal* sequences, let us look at (1) and (3) in the light of the above scheme.

3.1. The classical sampling formula. With the notation of the above section, take $d\mu(x) = dx$, $\Omega = \mathbb{R}$, $I = [-1, 1]$ and the kernel $K(x, t) = \frac{1}{\sqrt{2\pi}} e^{ixt}$, so the operator $\mathcal{K}$ is the Fourier transform.

The space $\mathcal{P}$ becomes the classical Paley-Wiener space PW. Now, take $N = \mathbb{Z}$ and the functions

$$P_n(t) = Q_n(t) = \phi_n(t) = \frac{1}{\sqrt{2}} e^{i\pi nt}, \quad n \in \mathbb{Z}.$$

Then, $S_n(x)$ is

$$S_n(x) = \widetilde{\mathcal{K}}(\chi_I \overline{Q_n})(x) = \int_{-1}^{1} \frac{e^{ixt}}{\sqrt{2\pi}} \frac{e^{-i\pi nt}}{\sqrt{2}}\, dt = \frac{\sin(x - \pi n)}{\sqrt{\pi}(x - \pi n)}.$$

From this expression, by using (12) we obtain (1).

Moreover, using the identity

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin(x - \pi n)}{x - \pi n} f(x)\, dx = f(\pi n), \qquad f \in PW,$$

and (13), we deduce the classical Whittaker-Shannon-Koltel'nikov sampling theorem

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(x - \pi n)}{x - \pi n} f(\pi n).$$

3.2. Gegenbauer's plane wave expansion. As in the previous case, take $d\mu(x) = dx$, $\Omega = \mathbb{R}$, $I = [-1, 1]$, the kernel $K(x, t) = \frac{1}{\sqrt{2\pi}} e^{ixt}$, $\mathcal{K}$ the Fourier transform, and $\mathcal{P} = PW$.

But, this time, consider $N = \mathbb{N} \cup \{0\}$ and, using $C_n^\beta(t)$ to denote the Gegenbauer polynomial of order $\beta > -1/2$ (with the usual trick of employing the Chebyshev polynomials if $\beta = 0$), take the biorthonormal system

$$P_n(t) = C_n^\beta(t),$$
$$Q_n(t) = (1 - t^2)^{\beta - 1/2} C_n^\beta(t)/h_n$$

with

$$h_n = \int_{-1}^{1} C_n^\beta(t)^2 (1-t^2)^{\beta-1/2}\, dt = \frac{\pi^{1/2}\Gamma(\beta+1/2)\Gamma(2\beta+n)}{\Gamma(\beta)\Gamma(2\beta)(n+\beta)n!}.$$

Using the integral

$$\int_{\mathbb{R}} e^{-ixt} x^{-\beta} J_{\beta+n}(x)\, dx = \frac{2^{-\beta+1}\pi^{1/2}\Gamma(2\beta)(-1)^n i^n n!}{\Gamma(\beta+1/2)\Gamma(2\beta+n)} (1-t^2)^{\beta-1/2} C_n^\beta(t)\chi_{[-1,1]}(t)$$

(see [13, Ch. 3.3, (9), p. 123]) we deduce that

$$S_n(x) = 2^{\beta-1/2}\pi^{-1/2} i^n \Gamma(\beta)(\beta+n) x^{-\beta} J_{\beta+n}(x).$$

Then, (12) becomes (3).

Moreover, every function $f \in PW$ admits an expansion in a uniformly Fourier-Neumann series of the form

$$f(x) = 2^{\beta-1/2}\pi^{-1/2}\Gamma(\beta) \sum_{n=0}^{\infty} c_n(f) i^n (\beta+n) x^{-\beta} J_{\beta+n}(x),$$

with

$$c_n(f) = \int_{\mathbb{R}} f(t) \mathcal{K}(\chi_{[-1,1]} C_n^\beta)(t)\, dt,$$

corresponding to the expansion (13). A more explicit expression (see (33)) for the coefficients $c_n(f)$ will be given in the next section for some values of β. Finally, since

$$\|K(x,\cdot)\|_{L^2(I,dx)} = \frac{1}{\sqrt{2\pi}} \left\| e^{ix\cdot} \right\|_{L^2([-1,1],dx)} = \frac{1}{\sqrt{\pi}},$$

Remark 2 automatically ensure that the abovementioned expansions converge uniformly on the real line.

4. The Dunkl kernel on the real line

4.1. The Dunkl transform. For $\alpha > -1$, let J_α denote the Bessel function of order α and, for complex values of the variable z, let

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}$$

($\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α; see [30]). Moreover, let us take

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(z), \qquad z \in \mathbb{C}.$$

The Dunkl operators on $\mathbb{R}^n$ are differential-difference operators associated with some finite reflection groups (see [9]). We consider the Dunkl operator Λ_α, $\alpha \geq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x} \left(\frac{f(x) - f(-x)}{2} \right). \tag{14}$$

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), & x \in \mathbb{R}, \\ f(0) = 1 \end{cases} \tag{15}$$

has $E_\alpha(\lambda x)$ as its unique solution (see [10] and [21]); this function is called the Dunkl kernel. For $\alpha = -1/2$, it is clear that $\Lambda_{-1/2} = d/dx$, and $E_{-1/2}(\lambda x) = e^{\lambda x}$.

Let $d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}\,dx$ and write

$$E_\alpha(ix) = 2^\alpha \Gamma(\alpha+1) \left(\frac{J_\alpha(x)}{x^\alpha} + \frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, xi \right). \tag{16}$$

In a similar way to the Fourier transform (which is the particular case $\alpha = -1/2$), the Dunkl transform of order $\alpha \geq -1/2$ is given by

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-iyx)\, d\mu_\alpha(x), \quad y \in \mathbb{R}, \tag{17}$$

for $f \in L^1(\mathbb{R}, d\mu_\alpha)$. By means of the Schwartz class $\mathcal{S}(\mathbb{R})$, the definition is extended to $L^2(\mathbb{R}, d\mu_\alpha)$ in the usual way. In [21], it is showed that $\mathcal{F}_\alpha$ is an isometric isomorphism on $L^2(\mathbb{R}, d\mu_\alpha)$ and that

$$\mathcal{F}_\alpha^{-1} f(y) = \mathcal{F}_\alpha f(-y)$$

for functions such that $f, \mathcal{F}_\alpha f \in L^1(\mathbb{R}, d\mu_\alpha)$.

From Fubini's theorem, it follows that the Dunkl transform satisfy the multiplication formula

$$\int_{\mathbb{R}} u(y) \mathcal{F}_\alpha v(y)\, d\mu_\alpha(y) = \int_{\mathbb{R}} \mathcal{F}_\alpha u(y) v(y)\, d\mu_\alpha(y). \tag{18}$$

Finally, let us take into account that the Dunkl transform $\mathcal{F}_\alpha$ can also be defined in $L^2(\mathbb{R}, d\mu_\alpha)$ for $-1 < \alpha < -1/2$, although the expression (17) is no longer valid for $f \in L^1(\mathbb{R}, d\mu_\alpha)$ in general. However, it preserves the same properties in $L^2(\mathbb{R}, d\mu_\alpha)$; see [24] for details. This allows us to extend our study to the case $\alpha > -1$.

4.2. The sampling theorem related to the Dunkl transform. In our general set up developed in subsection 2.2, let us start taking $\Omega = \mathbb{R}$, $I = [-1,1]$, $d\mu = d\mu_\alpha$, with $\alpha > -1$, and $L^2(I, d\mu) = L^2([-1,1], d\mu_\alpha)$. On this space, we consider the kernel $K(x,t) = E_\alpha(ixt)$, so the corresponding operator is $\mathcal{K} = \mathcal{F}_\alpha$, i.e., the abovementioned Dunkl transform.

Now, as usual in sampling theory, we take the space of Paley-Wiener type that, under our setting, is defined as

$$PW_\alpha = \left\{ f \in L^2(\mathbb{R}, d\mu_\alpha) : f(x) = \int_{-1}^{1} u(t) E_\alpha(ixt)\, d\mu_\alpha(t),\ u \in L^2([-1,1], d\mu_\alpha) \right\} \tag{19}$$

endowed with the norm of $L^2(\mathbb{R}, d\mu_\alpha)$. (This space is characterized in [4, Theorem 5.1] as being the space of entire functions of exponential type 1 that belong to $L^2(\mathbb{R}, d\mu_\alpha)$ when restricted to the real line.) Then, take, of course $\mathcal{P} = PW_\alpha$.

It is well-known that the Bessel function $J_{\alpha+1}(x)$ has an increasing sequence of positive zeros $\{s_n\}_{n\ge1}$. Consequently, the real function $\operatorname{Im}(E_\alpha(ix)) = \frac{x}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(ix)$ is odd and it has an infinite sequence of zeros $\{s_n\}_{n\in\mathbb{Z}}$ (with $s_{-n} = -s_n$ and $s_0 = 0$). Then, following [8], let us define the functions

$$e_{\alpha,n}(t) = d_n E_\alpha(is_n t), \quad n \in \mathbb{Z}, \quad t \in [-1,1], \tag{20}$$

where

$$d_n = \frac{2^{\alpha/2}(\Gamma(\alpha+1))^{1/2}}{|\mathcal{I}_\alpha(is_n)|}, \quad n \neq 0, \qquad d_0 = 2^{(\alpha+1)/2}(\Gamma(\alpha+2))^{1/2}.$$

With this notation, the sequence of functions $\{e_{\alpha,n}\}_{n\in\mathbb{Z}}$ is a complete orthonormal system in $L^2([-1,1], d\mu_\alpha)$, for $\alpha > -1$. Thus, let us take $N = \mathbb{Z}$ and $P_n(t) = Q_n(t) = e_{\alpha,n}(t)$.

On the other hand, let us use that, for $x, y \in \}$, $x \neq y$ and $\alpha > -1$, we have

$$\int_{-1}^{1} E_\alpha(ixt)\overline{E_\alpha(iyt)}\, d\mu_\alpha(t) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(iy) - y\mathcal{I}_{\alpha+1}(iy)\mathcal{I}_\alpha(ix)}{x-y} \tag{21}$$

(the proof can be found in [5] or [8]). Then,

$$\begin{aligned} S_n(x) = \widetilde{\mathcal{K}}(\chi_{[-1,1]}\overline{Q_n})(x) &= \int_{-1}^{1} E_\alpha(ixt)\overline{e_{\alpha,n}(t)}\, d\mu_\alpha(t) \\ &= \frac{d_n}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(is_n) - s_n\mathcal{I}_{\alpha+1}(is_n)\mathcal{I}_\alpha(ix)}{x-s_n} \\ &= \frac{d_n}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(is_n)}{x-s_n} \end{aligned}$$

because $\mathcal{I}_{\alpha+1}(is_n) = 0$. Consequently,

$$\begin{aligned} E_\alpha(ixt) &= \sum_{n\in\mathbb{Z}} e_{\alpha,n}(t) \frac{d_n}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(is_n)}{x-s_n} \\ &= \mathcal{I}_{\alpha+1}(ix) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{E_\alpha(is_nt)}{2(\alpha+1)\mathcal{I}_\alpha(is_n)} \frac{x\mathcal{I}_{\alpha+1}(ix)}{x-s_n}, \end{aligned}$$

which corresponds to the formula (12) in Theorem 1.

Finally, the formula (13) in Theorem 1 says that, if $f \in PW_\alpha$, then f has the representation

$$f(x) = f(s_0)\mathcal{I}_{\alpha+1}(ix) + \sum_{n\in\mathbb{Z}\setminus\{0\}} f(s_n) \frac{x\mathcal{I}_{\alpha+1}(ix)}{2(\alpha+1)\mathcal{I}_\alpha(is_n)(x-s_n)}, \tag{22}$$

that converges in the norm of $L^2(\mathbb{R}, d\mu_\alpha)$. This is so because

$$\frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \int_{\mathbb{R}} \frac{tI_{\alpha+1}(it)I_\alpha(is_n)}{t-s_n} f(t)\, d\mu_\alpha(t) = f(s_n).$$

Moreover, by using L'Hopital rule in (21), it is not difficult to check that

$$\begin{aligned} \left\| \frac{E_\alpha(x\cdot)}{(x\cdot)^\alpha} \right\|^2_{L^2([-1,1],d\mu_\alpha)} &= \int_{-1}^{1} |E_\alpha(ixr)|^2\, d\mu_\alpha(r) \\ &= \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \Big(\frac{x^2}{2(\alpha+1)} \mathcal{I}^2_{\alpha+1}(ix) \\ &\qquad - (2\alpha+1)\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_\alpha(ix) + 2(\alpha+1)\mathcal{I}^2_\alpha(ix) \Big), \end{aligned}$$

and this norm is bounded on every compact set on the real line. So, by Remark 2, the series (22) converges uniformly in compact subsets of $\mathbb{R}$. (22)

is the sampling theorem related to the Dunkl transform that has been established in [8].

4.3. Fourier-Neumann type expansion. Following [11, Definition 1.5.5, p. 27], let us introduce the generalized Gegenbauer polynomials $C^{(\lambda,\nu)}(t)$ for $\lambda > -1/2$, $\nu \geq 0$ and $n \geq 0$ (the case $\nu = 0$ corresponding with the ordinary Gegenbauer polynomials; actually, for convenience with the notation of this paper, we are going to use $C_n^{(\beta+1/2,\alpha+1/2)}(x)$. In this way, for $\beta > -1$ and $\alpha \geq -1/2$, the generalized Gegenbauer polynomials are defined by

$$C_{2n}^{(\beta+1/2,\alpha+1/2)}(t) = (-1)^n \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(1-2t^2), \tag{23}$$

$$C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t) = (-1)^n \frac{(\alpha+\beta+1)_{n+1}}{(\alpha+1)_{n+1}} t P_n^{(\alpha+1,\beta)}(1-2t^2), \tag{24}$$

where in the coefficients we are using the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. Note that there is not problem to extend the definition of the generalized Gegenbauer polynomials taking $\alpha > -1$, so we will assume this situation.

From the L^2-norm of the Jacobi polynomials (see [14, Ch. 16.4, (5), p. 285]), it is easy to find

$$\begin{aligned} h_{2n}^{(\beta,\alpha)} &= \int_{-1}^{1} \left[C_{2n}^{(\beta+1/2,\alpha+1/2)}(t)\right]^2 (1-t^2)^\beta \, d\mu_\alpha(t) \\ &= \frac{1}{2^{\alpha+1}} \frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+1)^2\Gamma(\alpha+n+1)\, n!}, \end{aligned} \tag{25}$$

$$\begin{aligned} h_{2n+1}^{(\beta,\alpha)} &= \int_{-1}^{1} \left[C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t)\right]^2 (1-t^2)^\beta \, d\mu_\alpha(t) \\ &= \frac{1}{2^{\alpha+1}} \frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+2)}{(\alpha+\beta+2n+2)\Gamma(\alpha+\beta+1)^2\Gamma(\alpha+n+2)\, n!}. \end{aligned} \tag{26}$$

Finally, given $\alpha > -1$, we define the functions by

$$\mathcal{J}_{\alpha,n}(x) = \frac{J_{\alpha+n+1}(x)}{x^{\alpha+1}}, \qquad x \in \mathbb{R}, \quad n = 0, 1, 2, \dots;$$

as these functions arise in Fourier-Neumann series, we will allude to $\mathcal{J}_{\alpha,n}(x)$ with the name of Neumann funtions.† From the identities

$$\int_0^\infty \frac{J_a(x)J_b(x)}{x}\,dx = \frac{2}{\pi}\frac{\sin((b-a)\pi/2)}{b^2-a^2}, \quad a>0,\ b>0,\ a\neq b,$$

$$\int_0^\infty \frac{J_a(x)^2}{x}\,dx = \frac{1}{2a}, \quad a>0,$$

and taking into account that $\mathcal{J}_{\alpha,n}(x)$ is even or odd according n is even or odd, respectively, it follows that $\{\mathcal{J}_{\alpha,n}\}_{n\geq 0}$ is an orthogonal system on $L^2(\mathbb{R}, d\mu_\alpha)$, namely,

$$\int_{\mathbb{R}} \mathcal{J}_{\alpha,n}(x)\mathcal{J}_{\alpha,m}(x)\,d\mu_\alpha(x) = \frac{\delta_{n,m}}{2^{\alpha+1}\Gamma(\alpha+1)(\alpha+n+1)}.$$

Generalized Gegenbauer polynomials and Neumann functions are the main ingredients to obtain the Dunkl analogue of Gegenbauer's expansion of the plane wave. To establish this result we need a relation between them. By using the notation

$$\mathcal{P}_n^{(\alpha,\beta)}(t) = C_n^{(\beta+1/2,\alpha+1/2)}(t), \tag{27}$$

$$\mathcal{Q}_n^{(\alpha,\beta)}(t) = \left(h_n^{(\beta,\alpha)}\right)^{-1}(1-t^2)^\beta C_n^{(\beta+1/2,\alpha+1/2)}(t) \tag{28}$$

(where $h_n^{(\beta,\alpha)}$ is given in (25) and (26)), this relation is given in the following lemma that, moreover, can have independent interest:

Lemma 1. *Let $\alpha,\beta>-1$, $\alpha+\beta>-1$, and $k=0,1,2,\dots$. The Dunkl transform of order α of $\mathcal{J}_{\alpha+\beta,k}(x)$ is*

$$\mathcal{F}_\alpha(\mathcal{J}_{\alpha+\beta,k})(t) = \frac{(-i)^k}{2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)(\alpha+\beta+k+1)}\,\mathcal{Q}_k^{(\alpha,\beta)}(t)\chi_{[-1,1]}(t). \tag{29}$$

Moreover, if $\beta<1$, we have‡

$$\mathcal{F}_\alpha(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,k})(t) = \frac{2^\beta\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)}(-i)^k\mathcal{P}_k^{(\alpha,\beta)}(t), \quad t\in[-1,1]. \tag{30}$$

We postpone the proof of the previous lemma to subsection 7.2. Now, we have all the tools to prove

†In the literature, the name "Neumann funtions" is sometimes used to call the Bessel functions of the second kind $Y_\alpha(x)$, but these functions will not arise in this paper.

‡Observe that nothing is said outside the interval $[-1,1]$; it does not means that this expression vanishes for $|t|>1$, that is not true when $\beta\neq 0$.

Theorem 2. *Let $\alpha, \beta > -1$ and $\alpha + \beta > -1$. Then for each $x \in \mathbb{R}$ the following expansion holds in $L^2([-1,1], d\mu_\alpha)$:*

$$E_\alpha(ixt) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)\sum_{n=0}^{\infty} i^n(\alpha+\beta+n+1)\mathcal{J}_{\alpha+\beta,n}(x)C_n^{(\beta+1/2,\alpha+1/2)}(t). \tag{31}$$

Moreover, for $\beta < 1$ and $f \in PW_\alpha$, we have the orthogonal expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(f)(\alpha+\beta+n+1)\mathcal{J}_{\alpha+\beta,n}(x) \tag{32}$$

with

$$a_n(f) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)\int_{\mathbb{R}} f(t)\mathcal{J}_{\alpha+\beta,n}(t)\,d\mu_{\alpha+\beta}(t). \tag{33}$$

Furthermore, the series converges uniformly in compact subsets of $\mathbb{R}$.

Proof: In the biorthogonal set up given in subsection 2.2, let $\Omega = \mathbb{R}$, $I = [-1,1]$, the space $L^2(I, d\mu) = L^2([-1,1], d\mu_\alpha)$, and the kernel $K(x,t) = E_\alpha(ixt)$, from which the operator $\mathcal{K}$ becomes the Dunkl transform $\mathcal{F}_\alpha$ (and $\widetilde{K} = \mathcal{F}_\alpha^{-1}$). Also, consider the Paley-Wiener space $\mathcal{P} = PW_\alpha$ (see (19)). Finally, for $N = \mathbb{N} \cup \{0\}$, take the biorthonormal system given by $P_n(t) = \mathcal{P}_n^{(\alpha,\beta)}(t)$ and $Q_n(t) = \mathcal{Q}_n^{(\alpha,\beta)}(t)$ as in (27) and (28). From (29), we have

$$S_n(x) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)i^n(\alpha+\beta+n+1)\mathcal{J}_{\alpha+\beta,n}(x).$$

Under this situation, the formula (12) in Theorem 1 gives (31).

Now, let us consider

$$T_n(x) = \overline{\mathcal{K}(\chi_{[-1,1]}P_n)(x)} = \overline{\mathcal{F}_\alpha(\chi_{[-1,1]}\mathcal{P}_n^{(\alpha,\beta)})(x)}.$$

Then, the identity given by (13) becomes

$$f(x) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)\sum_{n=0}^{\infty} c_n(f)i^n(\alpha+\beta+n+1)\mathcal{J}_{\alpha+\beta,n}(x), \quad f \in PW_\alpha,$$

with

$$c_n(f) = \int_{\mathbb{R}} f(t)\mathcal{F}_\alpha(\chi_{[-1,1]}\mathcal{P}_n^{(\alpha,\beta)})(t)\,d\mu_\alpha(t).$$

Let us see that, when $\beta < 1$, the coefficient $c_n(f)$ can be written as

$$c_n(f) = (-i)^n\int_{\mathbb{R}} f(t)\mathcal{J}_{\alpha+\beta,n}(t)\,d\mu_{\alpha+\beta}(t),$$

which implies (33) with $a_n(f) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1) i^n c_n(f)$.

Indeed, if we consider u such that $f = \mathcal{F}_\alpha^{-1}(u\chi_{[-1,1]})$ and use the multiplication formula (18), we write

$$c_n(f) = \int_{-1}^{1} u(x)\mathcal{P}_n^{(\alpha,\beta)}(x)\, d\mu_\alpha(x).$$

Now, by (30) and the multiplication formula again, we have

$$\begin{aligned} c_n(f) &= \int_{-1}^{1} u(x)\frac{i^n\Gamma(\alpha+1)}{2^\beta\Gamma(\alpha+\beta+1)}\mathcal{F}_\alpha(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,n})(x)\, d\mu_\alpha(x) \\ &= i^n \int_{\mathbb{R}} \mathcal{F}_\alpha(u\chi_{[-1,1]})(t)\mathcal{J}_{\alpha+\beta,n}(t)\, d\mu_{\alpha+\beta}(t). \end{aligned}$$

It is clear that $\mathcal{F}_\alpha(u\chi_{[-1,1]})(t) = f(-t)$, so the change of variable t by $-t$ gives

$$c_n(f) = (-i)^n \int_{\mathbb{R}} f(t)\mathcal{J}_{\alpha+\beta,n}(t)\, d\mu_{\alpha+\beta}(t)$$

because $\mathcal{J}_{\alpha+\beta,n}(-t) = (-1)^n\mathcal{J}_{\alpha+\beta,n}(t)$. ∎

Remark 3. Clearly, formulas (3) and (31) are equivalents for $\alpha \geq -1/2$. The proof in one direction is clear, just by specializing the parameters. To obtain (31) from (3), we can use the intertwining operator

$$V_\alpha g(x) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1/2)\Gamma(\alpha+3/2)}\int_{-1}^{1} g(sx)(1-s)^{\alpha-1/2}(1+s)^{\alpha+1/2}\, ds$$

(see [11, Definition 1.5.1, p. 24], we change the parameter μ in the definition given in [11] by $\alpha+1/2$), defined for $\alpha \geq -1/2$. With this notation we have

$$V_\alpha C_n^{\alpha+\beta+1}(x) = C_n^{(\beta+1/2,\alpha+1/2)}(x)$$

and

$$V_\alpha e^{i\cdot}(x) = E_\alpha(ix).$$

In this way, applying V_α to (3) (with $\alpha+\beta+1$ instead of β) we get (31). This idea has been used for the higher rank in [25].

4.4. Consequences for the Hankel transform. For $\alpha > -1$, consider the so-called modified Hankel transform H_α, that is

$$H_\alpha f(y) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha} f(y) x^{2\alpha+1}\, dx, \quad x > 0. \tag{34}$$

The kernel $E_\alpha(ixt)$ of the Dunkl transform (17) can be written in term of the Bessel functions of order α and $\alpha+1$, and this clearly allows to study the Hankel transform as a simple consequence of the Dunk transform. In particular, if we have a function $f \in L^2((0,\infty), x^{2\alpha+1}\, dx)$, we can take the even extension $f(|\cdot|) \in L^2(\mathbb{R}, d\mu_\alpha)$. Then, using that $J_\alpha(x)/x^\alpha$ is even and $J_{\alpha+1}(x)/x^\alpha$ is odd, write (34) as

$$H_\alpha f(y) = \mathcal{F}_\alpha(f(|\cdot|))(y).$$

The Paley-Wiener space for the Hankel transform is given by

$$PW'_\alpha = \Bigg\{ f \in L^2((0,\infty), x^{2\alpha+1}\, dx) : f(t) = \int_0^1 u(x) \frac{J_\alpha(xt)}{(xt)^\alpha} x^{2\alpha+1}\, dx, \\ u \in L^2((0,1), x^{2\alpha+1}\, dx) \Bigg\};$$

also, note that, if $f \in PW'_\alpha$, then both the even extension $f(|\cdot|)$ and the odd extension $\operatorname{sgn}(\cdot) f(|\cdot|)$ belong to PW_α.

So, let us adapt the sampling formula of subsection 4.2 and the Theorem 2 of subsection 4.3 to the context of the Hankel transform.

4.4.1. *The sampling formula for the Hankel transform.* For $f \in PW'_\alpha$, taking its even extension $f(|\cdot|)$, using that $s_{-n} = -s_n$ and grouping the summands corresponding to $1/(x - s_n)$ and $1/(x + s_n)$ in (22), we get

$$f(x) = f(s_0)\mathcal{I}_{\alpha+1}(ix) + \sum_{n=1}^\infty f(s_n) \frac{\mathcal{I}_{\alpha+1}(ix)}{(\alpha+1)\mathcal{I}_\alpha(is_n)} \frac{x^2}{x^2 - s_n^2}.$$

Similarly, with the odd extension of f, (22) becomes

$$f(x) = \sum_{n=1}^\infty f(s_n) \frac{\mathcal{I}_{\alpha+1}(ix)}{(\alpha+1)\mathcal{I}_\alpha(is_n)} \frac{s_n^2}{x^2 - s_n^2}.$$

The latter identity corresponds to the well-known Higgins sampling theorem for the Hankel transform [17].

4.4.2. *A version of the Theorem 2 for the Hankel transform.* Let us observe that

$$\frac{J_\alpha(xt)}{(xt)^\alpha} = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\big(E_\alpha(ixt) + \overline{E_\alpha(ixt)}\big).$$

From this, it is very easy to adapt (31) to the new context, and to write it in terms of Jacobi polynomials by using (23). Given $f \in PW'_\alpha$, let us consider its even extension $f(|\cdot|) \in PW_\alpha$. Applying (32) to this even function, it becomes an expansion that only contains $\mathcal{J}_{\alpha+\beta,2n}(x) = J_{\alpha+\beta+2n+1}(x)/x^{\alpha+\beta+1}$ (i.e., only with even indexes).

Thus, the results corresponding to the Hankel transforms can be summarized in this way:

Corollary 3. *Let $\alpha, \beta > -1$ and $\alpha+\beta > -1$. Then for each $x \in (0,\infty)$ the following expansion holds in $L^2((0,1), x^{2\alpha+1}\,dx)$:*

$$\frac{J_\alpha(xt)}{(xt)^\alpha} = \sum_{n=0}^\infty \frac{2^{\beta+1}(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} \mathcal{J}_{\alpha+\beta,2n}(x) P_n^{(\alpha,\beta)}(1-2t^2).$$

Moreover, for $\beta < 1$ and $f \in PW'_\alpha$, we have the orthogonal expansion

$$f(x) = \sum_{n=0}^\infty a_n(f)(\alpha+\beta+2n+1)\mathcal{J}_{\alpha+\beta,2n}(x)$$

with

$$a_n(f) = 2\int_0^\infty f(t)\mathcal{J}_{\alpha+\beta,2n}(t)\, t^{2\alpha+2\beta+1}\,dt.$$

Furthermore, the series converges uniformly in compact subsets of $(0,\infty)$.

Let us conclude observing than the L^p convergence of the orthogonal series that appear in the previous corollary has been studied in the papers [29, 7] for functions in an appropriate L^p extension of the Paley-Wiener space.

5. Diagonalization of the right inverse of Dunkl operator

In [20], Ismail and Zhang study the eigenfunctions and the eigenvalues of the right inverse of the derivative operator, the main tool is a suitable expansion of the corresponding plane wave. The aim of this section is the analysis of this question for the Dunkl operator (14) using the expansion (31). This is a very natural extension, because the ordinary derivative corresponds to the Dunkl operator Λ_α with $\alpha = -1/2$.

To simplify the notation, let us introduce the measure

$$d\mu_{\beta,\alpha}(t) = (1-t^2)^{\beta}\, d\mu_{\alpha}(t).$$

It is not difficult to prove, using the appropriate recurrence relations for the Jacobi polynomials, that the Dunkl operator over the generalized Gegenbauer operator satisfies

$$\Lambda_{\alpha} C_n^{(\beta+1/2,\alpha+1/2)}(t) = 2(\alpha+\beta+1) C_{n-1}^{(\beta+3/2,\alpha+1/2)}(t). \tag{35}$$

Motivated by the identity (35), we define the right inverse operator of Λ_α over $L^2((-1,1), d\mu_{\beta+1,\alpha})$ by

$$T_{\beta,\alpha} g(t) = \sum_{n=1}^{\infty} \frac{g_{n-1}}{2(\alpha+\beta+1)}\, C_n^{(\beta+1/2,\alpha+1/2)}(t),$$

where g has the expansion

$$g(t) = \sum_{n=0}^{\infty} g_n C_n^{(\beta+3/2,\alpha+1/2)}(t),$$

with

$$g_n = \left(h_n^{(\beta+1,\alpha)}\right)^{-1} \int_{-1}^{1} g(r) C_n^{(\beta+3/2,\alpha+1/2)}(r)\, d\mu_{\beta+1,\alpha}(r)$$

(the norms $h_n^{(\beta+1,\alpha)}$ are given in (25) and (26)). Then, we can write

$$T_{\beta,\alpha} g(t) = \int_{-1}^{1} g(r) K_{\beta,\alpha}(t,r)\, d\mu_{\beta+1,\alpha}(r),$$

with

$$K_{\beta,\alpha}(t,r) = \frac{1}{2(\alpha+\beta+1)} \sum_{n=1}^{\infty} \frac{C_n^{(\beta+1/2,\alpha+1/2)}(t) C_{n-1}^{(\beta+3/2,\alpha+1/2)}(r)}{h_{n-1}^{(\beta+1,\alpha)}}.$$

We want to diagonalize $T_{\beta,\alpha}$, to this end we have to find values $\lambda \in \mathbb{C}$ and functions $g_\lambda \in L^2((-1,1), d\mu_{\beta,\alpha}) \cap L^2((-1,1), d\mu_{\beta+1,\alpha})$ such that

$$\lambda g_\lambda(t) = T_{\beta,\alpha} g_\lambda(t), \tag{36}$$

where the expansion of g in $\{C_n^{(\beta+1/2,\alpha+1/2)}\}_{n\ge 0}$ is

$$g_\lambda(t) \sim \sum_{n=1}^{\infty} a_n(\lambda) C_n^{(\beta+1/2,\alpha+1/2)}(t). \tag{37}$$

To find the eigenvalues and the eigenfunctions we start obtaining a recurrence relation for the coefficients $\{a_n(\lambda)\}_{n\geq 1}$. This is done by using an expression to write $(1-r^2)C_{n-1}^{(\beta+3/2,\alpha+1/2)}(r)$ in terms of $C_{n-1}^{(\beta+1/2,\alpha+1/2)}(r)$ and $C_{n+1}^{(\beta+1/2,\alpha+1/2)}(r)$. This relation can be deduced from the identity for the Jacobi polynomials (see [1, p. 782, formula 22.7.16])

$$P_n^{(a,b+1)}(z) = \frac{2(n+b+1)P_n^{(a,b)}(z) + 2(n+1)P_{n+1}^{(a,b)}(z)}{(2n+a+b+2)(1+z)}. \tag{38}$$

Indeed, taking $z = 1-2t^2$ in (38) and using the definition of the generalized Gegenbauer polynomials, we have

$$(\alpha+\beta+1)(1-r^2)C_{n-1}^{(\beta+3/2,\alpha+1/2)}(r) = A_n C_{n-1}^{(\beta+1/2,\alpha+1/2)}(r) - B_n C_{n+1}^{(\beta+1/2,\alpha+1/2)}(r), \tag{39}$$

with

$$A_n = \begin{cases} \dfrac{(\beta+k+1)(\alpha+\beta+k+1)}{\alpha+\beta+2k+2}, & \text{if } n = 2k+1, \\ \dfrac{(\beta+k)(\alpha+\beta+k+1)}{\alpha+\beta+2k+1}, & \text{if } n = 2k, \end{cases}$$

$$B_n = \begin{cases} \dfrac{(k+1)(\alpha+k+1)}{\alpha+\beta+2k+2}, & \text{if } n = 2k+1, \\ \dfrac{k(\alpha+k+1)}{\alpha+\beta+2k+1}, & \text{if } n = 2k. \end{cases}$$

From (39) we have the decomposition

$$\begin{aligned} (1-r^2)K_{\beta,\alpha}(t,r) &= \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=1}^{\infty} A_n \frac{C_n^{(\beta+1/2,\alpha+1/2)}(t)C_{n-1}^{(\beta+1/2,\alpha+1/2)}(r)}{h_{n-1}^{(\beta+1,\alpha)}} \\ &\quad - \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=1}^{\infty} B_n \frac{C_n^{(\beta+1/2,\alpha+1/2)}(t)C_{n+1}^{(\beta+1/2,\alpha+1/2)}(r)}{h_{n-1}^{(\beta+1,\alpha)}}. \end{aligned}$$

With the previous identity, we get

$$\begin{aligned} T_{\beta,\alpha}g_\lambda(t) &= \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=2}^{\infty} a_{n-1}(\lambda) A_n \frac{h_{n-1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}} C_n^{(\beta+1/2,\alpha+1/2)}(t) \\ &\quad - \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=1}^{\infty} a_{n+1}(\lambda) B_n \frac{h_{n+1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}} C_n^{(\beta+1/2,\alpha+1/2)}(t). \end{aligned}$$

In this way, identifying the coefficients in both side of (36), we obtain the recurrence relation

$$\lambda a_n(\lambda) = \frac{1}{2(\alpha+\beta+1)^2}\left(a_{n-1}(\lambda)A_n\frac{h_{n-1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}} - a_{n+1}(\lambda)B_n\frac{h_{n+1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}}\right), \qquad n > 1,$$

and

$$\lambda a_1(\lambda) = -a_2(\lambda)\frac{B_1}{2(\alpha+\beta+1)^2}\frac{h_2^{(\beta,\alpha)}}{h_0^{(\beta+1,\alpha)}},$$

which, applying (25) and (26), becomes

$$\lambda a_n(\lambda) = \frac{a_{n-1}(\lambda)}{2(\alpha+\beta+n)} - \frac{a_{n+1}(\lambda)}{2(\alpha+\beta+n+2)}, \qquad n > 1, \tag{40}$$

and

$$\lambda a_1(\lambda) = -\frac{a_2(\lambda)}{2(\alpha+\beta+3)}. \tag{41}$$

Now, we can prove the following

Theorem 4. *For $\alpha, \beta > -1$, and $\alpha+\beta > -1$, let $R_{\beta+1/2,\alpha+1/2}$ be the closure of the span of $\{C_n^{(\beta+1/2,\alpha+1/2)}\}_{n\ge 1}$ in $L^2((-1,1), d\mu_{\beta+1,\alpha})$; then*

$$L^2((-1,1), d\mu_{\beta+1,\alpha}) = R_{\beta+1/2,\alpha+1/2} \oplus R_{\beta+1/2,\alpha+1/2}^{\perp}, \tag{42}$$

where

$$R_{\beta+1/2,\alpha+1/2}^{\perp} = \begin{cases} \operatorname{span}\{(1-t^2)^{-1}\}, & \text{for } \beta > 0, \\ \{0\}, & \text{for } 0 \ge \beta > -1. \end{cases} \tag{43}$$

Furthermore, if we let $g_\lambda(x) \in R_{\beta+1/2,\alpha+1/2}$ have the orthogonal expansion (37)*, then the eigenvalue equation* (36) *holds if and only if*

$$\sum_{n=1}^{\infty} |a_n(\lambda)|^2 n^{2\beta-1} < \infty. \tag{44}$$

Proof: It is clear that $L^2((-1,1), d\mu_{\beta,\alpha}) \subset L^2((-1,1), d\mu_{\beta+1,\alpha})$ and $T_{\beta,\alpha}$ maps $L^2((-1,1), d\mu_{\beta+1,\alpha})$ into $L^2((-1,1), d\mu_{\beta,\alpha})$. Moreover, $T_{\beta,\alpha}$ is a bounded operator and its norm is controlled by a constant M, given by

$$M^2 = \frac{1}{4(\alpha+\beta+1)^2}\sup_{n\ge 0}\frac{h_{n+1}^{(\beta,\alpha)}}{h_n^{(\beta+1,\alpha)}}.$$

Indeed,

$$\|T_{\beta,\alpha}g\|^2_{L^2((-1,1),d\mu_{\beta,\alpha})} = \sum_{n=1}^{\infty} \frac{|g_{n-1}|^2}{4(\alpha+\beta+1)^2} h_n^{(\beta,\alpha)} \leq M^2 \sum_{n=1}^{\infty} |g_n|^2 h_n^{(\beta+1,\alpha)} = M^2 \|g\|_{L^2((-1,1),d\mu_{\beta+1,\alpha})}.$$

Note that M is finite because

$$\frac{h_{2k+1}^{(\beta,\alpha)}}{h_{2k}^{(\beta+1,\alpha)}} = \frac{(\alpha+\beta+1)^2}{(\beta+k+1)(\alpha+k+1)} \qquad \text{and} \qquad \frac{h_{2k}^{(\beta,\alpha)}}{h_{2k-1}^{(\beta+1,\alpha)}} = \frac{(\alpha+\beta+1)^2}{k(\alpha+\beta+k+1)}.$$

In this way, we can deduce that $R_{\beta+1/2,\alpha+1/2}$ is an invariant subspace for $T_{\beta,\alpha}$ and the decomposition (42) holds. Now, each function $f \in R^{\perp}_{\beta+1/2,\alpha+1/2}$ will satisfy the conditions $\|f\|_{L^2((-1,1),d\mu_{\beta+1,\alpha})} < \infty$ and

$$\int_{-1}^{1} f(r) C_n^{(\beta+1/2,\alpha+1/2)}(r)\, d\mu_{\beta+1,\alpha}(r) = 0, \qquad n = 1, 2, \dots. \tag{45}$$

Using that $\{C_n^{(\beta+1/2,\alpha+1/2)}\}_{n\geq 0}$ is a complete and orthogonal system in $L^2((-1,1),d\mu_{\beta,\alpha})$ and $d\mu_{\beta+1,\alpha}(r) = (1-r^2)\, d\mu_{\beta,\alpha}(r)$, from (45) we get

$$(1-r^2) f(r) = K$$

for a certain constant K. So, taking into account that $f \in L^2((-1,1),d\mu_{\beta+1,\alpha})$, we conclude (43).

Let g_λ be a function having the expansion (37) and verifying (36). Then, using that $g_\lambda \in L^2((-1,1),d\mu_{\beta,\alpha})$

$$\|g_\lambda\|^2_{L^2((-1,1),d\mu_{\beta,\alpha})} = \sum_{n=1}^{\infty} |a_n(\lambda)|^2 h_n^{(\beta,\alpha)} < \infty$$

and this imply (44) because $h_n^{(\beta,\alpha)} \sim n^{2\beta-1}$.

On other hand, let us suppose that (44) and (37) hold. To prove that g_λ is a solution of (36), we need rewrite the function g_λ in terms of $C_n^{(\beta+3/2,\alpha+1/2)}$. To this end, we use the identity for the Jacobi polynomials [1, p. 782, formula 22.7.19]

$$(2n+a+b) P_n^{(a,b-1)}(z) = (n+a+b) P_n^{(a,b)}(z) + (n+a) P_{n-1}^{(a,b)}(z)$$

to produce

$$C_n^{(\beta+1/2,\alpha+1/2)}(t) = \frac{\alpha+\beta+1}{\alpha+\beta+n+1}(C_n^{(\beta+3/2,\alpha+1/2)}(t) - C_{n-2}^{(\beta+3/2,\alpha+1/2)}(t)).$$

So, we find

$$g_\lambda(t) = \sum_{n=1}^{\infty} \frac{\alpha+\beta+1}{\alpha+\beta+n+1} a_n(\lambda) C_n^{(\beta+3/2,\alpha+1/2)}(t) \\ - \sum_{n=1}^{\infty} \frac{\alpha+\beta+1}{\alpha+\beta+n+1} a_n(\lambda) C_{n-2}^{(\beta+3/2,\alpha+1/2)}(t). \quad (46)$$

Thus

$$\|g_\lambda\|^2_{L^2((-1,1),d\mu_{\beta+1,\alpha})} \sim \sum_{n=1}^{\infty} |a_n(\lambda)|^2 n^{2\beta-1} < \infty$$

and $g_\lambda \in L^2((-1,1), d\mu_{\beta+1,\alpha})$. Moreover, from (46), with (40) and (41), we can check that g_λ is an eigenfunction of $T_{\beta,\alpha}$. ■

It is clear, from (40) and (41), that $a_1(\lambda) \neq 0$ (in other case $a_n(\lambda) = 0$ for $n > 1$). So this is a multiplicative factor and can be factored out. To verify the condition (44), we have to renormalize the sequence $\{a_n(\lambda)\}_{n\geq 1}$. Set

$$a_n(\lambda) = i^{n-1} \frac{(\alpha+\beta+n+1)}{(\alpha+\beta+2)} b_{n-1}(i\lambda) a_1(\lambda). \quad (47)$$

Then, using the relations (40) and (41), we can check that

$$2w(\alpha+\beta+n+2) b_n(w) = b_{n-1}(w) + b_{n+1}(w), \qquad n \geq 1, \quad (48)$$

where $w = i\lambda$, $b_{-1}(w) = 0$, and $b_0(w) = 1$. If $R_{n,a}(z)$ denotes the Lommel polynomials, we define $h_{n,a}(z) = R_{n,a}(1/z)$, which are known as modified Lommel polynomials. Lommel polynomials satisfy the three terms recurrence relation (see [30, p. 299])

$$\frac{2(n+a)}{z} R_{n,a}(z) = R_{n-1,a}(z) + R_{n+1,a}(z).$$

In this way, we deduce that $b_n(w) = h_{n,\alpha+\beta+2}(w)$.

Let us identify the values verifying (44).

Theorem 5. *The convergence condition* (44) *holds if and only if* λ *is purely imaginary,* $\lambda \neq 0$*, and* $J_{\alpha+\beta+1}(i/\lambda) = 0$.

Proof: By Hurwitz' theorem [30, p. 302], we have

$$h_{n,a}(z) \sim \Gamma(n+a)(2z)^{n+a-1} J_{a-1}(1/z).$$

This fact, taking into account (47) and that $b_n(i\lambda) = h_{n-1,\alpha+\beta+2}(i\lambda)$, show that in order for (44) to hold it is necessary that $J_{\alpha+\beta+1}(i/\lambda) = 0$ or possibly $\lambda = 0$. For $\lambda = 0$ we can deduce, by (48), that $b_{2n+1}(0) = 0$ and $b_{2n}(0) = (-1)^n$, then (44) does not hold.

To deduce the sufficiency of $J_{\alpha+\beta+1}(i/\lambda) = 0$ we need the identity [30, pp. 295–295]

$$J_{a+n}(z) = R_{n,a}(z)J_a(z) - R_{n-1,a+1}(z)J_{a-1}(z).$$

From this relation we have

$$-J_{\alpha+\beta+n+1}(i/\lambda) = h_{n-1,\alpha+\beta+2}(\lambda/i)J_{\alpha+\beta}(i/\lambda) \tag{49}$$

when $J_{\alpha+\beta+1}(i/\lambda) = 0$. In this way, by the asymptotic

$$J_{a+n}(z) \sim \frac{(2z)^{-n-a}}{\Gamma(a+n+1)}, \qquad n \to \infty,$$

we can conclude that (44) is satisfied. ∎

Remember that the Bessel function J_a has an increasing sequence of positive zeros. In what follows we denote them by $j_{a,k}$, for $k \geq 1$.

Now, we can obtain the eigenvalues and the eigenfunctions of $T_{\beta,\alpha}$.

Theorem 6. *Let $\alpha, \beta > -1$, and $\alpha+\beta > -1$. Then the eigenvalues of the integral operator $T_{\beta,\alpha}$ are $\{\pm i/j_{\alpha+\beta+1,k}\}_{k\geq 1}$ and the eigenfunctions $g_{\pm i/j_{\alpha+\beta+1,k}}(t)$ have the series expansion in terms of the generalized Gegenbauer polynomials*

$$\sum_{n=1}^{\infty} (\mp i)^{n-1} \frac{(\alpha+\beta+n+1)}{(\alpha+\beta+2)} C_n^{(\beta+1/2,\alpha+1/2)}(t) h_{n-1,\alpha+\beta+2}\left(\frac{1}{j_{\alpha+\beta+1,k}}\right). \tag{50}$$

Moreover,

$$g_{\pm i/j_{\alpha+\beta+1,k}}(t) = \mp i \left(\frac{j_{\alpha+\beta+1,k}}{2}\right)^{\alpha+\beta+1} \frac{E_\alpha(\mp t j_{\alpha+\beta+1,k})}{\Gamma(\alpha+\beta+1)(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})}. \tag{51}$$

Proof: The eigenvalues follow immediately from Theorem 5. The expression (50) for the eigenfunctions is a consequence of (37), (47), the fact $b_n(i\lambda) = h_{n-1,\alpha+\beta+2}(i\lambda)$ and the identity

$$h_{n-1,\alpha+\beta+2}\left(i\frac{\pm i}{j_{\alpha+\beta+1,k}}\right) = (\mp 1)^{n-1} h_{n-1,\alpha+\beta+2}\left(\frac{1}{j_{\alpha+\beta+1,k}}\right),$$

which is obtained using that $R_{n,a}(-z) = (-1)^n R_{n,a}(a)$.

Let us prove (51). Taking $\lambda = \pm i/j_{\alpha+\beta+1,k}$ in (49), we obtain that

$$J_{\alpha+\beta+n+1}(j_{\alpha+\beta+1,k}) = -h_{n-1,\alpha+\beta+2}\left(\frac{1}{j_{\alpha+\beta+1,k}}\right) J_{\alpha+\beta}(j_{\alpha+\beta+1,k}).$$

From the previous identity, (50) becomes

$$\begin{aligned}
g_{\pm i/j_{\alpha+\beta+1,k}}(t) &= \frac{-1}{(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})} \\
&\qquad \times \sum_{n=1}^{\infty} (\mp i)^{n-1}(\alpha+\beta+n+1)C_n^{(\beta+1/2,\alpha+1/2)}(t) J_{\alpha+\beta+n+1}(j_{\alpha+\beta+1,k}) \\
&= \frac{\mp i\, j_{\alpha+\beta+1,k}^{\alpha+\beta+1}}{(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})} \\
&\qquad \times \sum_{n=0}^{\infty} i^n(\alpha+\beta+n+1)C_n^{(\beta+1/2,\alpha+1/2)}(t)\mathcal{J}_{\alpha+\beta,n}(\mp j_{\alpha+\beta+1,k}) \\
&= \mp i \left(\frac{j_{\alpha+\beta+1,k}}{2}\right)^{\alpha+\beta+1} \frac{E_\alpha(\mp t j_{\alpha+\beta+1,k})}{\Gamma(\alpha+\beta+1)(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})}
\end{aligned}$$

where in the last step we have used (31). ∎

6. Expansions associated to q-special functions

6.1. Preliminaries on q-special functions. We follow the standard notations (see [22]). Choose a number q such that $0 < q < 1$. The notational conventions from [16]

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad (a_1,\dots,a_m;q)_n = \prod_{l=1}^{m}(a_l;q)_n, \quad |q|<1,$$

where $n = 1, 2, \dots,$ will be used. The symbol ${}_{r+1}\phi_r$ stands for the function

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix}\,\middle|\, q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n} z^n.$$

We will also require the definition of the q-integral. The q-integral in the interval $(0,a)$ is defined as

$$\int_0^a f(t)\,d_qt = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

and in the interval $(0,\infty)$ as

$$\int_0^\infty f(t)\,d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n.$$

This can be extended to the whole real line in a obvious way.

The *third Jackson q-Bessel function* $J_\nu^{(3)}$ is defined by the power series

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}x^\nu \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n(q;q)_n}\,x^{2n}.$$

Throughout this paper, when no confusion is possible, we will drop the superscript and write simply

$$J_\nu(x;q) = J_\nu^{(3)}(x;q).$$

For $x\in(0,1)$, the *little q-Jacobi* polynomials are defined for $\alpha,\beta>-1$ by

$$p_n(x;q^\alpha,q^\beta;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1}\end{matrix}\,\middle|\, q;qx\right).$$

They satisfy the following discrete orthogonality relation (see the last line of section 0.8 in [22]):

$$\begin{aligned}\int_0^1 &\frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}x^\alpha p_n(x;q^\alpha,q^\beta;q)p_m(x;q^\alpha,q^\beta;q)\,d_qx \\ &= (1-q)\frac{(q,q^{\alpha+\beta+2};q)_\infty}{(q^{\alpha+1},q^{\beta+1};q)_\infty}\frac{(1-q^{\alpha+\beta+1})}{(1-q^{2n+\alpha+\beta+1})}\frac{(q,q^{\beta+1};q)_n}{(q^{\alpha+1},q^{\alpha+\beta+1};q)_n}q^{n(\alpha+1)}\delta_{m,n}.\end{aligned}$$

For our purposes we need to rewrite this orthogonality. We will use the polynomials $p_n^{(\alpha,\beta)}$ normalized as follows

$$p_n^{(\alpha,\beta)}(x;q) = q^{-\frac{n(\alpha+1)}{2}}\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,p_n(x;q^\alpha,q^\beta;q). \tag{52}$$

This normalization is such that

$$\lim_{q\to1} p_n^{(\alpha,\beta)}(x;q) = P_n^{(\alpha,\beta)}(1-2x),$$

where $P_n^{(\alpha,\beta)}$ are the classical Jacobi polynomials (see [18, p. 478]). It will be convenient to replace q by q^2 in the above orthogonality. Then, from the definition of the q-integral we obtain the identity

$$\int_0^1 f(x;q^2)\,d_{q^2}x = (1+q)\int_0^1 xf(x^2;q^2)\,d_qx,$$

and use it in order to obtain the following:

$$\int_0^1 \frac{(q^2x^2;q^2)_\infty}{(q^{2\beta+2}x^2;q^2)_\infty} p_n^{(\alpha,\beta)}(x^2;q^2)p_m^{(\alpha,\beta)}(x^2;q^2)x^{2\alpha+1}\,d_qx$$
$$= \frac{(1-q)}{(1-q^{4n+2\alpha+2\beta+2})}\frac{(q^{2+2n},q^{2\alpha+2\beta+2+2n};q^2)_\infty}{(q^{2\alpha+2+2n},q^{2\beta+2+2n};q^2)_\infty}\delta_{m,n}. \quad (53)$$

6.2. Kernels involving q-Bessel functions. A generalized q-exponential kernel (in the spirit of the kernel for the Dunkl transform) can be defined in terms of q-Bessel. Indeed, we can consider the function

$$E_\alpha(ix;q^2) = \frac{(q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}\left(\frac{J_\alpha(x;q^2)}{x^\alpha} + \frac{J_{\alpha+1}(x;q^2)}{x^{\alpha+1}}\,xi\right). \quad (54)$$

Taking the measure

$$d\mu_{q,\alpha}(x) = \frac{1}{2(1-q)}\frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}|x|^{2\alpha+1}\,d_qx,$$

in a similar way to the Dunkl transform, for $\alpha \geq -1/2$ we can define the following q-integral transform:

$$\mathcal{F}_{\alpha,q}f(y) = \int_{-\infty}^{\infty} f(x)E_\alpha(-iyx;q^2)\,d\mu_{q,\alpha}(x), \quad y \in \{\pm q^k\}_{k\in\mathbb{Z}}, \quad (55)$$

for $f \in L^1(\mathbb{R}, d\mu_{q,\alpha})$. The case $\alpha = -\frac{1}{2}$ provides a q-analogue of the Fourier transformation. In this special case, an inversion theory of this transform has been derived in [26] using the results of [23]. For odd functions our q-analogue of the Dunkl transform becomes in a q-analogue of the Hankel transform

$$H_{\alpha,q}f(x) = \int_0^\infty \frac{J_\alpha(xy;q^2)}{(xy)^\alpha}\,f(y)\,d\omega_{q,\alpha}(y), \quad x > 0,$$

where $d\omega_{q,\alpha}(y) = \frac{y^{2\alpha+1}}{1-q}\,d_qy$. This is the transform studied by Koornwinder and Swarttouw [23], up to a small modification. By the results in [23] we have the inversion formula

$$f(q^n) = H_{\alpha,q}(H_{\alpha,q}f)(q^n).$$

In particular, $H_{\alpha,q}$ is an isometric transformation in $L^2((0,\infty), d\omega_{q,\alpha})$.

For $\mathcal{F}_{\alpha,q}$, using again the arguments in [23], it is easy to check that $\mathcal{F}_{\alpha,q}^{-1}f(y) = \mathcal{F}_{\alpha,q}f(-y)$. Moreover, we have the multiplicative formula

$$\int_{-\infty}^{\infty} u(y)\mathcal{F}_{\alpha,q}v(y)\, d\mu_{q,\alpha}(y) = \int_{-\infty}^{\infty} \mathcal{F}_{\alpha,q}u(y)v(y)\, d\mu_{q,\alpha}(y), \tag{56}$$

and $\mathcal{F}_{\alpha,q}$ is an isometry on $L^2(\mathbb{R}, d\mu_{q,\alpha})$. As in the case of the Dunkl transform, we can consider the parameter $\alpha > -1$.

6.3. Generalized little q-Gegenbauer polynomials. To construct the plane wave expansion for the kernel (54) q-analogue of the Dunkl transform, we need introduce the appropriate q-analogues of the generalized Gegenbauer polynomials.

We define the generalized little q-Gegenbauer polynomials by

$$C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2) = (-1)^n \frac{(q^{2\alpha+2\beta+2};q^2)_n}{(q^{2\alpha+2};q^2)_n}\, p_n^{(\alpha,\beta)}(t^2;q^2),$$

$$C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t;q^2) = (-1)^n \frac{(q^{2\alpha+2\beta+2};q^2)_{n+1}}{(q^{2\alpha+2};q^2)_{n+1}}\, tp_n^{(\alpha+1,\beta)}(t^2;q^2),$$

where the polynomials $p_n^{(\alpha,\beta)}$ are defined by (52) in terms of the little q-Jacobi polynomials. Using (53) we obtain

$$\int_{-1}^{1} C_k^{(\beta+1/2,\alpha+1/2)}(t;q^2) C_j^{(\beta+1/2,\alpha+1/2)}(t;q^2) \frac{(q^2t^2;q^2)_\infty}{(q^{2\beta+2}t^2;q^2)_\infty}\, d\mu_{q,\alpha}(t) = h_{k,q}^{(\beta,\alpha)}\delta_{k,j},$$

where

$$\begin{aligned} h_{2n,q}^{(\beta,\alpha)} &= \int_{-1}^{1} \left[C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\right]^2 \frac{(q^2t^2;q^2)_\infty}{(q^{2\beta+2}t^2;q^2)_\infty}\, d\mu_{q,\alpha}(t) \\ &= \frac{1}{(1-q^{4n+2\alpha+2\beta+2})} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q^{2\alpha+2\beta+2};q^2)_n}{(q^{2\alpha+2};q^2)_n} \frac{(q^{2+2n}, q^{2\alpha+2\beta+2};q^2)_\infty}{(q^{2\beta+2+2n};q^2)_\infty}, \end{aligned} \tag{57}$$

$$\begin{aligned} h_{2n+1,q}^{(\beta,\alpha)} &= \int_{-1}^{1} \left[C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\right]^2 \frac{(q^2t^2;q^2)_\infty}{(q^{2\beta+2}t^2;q^2)_\infty}\, d\mu_{q,\alpha}(t) \\ &= \frac{1}{(1-q^{4n+2\alpha+2\beta+4})} \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q^{2\alpha+2\beta+2};q^2)_{n+1}(q^{2\alpha+2\beta+2}, q^{2+2n};q^2)_\infty}{(q^{2\alpha+2};q^2)_{n+1}(q^{2\alpha+2}, q^{2\beta+2+2n};q^2)_\infty}. \end{aligned} \tag{58}$$

We will also consider the little q-ultraspherical polynomials defined as

$$C_n^{\beta}(t;q^2) = C_n^{(\beta,0)}(t;q^2). \tag{59}$$

6.4. q-Fourier-Neumann type series. Now, given $\alpha > -1$, we define the q-Neumann functions by

$$\mathcal{J}_{\alpha,n}(x;q^2) = \frac{J_{\alpha+n+1}(xq^{[\frac{n}{2}]};q^2)}{x^{\alpha+1}},$$

where $\left[\frac{n}{2}\right]$ denotes the biggest integer less or equal than $\frac{n}{2}$. The identity

$$\begin{aligned}\int_0^\infty & x^{-\lambda} J_\mu(q^m x;q^2) J_\nu(q^n x;q^2)\, d_q x \\ &= (1-q)q^{n(\lambda-1)+(m-n)\mu} \frac{(q^{1+\lambda+\nu-\mu}, q^{2\mu+2};q^2)_\infty}{(q^{1-\lambda+\nu+\mu}, q^2;q^2)_\infty} \\ &\quad \times {}_2\phi_1 \left(\left. \begin{matrix} q^{1-\lambda+\mu+\nu}, q^{1-\lambda+\mu-\nu} \\ q^{2\mu+2} \end{matrix} \right| q^2; q^{2m-2n+1+\lambda+\nu-\mu} \right),\end{aligned} \tag{60}$$

was established in [23] and it is valid for $-1 < \operatorname{Re}\lambda < \operatorname{Re}(\mu+\nu+1)$, m and n nonnegative integers. By setting $\lambda = 1$, $\nu = \alpha+2n+1$, and $\mu = \alpha+2m+1$ and in (60), it is clear that, for $n, m = 0, 1, 2, \dots$,

$$\int_0^\infty J_{\alpha+2n+1}(q^n x;q^2) J_{\alpha+2m+1}(q^m x;q^2)\, \frac{d_q x}{x} = \frac{1-q}{1-q^{2\alpha+4m+2}} \delta_{n,m}.$$

From this fact, taking into account that $\mathcal{J}_{\alpha,n}(x;q)$ is even or odd according n is even or odd, respectively, we have that $\{\mathcal{J}_{\alpha,n}(x;q)\}_{n\geq 0}$ is an orthogonal system on $L^2(\mathbb{R}, d\mu_{\alpha,q}(x))$, namely

$$\int_{-\infty}^\infty \mathcal{J}_{\alpha,n}(x;q^2)\mathcal{J}_{\alpha,m}(x;q^2)\, d\mu_{q,\alpha}(x) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \frac{1}{(1-q^{2\alpha+2m+2})} \delta_{n,m},$$

for $n, m = 0, 1, 2, \dots$.

Now, we need a q-analogue of Lemma 1. To this end, we define the functions

$$\mathcal{Q}_n^{(\alpha,\beta)}(t;q^2) = (h_{n,q}^{(\beta,\alpha)})^{-1} \frac{(x^2q^2;q^2)_\infty}{(x^2q^{2+2\beta};q^2)_\infty} C_n^{(\beta+1/2,\alpha+1/2)}(t;q^2), \tag{61}$$

$$\mathcal{P}_n^{(\alpha,\beta)}(t;q^2) = C_n^{(\beta+1/2,\alpha+1/2)}(t;q^2). \tag{62}$$

Lemma 2. *Let $\alpha, \beta > -1$, $\alpha+\beta > -1$, and $k = 0, 1, 2, \ldots$. Then*

$$\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,k}(\,\cdot\,; q^2))(t) = \frac{(-i)^k q^{[\frac{k}{2}]\beta}}{(1-q^{2k+2\alpha+2\beta+2})} \frac{(q^{2\alpha+2\beta+2}; q^2)_\infty}{(q^2; q^2)_\infty} \mathcal{Q}_k^{(\alpha,\beta)}(t; q^2) \chi_{[-1,1]}(t), \tag{63}$$

for $t \in \{\pm q^k\}_{k\in\mathbb{Z}}$. Moreover, if $\beta < 1$, we have

$$\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta} \mathcal{J}_{\alpha+\beta,k}(\,\cdot\,; q^2))(t) = q^{-[\frac{k}{2}]\beta} (-i)^k \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^{2+2\alpha+2\beta}; q^2)_\infty} \mathcal{P}_k^{(\alpha,\beta)}(t; q^2), \tag{64}$$

for $t \in \{\pm q^k\}_{k\in\mathbb{Z}} \cap [-1, 1]$

The proof of Lemma 2 is contained in subsection 7.4.

To give our result in this setting we have to define the corresponding Paley-Wiener space

$$PW_{\alpha,q} = \left\{ f \in L^2(\mathbb{R}, d\mu_{q,\alpha}) : f(t) = \int_{-1}^{1} u(x) E_\alpha(ixt; q^2)\, d\mu_{q,\alpha}(x),\ u \in L^2(\mathbb{R}, d\mu_{q,\alpha}) \right\}. \tag{65}$$

Theorem 7. *Let $\alpha, \beta > -1$ and $\alpha + \beta > -1$. Then for each $x \in \{\pm q^k\}_{k\in\mathbb{Z}}$ the following expansion holds in $L^2([-1, 1], d\mu_{q,\alpha})$:*

$$E_\alpha(ixt; q^2) = \frac{(q^2; q^2)_\infty}{(q^{2+2\alpha+2\beta}; q^2)_\infty} \sum_{n=0}^{\infty} i^n (1 - q^{2\alpha+2\beta+2n+2}) \mathcal{J}_{\alpha+\beta,n}(x; q^2) C_n^{(\beta+1/2,\alpha+1/2)}(t; q^2). \tag{66}$$

Moreover, for $\beta < 1$ and $f \in PW_{\alpha,q}$, we have the orthogonal expansion

$$f(x) = \sum_{n=0}^{\infty} a_n(f)(1 - q^{2\alpha+2\beta+2n+2}) \mathcal{J}_{\alpha+\beta,n}(x; q^2) \tag{67}$$

with

$$a_n(f) = \frac{(q^2; q^2)_\infty}{(q^{2+2\alpha+2\beta}; q^2)_\infty} \int_{\mathbb{R}} f(t) \mathcal{J}_{\alpha+\beta,n}(t; q^2)\, d\mu_{q,\alpha+\beta}(t). \tag{68}$$

Furthermore, the series converges uniformly in compact subsets of $\mathbb{R}$.

Proof: We proceed as in in the proof of Theorem 2 by using the appropriate modifications. In the biorthogonal set up given in subsection 2.2, let $\Omega = \mathbb{R}$, $I = [-1, 1]$, the space $L^2(I, d\mu) = L^2([-1, 1], d\mu_{q,\alpha})$, and the kernel $K(x, t) = E_\alpha(ixt; q^2)$, so $\mathcal{K}$ becomes the q-analogous of the Dunkl transform $\mathcal{F}_{\alpha,q}$ defined in (55) (and $\widetilde{K} = \mathcal{F}_{\alpha,q}^{-1}$). Also, consider the Paley-Wiener space $\mathcal{P} = PW_{\alpha,q}$ (see (65)). Finally, for $N = \mathbb{N} \cup \{0\}$, take the biorthonormal system given by $P_n(t) = \mathcal{P}_n^{(\alpha,\beta)}(t; q^2)$ and $Q_n(t) = \mathcal{Q}_n^{(\alpha,\beta)}(t; q^2)$ as in (62) and (61). To conclude the result we use the same arguments as in Theorem 2 but using Lemma 2 instead of Lemma 1. ■

Remark 4. Setting $\alpha = -\frac{1}{2}$ and replacing β by $\beta - \frac{1}{2}$, in (68) we obtain the expansion for the q-exponential function studied in [26] in terms of the little q-ultraspherical polynomials defined in (59):

$$e(ixt; q^2) = \frac{(q^2; q^2)_\infty}{(q^{2\beta}; q^2)_\infty} x^{-\beta} \sum_{n=0}^{\infty} i^n (1 - q^{2n+2\beta}) J_{\beta+n}(xq^{[\frac{n}{2}]}; q^2) C_n^\beta(t; q^2).$$

Remark 5. As in section 4, it is also possible to derive, from the above theorem, some formulas involving the kernel of the q-Hankel transform of [23]. In particular, the q-Bessel analogue of the Paley-Wiener space which is the domain of the sampling theorem in [2], can be spanned by systems of q-Neumann functions. The details are similar as before and, in order to keep the paper within a reasonable length, we do not write down the formulas.

7. Technical lemmas

The main goal of this section is to prove Lemma 1 and Lemma 2, they are key in our study about the Dunkl transform on the real line and its analogue $\mathcal{F}_{\alpha,q}$. The proofs of Lemma 1 and Lemma 2 are contained in subsections 7.2 and 7.4, respectively. With this target, we need to establish some previous formulas. They are given in the subsections 7.1 and 7.3.

7.1. Some integrals involving Bessel functions. For the sake of completeness, let us start proving two identities that express some integrals involving the product of two Bessel functions in terms of Jacobi polynomials. This kind of integrals are usually written in terms of hypergeometric ${}_2F_1$ functions, however their expression as Jacobi polynomials are not easily found in the literature. For instance, it does not appear in the standard references [12, 30, 14].

Lemma 3. *For $\alpha, \beta > -1$ with $\alpha+\beta > -1$, and $n = 0, 1, 2, \dots$, let us define*

$$I_-(\alpha,\beta,n)(t) = t^{-\alpha} \int_0^\infty x^{-\beta} J_{\alpha+\beta+2n+1}(x) J_\alpha(xt)\, dx,$$

$$I_+(\alpha,\beta,n)(t) = t^{-\alpha} \int_0^\infty x^{\beta} J_{\alpha+\beta+2n+1}(x) J_\alpha(xt)\, dx.$$

Then, we have

$$I_-(\alpha,\beta,n)(t) = 2^{-\beta} \tfrac{\Gamma(n+1)}{\Gamma(\beta+n+1)} (1-t^2)^\beta P_n^{(\alpha,\beta)}(1-2t^2)\chi_{[0,1]}(t), \quad t \in (0,\infty). \tag{69}$$

Assume further $\beta < 1$; then,

$$I_+(\alpha,\beta,n)(t) = 2^{\beta}\, \tfrac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\beta)}(1-2t^2), \quad t \in (0,1). \tag{70}$$

Proof: We use the formula

$$\int_0^\infty x^{-\lambda} J_\mu(ax) J_\nu(bx)\, dx = \frac{b^\nu a^{\lambda-\nu-1}\Gamma(\frac{\mu+\nu-\lambda+1}{2})}{2^\lambda \Gamma(\nu+1)\Gamma(\frac{\lambda+\mu-\nu+1}{2})}\, {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right), \tag{71}$$

valid when $0 < b < a$ and $-1 < \lambda < \mu+\nu+1$; here, ${}_2F_1$ denotes the hypergeometric function (see [14, Ch. 8.11, (9), p. 48] or [30, Ch. XIII, 13.4 (2), p. 401]).

Then, let us start with (69). Taking $a = 1$ and $t = b$ in (71), and making the corresponding changes of variable and parameters ($\nu = \alpha$, $\mu = \alpha+\beta+2n+1$, $\lambda = \beta$) we get

$$I_-(\alpha,\beta,n)(t) = \frac{\Gamma(\alpha+n+1)}{2^\beta\, \Gamma(\alpha+1)\, \Gamma(\beta+n+1)}\, {}_2F_1(\alpha+n+1, -n-\beta; \alpha+1; t^2),$$

which is valid for $\alpha > -1$ and $\beta > -1$ in the interval $0 < t < 1$. Moreover, we have

$${}_2F_1(\alpha+n+1, -n-\beta; \alpha+1; t) = (1-t)^\beta\, {}_2F_1(-n, \alpha+\beta+n+1; \alpha+1; t),$$

where $\alpha, \beta > -1$, $n = 0, 1, 2, \dots$, and

$$P_n^{(\alpha,\beta)}(y) = \tfrac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\, {}_2F_1(-n, \alpha+\beta+n+1; \alpha+1; \tfrac{1-y}{2}), \tag{72}$$

whenever $\alpha, \beta > -1$ and $-1 < y < 1$. Therefore,

$$I_-(\alpha,\beta,n)(t) = 2^{-\beta}\, \tfrac{\Gamma(n+1)}{\Gamma(\beta+n+1)}\, (1-t^2)^\beta P_n^{(\alpha,\beta)}(1-2t^2), \quad t \in (0,1).$$

Now, we are going to evaluate $I_-(\alpha,\beta,n)(t)$ for $t>1$. To do that, let us take $t=a$, $b=1$, $\nu=\alpha+\beta+2n+1$, $\mu=\alpha$, and $\lambda=\beta$ in (71). In this way, $\frac{1}{2}(\lambda+\mu-\nu+1)=0,-1,-2,\dots$, so the coefficient $1/\Gamma(\frac{1}{2}(\lambda+\mu-\nu+1))$ vanishes and we get $I_-(\alpha,\beta,n)(t)=0$.

Finally, let us prove the second part of the lemma. To this end, we take, in (71), $a=1$ and $t=b$, with parameters $\lambda=-\beta$, $\mu=\alpha+\beta+2n+1$ and $\nu=\alpha$. Then, for $\beta<1$, $\alpha+\beta>-1$, and $0<t<1$ we get

$$I_+(\alpha,\beta,n)(t)=\frac{2^\beta\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+1)\,\Gamma(n+1)}\,{}_2F_1(\alpha+\beta+n+1,-n;\alpha+1;t^2).$$

Then, by using (72), it follows (70). ■

7.2. Proof of Lemma 1. We start evaluating $\mathcal{F}_\alpha(\mathcal{J}_{\alpha+\beta,k})(t)$ for $\alpha>-1$ and $\alpha+\beta>-1$.

By definition,

$$\mathcal{F}_\alpha(\mathcal{J}_{\alpha+\beta,k})(t)=\frac{1}{2}\int_{\mathbb{R}}\frac{J_{\alpha+\beta+k+1}(x)}{x^{\alpha+\beta+1}}\left(\frac{J_\alpha(xt)}{(xt)^\alpha}-\frac{J_{\alpha+1}(xt)}{(xt)^{\alpha+1}}\,xti\right)|x|^{2\alpha+1}\,dx.$$

For the case $k=2n$, by decomposing on even and odd functions, we can write

$$\mathcal{F}_\alpha(\mathcal{J}_{\alpha+\beta,2n})(t)=\int_0^\infty\frac{J_{\alpha+\beta+2n+1}(x)}{x^{\alpha+\beta+1}}\,\frac{J_\alpha(xt)}{(xt)^\alpha}\,x^{2\alpha+1}\,dx. \tag{73}$$

Then, for $t>0$, by using (69) in Lemma 3, (23) and (25), it follows that

$$\begin{aligned}\mathcal{F}_\alpha(\mathcal{J}_{\alpha+\beta,2n})(t)&=t^{-\alpha}\int_0^\infty x^{-\beta}J_{\alpha+\beta+2n+1}(x)J_\alpha(xt)\,dx\\&=\frac{\Gamma(n+1)}{2^\beta\,\Gamma(\beta+n+1)}\,(1-t^2)^\beta P_n^{(\alpha,\beta)}(1-2t^2)\chi_{[0,1]}(t)\\&=(-1)^n\,\frac{\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(\alpha+n+1)}{2^\beta\,\Gamma(\alpha+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}\,(1-t^2)^\beta C_{2n}^{(\beta+1/2,\alpha+1/2)}(t)\chi_{[0,1]}(t)\\&=\frac{i^k}{2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)(\alpha+\beta+k+1)}\,\mathcal{Q}_k^{(\alpha,\beta)}(t)\chi_{[0,1]}(t).\end{aligned}$$

For $t<0$, let us make, in (73), the change $t_1=-t$, use the evenness of the function $J_\alpha(z)/z^\alpha$, proceed as in the case $t>0$, and undo the change. Then, we get

$$\mathcal{F}_\alpha(\mathcal{J}_{\alpha+\beta,2n})(t)=\frac{i^k}{2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)(\alpha+\beta+k+1)}\,\mathcal{Q}_k^{(\alpha,\beta)}(t)\chi_{[-1,0]}(t).$$

Thus, (29) for even k is proved. The case $k = 2n+1$ is completely similar.

Proceeding in the same way, the formula (30) follows from (70).

7.3. Some integrals involving q-Bessel functions. In this subsection we prove a lemma analogous to Lemma 3 but involving q-Bessel functions.

We will use the transformation

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; z\right) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix} \,\middle|\, q; abz/c\right), \tag{74} $$

subject to the conditions $|z| < 1$, $|abz| < |c|$. (74) is formula (12.5.3) in [18]. We also make repeated use of the obvious identity $(a;q)_\infty = (a;q)_n(aq^n;q)_\infty$.

Lemma 4. *For $\alpha, \beta > -1$ with $\alpha+\beta > -1$, and $n = 0, 1, 2, \dots$, let us define*

$$ I_-(\alpha,\beta,n)(t,q) = \frac{t^{-\alpha}}{1-q}\int_0^\infty x^{-\beta} J_\alpha(xt;q^2) J_{\alpha+\beta+2n+1}(q^n x;q^2)\, d_q x $$

and

$$ I_+(\alpha,\beta,n)(t,q) = \frac{t^{-\alpha}}{1-q}\int_0^\infty x^{\beta} J_\alpha(xt;q^2) J_{\alpha+\beta+2n+1}(q^n x;q^2)\, d_q x. $$

Then we have, for $t \in \{q^m\}_{m\in\mathbb{Z}}$,

$$ I_-(\alpha,\beta,n)(t,q) = q^{n\beta}\frac{(q^{2+2\beta+2n};q^2)_\infty}{(q^{2+2n};q^2)_\infty}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2+2\beta};q^2)_\infty}\, p_n^{(\alpha,\beta)}(t^2;q^2)\chi_{[0,1]}(t). \tag{75} $$

Assume further $\beta < 1$; then

$$ I_+(\alpha,\beta,n)(t,q) = q^{-n\beta}\frac{(q^{2\alpha+n+2};q^2)_\infty}{(q^{2+2n+2\alpha+2\beta};q^2)_\infty}\, p_n^{(\alpha,\beta)}(t^2;q^2), \qquad t \in \{q^m\}_{m\in\mathbb{Z}} \cap (0,1). \tag{76} $$

Proof: We start evaluating I_- for $t \in \{q^m\}_{m\in\mathbb{Z}} \cap (0,1)$. To this end, we take in (60) $q^m = t$, $\mu = \alpha$, $\nu = \alpha+\beta+2n+1$, and $\lambda = \beta$. Then,

$$ \begin{aligned} &I_-(\alpha,\beta,n)(t,q) \\ &\quad = q^{n(\beta-\alpha-1)}\frac{(q^{2+2\beta+2n}, q^{2\alpha+2};q^2)_\infty}{(q^{2+2n+2\alpha}, q^2;q^2)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{2+2\alpha+2n}, q^{-2n-2\beta} \\ q^{2\alpha+2} \end{matrix} \,\middle|\, q^2; t^2q^{2+2\beta}\right). \end{aligned} $$

Now, applying formula (74) and the definition of $p_n^{(\alpha,\beta)}$ in terms of the little q-Jacobi polynomials, we have

$$
\begin{aligned}
&I_-(\alpha,\beta,n)(t,q)\\
&\qquad = q^{n(\beta-\alpha-1)}\frac{(q^{2+2\beta+2n},q^{2\alpha+2};q^2)_\infty}{(q^{2+2n+2\alpha},q^2;q^2)_\infty}\\
&\qquad\quad \times \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2+2\beta};q^2)_\infty}\,{}_2\phi_1\left(\begin{matrix} q^{-2n},q^{2+2\alpha+2\beta+2n}\\ q^{2\alpha+2}\end{matrix}\,\middle|\, q^2;t^2q^2\right)\\
&\qquad = q^{n(\beta-\alpha-1)}\frac{(q^{2+2\beta+2n},q^{2\alpha+2};q^2)_\infty}{(q^{2+2n+2\alpha},q^2;q^2)_\infty}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2+2\beta};q^2)_\infty}\,p_n(t^2;q^{2\alpha};q^{2\beta};q^2)\\
&\qquad = q^{n\beta}\frac{(q^{2+2\beta+2n};q^2)_\infty}{(q^{2+2n};q^2)_\infty}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2+2\beta};q^2)_\infty}\,p_n^{(\alpha,\beta)}(t^2;q^2).
\end{aligned}
$$

To evaluate I_- when $t>1$, we consider $q^n=t$, $\mu=\alpha+\beta+2n+1,\nu=\alpha$, and $\lambda=\beta$ in (60). In this way, $1+\lambda+\nu-\mu=-2n$, resulting on a factor $(q^{-2n};q^2)_\infty=0$ on the numerator. This gives

$$
I_-(\alpha,\beta,n)(t,q)=0, \qquad t\in\{q^m\}_{m\in\mathbb{Z}}\cap(1,\infty).
$$

So, the proof of (75) is completed.

To prove the second part of the lemma, we choose again the parameters $\mu=\alpha$ and $\nu=\alpha+\beta+2n+1$ but take $\lambda=-\beta$ in (60). This results in

$$
\begin{aligned}
I_+(\alpha,\beta,n)(t,q) &= q^{-n(\beta+\alpha+1)}(1-q)\frac{(q^{2+2n},q^{2\alpha+2};q^2)_\infty}{(q^{2+2n+2\alpha+2\beta},q^2;q^2)_\infty}\\
&\qquad \times {}_2\phi_1\left(\begin{matrix} q^{-2n},q^{2+2\alpha+2\beta+2n}\\ q^{2\alpha+2}\end{matrix}\,\middle|\, q^2;t^2q^{2m+2\beta+2}\right)\\
&= q^{-n(\beta+\alpha+1)}\frac{(q^{2+2n},q^{2\alpha+2};q^2)_\infty}{(q^{2+2n+2\alpha+2\beta},q^2;q^2)_\infty}\,p_n(t^2;q^{2\alpha};q^{2\beta};q^2)\\
&= q^{-n\beta}\frac{(q^{2\alpha+n+2};q^2)_\infty}{(q^{2+2n+2\alpha+2\beta};q^2)_\infty}\,p_n^{(\alpha,\beta)}(t^2;q^2).
\end{aligned}
$$

In this manner, we have proved (76) and the proof of the lemma is finished. ■

7.4. Proof of Lemma 2. Let us analyze the case $k = 2n$ for (63). By decomposing on even and odd functions we can write

$$\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) = \frac{1}{(1-q)}\int_0^\infty \frac{J_{\alpha+\beta+2n+1}(q^n x;q^2)}{x^{\alpha+\beta+1}}\frac{J_\alpha(xt;q^2)}{(xt)^\alpha}x^{2\alpha+1}\,d_q x. \tag{77}$$

Then, for $t > 0$, $\alpha, \beta > -1$, and $\alpha + \beta > -1$, by using (75) and $(a;q)_\infty = (a;q)_n(aq^n;q)_\infty$, it is verified that

$$\begin{aligned}\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) &= q^{n\beta}\frac{(q^{2+2\beta+2n},q^2)_\infty}{(q^{2+2n};q^2)_\infty}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2+2\beta};q^2)_\infty}\,p_n^{(\alpha,\beta)}(t^2;q^2)\chi_{[0,1]}(t)\\ &= (-1)^n q^{n\beta}\frac{(q^{2\alpha+2};q^2)_n}{(q^{2\alpha+2\beta+2};q^2)_n}\frac{(q^{2+2\beta+2n};q^2)_\infty}{(q^{2+2n};q^2)_\infty}\\ &\quad\times\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2+2\beta};q^2)_\infty}C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\chi_{[0,1]}(t)\\ &= \frac{(-1)^n q^{n\beta}}{(1-q^{4n+2\alpha+2\beta+2})}\frac{(q^{2\alpha+2\beta+2};q^2)_\infty}{(q^2;q^2)_\infty}\mathcal{Q}_{2n}^{(\alpha,\beta)}(t;q^2)\chi_{[0,1]}(t).\end{aligned}$$

For $t < 0$, let us make in (77) the change $t_1 = -t$, use the evenness of the function $J_\alpha(z)/z^\alpha$, proceed as in the case $t > 0$, and undo the change. Then, for $k = 2n$, we get

$$\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,k}(\,\cdot\,;q^2))(t) = \frac{(-i)^k q^{\frac{k}{2}\beta}}{(1-q^{2k+2\alpha+2\beta+2})}\frac{(q^{2\alpha+2\beta+2};q^2)_\infty}{(q^2;q^2)_\infty}\mathcal{Q}_k^{(\alpha,\beta)}(t;q^2)\chi_{[-1,1]}(t).$$

The case $k = 2n + 1$ works in a similar way.

Now, we are going to prove (64). Again let us analyze the case $k = 2n$. By decomposing on even and odd functions we can write

$$\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) = \int_0^\infty x^{2\beta}\frac{J_{\alpha+\beta+2n+1}(q^n x;q^2)}{x^{\alpha+\beta+1}}\frac{J_\alpha(xt;q^2)}{(xt)^\alpha}x^{2\alpha+1}\,d_q x$$

Then, if $\beta < 1$ and $0 < t < 1$, we can use (76) to obtain

$$\begin{aligned}\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) &= q^{-n\beta}\frac{(q^{2+2n};q^{2\alpha+1};q^2)_\infty}{(q^{2+2n+2\alpha+2\beta};q^2;q^2)_\infty}p_n(x^2;q^{2\alpha};q^{2\beta};q^2)\\ &= q^{-n\beta}\frac{(q^{2\alpha+n+2};q^2)_\infty}{(q^{2+2n+2\alpha+2\beta};q^2)_\infty}\,p_n^{(\alpha,\beta)}(x^2;q^2)\\ &= (-1)^n q^{-n\beta}\frac{(q^{2\alpha+2};q^2)_\infty}{(q^{2+2\alpha+2\beta};q^2)_\infty}C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\end{aligned}$$

$$= (-1)^n q^{-n\beta} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^{2+2\alpha+2\beta}; q^2)_\infty} \mathcal{P}_{2n}^{(\alpha,\beta)}(t; q^2).$$

For $t < 0$ we proceed as in the previous identity. Then, for $k = 2n$, we get

$$\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta} \mathcal{J}_{\alpha+\beta,k}(\,\cdot\,; q^2))(t) = q^{-\frac{k}{2}\beta}(-i)^k \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^{2+2\alpha+2\beta}; q^2)_\infty} \mathcal{P}_{k}^{(\alpha,\beta)}(t; q^2).$$

The case $k = 2n+1$ can be checked with the same arguments.

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Luís Daniel Abreu
CMUC, Department of Mathematics of University of Coimbra, School of Science and Technology (FCTUC) 3001-454 Coimbra, Portugal

E-mail address: `daniel@mat.uc.pt`
URL: `http://www.mat.uc.pt/ daniel/`

Óscar Ciaurri
Departamento de Matemáticas y Computación, Universidad de La Rioja, Calle Luis de Ulloa s/n, 26004 Logroño, Spain

E-mail address: `oscar.ciaurri@unirioja.es`

Juan Luis Varona
Departamento de Matemáticas y Computación, Universidad de La Rioja, Calle Luis de Ulloa s/n, 26004 Logroño, Spain

E-mail address: `jvarona@unirioja.es`
URL: `http://www.unirioja.es/cu/jvarona/`