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INTEGRO-DIFFERENTIAL IBVP VERSUS DIFFERENTIAL IBVP: STABILITY ANALYSIS

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ABSTRACT: The aim of this paper is the qualitative analysis from theoretical and numerical points of view of an integro-differential initial boundary value problem where the reaction term presents a certain memory effect. Stability results are established in both cases. As in certain cases the integro-differential initial boundary value problem can be seen as a differential initial boundary value problem, the results obtained for the integro-differential formulation are compared with the correspondent results stated for the differential initial boundary value problem. Numerical results illustrating the theoretical results are also included.

Keywords: Integro-differential problem, FitzHugh-Nagumo equation, Numerical methods, Stability

1. Introduction

In this paper we study, from theoretical and numerical point of view, the initial boundary values problem (IBVP)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \beta \int_0^t e^{-\gamma(t-s)} u(s)\, ds + f(u)\,, \quad t \in (0,T]\,, \tag{1}$$

where γ and β are positive constants,

$$u(a,t) = u_a(t),\ u(b,t) = u_b(t), \quad t \in (0,T]\,, \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in (a,b)\,. \tag{3}$$

In (1), f represents a nonlinear term depending on u. In certain applications f takes the form

$$f(u) = u(1-u)(u-\alpha)\,, \tag{4}$$

where α is a positive constant with $0 < \alpha < 1$.

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Equation (1) with f defined by (4) was studied for instance in [3] and [9] and arises naturally from the system of partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + f(u) - v \\ \dfrac{\partial v}{\partial t} = \beta u - \gamma v\,. \end{cases} \tag{5}$$

This system, with f defined by (4), is used to model the nerve impulse transmission and is known by FitzHugh-Nagumo equation. In this case u represents the membrane potential , x measures the distance along the axon, t represents the time, the term $u(1-u)(u-\alpha)$ is analogous to an instantaneous turning on of sodium permeability and v is a recovery variable and is analogous to the turning on of potassium permeability. In the last decade this problem has been largely considered in the literature and without be exhaustive we mention [1], [6], [7], [8], [10] and [11].

Stability estimates with respect to the L^2-norm of the solution of the integro-differential model (1), (2), (3) and of its "weight past in time" will be established in this paper. The approach used here was considered by the authors and their collaborators for instance in [2], [4] and [5]. As the integro-differential IBVP (1), (2), (3) can be seen as the differential IBVP (2), (3), (5), with convenient boundary and initial conditions on v, such stability estimates will be compared with the ones that will be deduced for the last problem.

This paper also focuses the study of finite difference methods (FDM) to solve numerically the IBVP (1), (2), (3) which can be seen as a fully piecewise linear finite element method. Stability results for such FDM are established being the stability estimates discrete versions of the correspondent estimates for the continuous model. As the integro-differential IBVP (1), (2), (3), is equivalent to the differential IBVP (2), (3), (5), the stability estimates for the FDM of the the differential IBVP are compared with the ones obtained for the discretizations of the integro-differential problem.

An inspection to the numerical methods studied enable us to conclude that their application leads to the need of great storage of information in each time level. However, as we will see, this disadvantage is apparent because the methods can be rewritten as three times level schemes.

The paper is organized as follows. In Section 2 the stability of the integro-differential problem (1),(2), (3) is studied. The correspondent study for the differential problem (2), (3), (5), is presented also in this section. FDM for the integro-differential problem are proposed in Section 3. In this section are also presented stability results for the FDM for IBVP (2), (3), (5). Finally, in Section 4 are included some numerical results illustrating the theoretical results.

2. Stability analysis

2.1. Integro-differential IBVP. By $L^2(a,b), H^1(a,b)$ and $H^1_0(a,b)$ we represent the usual spaces. The norm in $L^2(a,b)$ and $H^1(a,b)$ are denoted by $\|.\|$ and $\|.\|_1$, respectively. Let $L^2(0,T,H^1_0(a,b))$ be the space of functions u defined on $[a,b]\times[0,T]$ such that, for each $t\in[0,T]$, $u(t)\in H^1_0(a,b)$ and the norm

$$\int_0^T \|u(t)\|_1^2\,dt$$

is finite. By $u(t)$ we represent the x-function $u(.,t)$.

As our aim is to establish stability results, we assume that the boundary conditions (2) are homogeneous. Moreover the stability result is established for the weak solution $u\in L^2(0,T,H^1_0(a,b))$ such that, for $t\in(0,T]$, $\dfrac{\partial u}{\partial t}(t)\in L^2(a,b)$ and

$$\begin{cases} (\dfrac{\partial u}{\partial t}(t),w)+D(\dfrac{\partial u}{\partial x}(t),w')=-\beta(\displaystyle\int_0^t e^{-\gamma(t-s)}u(s)\,ds,w)+(f(u(t)),w), \\ \hfill \forall w\in H^1_0(a,b)\,, \\ u(0)=u_0\,. \end{cases} \tag{6}$$

The proof is based on the energy method.

Theorem 1. *Let f be such that $f(0)=0$ and f' is bounded. If $u\in L^2(0,T,H^1_0(a,b))$ satisfies (6) and is such that $\dfrac{\partial u}{\partial t}(t)\in L^2(a,b)$ then, for $t\in[0,T]$,*

(1)

$$\|u(t)\|^2+\beta\|\int_0^t e^{-\gamma(t-s)}\,u(s)\,ds\|^2\,ds+2D\int_0^t\|\frac{\partial u}{\partial x}(s)\|^2\,ds\le S_{i,g}(t)\|u_0\|^2, \tag{7}$$

with

$$S_{i,g}(t)=e^{2\,\max\{f'_{\sup},0\}t}\,, \tag{8}$$

(2)

$$\|u(t)\|^2 + \beta \| \int_0^t e^{-\gamma(t-s)}\, u(s)\, ds\|^2 \leq S_i(t)\, \|u_0\|^2, \tag{9}$$

with

$$S_i(t) = e^{2\, \max\{f'_{\sup} - \frac{D}{(b-a)^2}, -\gamma\}\, t}\,, \tag{10}$$

where $f'_{\sup} = \sup\limits_{y\in\mathbb{R}} f'(y)$.

Proof: Considering in (6) w replaced by $u(t)$ we easily get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + D\frac{d}{dt}\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\, ds &= -\beta\left(\int_0^t e^{-\gamma(t-s)} u(s)\, ds\,,\, u(t)\right) \\ &\quad + \Big(f(u(t)), u(t)\Big)\,. \end{aligned}$$

Due to the fact that $f(0) = 0$, we have $f(u(t)) = f'(\eta)\, u(t)$, for some η between 0 and $u(t)$, and so $\Big(f(u(t)), u(t)\Big) \leq f'_{\sup}\, \|u(t)\|^2$. As

$$\begin{aligned} \left(\int_0^t e^{-\gamma(t-s)}\, u(s)\, ds, u(t)\right) &= \gamma \| \int_0^t e^{-\gamma(t-s)}\, u(s)\, ds\|^2 \\ &\quad + \frac{1}{2}\frac{d}{dt}\| \int_0^t e^{-\gamma(t-s)}\, u(s)\, ds\|^2\,, \end{aligned}$$

we deduce

$$\begin{aligned} &\frac{d}{dt}\left(\|u(t)\|^2 + \beta\|\int_0^t e^{-\gamma(t-s)}\, u(s)\, ds\|^2 + 2D\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\, ds\right) \\ &\leq -2\gamma\beta\| \int_0^t e^{-\gamma(t-s)} u(s)\, ds\|^2 + 2f'_{\sup}\|u(t)\|^2. \end{aligned} \tag{11}$$

The inequality (11) enable us to conclude (7) with $S_{i,g}(t)$ given by (8).

As $\|u(t)\|^2 \leq (b-a)^2 \|\frac{\partial u}{\partial x}(t)\|^2$ the proof of inequality (9) follows the proof of (7).

■

Remark 1. *Theorem 1 was established under the condition that f' is bounded. This assumption can be weakened if the solution of (6) satisfies*

$$|u(x,t)| \leq L, \quad x \in [a,b],\; t \in [0,T]\,, \tag{12}$$

for some positive constant L. Supposing that u satisfies (12) then we only need to assume that f' is bounded in $[-L, L]$. Moreover, if f' is continuous then $f'_{\sup}$ should be replaced by $f'_{\max} := \max_{|y|\leq L} f'(y)$.

Remark 2. *From the inequality (7) we conclude that if $f'_{\sup} > 0$ then*

$$\begin{aligned}
&\|u(t)\|^2 \leq M,\\
&\|\int_0^t e^{-\gamma(t-s)}\, u(s)\, ds\|^2\, ds \leq M,\\
&\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\, ds \leq M,
\end{aligned}$$

for some positive constant M when we consider bounded time intervals. Moreover from the inequality (9) we also conclude that if

$$f'_{\sup} < \frac{D}{(b-a)^2}, \tag{13}$$

then

$$\begin{aligned}
&\|u(t)\|^2 \to 0\,,\\
&\|\int_0^t e^{-\gamma(t-s)}\, u(s)\, ds\,\|^2\, ds \to 0,
\end{aligned} \tag{14}$$

when $t \to +\infty$.

2.2. Differential IBVP. Let $L^2(0,T,L^2(a,b))$ be the space of functions v defined on $[a,b]\times[0,T]$ such that, for each t, $v(t) \in L^2(a,b)$ and

$$\int_0^T \|v(t)\|^2\, dt < \infty.$$

Let $u \in L^2(0,T,H_0^1(a,b))$ and $v \in L^2(0,T,L^2(a,b))$ be such that, for each time $t \in (0,T]$

$$v(a,t) = v(b,t) = 0,\ \frac{\partial u}{\partial t}(t), \frac{\partial v}{\partial t}(t) \in L^2(a,b)$$

and

$$\begin{cases} (\dfrac{\partial u}{\partial t}(t), w) + D(\dfrac{\partial u}{\partial x}(t), w') = (f(u(t)), w) - (v(t), w),\ \forall w \in H_0^1(a,b)\,, \\ (\dfrac{\partial v}{\partial t}(t), q) = \beta(u(t), q) - \gamma(v(t), q),\ \forall q \in L^2(a,b)\,, \\ u(0) = u_0\,, \\ v(0) = v_0\,. \end{cases} \tag{15}$$

In what follows an estimate for $\|u(t)\|^2 + \|v(t)\|^2$ is established being its proof based on the energy method.

Theorem 2. *Let f be such that $f(0) = 0$ and f' is bounded. If $u \in L^2(0,T,H_0^1(a,b))$ and $v \in L^2(0,T,L^2(a,b))$ satisfying (15) and are such that $\dfrac{\partial u}{\partial t}(t), \dfrac{\partial v}{\partial t}(t) \in L^2(a,b)$ then, for $t \in [0,T]$,*

(1)

$$\|u(t)\|^2 + \|v(t)\|^2 + 2D\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\,ds \le S_{d,g}(t)\left(\|u_0\|^2 + \|v_0\|^2\right), \tag{16}$$

with

$$S_{d,g}(t) = e^{\max\{|\beta-1|+2f'_{\sup},\,|\beta-1|-2\gamma,0\}\,t}, \tag{17}$$

(2)

$$\|u(t)\|^2 + \|v(t)\|^2 \le S_D(t)\left(\|u_0\|^2 + \|v_0\|^2\right), \tag{18}$$

with

$$S_D(t) = e^{\max\{|\beta-1|+2f'_{\sup}-2\frac{D}{(b-a)^2},\,|\beta-1|-2\gamma\}\,t}. \tag{19}$$

Proof: Taking in (15) w and q replaced by $u(t)$ and $v(t)$ respectively, we easily obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2 + \|v(t)\|^2\right) &= -D\|\frac{\partial u}{\partial x}(t)\|^2 + \Big(f(u(t)),\,u(t)\Big) \\ &\quad +(\beta-1)\Big(v(t),\,u(t)\Big) - \gamma\|v(t)\|^2\,. \end{aligned}$$

Following the proof of Theorem 1 we conclude the proof of Theorem 2. ■

Remark 3. *A remark analogous to Remak 1 can be stated for the smoothness of the reaction term f.*

Remark 4. *Let us suppose that we compute $v(t)$ from the second equation of (5) using the boundary and initial conditions such that (5) leads to (1). Looking to the stability factors $S_{d,g}(t)$ and $S_D(t)$ arising in the stability upper bounds established in Theorem 2, we observe that they are greater than the correspondent stability factors $S_{i,g}(t)$ and $S_i(t)$.*

Remark 5. *In the context of the previous remark, we are not able to conclude the convergence (14) when $f'_{\sup}$ satisfies (13).*

3. FDM

3.1. Integro-differential IBVP. Let us consider in $[a,b]$ the grid $I_h := \{x_i : i = 0,\dots,N\}$ with $x_0 = a$, $x_N = b$ and $x_i - x_{i-1} = h$. In $[0,T]$ we define a time grid $\{t_j : j = 0,\dots,M\}$ with $t_0 = 0$, $t_M = T$ and $t_{j+1} - t_j = \Delta t$.

We discretize the second partial derivative of u with respect to x in (1) using the second order centered finite-difference operator $D_{2,x}$ defined by

$$D_{2,x} w_h(x_i,t_j) := \frac{w(x_{i+1},t_j) - 2w(x_i,t_j) + w(x_{i-1},t_j)}{h^2}, \quad i = 1,\dots,N-1\,.$$

By D_{-x} and D_{-t} we denote the backward finite-difference operators defined by

$$D_{-x} w_h(x_i,t_j) := \frac{w_h(x_i,t_j) - w_h(x_{i-1},t_j)}{h}, \quad i = 1,\dots,N\,,$$
$$D_{-t} w_h(x_i,t_j) := \frac{w_h(x_i,t_j) - w_h(x_i,t_{j-1})}{\Delta t}, \quad j = 1,\dots,M\,.$$

The stability results will be established with respect to a norm that we present in what follows and that can be seen as a natural descritization of the L^2-norm. By $L^2_0(I_h)$ we denote the space of grid functions w_h defined in I_h such that $w_h(x_0) = w_h(x_M) = 0$. In $L^2_0(I_h)$ we consider the discrete inner product

$$(v_h, w_h)_h := h \sum_{i=1}^{N-1} v_h(x_i)\, w_h(x_i), \quad v_h, w_h \in L^2_0(I_h)\,, \tag{20}$$

and $\|.\|_h$ denotes the norm induced by the above inner product.

In this section we study the following FDM

$$
\begin{aligned}
&D_{-t}u_h^{n+1}(x_i) = D\, D_{2,x}u_h^{n+1}(x_i) - \beta\Delta t\sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)}u_h^j(x_i) + f\big(u_h^{n+1}(x_i)\big),\\
&i=1,...,N-1, \quad n=0,\dots,M-1,
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
&u_h^j(x_0) = u_a(t_j),\ u_h^j(x_N) = u_b(t_j), \quad j=1,\dots,M,\\
&u_h^0(x_i) = u_0(x_i), \quad i=0,\dots,N\,.
\end{aligned}
\tag{22}
$$

This method can be seen as a discretization of the integro-differential IBVP (1), (2), (3), when the spatial derivative is discretized by the operator $D_{2,x}$ and the right rectangular rule is considered on the discretization of integral term. However, method (21) can also be seen as a discretization of the weak differential problem (6). In order to show that, at least for homogeneous boundary conditions, let P_h be the piecewise linear interpolation operator induced by the grid I_h. Let ϕ_i, $i=1,\dots,N-1$, be the usual hat functions. Considering in the variational equality of (6) $u(t)$ and w replaced by

$$
P_hu_h(t) = \sum_{i=1}^{N-1} u_h(x_i,t)\phi_i(x)
$$

and

$$
P_hw_h = \sum_{i=1}^{N-1} w_h(x_i)\phi_i(x),
$$

respectively, we get

$$
\begin{aligned}
(\frac{\partial P_hu_h}{\partial t}(t), P_hw_h) + D(\frac{\partial P_hu_h}{\partial x}(t), P_hw_h') &= -\beta(\int_0^t e^{-\gamma(t-s)}P_hu_h(s)\,ds, P_hw_w)\\
&\quad +(f(P_hu_h(t)), P_hw_h)\,,
\end{aligned}
\tag{23}
$$

for all $w_h \in L^2_0(I_h)$. Replacing in (23) the inner product $(.,.)$ by the discrete inner product $(.,.)_h$ we obtain

$$\begin{aligned}(\frac{d}{dt}u_h(t), w_h)_h + D(D_{-x}u_h(t), D_{-x}w_h)_{h,+} &= -\beta(\int_0^t e^{-\gamma(t-s)}u_h(s)\,ds, w_w)_h \\ &\quad +(f(u_h(t)), w_h)_h\,,\end{aligned} \tag{24}$$

for all $w_h \in L^2_0(I_h)$, where

$$(D_{-x}u_h(t), D_{-x}w_h)_{h,+} = h\sum_{i=1}^{N} D_{-x}u_h(x_i,t)D_{-x}w_h(x_i).$$

Taking in (24) $t = t_{n+1}$ and considering the time derivative replaced by the backward finite difference operator and the integral term descritized by the right-hand side rectangular rule we obtain

$$\begin{aligned}(D_{-t}u_h^{n+1}, w_h)_h + D(D_{-x}u_h^{n+1}, D_{-x}w_h)_{h,+} &= -\beta\Delta t\sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)}(u_h^j, w_h)_h \\ &\quad +(f(u_h^{n+1}), w_h)_h\,,\end{aligned} \tag{25}$$

for all $w_h \in L^2_0(I_h)$.

Finally, the method (21)-(22) is established using the fact

$$(D_{-x}u_h^{n+1}, D_{-x}w_h)_{h,+} = -(D_{2,x}u_h^{n+1}, w_h)_h,$$

and replacing in (25) w_h by the grid function

$$w_{h,i}(x_j) = \left\{\begin{array}{l} 1\,,\ j = i \\ 0\,,\ j \neq i\,, \end{array}\right.$$

for $i = 1, \ldots, N-1$.

The previous considerations allow us to conclude that the numerical approximation computed with method (21),(22), $u_h^n(x_i), i = 1, \ldots, N, n = 1, \ldots, M-1$, is a finite difference approximation for $u(x_i, t_n)$, where u is solution of (1),(2),(3) being $P_h u_h^n$ an approximation for the weak solution defined by (6).

Theorem 3. *Let u_h^j be defined by (21), (22) with homogeneous boundary conditions. If the source term f is such that $f(0) = 0$ and f' is bounded,* *then*

$$\|u_h^{n+1}\|_h^2 + \beta \|\Delta t \sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)} u_h^j\|_h^2 \leq (S_i(\Delta t))^{n+1} \, \|u_0\|_h^2 \, , \tag{26}$$

where

$$S_i(\Delta t) = \frac{1}{\min\{1,\ 1+\Delta t\Big(\beta\Delta t + 2\Big(\frac{D}{(b-a)^2} - f'_{\sup}\Big)\Big)\}} \tag{27}$$

provided that Δt *satisfies*

$$1+\Delta t\Big(\beta\Delta t + 2\Big(\frac{D}{(b-a)^2} - f'_{\sup}\Big)\Big) > 0\, . \tag{28}$$

Proof:

(a) Let us consider (21) with $n \in \mathbb{N}$. Taking in (25) w_h replaced by u_h^{n+1} and combining the obtained equality with

$$\begin{aligned} 2(\sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)} u_h^j, u_h^{n+1})_h &= \| \sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)} u_h^j\|_h^2 \\ &\quad - e^{-2\gamma\Delta t} \| \sum_{j=1}^{n} e^{-\gamma(t_n-t_j)} u_h^j\|_h^2 + \|u_h^{n+1}\|_h^2 \end{aligned}$$

and with the discrete Friedrich-Poincaré inequality

$$\|u_h^{n+1}\|_h^2 \leq (b-a)^2 \|D_{-x} u_h^{n+1}\|_{h,+}^2,$$

where $\|D_{-x} u_h^{n+1}\|_{h,+}^2 = (D_{-x} u_h^{n+1}, D_{-x} u_h^{n+1})_{h,+}$, we deduce

$$\begin{aligned} &\|u_h^{n+1}\|_h^2 + \frac{\beta}{2}\Delta t^2 \|u_h^{n+1}\|_h^2 + \frac{\Delta t\, D}{(b-a)^2}\|u_h^{n+1}\|_h^2 + \frac{\beta}{2}\Delta t^2 \| \sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)} u_h^j\|_h^2 \\ &\leq \Big(u_h^n, u_h^{n+1}\Big)_h + e^{-2\gamma\Delta t}\, \frac{\beta}{2}\Delta t^2 \| \sum_{j=1}^{n} e^{-\gamma(t_n-t_j)} u_h^j\|_h^2 + \Delta t\Big(f(u_h^{n+1}), u_h^{n+1}\Big)_h . \end{aligned} \tag{29}$$

Using the Cauchy-Schwarz inequality and the estimate

$$\Big(f(u_h^{n+1}), u_h^{n+1}\Big)_h \leq f'_{\sup} \|u_h^{n+1}\|_h^2 \, ,$$

we obtain

$$\begin{aligned}
&\Big(1+\beta\Delta t^2+\frac{2\Delta t\, D}{(b-a)^2}-2\Delta t f'_{\sup}\Big)\|u_h^{n+1}\|_h^2+\beta\|\Delta t\sum_{j=1}^{n+1}e^{-\gamma(t_{n+1}-t_j)}u_h^j\|_h^2\\
&\leq \|u_h^n\|_h^2+e^{-2\gamma\Delta t}\beta\|\Delta t\sum_{j=1}^{n}e^{-\gamma(t_n-t_j)}u_h^j\|_h^2 .
\end{aligned} \tag{30}$$

The inequality (30) easily leads to

$$\|u_h^{n+1}\|_h^2+\beta\|\Delta t\sum_{j=1}^{n+1}e^{-\gamma(t_{n+1}-t_j)}u_h^j\|_h^2\leq S_i(\Delta t)\left(\|u_h^n\|_h^2+\beta\|\Delta t\sum_{j=1}^{n}e^{-\gamma(t_n-t_j)}u_h^j\|_h^2\right),$$

with $S_i(\Delta t)$ defined by (27) and provided that Δt satisfies (28).

(b) We now consider (21) with $n=0$. From (25) with $n=0$ and w_h replaced by u_h^1, following (a) it can be shown that

$$\Big(1+\Delta t\big(\beta\Delta t+\frac{2D}{(b-a)^2}-2f'_{\sup}\big)\Big)\|u_h^1\|_h^2+\beta\|\Delta t\, u_h^1\|_h^2\leq\|u_h^0\|_h^2 .$$

Finally, the inequality (26) follows from (a) and (b).

■

Remark 6. *Theorem 3 holds assuming that f' is bounded. This assumption can be weakened if $|u_h^n(x_i)|\leq L$ for all i and for all n, for some constant $L>0$. In this case, Theorem 3 can be established if f' is bounded in $[-L,L]$ being $f'_{\sup}=\sup_{|y|\leq L}f'(y)$. Moreover, f' is continuous in $[-L,L]$, then Theorem 3 holds with $f'_{\sup}$ replaced by $f'_{\max}=\max_{|y|\leq L}f'(y)$.*

Remark 7. *Theorem 3 is established provided that the time stepsize Δt satisfies (28). We consider in what follows practical conditions that imply the last stepsize restriction.*

If $f'_{\sup}, D$ and β are such that

$$\Big(\frac{D}{(b-a)^2}-f'_{\sup}\Big)^2<\beta, \tag{31}$$

or

$$\Big(\frac{D}{(b-a)^2}-f'_{\sup}\Big)^2>\beta \ \text{ and } \ \frac{D}{(b-a)^2}-f'_{\sup}\geq 0, \tag{32}$$

then (28) holds for all time stepsize. Otherwise, if

$$\Big(\frac{D}{(b-a)^2}-f'_{\sup}\Big)^2>\beta \ \ \textit{and} \ \ \frac{D}{(b-a)^2}-f'_{\sup}<0, \tag{33}$$

then Δt satisfies (28) if and only if

$$\Delta t<\Delta t_0:=\frac{1}{\beta}\Big(f'_{\sup}-\frac{D}{(b-a)^2}-\sqrt{\Big(\frac{D}{(b-a)^2}-f'_{\sup}\Big)^2-\beta}\Big). \tag{34}$$

Remark 8. i) *If (32) holds then* $\min\{1,\ 1+\Delta t\Big(\beta\Delta t+2\Big(\frac{D}{(b-a)^2}-f'_{\max}\Big)\Big)\}=1$ *and consequently $S_i(\Delta t)=1$. In this case (26) is equivalent to*

$$\|u_h^{n+1}\|_h^2+\beta\|\Delta t\sum_{j=1}^{n+1}e^{-\gamma(t_{n+1}-t_j)}u_h^j\|_h^2\leq\|u_0\|_h^2\,,$$

without any restriction on the time stepsize which means that the method (21) is unconditionally stable.

ii) *If (33) holds then for Δt satisfying*

$$\Delta t<\Delta t_1:=\min\{\frac{2}{\beta}\Big(f'_{\sup}-\frac{D}{(b-a)^2}\Big)\,,\Delta t_0\}, \tag{35}$$

the inequality (26) holds. Since $S_i(\Delta t)$ verifies

$$S_i(\Delta t)\leq 1+\Delta t\frac{2\Big(f'_{\sup}-\frac{D}{(b-a)^2}\Big)}{1+\Delta t_1\Big(\beta\Delta t_1+2\Big(\frac{D}{(b-a)^2}-f'_{\sup}\Big)\Big)}\,,$$

from (26) we obtain

$$\|u_h^{n+1}\|_h^2+\beta\|\Delta t\sum_{j=1}^{n+1}e^{-\gamma(t_{n+1}-t_j)}u_h^j\|_h^2\leq e^{(n+1)\Delta t\eta}\|u_0\|_h^2\,,$$

with $\eta=\dfrac{2\Big(f'_{\sup}-\frac{D}{(b-a)^2}\Big)}{1+\Delta t_1\Big(\beta\Delta t_1+2\Big(\frac{D}{(b-a)^2}-f'_{\sup}\Big)\Big)}$. *In this case the method (21) is stable for $\Delta t\in(0,\Delta t_1)$ with Δt_1 defined by (35).*

Remark 9. *In the computation of the numerical solution at time level $n+1$ with the method (21) we use $u_h^j, j=0,\ldots,n$. Consequently, if n increases* *then increases the storage of needed information. Nevertheless this disadvantage is only apparent because the method can be rewritten in the following form:*

$$D_{-t}u_h^1(x_i) = DD_{2,x}u_h^1(x_i) - \beta\Delta t e^{-\gamma\Delta t}u_h^1(x_i) + f(u_h^1(x_i)),\ i = 1,\dots,N-1,$$

$$\begin{aligned} D_{-t}u_h^{n+1}(x_i) &= DD_{2,x}u_h^{n+1}(x_i) - \beta\Delta t u_h^{n+1}(x_i) + f(u_h^{n+1}(x_i)) \\ &\quad + e^{-\gamma\Delta t}\Big(D_{-t}u_h^n(x_i) - DD_{2,x}u_h^n(x_i) - f(u_h^n(x_i))\Big),\ i = 1,\dots,N-1, \\ n &= 1,\dots,M-1. \end{aligned}$$

In the method (21),(22) the reaction term f is considered at time level t_{n+1}. Due to this fact, we need to solve a nonlinear system in each time step. When the reaction term f is nonstiff this term can be considered at time level t_n. In this case the previous method is replaced by the following one

$$\begin{aligned} &D_{-t}u_h^{n+1}(x_i) = D\,D_{2,x}u_h^{n+1}(x_i) - \beta\Delta t\sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)}u_h^j(x_i) + f\big(u_h^n(x_i)\big)\,, \\ &i = 1,\dots,N-1, \quad n = 0,\dots,M-1\,, \end{aligned} \tag{36}$$

that belongs to the well known class of IMEX methods.

For the IMEX method (36) holds a stability result analogous to Theorem 3. In fact, as

$$\Big(f(u_h^n), u_h^{n+1}\Big)_h \leq \frac{1}{2}\|u_h^n\|_h^2 + \frac{1}{2}f'^{\,2}_{\sup}\|u_h^{n+1}\|_h^2\,,$$

following the proof of this result, it can be shown that

$$\begin{aligned} &\min\{1 + \beta\Delta t^2 + \frac{2D\Delta t}{(b-a)^2} - f'^{\,2}_{\sup}\Delta t, 1\}\left(\|u_h^n\|_h^2 + \beta\|\Delta t\sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)}u_h^j\|_h^2\right) \\ &\leq \max\{1+\Delta t, 1\}\left(\|u_h^n\|_h^2 + \beta\|\Delta t\sum_{j=1}^{n} e^{-\gamma(t_n-t_j)}u_h^j\|_h^2\right)\,, \end{aligned}$$

which is equivalent to

$$\|u_h^n\|_h^2 + \beta \|\Delta t \sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)} u_h^j\|_h^2$$
$$\leq S_{i,e}(\Delta t) \left(\|u_h^n\|_h^2 + \beta \|\Delta t \sum_{j=1}^{n} e^{-\gamma(t_n-t_j)} u_h^j\|_h^2 \right),$$

with

$$S_{i,e}(\Delta t) = \frac{1+\Delta t}{\min\{1+\Delta t\big(\beta\Delta t + \frac{2D}{(b-a)^2} - f'^{\,2}_{\sup}\big), 1\}}, \tag{37}$$

provided that the time stepsize Δt satisfies

$$1+\Delta t\big(\beta\Delta t + \frac{2D}{(b-a)^2} - f'^{\,2}_{\sup}\big) > 0. \tag{38}$$

The stability result for the IMEX method (36) is established now.

Theorem 4. *Let u_h^j be defined by (36) with homogeneous boundary conditions. Under the assumptions of Theorem 3 we have*

$$\|u_h^{n+1}\|_h^2 + \beta \|\Delta t \sum_{j=1}^{n+1} e^{-\gamma(t_{n+1}-t_j)} u_h^j\|_h^2 \leq \big(S_{i,e}(\Delta t)\big)^{n+1} \|u_h^0\|_h^2, \tag{39}$$

provided that Δt satisfies (38) and with $S_{i,e}(\Delta t)$ defined by (37).

Remark 10. i) *For the IMEX method (36) hold remarks analogous to Remark 6 and 7 with $f'_{\max}$ and $f'_{\sup}$ replaced by $\frac{1}{2}f'^{\,2}_{\max}$ and $\frac{1}{2}f'^{\,2}_{\sup}$, respectively.*

ii) *Remark 8 also holds for the IMEX method (36) with $S_i(\Delta t)$ and $f'_{\sup}$ replaced by $S_{i,e}(\Delta t)$ and $\frac{1}{2}f'^{\,2}_{\sup}$, respectively. For $S_{i,e}$ holds*

$$S_{i,e}(\Delta t) \leq 1 + \Delta t \frac{1+2\Big(\frac{1}{2}f'^{\,2}_{\sup} - \frac{D}{(b-a)^2}\Big)}{1+\Delta t_1\Big(\beta\Delta t_1 + 2\Big(\frac{D}{(b-a)^2} - \frac{1}{2}f'^{\,2}_{\sup}\Big)\Big)},$$

where Δt_1 is defined by (35) with $f'_{\sup}$ replaced by $\frac{1}{2}f'^{\,2}_{\sup}$.

3.2. Differential IBVP. We mentioned above that the integro-differential IBVP (1), (2), (3) is equivalent, for certain initial and boundary conditions, to the differential IBVP (2),(3), (5). Then we can solve computationally the first problem considering numerical methods for the second one.

In this section we study the implicit FDM

$$\begin{cases} D_{-t}u_h^{n+1}(x_i) = DD_{2,x}u_h^{n+1}(x_i) - v_h^{n+1}(x_i) + f\left(u_h^{n+1}(x_i)\right) \\ \\ D_{-t}v_h^{n+1}(x_i) = \beta u_h^{n+1}(x_i) - \gamma v_h^{n+1}(x_i)\,, \end{cases} \tag{40}$$

$i = 1, \dots, N-1$, $n = 0, \dots, M-1$, and

$$\begin{cases} u_h^j(x_0) = u_a(t_j),\ u_h^j(x_N) = u_b(t_j), \quad j = 1, \dots, M, \\ v_h^j(x_0) = v_a(t_j),\ v_h^n(x_N) = v_b(t_j), \quad j = 1, \dots, M, \\ u_h^0(x_i) = u_0(x_i),\ v_h^0(x_i) = v_0(x_i), \quad i = 1, \dots, N-1, \end{cases} \tag{41}$$

where $f(u)$ represents the reaction, not necessarily defined by $f(u) = u(1-u)(u-\alpha)$. This methods can be seen as a discretization of the differential problem (2), (3), (5) or it can be used to compute an approximation to the weak solution of the variational problem (15) because, for homogeneous boundary conditions, u_h^n, v_h^n satisfy

$$\begin{cases} (D_{-t}u_h^{n+1}, w_h)_h + D(D_{-x}u_h^{n+1}, D_{-x}w_h)_{h,+} = -(v_h^{n+1}, w_h)_h + (f(u_h^{n+1}), w_h)_h, \\ (D_{-t}v_h^{n+1}, q_h)_h = \beta(u_h^{n+1}, q_h)_h - \gamma(v_h^{n+1}, q_h)_h\,, \end{cases} \tag{42}$$

for all $w_h, q_h \in L_0^2(I_h)$, with u_h^0, v_h^0 defined as in (41).

Theorem 5. *Let u_h^j, v_h^j, $j = 1, \dots, M$, be defined by (40), (41) with homogeneous boundary conditions. If $f(0) = 0$ and f' is bounded then*

$$\|u_h^{n+1}\|_h^2 + \|v_h^{n+1}\|_h^2 \le \Big(S_d(\Delta t)\Big)^{n+1} \Big(\|u_h^0\|_h^2 + \|v_h^0\|_h^2\Big), \tag{43}$$

with the stability factor $S_d(\Delta t)$ defined by

$$S_d(\Delta t) = \frac{1}{\min\Big\{1 + \Delta t(\frac{2D}{(b-a)^2} - 2f'_{\sup} - |\beta - 1|),\, 1 + \Delta t(2\gamma - |\beta - 1|)\Big\}}. \tag{44}$$

and provided that Δt satisfies

$$1 + \Delta t(\frac{2D}{(b-a)^2} - 2f'_{\sup} - |\beta - 1|) > 0 \quad \textit{and} \ 1 + \Delta t(2\gamma - |\beta - 1|) > 0\,. \tag{45}$$

Proof: Let us consider (42) with w_h and q_h replaced by u_h^{n+1} and v_h^{n+1}, respectively. It is easy to show that the following inequality holds

$$\frac{1}{2}\|u_h^{n+1}\|_h^2 + (\frac{1}{2}+\gamma\Delta t)\|v_h^{n+1}\|_h^2 \leq \frac{1}{2}\|u_h^{n}\|_h^2 + \frac{1}{2}\|v_h^{n}\|_h^2 - \Delta t\|D_{-x}u_h^{n+1}\|_{h,+}^2$$
$$+ \Delta t(f(u_h^{n+1}), u_h^{n+1}) + \Delta t(\beta-1)(u_h^{n+1}, v_h^{n+1}). \tag{46}$$

Considering in (46) the discrete Friedrich-Poincaré inequality

$$\|u_h^{n+1}\|_h^2 \leq (b-a)^2\|D_{-x}u_h^{n+1}\|_{h,+}^2$$

and the inequality $(f(u_h^{n+1}), u_h^{n+1}) \leq f'_{\sup}\|u_h^{n+1}\|,$ we obtain

$$\Big(1+\Delta t(\frac{2D}{(b-a)^2} - 2f'_{\sup} - |\beta-1|)\|u_h^{n+1}\|_h^2 + \Big(1+\Delta t(2\gamma-|\beta-1|)\Big)\|v_h^{n+1}\|_h^2$$
$$\leq \|u_h^n\|_h^2 + \|v_h^n\|_h^2$$

witch allow us to conclude (43) provided that the time stepsize satisfies (45). ■

Remark 11. *Theorem 5 was established assuming that the reaction term f' is bounded. This assumption can be weakened if we suppose that the numerical solution $u_h^n(x_i)$ is bounded for all i and n.*

Remark 12. *If the coefficients D, γ and the reaction term t are such that*

$$\frac{D}{(b-a)^2} - f'_{\sup} - \gamma < 0, \tag{47}$$

and Δt_0 is fixed satisfying the following restriction

$$1+\Delta t_0(\frac{2D}{(b-a)^2} - 2f'_{\sup} - |\beta-1|) > 0, \tag{48}$$

then (43) holds, for $\Delta t \in (0, \Delta t_0]$, with $S_d(\Delta t)$ such that

$$S_d(\Delta t) = 1 + \Delta t\,\eta \tag{49}$$

and

$$\eta = \frac{2f'_{\sup} - \frac{2D}{(b-a)^2} + |\beta-1|}{1+\Delta t_0(\frac{2D}{(b-a)^2} - 2f'_{\sup} - |\beta-1|)}.$$

Consequently we have

$$\|u_h^{n+1}\|_h^2 + \|v_h^{n+1}\|_h^2 \leq e^{(n+1)\Delta t\eta}\Big(\|u_h^0\|_h^2 + \|v_h^0\|_h^2\Big),$$

for $\Delta t \in (0, \Delta t_0]$, *which means that the implicit method (40) is conditionally stable.*

Otherwise, if (47) does not holds then (43) holds with

$$S_d(\Delta t) = 1 - \frac{\Delta t\big(2\gamma - |1-\beta|\big)}{1 + \Delta t_0\big(2\gamma - |1-\beta|\big)},$$

for $\Delta t \in (0, \Delta t_0]$ *where* Δt_0 *satisfies* $1 + \Delta t_0\big(2\gamma - |1-\beta|\big) > 0$.

and without any restriction on the stepsize Δt. *Consequently we have*

$$\|u_h^{n+1}\|_h^2 + \|v_h^{n+1}\|_h^2 \le \|u_h^0\|_h^2 + \|v_h^0\|_h^2,$$

which means that the implicit method (40) is conditionally stable.

If the reaction term in nonstiff then the term f can be considered evaluated at u_h^n, and the method (40) is replaced by the IMEX method

$$\begin{cases} D_{-t}u_h^{n+1}(x_i) = D\,D_{2,x}u_h^{n+1}(x_i) - v_h^{n+1}(x_i) + f\left(u_h^n(x_i)\right) \\ D_{-t}v_h^{n+1}(x_i) = \beta u_h^{n+1}(x_i) - \gamma v_h^{n+1}(x_i)\,, \end{cases} \tag{50}$$

$i = 1, \dots, N-1$, $n = 0, \dots, M-1$. Using the IMEX method (50) we reduce the computational cost of method (40) because in each time step we only need to solve a linear system.

For the IMEX method (50) holds a result analogous to Theorem 5 with $f'_{\sup}$ replaced by $\frac{1}{2}f'^{\,2}_{\sup}$, that is

$$1 + \Delta t\Big(\frac{2D}{(b-a)^2} - f'^{\,2}_{\sup} - |\beta - 1|\Big) > 0 \;\text{ and }\; 1 + \Delta t(2\gamma - |\beta - 1|) > 0\,. \tag{51}$$

Moreover Remarks 11 and 12 also hold with the previous modifications.

4. Numerical results

In this section our aim is to illustrate the theoretical results obtained in the last section. Let SC_i and $SC_{i,e}$ be defined by

$$SC_i = \max\{0,\, 1 + \Delta t(\beta\Delta t + \frac{2D}{(b-a)^2} - 2f'_{\sup})\}$$

and

$$SC_{i,e} = \max\{0,\, 1 + \Delta t(\beta\Delta t + \frac{2D}{(b-a)^2} - f'^{\,2}_{\sup})\}.$$

These two terms arise in the definitions of the stability conditions (28) and (38) and allow us to compare these conditions when the implicit method (21) and the IMEX method (36) are used.

By SC_d and $SC_{d,e}$ we represent the following quantities

$$SC_d = \max\{0, \min\{1+\Delta t(\frac{2D}{(b-a)^2} - 2f'_{\sup} - |\beta-1|),\, 1+\Delta t(2\gamma - |\beta-1|)\}\},$$

$$SC_{d,e} = \max\{0, \min\{1+\Delta t(\frac{2D}{(b-a)^2} - f'^{\,2}_{\sup} - |\beta-1|),\, 1+\Delta t(2\gamma - |\beta-1|)\}\}$$

which arise in the stability restrictions (45) and (51) for the methods (40) and (50), respectively.

In what follows we measure the stability conditions (28), (38), (45) and (51) using $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$. We consider $D = 1$, $\beta = 0.008$, $\gamma = 0.02032$, $(a,b) = (0,100)$ and different values of $f'_{\sup}$.

In Figure 1 we plot $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$ for $f'_{\sup} = -1$. The conditions (38), (45) for the methods (36) and (40), respectively, have the same behaviour. The condition (51) for the method (50) is the strongest condition being the condition (28) the weaker one.

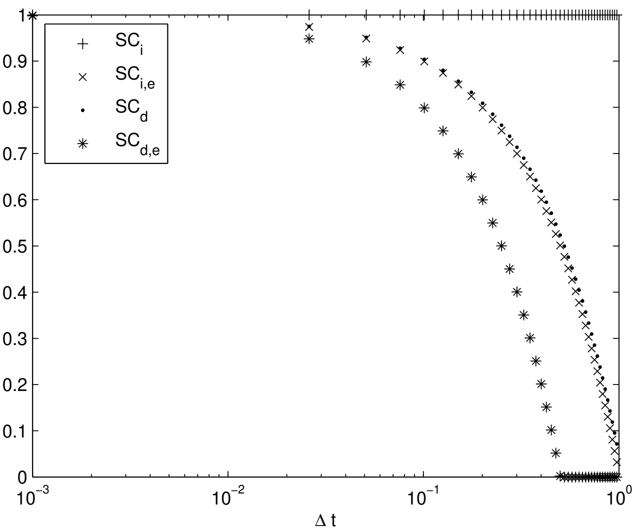

FIGURE 1. $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$ for $f'_{\sup} = -1$

In Figure 2 we plot $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$ for $f'_{\sup} = 0.5$. In this case the IMEX method (36) presents the weaker stability condition and the strongest condition is presented by the implicit method (40).

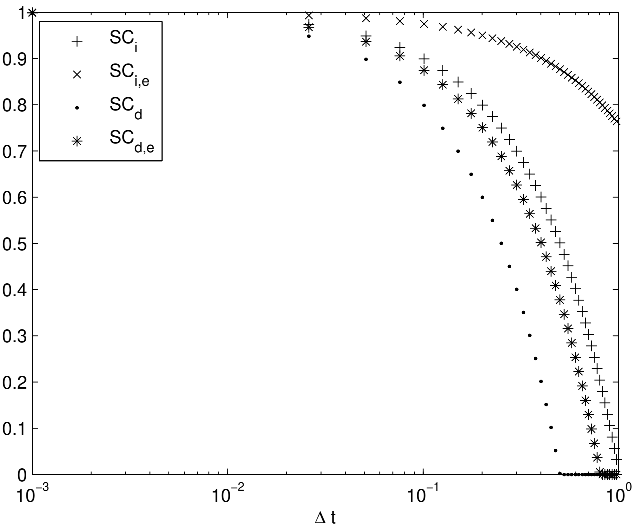

FIGURE 2. $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$ for $f'_{\sup} = 0.5$

The functions $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$ for $f'_{\sup} = 5$ are plotted in Figure 3. The stability restrictions for the two methods (28) and (40) with the implicit reaction term present the same behaviour while the stability restrictions for the methods (36) and (50) have the same behaviour.

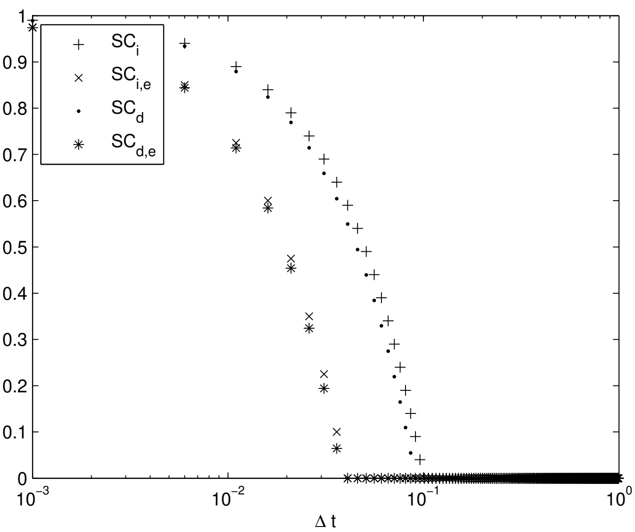

FIGURE 3. $SC_i, SC_{i,e}, SC_d$ and $SC_{d,e}$ for $f'_{\sup} = 5$.

From Figures 1, 2 and 3 we observe that the stability restrictions for the methods for the IBVP in the integro-differential form are weaker than the stability conditions for the correspondent methods for the IBVP in the differential form.

We now illustrate the behaviour of the stability factors $S_i(\Delta t)$ and $S_{i,e}(\Delta t)$ defined by (27), (37), respectively, for the methods (21) and (36) used for the IBVP in the integro-differential form. In what follows we also consider the behaviour of the stability factors $S_d(\Delta t)$ defined by (44) and $S_{d,e}(\Delta t)$ defined by

$$SC_{d,e}(\Delta t) = \frac{1+\Delta t}{\min\{1+\Delta t(\frac{2D}{(b-a)^2} - f'^{\,2}_{\text{sup}} - |\beta - 1|),\, 1+\Delta t(2\gamma - |\beta-1|)\}}$$

for the methods (40) and (50), respectively, that were considered for the IBVP in the differential form.

In Figure 4 we plot $S_i(\Delta t), S_{i,e}(\Delta t), S_{d,e}(\Delta t)$ and $SC_{d,e}$ for $f'_{\text{sup}} = -1$. The stability factors of the implicit methods are lower than the stability factors of the correspondent IMEX methods. Moreover, when the parameter f'_{sup} increases the same behaviour can be observed as it can be seen in Figure 5. However the methods (40), (50) for the IBVP in the differential form present higher stability factors. For greater values of f'_{sup} the implicit methods (21) and (40) present lower stability factors. This behaviour is well illustrated in Figure 6.

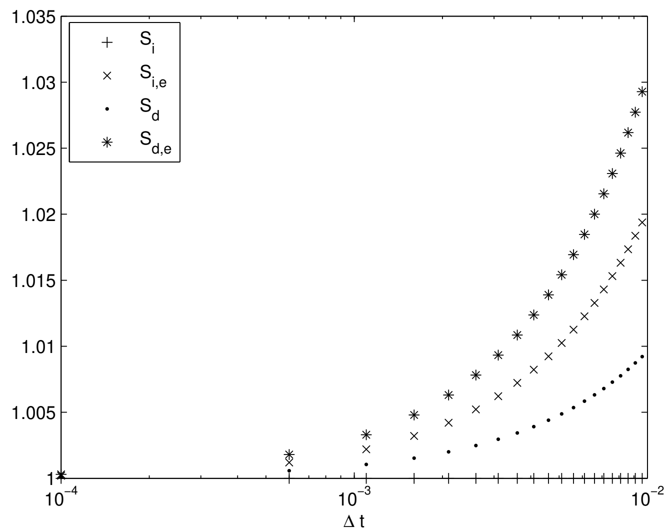

FIGURE 4. The stability factors $S_i(\Delta t), S_{i,e}(\Delta t), S_d(\Delta t)$ and $S_{d,e}(\Delta t)$ for $f'_{\text{sup}} = -1$.

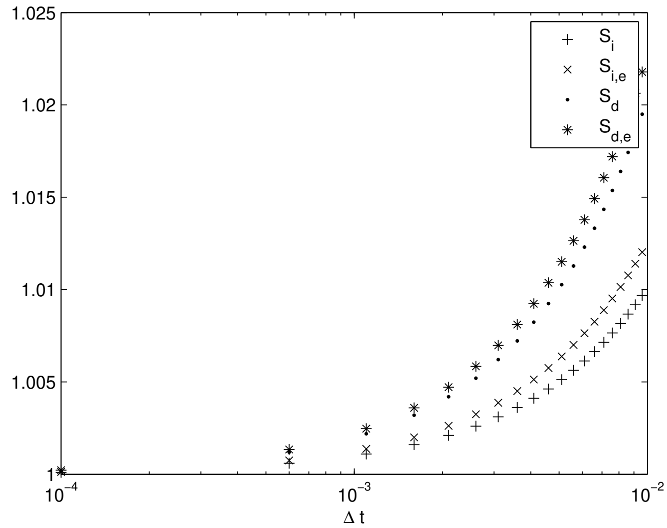

FIGURE 5. The stability factors $S_i(\Delta t), S_{i,e}(\Delta t), S_d(\Delta t)$ and $S_{d,e}(\Delta t)$ for $f'_{\sup} = 0.5$.

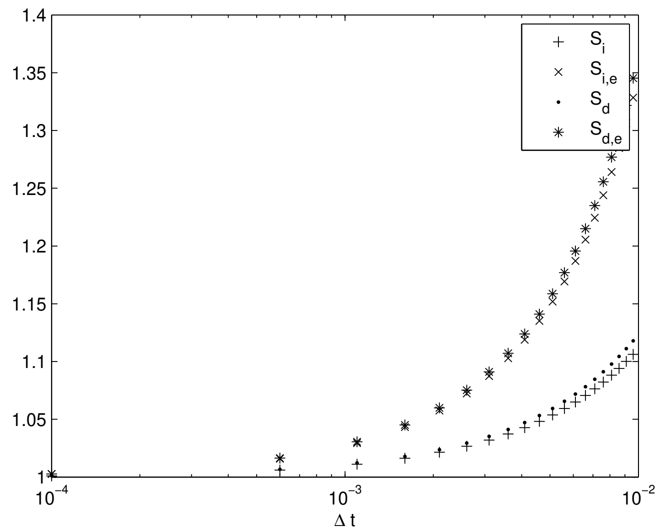

FIGURE 6. The stability factors $S_i(\Delta t), S_{i,e}(\Delta t), S_d(\Delta t)$ and $S_{d,e}(\Delta t)$ for $f'_{\sup} = 5$.

Finally, we illustrate the performance of the methods considered for the IBVP in the integro-differential form. In Figures 7 we plot the numerical solutions obtained with the implicit method (21) and with the IMEX method (36) for $f(u) = u(1-u)(u-\alpha)$, $D = 1$, $\beta = 0.008$, $\gamma = 0.0203$, $\alpha = 0.1$, $(a, b) =$

$(0,100)$, $T = 200$, $h = 1$, $\Delta t = 4$ and $u_0(x) = 0, x \in (0,100), u(0,t) = 1, t > 0, u(100,t) = 0, t > 0$.

As $f'_{\sup} < 0$ the stability condition (28) is verified for the previous parameters being the condition (38) violated, which leads to a pathologic behaviour of the method (36) as it can be seen in Figures 7 and 8. In this last figure we plot time and space cuts of the IMEX solution presented in Figure 7.

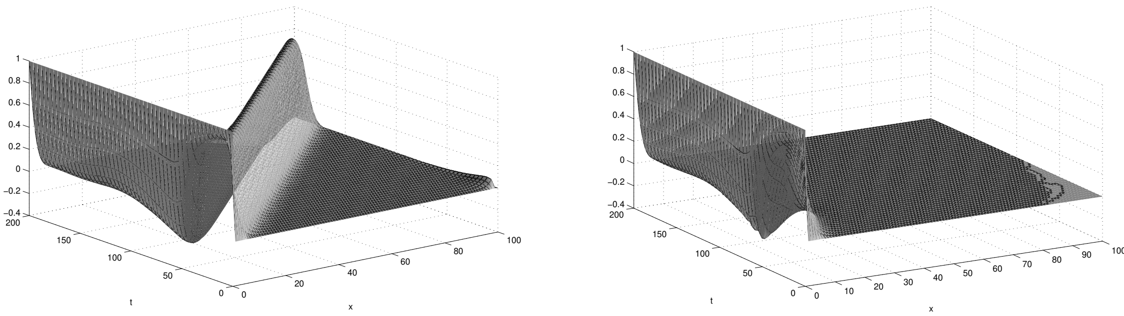

FIGURE 7. The implicit solution (left side) and the IMEX solution (right side) defined by (21) and (36), respectively.

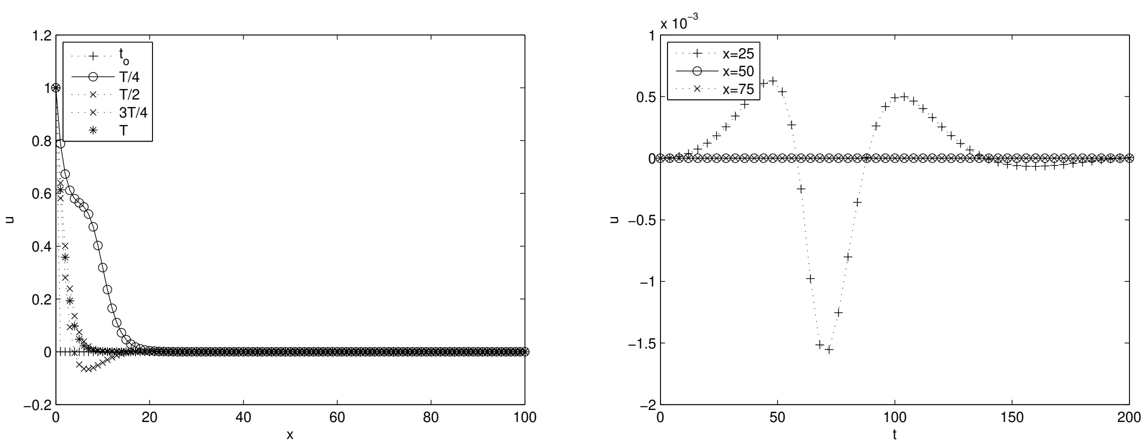

FIGURE 8. The IMEX solution defined by (36).

5. Conclusions

In this paper we studied the stability of the IBVP (1), (2), (3) and IBVP (5), (2), (3). For convenient initial and boundary conditions these two IBVPs are equivalent. In this case, in Theorems 1 and 2 are obtained estimates for

$$E_{1,\sigma}(u(t)) = \|u(t)\|^2 + \sigma \| \int_0^t e^{-\gamma(t-s)} u(s)\, ds\|^2 + 2D \int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\, ds,$$

for $\sigma = \beta, \sigma = \beta^2$.

If $\beta > 1$ the upper bound obtained using the differential form of the IBVP is greater than the upper bound obtained using the integro-differential form provided that $|\beta - 1| - 2\gamma \geq 0$. If $\beta = 1$ then the two results give the same upper bound. Finally, if $0 < \beta < 1$ and $|\beta - 1| - 2\gamma \geq 0$, then the upper bound given by the second result is greater than the one given by the first result.

For

$$E_{2,\sigma}(u(t)) = \|u(t)\|^2 + \sigma \| \int_0^t e^{-\gamma(t-s)} u(s)\, ds \|^2,$$

with $\sigma = \beta, \sigma = \beta^2$, hold conclusions analogous to those established for $E_{1,\sigma}(u(t))$, for $\sigma = \beta, \beta^2$.

From the previous considerations we conclude that the integro-differential form (1), (2), (3) leads to more accurate estimates than the differential form (5), (2), (3) for convenient initial and boundary conditions for v.

Let us compare now the stability properties of the methods (21) and (36) for the integro-differential IVBP (1), (2), (3), with the stability properties of the methods (40) and (50) for the IBVP (2), (3), (5).

In what concerns the stability restrictions (28) and (38) for the methods (21) and (36) we remark that for lower value of f'_{sup} the two conditions have the same behaviour which means that the stability behaviour of both methods will be the same. When f' increases the first method will be more stable than the IMEX method because in the condition (38) arises the term $-f'^{\,2}_{\text{sup}}$ which is negative.

Considering now the methods (21) and (36), for the integro-differential IVBP (1), (2), (3) and the methods (40) and (50), for the IBVP (2), (3), (5), we conclude that the first group is more stable than the second one.

As the stability condition (51) for the method (50) can be obtained from (45) replacing f'_{sup} by $\frac{1}{2} f'^{\,2}_{\text{sup}}$ we conclude that the two methods (40) and (50) have the same stability behaviour for lower values of f'_{sup}, being the implicit method (40) more stable than the IMEX method (50) for higher values of this parameter.

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