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ON THE UNIFORMIZATION OF L-VALUED FRAMES

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Abstract: This note discusses the appropriate way of uniformizing the notion of an L-valued frame introduced by A. Pultr and S. Rodabaugh in [*Lattice-valued frames, functor categories, and classes of sober spaces*, Chapter 6 of *Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Kluwer, 2003]. It covers the case of a completely distributive lattice L (which is, in a certain sense, the most general one) and studies the corresponding category of uniform L-valued frames.

Keywords: L-valued frames; L-Frm; L-topological spaces; L-Top; uniform L-valued frames; iota functors; upper/lower forgetful functors.

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1. Introduction

L-valued frames are structures of increasing interest for fuzzy topology (see [14], [8], [15], [16], [17], [6], [7]). They were introduced by Pultr and Rodabaugh in [14] for the case of a complete chain L and were recently extended for the more general case of a completely distributive lattice L by Gutiérrez García, Höhle and de Prada Vicente [7]. This paper is a continuation of our previous paper [6], and has its roots in the convenience of finding (after [7]) an appropriate notion of a uniform L-valued frame. There are two obvious candidates for it: the most direct one, provided by the direct approach of uniformizing the L-topologies as a frame, and the one suggested by [14], based on the concept of an L-valued frame. In the previous [6] we chose the latter (that would eventually provide a nice categorical picture of the categories at hand) since the former reveals to make no sense after the following observation of Pultr and Rodabaugh in [14]:

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> *(...) we envisage potential applications of the notion of an L-valued frame in the linear case. One of the questions of interest in fuzzy topology concerns well-founded definitions of the structures of the uniformity type. The case of a complete chain L, important historically and still very important for applications, does not allow the direct uniformization of an L-topology τ as a frame: a uniformity on τ induces a uniformity on L (as observed by Banaschewski); and since the only linearly ordered frame admitting a uniformity is the two point Boolean algebra $2 = \{0 < 1\}$ we would be left with the crisp case. One can, however, think of definitions based on the concept of an L-valued frame which would be more satisfactory.*

However, as we now have found with the following example, this observation is not true in general (it holds only for the stratified case): Let L be a non-uniformizable frame, for example $L = [0, 1]$ or any linearly ordered frame different from $2 = \{0 < 1\}$. Let $1_\varnothing$ and 1_X denote, respectively, the bottom and top elements in L^X. Clearly enough $\tau = \{1_\varnothing, 1_X\}$ is a uniformizable L-topology on X. If the observation above would be true, then there would exist a uniformity on the frame L, a contradiction.

So, this places again the former alternative as the most natural candidate for a good definition of a uniform L-valued frame and prompts for its study. This is the problem that we address in this paper, having in mind the treatment of the case of a completely distributive lattice L, following the lines of [7]. As it is shown in [7], the case of a completely distributive lattice cannot, in a certain sense, be weakened.

2. Preliminaries and notation

For standard notions and facts from category theory used here we refer to [1]. As a general reference to frames we suggest [11] or [13].

2.1. Uniform frames. A *frame* is a complete lattice A satisfying the distributive law

$$\forall a \in A,\ \forall S \subseteq A, \quad a \wedge (\bigvee S) = \bigvee\{a \wedge b \mid b \in S\}.$$

Given two frames A and B, a *frame homomorphism* $h : A \to B$ is a mapping preserving all joins and finite meets. The category of frames and frame homomorphism will be denoted by Frm.

Let A be a frame. A set $U \subseteq A$ is a *cover* of A if $\bigvee U = 1$. The set of covers of A can be preordered: a cover U *refines* a cover V, written $U \leq V$, if for each $a \in U$ there is a $b \in V$ with $a \leq b$.

For $a \in A$, the element $st(a, U) = \bigvee\{b \in U \mid b \wedge a \neq 0\}$ is called the *star of a in U*. Further, for a family $\mathcal{U}$ of covers of A, we write

$$a \overset{\mathcal{U}}{\lhd} b \text{ if there exists } U \in \mathcal{U} \text{ such that } st(a, U) \leq b.$$

Remark 2.1. The following useful properties are easy to check:
(i) $a \overset{\mathcal{U}}{\lhd} b \implies a \leq b$,

(ii) $a \leq b \overset{\mathcal{U}}{\lhd} c \leq d \implies a \overset{\mathcal{U}}{\lhd} d$,

(iii) $a \overset{\mathcal{U}}{\lhd} b,\ c \overset{\mathcal{U}}{\lhd} d \implies a \wedge c \overset{\mathcal{U}}{\lhd} b \wedge d$.

A family $\mathcal{U}$ of covers of A is a *uniformity basis* [12] provided that:

(U1) $\mathcal{U}$ is a filter basis of the preordered set $(Cov(A), \leq)$ of all covers of A.
(U2) Every $U \in \mathcal{U}$ has a star-refinement, i.e., for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with

$$st(V) := \{st(a, V) \mid a \in V\} \leq U.$$

(U3) For every $a \in A$, $a = \bigvee\{b \in A \mid b \overset{\mathcal{U}}{\lhd} a\}$.

Also, we understand by a *uniformity subbasis* a family of covers of A such that finite meets of elements of the family constitute a uniformity basis.

A *uniformity* on A is a filter $\mathcal{U}$ of covers of A generated by some uniformity basis. The pair $(A, \mathcal{U})$ is then called a *uniform frame*. Let $(A, \mathcal{U})$ and $(B, \mathcal{V})$ be uniform frames. A frame homomorphism $h : A \to B$ is a *uniform homomorphism* if, for every $U \in \mathcal{U}$, $h[U] = \{h(a) \mid a \in U\} \in \mathcal{V}$. We denote by UFrm the category of uniform frames and uniform homomorphisms.

2.2. The iota functor $\iota_L^{\top} : L\text{-}\mathsf{Top} \to \mathsf{Top}$. Let L denote a complete lattice. An element $p \in L$ is called *prime* if for each $a, b \in L$ with $a \wedge b \leq p$ either $a \leq p$ or $b \leq p$. As in [4], we denote by $\mathrm{PRIME}\, L$ the set of all prime elements of L and $\mathrm{Spec}\, L = \mathrm{PRIME}\, L \setminus \{1\}$.

Following [7], for each $\alpha \in L$ we denote by $\Uparrow\alpha$ the set $\{\beta \in L : \alpha \lessdot \beta\}$, where $\lessdot$ is the opposite relation of the way-below relation in the lattice L^{op}:

$$\alpha \lessdot \beta \equiv \text{for all } S \subseteq L \text{ such that } \bigwedge S \leq \alpha \text{ there exist } \gamma_1, \ldots \gamma_n \in S : \bigwedge_{i=1}^{n} \gamma_i \leq \beta.$$

An *L-valued topological space* [2, 5] (shortly, an *L-topological space*) is a pair (X, τ) consisting of a set X and a subset τ of L^X (the *L-valued topology* or *L-topology* on the set X), containing $1_\varnothing$ and 1_X and closed under finite meets and arbitrary joins (where meets and joins in L^X are defined pointwisely).

Given two *L-valued topological spaces* $(X, \tau_1), (Y, \tau_2)$ a map $f : X \to Y$ is an *L-continuous map* if the correspondence $b \mapsto f^{\leftarrow}(b) = b \circ f$ maps τ_2 into τ_1. The resulting category will be denoted by L-Top.

Of course, when $L = \mathsf{2}$, an L-topological space is precisely a topological space and there is an isomorphism between Top and L-Top, via the characteristic functor (the one associating to each subset its characteristic function and leaving morphisms unchanged). If L is a frame then the L-topologies, being subframes of the frame L^X, are frames as well.

The well-known iota functor ι_L, originally introduced by Lowen [10] for $L = [0, 1]$ and later on extended by Kubiak [9] to an arbitrary complete lattice, was the original motivation to define chain-valued frames and the corresponding category in [14]. It constructs at the fibre level, for each L-topology, a traditional topology with subbasis the family of all level sets for all members of the L-topology:

For each set X and each $\alpha \in L$, the *α-level mapping* $\iota_\alpha : L^X \to \mathsf{2}^X$ is defined by

$$\iota_\alpha(a) = [a \not\leq \alpha] := \{x \in X \mid a(x) \not\leq \alpha\}, \quad \text{for each } a \in L^X.$$

Now, given an L-topology τ on X, we consider the associated crisp topology

$$\iota_L^{\mathsf{T}}(\tau) = \langle \{\iota_\alpha(\tau) \mid \alpha \in L\} \rangle = \langle \{\iota_\alpha(a) \mid a \in \tau, \alpha \in L\} \rangle.$$

Remark 2.2. Notice that when L is a completely distributive lattice, the collection $\{\iota_p(a) \mid a \in \tau, p \in \operatorname{Spec} L\}$ is also a subbase of the topology $\iota_L^{\mathsf{T}}(\tau)$.

The correspondence $(X, \tau) \mapsto (X, \iota_L^{\mathsf{T}}(\tau))$ defines a functor $\iota_L^{\mathsf{T}} : L\text{-}\mathsf{Top} \to \mathsf{Top}$: for each L-continuous map $f : (X, \tau_1) \to (Y, \tau_2)$, $f : (X, \iota_L^{\mathsf{T}}(\tau_1)) \to (Y, \iota_L^{\mathsf{T}}(\tau_2))$ is continuous, since $b \circ f \in \tau_1$ and $f^{-1}[\iota_\alpha(b)] = \iota_\alpha(b \circ f)$ for every $b \in \tau_2$.

Note that whenever L is a frame the mapping $\iota_p : \tau \to \iota_L^{\top}(\tau)$ is a frame homomorphism for each $p \in \operatorname{Spec} L$ (see Corollary 3.5 bellow; this is not true in general if p fails to be prime). Consequently we can consider the system of frame homomorphisms

$$\big(\iota_p : \tau \to \iota_L^{\top}(\tau) \mid p \in \operatorname{Spec} L\big). \tag{2.2.1}$$

2.3. L-valued frames. ¿From now on let L denote a completely distributive lattice. Recall that any completely distributive lattice is a spatial frame (i.e. a frame isomorphic to the lattice of open sets of some topological space) and therefore in any completely distributive lattice each element is a meet of primes.

An *L-valued frame* (shortly, an *L-frame*) is a system

$$\mathfrak{A} = \big(\varphi_p^{\mathfrak{A}} : A^{\mathfrak{u}} \to A^{\mathfrak{l}} \mid p \in \operatorname{Spec} L\big)$$

of frame homomorphisms ($A^{\mathfrak{u}}$ is the *upper frame* and $A^{\mathfrak{l}}$ is the *lower frame*) satisfying the following conditions:

(F0) For every $p \in \operatorname{Spec} L$, $\varphi_p^{\mathfrak{A}} = \bigvee\{\varphi_q^{\mathfrak{A}} \mid q \in {\Uparrow} p \cap \operatorname{Spec} L\}$.

(F1) $A^{\mathfrak{l}} = \langle \bigcup_{p \in \operatorname{Spec} L} \varphi_p^{\mathfrak{A}}(A^{\mathfrak{u}})\rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ then $\varphi_p^{\mathfrak{A}}(a) \neq \varphi_p^{\mathfrak{A}}(b)$ for some $p \in \operatorname{Spec} L$. (collectionwise monomorphic)

L-frames were introduced by Pultr and Rodabaugh [14] for the case of a complete chain L and extended by Gutiérrez García, Höhle and de Prada Vicente [7] for the case of a completely distributive lattice.

Given two L-frames $\mathfrak{A}$ and $\mathfrak{B}$ an *L-frame homomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ is an ordered pair of frame morphisms

$$(h^{\mathfrak{u}} : A^{\mathfrak{u}} \to B^{\mathfrak{u}}, h^{\mathfrak{l}} : A^{\mathfrak{l}} \to B^{\mathfrak{l}})$$

satisfying

$$\forall p \in \operatorname{Spec} L,\ h^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}} = \varphi_p^{\mathfrak{B}} \circ h^{\mathfrak{u}}.$$

The resulting category, with composition and identities component-wise in Frm, is denoted by L-Frm.

Note that for each L-topological space (X, τ) with L completely distributive, the system (2.2.1) of frame homomorphisms defines an L-frame [7].

This was the motivating example for the notion of an L-frame.

We recall also the *upper*, resp. *lower*, forgetful functors

$$\mathcal{U}^{\mathfrak{u}} : L\text{-}\mathsf{Frm} \to \mathsf{Frm}, \quad \mathcal{U}^{\mathfrak{l}} : L\text{-}\mathsf{Frm} \to \mathsf{Frm}$$

defined by sending $(\varphi_p^{\mathfrak{A}} : A^{\mathfrak{u}} \to A^{\mathfrak{l}} \mid p \in \operatorname{Spec} L)$ to $A^{\mathfrak{u}}$, resp. $A^{\mathfrak{l}}$, and $(h^{\mathfrak{u}}, h^{\mathfrak{l}})$ to $h^{\mathfrak{u}}$, resp. $h^{\mathfrak{l}}$.

3. An extension of the iota functor

Let L-UTop denote the category whose objects are of the form $(X, \tau_X, \mathcal{U}_X)$ where τ_X is an L-topology on X and $\mathcal{U}_X$ is a uniformity on the frame τ_X, and morphisms $f : (X, \tau_X, \mathcal{U}_X) \to (Y, \tau_Y, \mathcal{U}_Y)$ are the L-continuous functions $f : (X, \tau_X) \to (Y, \tau_Y)$ for which $f^{\leftarrow} : (\tau_Y, \mathcal{U}_Y) \to (\tau_X, \mathcal{U}_X)$ is a uniform homomorphism.

In the particular case $L = \mathsf{2}$ the objects of $\mathsf{2}$-UTop are topological spaces $(X, \mathcal{T}_X)$ endowed with a uniformity on the spatial frame $\mathcal{T}_X$.

One can then try to extend the iota functor for L-UTop:

$$\iota_L^{\mathsf{TUF}} : L\text{-}\mathsf{UTop} \to \mathsf{2}\text{-}\mathsf{UTop}.$$

We will do that by defining how it acts on the additional structure (the uniformity on the L-topology) since in the L-topology it will act precisely as the iota functor already defined in Subsection (2.2).

For each $p \in \operatorname{Spec} L$ and each $\mathcal{A} \subseteq L^X$, let $\iota_p[\mathcal{A}] = \{\iota_p(a) : a \in \mathcal{A}\} \subseteq \mathsf{2}^X$.

We state without proof some basic facts satisfied by the maps $\{\iota_p \mid p \in \operatorname{Spec} L\}$:

Lemma 3.1. *Let $\mathcal{A} \subseteq L^X$, $f : X \to Y$, $a \in L^X$, $b \in L^Y$ and $p, q \in \operatorname{Spec} L$. Then:*

(1) $\iota_p(\bigvee \mathcal{A}) = \bigcup_{a \in \mathcal{A}} \iota_p(a)$.
(2) $\iota_p(\bigwedge \mathcal{A}) = \bigcap_{a \in \mathcal{A}} \iota_p(a)$ *whenever $\mathcal{A}$ is finite.*
(3) $p \leq q \Rightarrow \iota_q(a) \subseteq \iota_p(a)$.
(4) $f^{-1}(\iota_p(b)) = \iota_p(f^{\leftarrow}(b))$.

We have now the following, which is easy to check (cf. [6]):

Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be covers of the frame L^X and $p \in \operatorname{Spec} L$. Then:*

(1) $\iota_p[\mathcal{A}]$ *is a cover of X.*
(2) *If $\mathcal{A} \leq \mathcal{B}$ then $\iota_p[\mathcal{A}] \leq \iota_p[\mathcal{B}]$. Hence $\iota_p[\mathcal{A} \wedge \mathcal{B}] \leq \iota_p[\mathcal{A}]$ and $\iota_p[\mathcal{A} \wedge \mathcal{B}] \leq \iota_p[\mathcal{B}]$.*
(3) $\operatorname{st}(\iota_p(a), \iota_p[\mathcal{A}]) \subseteq \iota_p[\operatorname{st}(a, \mathcal{A})]$.
(4) *If $\operatorname{st}(\mathcal{A}) \leq \mathcal{B}$ then $\operatorname{st}(\iota_p[\mathcal{A}]) \leq \iota_p[\mathcal{B}]$.*

Corollary 3.3. *Let* $(X,\tau_X,\mathcal{U}_X)\in L\text{-}\mathsf{UTop}$. *The family*

$$\{\iota_p[U]\mid U\in\mathcal{U}_X \text{ and } p\in\operatorname{Spec}L\}$$

generates (as a subbase) a uniformity $\iota_L^{\mathsf{UF}}(\mathcal{U}_X)$ *on the frame* $\iota_L^{\mathsf{T}}(\tau_X)$ *whose base is*

$$\{\iota_p[U]\wedge\iota_q[U]\mid U\in\mathcal{U}_X \text{ and } p,q\in\operatorname{Spec}L\}.$$

Proof: It follows from Lemma 3.1 and the properties in Proposition 3.2. Regarding condition (U3) notice that, by Proposition 3.2(3), $b\overset{\mathcal{U}_X}{\lhd}a$ implies that $\iota_p(b)\overset{\iota_L^{\mathsf{UF}}(\mathcal{U}_X)}{\lhd}\iota_p(a)$, thus we have $\iota_p(a)=\bigcup\big\{\iota_p(b)\mid\iota_p(b)\overset{\iota_L^{\mathsf{UF}}(\mathcal{U}_X)}{\lhd}\iota_p(a)\big\}$ for every $p\in\operatorname{Spec}L$ and every $a\in\tau_X$, which suffices to conclude (U3). Indeed: for any element H of $\iota_L^T(\tau_X)$, there are $a_i,b_i\in\tau_X$ and $p_i,q_i\in\operatorname{Spec}L$ such that (cf. Remark 2.2)

$$\begin{aligned}
H &= \textstyle\bigcup_{i\in I}(\iota_{p_i}(a_i)\cap\iota_{q_i}(b_i))\\
&= \textstyle\bigcup_{i\in I}\Big(\bigcup\big\{\iota_{p_i}(c)\mid\iota_{p_i}(c)\overset{\iota_L^{\mathsf{UF}}(\mathcal{U}_X)}{\lhd}\iota_{p_i}(a_i)\big\}\cap\bigcup\big\{\iota_{q_i}(d)\mid\iota_{q_i}(d)\overset{\iota_L^{\mathsf{UF}}(\mathcal{U}_X)}{\lhd}\iota_{q_i}(b_i)\big\}\Big)\\
&\subseteq \textstyle\bigcup_{i\in I}\bigcup\big\{\iota_{p_i}(c)\cap\iota_{q_i}(d)\mid\iota_{p_i}(c)\cap\iota_{q_i}(d)\overset{\iota_L^{\mathsf{UF}}(\mathcal{U}_X)}{\lhd}\iota_{p_i}(a_i)\cap\iota_{q_i}(b_i)\big\}\\
&\subseteq \textstyle\bigcup\big\{G\in\iota_L^T(\tau_X)\mid G\overset{\iota_L^{\mathsf{UF}}(\mathcal{U}_X)}{\lhd}H\big\}\ \subseteq\ H,
\end{aligned}$$

where the last inclusions follow from Remark 2.1. $\square$

The following is obvious:

Corollary 3.4. *If* $f:(X,\tau_X,\mathcal{U}_X)\to(Y,\tau_Y,\mathcal{U}_Y)$ *is a morphism of* $L\text{-}\mathsf{UTop}$ *then* $f:(X,\iota_L^{\mathsf{T}}(\tau_X),\iota_L^{\mathsf{UF}}(\mathcal{U}_X))\to(Y,\iota_L^{\mathsf{T}}(\tau_Y),\iota_L^{\mathsf{UF}}(\mathcal{U}_Y))$ *is a morphism of* $2\text{-}\mathsf{UTop}$. $\square$

Consequently, we have a functor $\iota_L^{\mathsf{TUF}}:L\text{-}\mathsf{UTop}\to 2\text{-}\mathsf{UTop}$ given on objects by

$$\iota_L^{\mathsf{TUF}}(X,\tau_X,\mathcal{U}_X)=(X,\iota_L^{\mathsf{T}}(\tau_X),\iota_L^{\mathsf{UF}}(\mathcal{U}_X))\qquad\text{for each } (X,\tau_X,\mathcal{U}_X)\in L\text{-}\mathsf{UTop}$$

and, on morphisms, for each $f:(X,\tau_X,\mathcal{U}_X)\to(Y,\tau_Y,\mathcal{U}_Y)$, $\iota_L^{\mathsf{TUF}}(f)=f$.

Hence the diagram

$$\begin{tikzcd} L\text{-}\mathsf{UTop} \arrow[r, "\iota_L^{\mathsf{TUF}}"] \arrow[d, "F_L^{\mathsf{UF}}"'] & 2\text{-}\mathsf{UTop} \arrow[d, "F^{\mathsf{UF}}"] \\ L\text{-}\mathsf{Top} \arrow[r, "\iota_L^{\mathsf{T}}"] & \mathsf{Top} \end{tikzcd}$$

commutes (where F_L^{UF} and F^{UF} denote, respectively, the forgetful functors).

Corollary 3.5. *For each* $(X, \tau_X, \mathcal{U}_X) \in L\text{-}\mathsf{UTop}$ *and each* $p \in \operatorname{Spec} L$ *the mapping* $\iota_p : (\tau_X, \mathcal{U}_X) \to (\iota_L^{\mathsf{T}}(\tau_X), \iota_L^{\mathsf{UF}}(\mathcal{U}_X))$ *is a uniform homomorphism.*

Proof: By Lemma 3.1(1) and (2), each ι_p is a frame homomorphism. Clearly, it is moreover uniform, that is, for every $U \in \mathcal{U}_X$, $\iota_p[U] \in \iota_L^{\mathsf{UF}}(\mathcal{U}_X)$. $\square$

4. L-valued uniform frames

Let us consider the following system of uniform homomorphisms:

$$\big(\iota_p : (\tau_X, \mathcal{U}_X) \to (\iota_L^{\mathsf{T}}(\tau_X), \iota_L^{\mathsf{UF}}(\mathcal{U}_X)) \ \mid \ p \in \operatorname{Spec} L\big).$$

Remarks 4.1. (1) As it was shown in [7], for each $p \in \operatorname{Spec} L$,

$$\iota_p = \bigvee \{\iota_q \mid q \in {\Uparrow} p \cap \operatorname{Spec} L\}. \tag{4.1.1}$$

In particular, the assignment $p \mapsto \iota_p$ is antitone.

(2) We have

$$(\iota_L^{\mathsf{T}}(\tau_X), \iota_L^{\mathsf{UF}}(\mathcal{U}_X)) = \Big\langle \bigvee_{p \in \operatorname{Spec} L} \iota_p(\tau_X, \mathcal{U}_X) \Big\rangle, \tag{4.1.2}$$

where, by the previous expression we mean that the topology $\iota_L^{\mathsf{T}}(\tau_X)$ is generated by the subbase $\{\iota_p(a) \mid a \in \tau_X \text{ and } p \in \operatorname{Spec} L\}$ and the uniformity $\iota_L^{\mathsf{UF}}(\mathcal{U}_X)$ is generated by the subbase $\{\iota_p[U] \mid U \in \mathcal{U}_X \text{ and } p \in \operatorname{Spec} L\}$.

(3) For each pair of distinct $a, b \in \tau_X$ there exists $x \in X$ such that $a(x) \neq b(x)$, hence there exists $p \in \operatorname{Spec} L$ such that either $a(x) \leq p$ and $b(x) \not\leq p$ or $a(x) \not\leq p$ and $b(x) \leq p$ and so $[a \not\leq p] \neq [b \not\leq p]$. It follows that

$$\text{if } a \neq b \in \tau_X \text{ then } \iota_p(a) \neq \iota_p(b) \text{ for some } p \in \operatorname{Spec} L. \tag{4.1.3}$$

(4) As a consequence of the previous comments, the system of uniform homomorphisms

$$\big(\iota_p : (\tau_X, \mathcal{U}_X) \to (\iota_L^{\mathsf{T}}(\tau_X), \iota_L^{\mathsf{UF}}(\mathcal{U}_X)) \mid p \in \operatorname{Spec} L\big)$$

satisfies conditions (4.1.1), (4.1.2) and (4.1.3) above.

Proposition 4.2. *Let $(A,\mathcal{U})$ be a uniform frame and $(X,\mathcal{T}_X,\mathcal{U}_X) \in$ 2-UTop. Let $(\varphi_p : (A,\mathcal{U}) \to (\mathcal{T}_X,\mathcal{U}_X) \mid p \in \operatorname{Spec} L)$ be a system of uniform homomorphisms satisfying:*

- *for each $p \in \operatorname{Spec} L$, $\varphi_p = \bigvee\{\varphi_q \mid q \in {\uparrow}p \cap \operatorname{Spec} L\}$.* (4.2.1)
- $(\mathcal{T}_X,\mathcal{U}_X) = \langle \bigvee_{p\in\operatorname{Spec} L} \varphi_p((A,\mathcal{U}))\rangle$. (4.2.2)
- *if $a \neq b \in A$ then $\varphi_p(a) \neq \varphi_p(b)$ for some $p \in \operatorname{Spec} L$.* (4.2.3)

Then there is a uniform frame $(B,\mathcal{V})$ and a uniform isomorphism $h : (A,\mathcal{U}) \to (B,\mathcal{V})$ satisfying the following:

(1) *B is an L-topology on X (hence $(X,B,\mathcal{V}) \in L$-UTop).*
(2) $\iota_L^{\mathsf{TUF}}(X,B,\mathcal{V}) = (X,\mathcal{T}_X,\mathcal{U}_X)$.
(3) *For each $p \in \operatorname{Spec} L$, $\iota_p \circ h = \varphi_p$.*

Proof: The proof follows the lines of Proposition 2.5 in [7].

Given $a \in A$, let $h(a) \in L^X$ be the L-valued function induced by the family $\{\varphi_p(a) \mid p \in \operatorname{Spec} L\}$, that is,

$$h(a)(x) = \bigwedge\{p \in \operatorname{Spec} L \mid x \notin \varphi_p(a)\} \quad \text{for every } x \in X.$$

Take $B = \{h(a) \mid a \in A\}$ and $\mathcal{V} = \{h[U] : U \in \mathcal{U}\}$. It is easy to check (see [7, Proposition 2.5]) that condition (4.2.1) implies that $h(a)(x) \le p \iff x \notin \varphi_p(a)$ and so $\iota_p(h(a)) = [h(a) \not\le p] = \varphi_p(a)$ for every $a \in A$ and $p \in \operatorname{Spec} L$, i.e., $\iota_p \circ h = \varphi_p$ for every $p \in \operatorname{Spec} L$.

We now check that h is injective: Given $a \neq b \in A$, by (4.2.3) we have $\varphi_p(a) \neq \varphi_p(b)$ for some $p \in \operatorname{Spec} L$. We can assume, without loss of generality, that there exists $x \in \varphi_p(a)$ such that $x \notin \varphi_p(b)$. Then $h(b)(x) \le p$ and $h(a)(x) \not\le p$ which implies $h(b) \neq h(a)$.

Moreover, h is a frame homomorphism. Indeed:
Let $a, b \in A$, $x \in X$ and $p \in \operatorname{Spec} L$. We have

$$\begin{aligned} h(a\wedge b)(x) \le p &\iff x \notin \varphi_p(a\wedge b) = \varphi_p(a) \cap \varphi_p(b) \\ &\iff x \notin \varphi_p(a) \text{ or } x \notin \varphi_p(b) \\ &\iff h(a)(x) \le p \text{ or } h(b)(x) \le p \\ &\iff h(a)(x) \wedge h(b)(x) \le p \end{aligned}$$

and therefore, since L is a spatial frame, we conclude that $h(a \wedge b) = h(a) \wedge h(b)$. Further, $h(1) = 1$, since $\varphi_p(1) = X$ for every $p \in \operatorname{Spec} L$. Thus h preserves finite meets.

On the other hand, given $\{a_i\}_{i \in I} \subseteq A$, $x \in X$ and $p \in \operatorname{Spec} L$ we have

$$\begin{aligned} h(\textstyle\bigvee_{i \in I} a_i)(x) \leq p &\iff x \notin \varphi_p(\textstyle\bigvee_{i \in I} a_i) = \textstyle\bigvee_{i \in I} \varphi_p(a_i) \\ &\iff x \notin \varphi_p(a_i) \text{ for all } i \in I \\ &\iff h(a_i)(x) \leq p \text{ for all } i \in I \\ &\iff \textstyle\bigvee_{i \in I} h(a_i)(x) \leq p \end{aligned}$$

and thus, once again because L is a spatial frame, we may conclude that $h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i)$.

Hence $B = h(A)$ is a subframe of L^X, i.e. an L-topology on X, and therefore, $h : A \to B$ is a frame isomorphism. The latter fact makes straightforward the proof that $\mathcal{V} = \{h[U] : U \in \mathcal{U}\}$ is a uniformity on B and that $h : (A, \mathcal{U}) \to (B, \mathcal{V})$ is a uniform isomorphism, since h and h^{-1} are trivially uniform.

Finally, it follows from (4.2.2) that

$$\begin{aligned} \iota_L^{\mathsf{T}}(B) &= \langle \{\iota_p(h(a)) \mid a \in A,\ p \in \operatorname{Spec} L\} \rangle \\ &= \langle \{\varphi_p(a) \mid a \in A,\ p \in \operatorname{Spec} L\} \rangle = \mathcal{T}_X \end{aligned}$$

and

$$\begin{aligned} \iota_L^{\mathsf{UF}}(\mathcal{V}) &= \langle \{\iota_p[h(U)] \mid U \in \mathcal{U},\ p \in \operatorname{Spec} L\} \rangle \\ &= \langle \{\varphi_p[U] \mid U \in \mathcal{U},\ p \in \operatorname{Spec} L\} \rangle = \mathcal{U}_X, \end{aligned}$$

i.e., $\iota_L^{\mathsf{TUF}}(X, B, \mathcal{V}) = (X, \mathcal{T}_X, \mathcal{U}_X)$. $\square$

Proposition 4.2 leads immediately to the following definition:

Definition 4.3. An *L-valued uniform frame* $\mathfrak{A}$ is a system

$$\mathfrak{A} = \big(\varphi_p^{\mathfrak{A}} : (A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}}) \to (A^{\mathfrak{l}}, \mathcal{U}^{\mathfrak{l}}) \mid p \in \operatorname{Spec} L\big)$$

of uniform homomorphisms ($(A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}})$ is the *upper uniform frame* and $(A^{\mathfrak{l}}, \mathcal{U}^{\mathfrak{l}})$ is the *lower uniform frame*) satisfying the following conditions:

(UF0) For every $p \in \operatorname{Spec} L$, $\varphi_p^{\mathfrak{A}} = \bigvee \{\varphi_q^{\mathfrak{A}} \mid q \in \Uparrow p \cap \operatorname{Spec} L\}$.

(UF1) $(A^{\mathfrak{l}}, \mathcal{U}^{\mathfrak{l}}) = \langle \bigvee_{p \in \operatorname{Spec} L} \varphi_p^{\mathfrak{A}}((A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}})) \rangle$.

(UF2) If $a \neq b$ then $\varphi_p^{\mathfrak{A}}(a) \neq \varphi_p^{\mathfrak{A}}(b)$ for some $p \in \operatorname{Spec} L$.

An *L-uniform homomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ is an ordered pair of uniform homomorphisms

$$(h^{\mathfrak{u}} : (A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}}) \to (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}}), h^{\mathfrak{l}} : (A^{\mathfrak{l}}, \mathcal{U}^{\mathfrak{l}}) \to (B^{\mathfrak{l}}, \mathcal{V}^{\mathfrak{l}}))$$

satisfying

$$\forall p \in \operatorname{Spec} L,\ h^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}} = \varphi_p^{\mathfrak{B}} \circ h^{\mathfrak{u}}.$$

The resulting category, with composition and identities component-wise in UFrm, is denoted by L-UFrm.

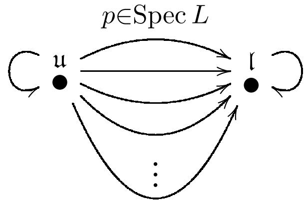
Remarks 4.4. (1) Note that any L-valued uniform frame is, in particular, an L-valued frame.

(2) For $L = 2$, we have exactly one uniform homomorphism $\varphi_0^{\mathfrak{A}}$ which automatically satisfies (UF0): conditions (UF1) and (UF2) imply that $\varphi_0^{\mathfrak{A}}$ is an isomorphism $(A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}}) \to (A^{\mathfrak{l}}, \mathcal{U}^{\mathfrak{l}})$. Thus a 2-uniform frame is a pair of isomorphic uniform frames. Further, each 2-uniform homomorphism is a pair of uniform homomorphisms $(h^{\mathfrak{u}}, h^{\mathfrak{l}})$ such that each factors through the other via isomorphisms. Each 2-uniform frame is a functor $\mathbb{2} \to \mathsf{UFrm}$ where $\mathbb{2}$ denotes the category

$$\begin{tikzcd} \overset{\mathfrak{u}}{\bullet} \arrow[loop left] \arrow[rr] & & \overset{\mathfrak{l}}{\bullet} \arrow[loop right] \end{tikzcd}$$

It is then easy to conclude that the category 2-UFrm of 2-valued uniform frames is the category $\mathsf{UFrm}^{\mathbb{2}}$ of functors $\mathbb{2} \to \mathsf{UFrm}$ and natural transformations between such functors. Consequently, 2-UFrm is equivalent to the category UFrm, since $\mathsf{UFrm}^{\mathbb{2}}$ is a category equivalent to UFrm via functors $F : \mathsf{UFrm}^{\mathbb{2}} \to \mathsf{UFrm}$, with $F((A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}}), (A^{\mathfrak{l}}, \mathcal{U}^{\mathfrak{l}})) = (A^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{u}})$ and $F(h^{\mathfrak{u}}, h^{\mathfrak{l}}) = h^{\mathfrak{u}}$, and $G : \mathsf{UFrm} \to \mathsf{UFrm}^{\mathbb{2}}$, with $G(A, \mathcal{U}) = ((A, \mathcal{U}), (A, \mathcal{U}))$ and $G(h) = (h, h)$.

(3) For a general completely distributive L, L-UFrm is a full subcategory of the category $\mathsf{UFrm}^{\mathfrak{L}}$ where $\mathfrak{L}$ denotes the category



(the morphisms between $\mathfrak{u}$ and $\mathfrak{l}$ are indexed by $\operatorname{Spec} L$).

Theorem 4.5. *The category L-UFrm is complete and cocomplete.*

Proof: Let UF_0 and UF_1 denote the categories defined by relaxing the axioms for objects in the definition of L-UFrm as follows: in UF_0 the objects are systems of uniform homomorphisms for which only (UF0) is required and in UF_1 the objects are systems of uniform homomorphisms for which only (UF0) and (UF1) are required.

Claim 1. *The category UF_0 is complete and cocomplete.*

Proof of Claim 1. Given any family of objects $\big(\mathfrak{A}_i\big)_{i\in I}$, with $\mathfrak{A}_i = (\varphi_p^{\mathfrak{A}_i} : (A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}}) \to (A_i^{\mathfrak{l}},\mathcal{U}_i^{\mathfrak{l}}) \mid p \in \operatorname{Spec} L)$, in UF_0, consider the uniform frame coproduct [18] $(A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}})$ of the $(A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}})$ (with injections $u_i^{\mathfrak{u}}$) and the uniform frame coproduct $(A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}})$ of the $(A_i^{\mathfrak{l}},\mathcal{U}_i^{\mathfrak{l}})$ (with injections $u_i^{\mathfrak{l}}$). Then, for each $p \in \operatorname{Spec} L$, there exists a unique $\varphi_p^{\mathfrak{A}} : (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \to (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}})$ such that $\varphi_p^{\mathfrak{A}} \circ u_i^{\mathfrak{u}} = u_i^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}_i}$ for every $i \in I$:

$$\begin{tikzcd}
(A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}}) \arrow[r, "u_i^{\mathfrak{u}}"] \arrow[d, "\varphi_p^{\mathfrak{A}_i}"'] & (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \arrow[d, dashed, "\varphi_p^{\mathfrak{A}}"] \\
(A_i^{\mathfrak{l}},\mathcal{U}_i^{\mathfrak{l}}) \arrow[r, "u_i^{\mathfrak{l}}"] & (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}})
\end{tikzcd}$$

Further, $\mathfrak{A} = (\varphi_p^{\mathfrak{A}} : (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \to (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}))_{p\in\operatorname{Spec} L}$ is in UF_0. Indeed, for each i,

$$\begin{aligned}
\varphi_p^{\mathfrak{A}} \circ u_i^{\mathfrak{u}} = u_i^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}_i} &= u_i^{\mathfrak{l}} \circ \bigvee\{\varphi_q^{\mathfrak{A}_i} \mid q \in {\uparrow} p \cap \operatorname{Spec} L\} \\
&= \bigvee\{u_i^{\mathfrak{l}} \circ \varphi_q^{\mathfrak{A}_i} \mid q \in {\uparrow} p \cap \operatorname{Spec} L\} \\
&= \bigvee\{\varphi_q^{\mathfrak{A}} \circ u_i^{\mathfrak{u}} \mid q \in {\uparrow} p \cap \operatorname{Spec} L\} \\
&= \big(\bigvee\{\varphi_q^{\mathfrak{A}} \mid q \in {\uparrow} p \cap \operatorname{Spec} L\}\big) \circ u_i^{\mathfrak{u}},
\end{aligned}$$

from which it readily follows that $\varphi_p^{\mathfrak{A}} = \bigvee\{\varphi_q^{\mathfrak{A}} \mid q \in {\uparrow} p \cap \operatorname{Spec} L\}$.

Finally, $(\mathfrak{A}, (u_i^{\mathfrak{u}}, u_i^{\mathfrak{l}})_{i\in I}) = ((\varphi_p^{\mathfrak{A}} : (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \to (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}))_{p\in\operatorname{Spec} L}, (u_i^{\mathfrak{u}}, u_i^{\mathfrak{l}})_{i\in I}))$ is the coproduct in UF_0 of the system of all $\mathfrak{A}_i$. Indeed, given any $\mathfrak{B} = (\varphi_p^{\mathfrak{B}} : (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}}) \to (B^{\mathfrak{l}}, \mathcal{V}^{\mathfrak{l}}))_{p\in\operatorname{Spec} L} \in \mathsf{UF}_0$ and any collection of L-uniform homomorphisms $\{h_i = (h_i^{\mathfrak{u}}, h_i^{\mathfrak{l}}) : \mathfrak{A}_i \to \mathfrak{B} \mid i \in I\}$, since $(A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}})$ is the coproduct in UFrm of the $(A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}})$, there exists a unique uniform homomorphism $h^{\mathfrak{u}} : (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \to (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}})$ such that $h^{\mathfrak{u}} \circ u_i^{\mathfrak{u}} = h_i^{\mathfrak{u}}$ for every $i \in I$. Similarly, there exists a unique uniform homomorphism $h^{\mathfrak{l}} : (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}) \to (B^{\mathfrak{l}},\mathcal{V}^{\mathfrak{l}})$ such that $h^{\mathfrak{l}} \circ u_i^{\mathfrak{l}} = h_i^{\mathfrak{l}}$ for every $i \in I$.

$$\begin{tikzcd}
(A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}}) \arrow[r, "u_i^{\mathfrak{u}}"] \arrow[rr, bend left, "h_i^{\mathfrak{u}}"] \arrow[d, "\varphi_p^{\mathfrak{A}_i}"'] & (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \arrow[r, dashed, "h^{\mathfrak{u}}"] \arrow[d, "\varphi_p^{\mathfrak{A}}"] & (B^{\mathfrak{u}},\mathcal{V}^{\mathfrak{u}}) \arrow[d, "\varphi_p^{\mathfrak{B}}"] \\
(A_i^{\mathfrak{l}},\mathcal{U}_i^{\mathfrak{l}}) \arrow[r, "u_i^{\mathfrak{l}}"] \arrow[rr, bend right, "h_i^{\mathfrak{l}}"'] & (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}) \arrow[r, dashed, "h^{\mathfrak{l}}"] & (B^{\mathfrak{l}},\mathcal{V}^{\mathfrak{l}})
\end{tikzcd}$$

Since, for each $i \in I$ and $p \in \operatorname{Spec} L$,

$$\varphi_p^{\mathfrak{B}} \circ h^{\mathfrak{u}} \circ u_i^{\mathfrak{u}} = \varphi_p^{\mathfrak{B}} \circ h_i^{\mathfrak{u}} = h_i^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}_i} = h^{\mathfrak{l}} \circ u_i^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}_i} = h^{l} \circ \varphi_p^{\mathfrak{A}} \circ u_i^{\mathfrak{u}},$$

then $\varphi_p^{\mathfrak{B}} \circ h^{\mathfrak{u}} = h^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{A}}$, and the pair $(h^{\mathfrak{u}}, h^{\mathfrak{l}})$ is an L-uniform homomorphism $\mathfrak{A} \to \mathfrak{B}$. It is clearly the unique L-uniform homomorphism $\mathfrak{A} \to \mathfrak{B}$ such that $(h^{\mathfrak{u}}, h^{\mathfrak{l}}) \circ (u_i^{\mathfrak{u}}, u_i^{\mathfrak{l}}) = (h_i^{\mathfrak{u}}, h_i^{\mathfrak{l}})$ for every $i \in I$.

In a similar way, one can construct the coequalizers, products and equalizers of UF_0 from the corresponding constructions in UFrm. $\square$

Claim 2. UF_1 *is mono-coreflective in* UF_0*. Consequently,* UF_1 *inherits colimits from* UF_0 *and the limits of* UF_1 *are the coreflections of limits of* UF_0*, and hence* UF_1 *is also complete and cocomplete.*

Proof of Claim 2. Let $\mathfrak{B} = (\varphi_p^{\mathfrak{B}} : (B^{\mathfrak{u}},\mathcal{V}^{\mathfrak{u}}) \to (B^{\mathfrak{l}},\mathcal{V}^{\mathfrak{l}}))_{p\in\operatorname{Spec} L} \in \mathsf{UF}_0$ be given and consider $\mathfrak{A} = (\varphi_p^{\mathfrak{A}} : (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \to (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}))_{p\in\operatorname{Spec} L}$ where $(A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) = (B^{\mathfrak{u}},\mathcal{V}^{\mathfrak{u}})$, $A^{\mathfrak{l}}$ is the subframe of $B^{\mathfrak{l}}$ generated by $\bigcup_{p\in\operatorname{Spec} L} \varphi_p^{\mathfrak{B}}[B^{\mathfrak{u}}]$, $\mathcal{U}^{\mathfrak{l}}$ is the uniformity contained in $\mathcal{V}^{\mathfrak{l}}$ generated by $\{\varphi_p^{\mathfrak{B}}[V] \mid p \in \operatorname{Spec} L, V \in \mathcal{V}^{\mathfrak{u}}\}$ and $\varphi_p^{\mathfrak{A}} = {\varphi_p^{\mathfrak{B}}}_{|\mathrm{cod}=\varphi_p^{\mathfrak{B}}[B^{\mathfrak{u}}]}$. It is immediate that $\mathfrak{A}$ satisfies (UF0) and (UF1). Further, define $h : \mathfrak{A} \to \mathfrak{B}$ by $h^{\mathfrak{u}} = \mathrm{id}_{B^{\mathfrak{u}}}$ and $h^{\mathfrak{l}} : A^{\mathfrak{l}} \hookrightarrow B^{\mathfrak{l}}$. It is straightforward to check that h is a monomorphism in UF_0 satisfying the required universal property. $\square$

Claim 3. UF_1 *is an* $(E,\mathcal{M})$*-category, for* E *the class* RegEpi *of regular epimorphisms and* $\mathcal{M}$ *the class* Mono-source *of mono-sources.*

Proof of Claim 3. By Proposition 15.13 of [1], it suffices to check that each source in UF_1 has a (RegEpi, Mono-source)-factorization.

So let $(h_i : \mathfrak{A} \to \mathfrak{A}_i)_{i\in I}$ be a mono-source in UF_1 and consider the product $((B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}}), p_i^{\mathfrak{u}})$ (resp. $((B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}}), p_i^{\mathfrak{u}})$) of the domains $(A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}})$ (resp. codomains $(A_i^{\mathfrak{l}},\mathcal{U}_i^{\mathfrak{l}})$) of the $\varphi_p^{\mathfrak{A}_i}$. Then there exists, for each $p \in \operatorname{Spec} L$, a unique $\varphi_p^{\mathfrak{B}} : (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}}) \to (B^{\mathfrak{l}}, \mathcal{V}^{\mathfrak{l}})$ such that $p_i^{\mathfrak{l}} \circ \varphi_p^{\mathfrak{B}} = \varphi_p^{\mathfrak{A}_i} \circ p_i^{\mathfrak{u}}$ for every $i \in I$.

The function $f^{\mathfrak{u}} : (A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \to (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}})$ (resp. $f^{\mathfrak{l}} : (A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}) \to (B^{\mathfrak{l}}, \mathcal{V}^{\mathfrak{l}})$) defined by $p_i^{\mathfrak{u}} \circ f^{\mathfrak{u}} = h_i^{\mathfrak{u}}$ for each $i \in I$ (resp. $p_i^{\mathfrak{l}} \circ f^{\mathfrak{l}} = h_i^{\mathfrak{l}}$ for each $i \in I$) has a factorization

$$(A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \xrightarrow{e^{\mathfrak{u}}} (C^{\mathfrak{u}}, \mathcal{W}^{\mathfrak{u}}) \xrightarrow{m^{\mathfrak{u}}} (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}})$$

(resp. $(A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}) \xrightarrow{e^{\mathfrak{l}}} (C^{\mathfrak{l}}, \mathcal{W}^{\mathfrak{l}}) \xrightarrow{m^{\mathfrak{l}}} (B^{\mathfrak{l}}, \mathcal{V}^{\mathfrak{l}})$) with $e^{\mathfrak{u}}$ and $e^{\mathfrak{l}}$ surjections and $m^{\mathfrak{u}}$ and $m^{\mathfrak{l}}$ injective functions.

Since, for each $c \in C^{\mathfrak{u}}$, $\varphi_p^{\mathfrak{B}}(m^{\mathfrak{u}}(c)) = \varphi_p^{\mathfrak{B}}(m^{\mathfrak{u}}(e^{\mathfrak{u}}(a))) = \varphi_p^{\mathfrak{B}}(f^{\mathfrak{u}}(a)) = f^{\mathfrak{l}}(\varphi_p^{\mathfrak{A}}(a)) = m^{\mathfrak{l}}(e^{\mathfrak{l}}(\varphi_p^{\mathfrak{A}}(a)))$ for some $a \in A^{\mathfrak{u}}$, we may define

$$\varphi_p^{\mathfrak{C}} : (C^{\mathfrak{u}}, \mathcal{W}^{\mathfrak{u}}) \to (C^{\mathfrak{l}}, \mathcal{W}^{\mathfrak{l}})$$

by $\varphi_p^{\mathfrak{C}}(c) = (m^{\mathfrak{l}})^{-1}(\varphi_p^{\mathfrak{B}}(m^{\mathfrak{u}}(c))) = e^{\mathfrak{l}}(\varphi_p^{\mathfrak{A}}(a))$ (easily seen to be well defined).

Finally, if we define $m_i^{\mathfrak{u}} : C^{\mathfrak{u}} \to A_i^{\mathfrak{u}}$ by $m_i^{\mathfrak{u}} = p_i^{\mathfrak{u}} \circ m^{\mathfrak{u}}$ and $m_i^{\mathfrak{l}} : C^{\mathfrak{l}} \to A_i^{\mathfrak{l}}$ by $m_i^{\mathfrak{l}} = p_i^{\mathfrak{l}} \circ m^{\mathfrak{l}}(a)$ we have the following diagram:

$$\begin{tikzcd}
(A^{\mathfrak{u}},\mathcal{U}^{\mathfrak{u}}) \arrow[r, "e^{\mathfrak{u}}"] \arrow[rr, bend left, "h_i^{\mathfrak{u}}"] \arrow[dr, "f^{\mathfrak{u}}"'] \arrow[ddd, "\varphi_p^{\mathfrak{A}}"'] & (C^{\mathfrak{u}}, \mathcal{W}^{\mathfrak{u}}) \arrow[r, "m_i^{\mathfrak{u}}"] \arrow[d, "m^{\mathfrak{u}}"] & (A_i^{\mathfrak{u}},\mathcal{U}_i^{\mathfrak{u}}) \arrow[ddd, "\varphi_p^{\mathfrak{A}_i}"] \\
& (B^{\mathfrak{u}}, \mathcal{V}^{\mathfrak{u}}) \arrow[ur, "p_i^{\mathfrak{u}}"'] \arrow[d, dashed, "\varphi_p^{\mathfrak{B}}"] & \\
& (B^{\mathfrak{l}}, \mathcal{V}^{\mathfrak{l}}) \arrow[dr, "p_i^{\mathfrak{l}}"] & \\
(A^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}}) \arrow[ur, "f^{\mathfrak{l}}"] \arrow[r, "e^{\mathfrak{l}}"'] \arrow[rr, bend right, "h_i^{\mathfrak{l}}"'] & (C^{\mathfrak{l}}, \mathcal{W}^{\mathfrak{l}}) \arrow[u, "m^{\mathfrak{l}}"'] \arrow[r, "m_i^{\mathfrak{l}}"'] & (A_i^{\mathfrak{l}},\mathcal{U}^{\mathfrak{l}})
\end{tikzcd}$$

It follows that $(h_i^{\mathfrak{u}}, h_i^{\mathfrak{l}}) = (m_i^{\mathfrak{u}} \circ e^{\mathfrak{u}}, m_i^{\mathfrak{l}} \circ e^{\mathfrak{l}})$ is a (RegEpi, Mono-source)-factorization of $(h_i)_{i\in I} = (h_i^{\mathfrak{u}}, h_i^{\mathfrak{l}})_{i\in I}$ in UF_1. $\square$

Claim 4. L-UFrm *is a full subcategory of* UF_1 *closed under the formation of mono-sources in* UF_1.

Proof of Claim 4. First, notice that for any mono-source $(h_i : \mathfrak{A} \to \mathfrak{A}_i)_{i\in I}$ in UF_1, $(h_i^{\mathfrak{u}})_{i\in I}$ is a mono-source in UFrm and, therefore, if $h_i^{\mathfrak{u}}(a) = h_i^{\mathfrak{u}}(b)$ for every $i \in I$ then $a = b$. Thus, if each $\mathfrak{A}_i$ belongs to L-UFrm and $a, b \in A^{\mathfrak{u}}$ then, for each $p \in \operatorname{Spec} L$,

$$\begin{aligned}
\varphi_p^{\mathfrak{A}}(a) = \varphi_p^{\mathfrak{A}}(b) \quad &\Longrightarrow \quad h_i^{\mathfrak{l}}(\varphi_p^{\mathfrak{A}}(a)) = h_i^{\mathfrak{l}}(\varphi_p^{\mathfrak{A}}(b)), \text{ for all } i \in I \\
&\Longleftrightarrow \quad \varphi_p^{\mathfrak{A}_i}(h_i^{\mathfrak{u}}(a)) = \varphi_p^{\mathfrak{A}_i}(h_i^{\mathfrak{u}}(b)), \text{ for all } i \in I \\
&\Longrightarrow \quad h_i^{\mathfrak{u}}(a) = h_i^{\mathfrak{u}}(b), \text{ for all } i \in I \\
&\Longrightarrow \quad a = b,
\end{aligned}$$

which shows that $\mathfrak{A}$ is also in L-UFrm. Hence L-UFrm is closed under the formation of mono-sources in UF_1. $\square$

In conclusion, L-UFrm is a full subcategory of an $(E, \mathcal{M})$-category UF_1, closed under the formation of $\mathcal{M}$-sources in UF_1. Hence, by Theorem 16.8 of [1], L-UFrm is E-reflective in UF_1. Consequently, L-UFrm inherits limits from UF_1 and the colimits of L-UFrm are the reflections of colimits of UF_1, and hence L-UFrm is complete and cocomplete. $\square$

References

[1] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Wiley Interscience Pure and Applied Mathematics, Wiley, 1990.

[2] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* 24 (1968) 182–190.

[3] J. Frith, The category of uniform frames, *Cahiers Topologie Géom. Différentielle Catég.* 31 (1990) 305–313.

[4] G. Gierz, et al., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.

[5] J.A. Goguen, The fuzzy Tychonoff theorem, *J. Math. Anal. Appl.* 43 (1973), 734–742.

[6] J. Gutiérrez García, I. Mardones-Pérez, J. Picado, M. A. de Prada Vicente, Uniform-type structures on lattice-valued spaces and frames, *Fuzzy Sets and Systems* 159 (2008) 2469–2487.

[7] J. Gutiérrez García, U. Höhle, M. A. de Prada Vicente, On lattice-valued frames: the completely distributive case, *Fuzzy Sets and Systems*, in press, doi: 10.1016/j.fss.2009.10.024.

[8] U. Höhle, S. E. Rodabaugh, *Appendix to Chapter 6: weakening the requirement that L be a complete chain*, in: S. E. Rodabaugh, E. P. Klement (Eds.) Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic, vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, pp. 189–197.

[9] T. Kubiak, *The topological modification of the L-fuzzy unit interval*, in: S. E. Rodabaugh, E. P. Klement, U. Höhle (Eds.), Applications of Category Theory To Fuzzy Subsets, Theory and Decision Library—Series B: Mathematical and Statistical Methods, vol. 14, Kluwer Academic Publishers, Boston, Dordrecht, London, 1992, pp. 275–305 (Chapter 11).
[10] R. Lowen, Fuzzy topological spaces and fuzzy compactness, *J. Math. Anal. Appl.* 56 (1976) 621–633.
[11] J. Picado, A. Pultr, A. Tozzi, *Locales*, in: M.C. Pedicchio, W. Tholen (Eds.), *Categorical Foundation – Special Topics in Order, Algebra and Sheaf Theory*, Encyclopedia of Mathematics and its Applications, Vol. 97, Cambridge Univ. Press, Cambridge, 2004, pp. 49–101 (Chapter 2).
[12] A. Pultr, Pointless uniformities I. Complete regularity, *Comment. Math. Univ. Carolinae* 25 (1984) 91-104.
[13] A. Pultr, *Frames*, in: *Handbook of Algebra*, Vol. 3, North-Holland, Amsterdam, 2003, pp. 791–857.
[14] A. Pultr, S.E. Rodabaugh, *Lattice-valued frames, functor categories, and classes of sober spaces*, in: S. E. Rodabaugh, E. P. Klement (Eds.) Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic, vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, pp. 153–187 (Chapter 6).
[15] A. Pultr, S. E. Rodabaugh, *Examples for different sobrieties in fixed-basis topology*, in: S. E. Rodabaugh, E. P. Klement (Eds.) Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic, vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, pp. 427–440 (Chapter 17).
[16] A. Pultr, S. E. Rodabaugh, Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations, *Fuzzy Sets and Systems* 159 (2008) 501–528.
[17] A. Pultr, S. E. Rodabaugh, Category theoretic aspects of chain-valued frames: Part II: Applications to lattice-valued topology, *Fuzzy Sets and Systems* 159 (2008) 529–558.
[18] J. Walters-Wayland, *Completeness and nearly fine uniform frames*, PhD Thesis, University of Cape Town, 1996.

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