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ON THE RELATION BETWEEN THE FULL KOSTANT-TODA LATTICE AND MULTIPLE ORTHOGONAL POLYNOMIALS

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Abstract: The correspondence between a high-order non symmetric difference operator with complex coefficients and the evolution of an operator defined by a Lax pair is established. The solution of the discrete dynamical system is studied, giving explicit expressions for the resolvent function and, under some conditions, the representation of the vector of functionals, associated with the solution for our integrable systems. The method of investigation is based on the evolutions of the matrical moments.

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1. Introduction

We consider the following special *full Kostant-Toda* system,

$$\left.\begin{array}{rcl} \dot{a}_n & = & b_n - b_{n-1} \\ \dot{b}_n & = & b_n(a_{n+1} - a_n) + c_n - c_{n-1} \\ \dot{c}_n & = & c_n(a_{n+2} - a_n) \end{array}\right\} , \quad n \in \mathbb{N}, \qquad (1)$$

where the dot means differentiation with respect to $t \in \mathbb{R}$ and we assume $b_0 \equiv 0$, $c_n \neq 0$. It is well known that these equations can be written as a *Lax pair* $\dot{J} = [J, J_-]$, where $[M, N] = MN - NM$ is the *commutator* of the operators M and N, and J, J_- are the operators which matrix representation is given, respectively, by

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$$J = \begin{pmatrix} a_1 & 1 & & \\ b_1 & a_2 & 1 & \\ c_1 & b_2 & a_3 & \ddots \\ 0 & c_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & & & \\ b_1 & 0 & & \\ c_1 & b_2 & 0 & \\ 0 & c_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

Here, and in the following, we suppress the explicit t-dependence for brevity. Also, we identify an operator and its matrix representation with respect to the canonical basis. For the sake of simplicity, we only consider a four banded matrix J in this work, but the method can be extended to higher order banded matrices J.

In [1] and [8] the authors considered some special cases of the systems studied here, and in [4] some finite full Kostant-Toda systems is considered and solved using bi-orthogonal systems of polynomials.

When J is a bounded operator, then it is possible to define the *resolvent operator*,

$$(zI - J)^{-1} = \sum_{n\geq 0} \frac{J^n}{z^{n+1}}, \quad |z| > \|J\| \tag{3}$$

(see [12, Th. 3, p. 211]). We denote by M_{ij} the 2×2 block, of any infinite matrix M, formed by the entries of rows $2i-1, 2i$ and columns $2j-1, 2j$. In this way, for each $n \in \mathbb{N}$, J^n can be written as a blocked matrix,

$$J^n = \begin{pmatrix} J^n_{11} & J^n_{12} & \cdots \\ J^n_{21} & J^n_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{4}$$

Moreover, we define $\mathcal{R}_J(z)$ as the main block of $(zI-J)^{-1}$, this is, $\mathcal{R}_J(z) := (zI-J)^{-1}_{11}$. Then, from (3) we have

$$\mathcal{R}_J(z) = \sum_{n\geq 0} \frac{J^n_{11}}{z^{n+1}}, \quad |z| > \|J\|. \tag{5}$$

As a consequence of the Lax pair representation, for (1) and other systems, the operator theory is a useful tool and a remarkable connection between the integrable systems and the approximation theory. Consider the sequence of polynomials $\{P_n\}$ given by the recurrence relation

$$\left.\begin{array}{r} c_{n-1}P_{n-2}(z) + b_nP_{n-1}(z) + (a_{n+1}-z)P_n(z) + P_{n+1}(z) = 0, n = 0,1,\ldots \\ P_0(z) = 1, \quad P_{-1}(z) = P_{-2}(z) = 0. \end{array}\right\} \tag{6}$$

Taking $\mathcal{B}_m(z) = (P_{2m}(z), P_{2m+1}(z))^T$, we can rewrite (6) as

$$\left.\begin{array}{r} C_n\,\mathcal{B}_{n-1}(z) + (B_{n+1} - zI_2)\mathcal{B}_n(z) + A\,\mathcal{B}_{n+1}(z) = 0\,, \quad n = 0, 1, \dots \\ \mathcal{B}_{-1}(z) = 0\,, \quad \mathcal{B}_0(z) = (1, z - a_1)^T \end{array}\right\} \tag{7}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \, C_n = \begin{pmatrix} c_{2n-1} & b_{2n} \\ 0 & c_{2n} \end{pmatrix}, \, B_n = \begin{pmatrix} a_{2n-1} & 1 \\ b_{2n-1} & a_{2n} \end{pmatrix}, \, n \in \mathbb{N}, \tag{8}$$

and C_0 is an arbitrary 2×2 matrix. In (7) and in the following, we denote for the sake of simplicity $(0, 0)^T$ and $0 \in \mathbb{R}$ in the same way. We recall that the polynomials $P_n(z)$ and the vectorial polynomials $\mathcal{B}_n(z)$ depend on $t \in \mathbb{R}$, when this dependence holds for the coefficients a_n, b_n, c_n.

Our main goal is to study the solutions of (1) in terms of the operator J and its associated vectorial polynomials $\mathcal{B}_n(z)$. We underline that, in a different context (cf. [2, Theorem 2]), the characterization of solutions of the integrable system considered there was established in terms of the derivative of the polynomials associated with J. In this work, using the sequence of vectorial polynomials $\{\mathcal{B}_n\}$, we extend that result. Our first result is the following.

Theorem 1. *Assume that the sequence $\{a_n\,, b_n\,, c_n\}\,, n \in \mathbb{N}$, is uniformly bounded, i.e. there exists $K \in \mathbb{R}_+$ such that $\max\{|a_n(t)|\,, |b_n(t)|\,, |c_n(t)|\} \leq M$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Assume, also, $c_n(t) \neq 0$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then, the following conditions are equivalent:*

(a) *$\{a_n\,, b_n\,, c_n\}\,, n \in \mathbb{N}$, is a solution of (1), this is,*

$$\dot{J} = [J, J_-]. \tag{9}$$

(b) *For each $n \in \mathbb{N} \cup \{0\}$ we have*

$$\frac{d}{dt} J_{11}^n = J_{11}^{n+1} - J_{11}^n B_1 + [J_{11}^n, (J_-)_{11}]. \tag{10}$$

(c) *For all $z \in \mathbb{C}$ such that $|z| > \|J\|$,*

$$\dot{\mathcal{R}}_J(z) = \mathcal{R}_J(z)(zI_2 - B_1) - I_2 + [\mathcal{R}_J(z), (J_-)_{11}]\,, \tag{11}$$

where $\mathcal{R}_J(z)$ is given by (5).

(d) *For each $n \in \mathbb{N} \cup \{0\}$, the polynomial $\mathcal{B}_n$ defined by (7) satisfies*

$$\dot{\mathcal{B}}_n(z) = -C_n\,\mathcal{B}_{n-1}(z) - D_n\,\mathcal{B}_n(z)\,, \tag{12}$$

where $D_n = \begin{pmatrix} 0 & 0 \\ b_{2n+1} & 0 \end{pmatrix}$.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients. It is well known that, given the recurrence relation (6), there exist two linear moment functionals u^1, u^2 from $\mathcal{P}$ to $\mathbb{C}$ verifying, for all $m \in \mathbb{N}$,

$$\begin{cases} u^i[z^j P_{2m}] = u^i[z^j P_{2m+1}] = 0 \,, \ j = 0, 1, \ldots, m-1 \,, \, i = 1, 2 \\ u^1[z^m P_{2m+1}] = 0 \end{cases} \tag{13}$$

(see [10, Th. 3.2], see also [5, 9]).

We consider the space $\mathcal{P}^2 = \{(q_1, q_2)^T : q_i \text{ polynomial}, \, i = 1, 2\}$ and the space $\mathcal{M}_{2\times 2}$ of (2×2)-matrices with complex entries.

Definition 1. If the functionals u^1, u^2 verify (13), then we say that the function $\mathcal{W} : \mathcal{P}^2 \to \mathcal{M}_{2\times 2}$ given by

$$\mathcal{W}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u^1[q_1] & u^2[q_1] \\ u^1[q_2] & u^2[q_2] \end{pmatrix} , \tag{14}$$

is a *vector of functionals* associated with the recurrence relation (6).

If $\mathcal{W}$ is a vector of functionals associated with the recurrence relation (6), then the following orthogonality relations are verified

$$\mathcal{W}(z^j \mathcal{B}_m) = \mathcal{O}_2 \,, \quad j = 0, 1, \ldots, m-1 \,, \tag{15}$$

where $\mathcal{O}_2$ denotes the 2×2 null matrix.

Definition 2. A function $\mathcal{W} : \mathcal{P}^2 \to \mathcal{M}_{2\times 2}$ verifying (15) is called *orthogonality vector of functionals* for the recurrence relation (7).

Since the above definitions, any vector of functionals associated with the recurrence relation (6) is always an orthogonality vector of functionals for the recurrence relation (7). As in the scalar case, it is possible to find more than one orthogonality vector of functionals. In fact, given a such function $\mathcal{W} : \mathcal{P}^2 \to \mathcal{M}_{2\times 2}$, and given any matrix $M \in \mathcal{M}_{2\times 2}$ it is enough to define $\mathcal{W}_M$ as

$$\mathcal{W}_M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} := \mathcal{W}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} M \,, \tag{16}$$

for having another orthogonality vector of functionals. In the following, we assume that $\mathcal{W}$ is a fixed vector of functionals associated with the recurrence relation (6) such that $\mathcal{W}(\mathcal{B}_0)$ is an invertible matrix.

We recall that, in (7), the matrix C_0 was arbitrary chosen. In the sequel we assume

$$C_0 = \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix}.$$

Take $M = (\mathcal{W}(\mathcal{B}_0))^{-1} C_0$ and define $\mathcal{U} = \mathcal{W}_M$ as in (16). Then, $\mathcal{U}(\mathcal{Q}) = \mathcal{W}(\mathcal{Q})(\mathcal{W}(\mathcal{B}_0))^{-1} C_0$ for any $\mathcal{Q} \in \mathcal{P}^2$ and, in particular,

$$\mathcal{U}(\mathcal{B}_0) = C_0\,. \tag{17}$$

Moreover, from (7) and (15),

$$\mathcal{U}(z^m \mathcal{B}_m) = C_m\, \mathcal{U}(z^{m-1}\mathcal{B}_{m-1})\,, \quad m \in \mathbb{N} \tag{18}$$

(see Lemma 1). Using (17), (18), and again (15), for each $m \in \mathbb{N} \cup \{0\}$ we arrive to

$$\mathcal{U}(z^j \mathcal{B}_m) = \begin{cases} \mathcal{O}_2\,, \; j = 0, 1, \ldots, m-1 \\ C_m C_{m-1} \cdots C_0\,, \; j = m\,. \end{cases} \tag{19}$$

We use the vectors $\mathcal{P}_m = \mathcal{P}_m(z) = \left(z^{2m}, z^{2m+1}\right)^T$ for each $m \in \mathbb{N} \cup \{0\}$. The following definition extends the corresponding one to the scalar case.

Definition 3. For each $m = 0, 1, \ldots$, the matrix $\mathcal{U}(z^m \mathcal{P}_0)$ is called *moment of order* m for the vector of functionals $\mathcal{U}$.

In particular, since $\mathcal{B}_0 = C_0\, \mathcal{P}_0$, we have

$$\mathcal{U}(\mathcal{P}_0) = I_2 \tag{20}$$

(see (17)).

We know $\mathcal{U} = \mathcal{U}\{t\}$ depends on t, besides this dependence is not explicitly written (as we said above). Then, it is possible to define the derivative of $\mathcal{U}$ as usual,

$$\frac{d\,\mathcal{U}}{dt} : \mathcal{P}^2 \longrightarrow \mathcal{M}_{2\times 2}$$

such that, for each $\mathcal{B} \in \mathcal{P}^2$,

$$\frac{d\,\mathcal{U}}{dt}(\mathcal{B}) = \lim_{\Delta t \to 0} \frac{\mathcal{U}\{t + \Delta t\}(\mathcal{B}) - \mathcal{U}\{t\}(\mathcal{B})}{\Delta t}\,.$$

Obviously, the usual properties for this kind of operators are verified. In particular,

$$\frac{d}{dt}\left(\mathcal{U}(\mathcal{B})\right) = \frac{d\,\mathcal{U}}{dt}(\mathcal{B}) + \mathcal{U}(\dot{\mathcal{B}})\,, \quad \mathcal{B} \in \mathcal{P}^2\,. \tag{21}$$

We use $\frac{d\mathcal{U}}{dt}$ and (21) below. Also, we will use the matrix function $\mathcal{R}_J$ given in (5). We define the *generating function of the moments* as

$$\mathcal{F}_J(z) = C_0^{-1}\,\mathcal{R}_J(z)\,C_0\,, \quad |z| > \|J\|\,. \tag{22}$$

Next, we have our second main result, related with Theorem 1. More precisely speaking, we will see that Theorem 1 follows directly from Theorem 2.

Theorem 2. *In the conditions of Theorem 1, assume $\dot{a}_1 = b_1$. Let $\mathcal{U}$ be given by (19). Then, the following assertions are equivalent:*

(e) $\{a_n\,, b_n\,, c_n\}\,,\ n \in \mathbb{N}\,,$ *is a solution of (1), this is, (9) holds.*

(f) *For each $n = 0, 1, \ldots,$ we have*

$$\frac{d}{dt}\mathcal{U}\left(z^n\mathcal{P}_0\right) = \mathcal{U}\left(z^{n+1}\mathcal{P}_0\right) - \mathcal{U}\left(z^n\mathcal{P}_0\right)\mathcal{U}\left(z\mathcal{P}_0\right)\,. \tag{23}$$

(g) *For all $\zeta \in \mathbb{C}$ such that $|\zeta| > \|J\|$,*

$$\dot{\mathcal{F}}_J(\zeta) = \mathcal{F}_J(\zeta)\left(\zeta I_2 - \mathcal{U}(z\mathcal{P}_0)\right) - I_2\,, \tag{24}$$

being $\mathcal{F}_J$ the generating function defined in (22).

(h) *For all $\mathcal{B} \in \mathcal{P}^2$ we have*

$$\left(\frac{d}{dt}\mathcal{U}\right)(\mathcal{B}) = \mathcal{U}(z\mathcal{B}) - \mathcal{U}(\mathcal{B})\mathcal{U}(z\mathcal{P}_0)\,. \tag{25}$$

(i) *For each $n = 0, 1, \ldots,$ we have (12).*

Moreover, we have other consequences of Theorem 1 and Theorem 2.

In the next result, $\int_0^t f(s)ds$ is understood to be the solution $X = X(t)$ of the *Cauchy problem*

$$\left.\begin{array}{r} \dot{X} = f(t) \\ X(0) = 0 \end{array}\right\}$$

in the suitable space. It is well-known that, in our conditions, there exists a unique solution of this problem (see, for instance, [6] and [11]).

Corollary 1. *Under the conditions of Theorem 1, if $\{a_n\,, b_n\,, c_n\}\,,\ n \in \mathbb{N}\,,$ is a solution of (1), then*

$$\mathcal{R}_J(z) = \exp(zt)\,C_0\,M(t,z)\,(N(t))^{-1}\,, \tag{26}$$

where

$$N(t) = \begin{pmatrix} \exp\left(\int_0^t a_1 ds\right) & \exp\left(\int_0^t a_1 ds\right)\int_0^t \exp\left(-\int_0^s (a_2 - a_1)dr\right) ds \\ 0 & \exp\left(\int_0^t a_2 ds\right) \end{pmatrix},$$

$$M(t,z) = -\int_0^t \exp(-zs)\, C_0^{-1} N(s) ds + (C_0(0))^{-1} \mathcal{R}_0(z)$$

(here, $C_0(0)$ and $\mathcal{R}_0(z)$ are, respectively, C_0 and $\mathcal{R}(z)$ for $t=0$).

Given a linear functional $u : \mathcal{P} \to \mathbb{C}$, we may define the new functional $e^{zt}u : \mathcal{P} \to \mathbb{C}$ as

$$\left(e^{zt}u\right)[z^j] = \sum_{k\geq 0} \frac{t^k}{k!} u[z^{k+j}] . \tag{27}$$

We denote by $\mathcal{U}_0 = (u_0^1, u_0^2)^T$ our vector of functionals $\mathcal{U}$ when $t=0$ and, similarly, by J_0 the triangular matrix given in (2) when $t=0$. If J_0 is a bounded matrix, then

$$|u_0^i[z^{k+j}]| \leq m_{ij} \|J_0\|^{k+j} , \quad i = 1, 2 , \quad k, j = 0, 1, \dots ,$$

and the right-hand side of (27) is well-defined for $u = u_0^i$, $i = 1, 2$ (see [7, Theorem 4]). In this case, we can define the vector of functionals $e^{zt}\,\mathcal{U}_0$ as

$$\left(e^{zt}\,\mathcal{U}_0\right)(\mathcal{B}) = \begin{pmatrix} \left(e^{zt}u_0^1\right)[q_1] & \left(e^{zt}u_0^2\right)[q_1] \\ \left(e^{zt}u_0^1\right)[q_2] & \left(e^{zt}u_0^2\right)[q_2] \end{pmatrix}$$

for each $\mathcal{B} = (p, q)^T \in \mathcal{P}^2$. As in [3, Theorem 3], we give a possible representation for the vector of functionals $\mathcal{U}$.

Corollary 2. *In the conditions of Theorem 2, and with the notation used in (16), assume that the vector of functionals $\mathcal{U}$ verifies*

$$\mathcal{U} = \left(e^{zt}\,\mathcal{U}_0\right)_M \tag{28}$$

for some $M \in \mathcal{M}_{2\times 2}$. Then, $\{a_n\, ,\, b_n\, ,\, c_n\}$, $n \in \mathbb{N}$, is a solution of (1).

In section 2 we show that the study of the system (1) can be reduced to the evolution of the main block of the matrix J^n, i.e. J_{11}^n. In section 3 we prove Theorem 2. The main idea is to express (1) in terms of the evolution of the moments. In section 4 we prove Theorem 1. The main feature of this result is the connections between the resolvent function, $\mathcal{R}_J$, and the evolution of the systems of vector polynomial, $\{\mathcal{B}_n\}$.

2. Auxiliary results

Next lemma was used for proving (18).

Lemma 1. *Let $\mathcal{W}$ be a vector of functionals associated with (6). Then*

$$\mathcal{W}(A_1\,\mathcal{Q}_1 + A_2\,\mathcal{Q}_2) = A_1\,\mathcal{W}(\mathcal{Q}_1) + A_2\,\mathcal{W}(\mathcal{Q}_2) \tag{29}$$

is verified for any $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{P}^2$ and $A_1, A_2 \in \mathcal{M}_{2\times 2}$.

Proof: It is sufficient to take into account that $\mathcal{W} : \mathcal{P}^2 \to \mathcal{M}_{2\times 2}$ is given by

$$\mathcal{W}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u[q_1] & v[q_1] \\ u[q_2] & v[q_2] \end{pmatrix}$$

when $u, v : \mathcal{P} \to \mathbb{C}$ are linear functionals. ∎

Lemma 2. *The orthogonality vector of functionals $\mathcal{U} : \mathcal{P}^2 \longrightarrow \mathcal{M}_{2\times 2}$ is determined by (19). This is, $\mathcal{U}$ is the unique vector of functionals associated with the recurrence relation (6) verifying (19).*

Proof: Given $(q_1, q_2)^T \in \mathcal{P}^2$, for each $i = 1, 2$ we can write

$$q_i(z) = \sum_{k=1}^{2} \alpha_{ik}^0 P_{k-1}(z) + \sum_{k=1}^{2} \alpha_{ik}^1 P_{1+k}(z) + \cdots + \sum_{k=1}^{2} \alpha_{ik}^m P_{2m+k-1}(z)\,,\ \alpha_{ik}^j \in \mathbb{C}\,,$$

where $m = \max\{m_1, m_2\}$ and $\deg(q_i) \le 2m_i + 1$ (we understand $\alpha_{ik}^j = 0$ when $j > m_i$). This is,

$$(q_1, q_2)^T = \sum_{j=0}^{m} E_j\,\mathcal{B}_j\,, \tag{30}$$

being $E_j = \left(\alpha_{ik}^j\right) \in \mathcal{M}_{2\times 2}\,,\ j = 0, \ldots, m$. From (29), if $\widetilde{\mathcal{U}} : \mathcal{P}^2 \longrightarrow \mathcal{M}_{2\times 2}$ is a vector of functionals associated with the recurrence relation (6), then

$$\widetilde{\mathcal{U}}\left(\begin{array}{c} q_1 \\ q_2 \end{array}\right) = \sum_{j=0}^{m} E_j\,\widetilde{\mathcal{U}}\,(\mathcal{B}_j)\,. \tag{31}$$

Moreover, if $\widetilde{\mathcal{U}}$ verifies (19) we have $\widetilde{\mathcal{U}}\,(\mathcal{B}_j) = \mathcal{U}\,(\mathcal{B}_j) = 0\,,\ j = 1, 2, \ldots\,,$ and also $\widetilde{\mathcal{U}}\,(\mathcal{B}_0) = \mathcal{U}\,(\mathcal{B}_0) = C_0\,.$ Therefore, from (31) we arrive to $\widetilde{\mathcal{U}} = \mathcal{U}\,.$ ∎

Next result shows that it is possible to recover the entries of matrix J using the orthogonality vector of functionals $\mathcal{U}$.

Lemma 3. *The entries of the matrix J are determined by the sequence $\{\mathcal{B}_n\}$ of vectorial polynomials.*

Proof: The entries of J are determined by the blocks C_n , B_n , $n \in \mathbb{N}$. Then, it is sufficient to express these blocks in terms of $\{\mathcal{B}_n\}$. Since C_k , $k = 0, 1, \dots$, are invertible matrices, also $\mathcal{U}\left(z^k \mathcal{B}_k\right)$ is invertible and, from (19),

$$C_n = \mathcal{U}\left(z^n \mathcal{B}_n\right) \left(\mathcal{U}\left(z^{n-1} \mathcal{B}_{n-1}\right)\right)^{-1} , \quad n = 1, 2, \dots .$$

On the other hand, from (7) and (19) we deduce

$$C_n \,\mathcal{U}\left(z^n \mathcal{B}_{n-1}\right) + B_{n+1} \,\mathcal{U}\left(z^n \mathcal{B}_n\right) - \mathcal{U}\left(z^{n+1} \mathcal{B}_n\right) = 0 , \quad n = 0, 1, \dots .$$

Then, for $n \in \mathbb{N}$ we have

$$B_n = \left(\mathcal{U}\left(z^n \mathcal{B}_{n-1}\right) - C_{n-1} \,\mathcal{U}\left(z^{n-1} \mathcal{B}_{n-2}\right)\right) \left(\mathcal{U}\left(z^{n-1} \mathcal{B}_{n-1}\right)\right)^{-1} ,$$

and the result follows. ■

Next, we determine the expression of the moment $\mathcal{U}\left(\mathcal{P}_n\right) = \mathcal{U}\left(x^n \,\mathcal{P}_0\right)$ in terms of the matrix J.

Lemma 4. *For each $n = 0, 1, \dots$ we have*

$$\mathcal{U}\left(z^n \,\mathcal{P}_0\right) = C_0^{-1} \,J_{11}^n \,C_0 . \tag{32}$$

Proof: We know that $\mathcal{U}\left(\mathcal{P}_0\right) = I_2$ (see (20)), then the result is verified for $n = 0$.

Another way to express (7) is

$$J \begin{pmatrix} \mathcal{B}_0(z) \\ \mathcal{B}_1(z) \\ \vdots \end{pmatrix} = z \begin{pmatrix} \mathcal{B}_0(z) \\ \mathcal{B}_1(z) \\ \vdots \end{pmatrix} .$$

Thus,

$$J^n \begin{pmatrix} \mathcal{B}_0(z) \\ \mathcal{B}_1(z) \\ \vdots \end{pmatrix} = z^n \begin{pmatrix} \mathcal{B}_0(z) \\ \mathcal{B}_1(z) \\ \vdots \end{pmatrix} , \quad n \in \mathbb{N} . \tag{33}$$

Comparing the first rows in (33), and taking into account (4) and the form of J,

$$\sum_{i \geq 1} J_{1i}^n \,\mathcal{B}_{i-1}(z) = J_{11}^n \,\mathcal{B}_0(z) + J_{12}^n \,\mathcal{B}_1(z) = z^n \mathcal{B}_0(z) .$$

Then, from (19), $\mathcal{U}\left(z^n \mathcal{B}_0\right) = J_{11}^n \,\mathcal{U}\left(\mathcal{B}_0\right)$, this is, $C_0 \,\mathcal{U}\left(z^n \mathcal{P}_0\right) = J_{11}^n \,C_0$ (see (17) and (20)), which is (32). ■

The following result concerns to solutions associated with the matrix J, non necessarily bounded.

Lemma 5. *If the sequence* $\{a_n\,,\,b_n\,,\,c_n\}\,,\,n\in\mathbb{N}\,,$ *is a solution of (1), then (10) is verified.*

Proof: Under the given conditions, we know that (9) holds. Then, it is very easy to verify

$$\frac{d}{dt}J^n = J^n J_- - J_- J^n$$

and, with the established notation,

$$\frac{d}{dt}J^n_{11} = (J^n J_-)_{11} - (J_- J^n)_{11}$$

From (2) and (4),

$$\left.\begin{array}{rcl}(J^n J_-)_{11} & = & J^n_{11}\,(J_-)_{11} + J^n_{12}\,(J_-)_{21}\\ (J_- J^n)_{11} & = & (J_-)_{11}\,J^n_{11}\,.\end{array}\right\}$$

Then,

$$\dot{J}^n_{11} = J^n_{11}\,(J_-)_{11} - (J_-)_{11}\,J^n_{11} + J^n_{12}\,(J_-)_{21}\,. \tag{34}$$

On the other hand, $J^{n+1}_{11} = J^n_{11} J_{11} + J^n_{12}\,(J_-)_{21}$, which, joint with (34), goes to (10). ■

3. Proof of Theorem 2

In the first place, we show (e)$\Rightarrow$ (f). Assume that (9) holds. Since Lemma 5, we have (10). Moreover, due to Lemma 4, $J^n_{11} = C_0\,\mathcal{U}(z^n\mathcal{P}_0)\,C_0^{-1}\,,\ n\in\mathbb{N}$, and, in particular, $B_1 = J_{11} = C_0\,\mathcal{U}(z\mathcal{P}_0)\,C_0^{-1}$. Also, $C_0^{-1}\,(J_-)_{11} = (J_-)_{11}\,C_0 = (J_-)_{11}$. Then, from (34),

$$\begin{aligned}\frac{d}{dt}\left(C_0\,\mathcal{U}(z^n\mathcal{P}_0)\,C_0^{-1}\right) &= C_0\,\mathcal{U}(z^{n+1}\mathcal{P}_0)\,C_0^{-1} - C_0\,\mathcal{U}(z^n\mathcal{P}_0)\mathcal{U}(z\mathcal{P}_0)\,C_0^{-1}\\ &\quad + C_0\,\mathcal{U}(z^n\mathcal{P}_0)\,(J_-)_{11} - (J_-)_{11}\,\mathcal{U}(z^n\mathcal{P}_0)\,C_0^{-1}\end{aligned} \tag{35}$$

On the other hand, taking derivatives (and denoting by $\dot{C}_0^{-1}$ the derivative of C_0^{-1}),

$$\begin{aligned}&\frac{d}{dt}\left(C_0\,\mathcal{U}(z^n\mathcal{P}_0)\,C_0^{-1}\right)\\ &\quad = \dot{C}_0\,\mathcal{U}(z^n\mathcal{P}_0)\,C_0^{-1} + C_0\,\mathcal{U}(z^n\mathcal{P}_0)\,\dot{C}_0^{-1} + C_0\,\frac{d}{dt}\left(\mathcal{U}(z^n\mathcal{P}_0)\right)\,C_0^{-1}\end{aligned} \tag{36}$$

Since $\dot{a}_1 = b_1$, we can verify $\dot{C}_0 = -\left(J_-\right)_{11}$, $\dot{C}_0^{-1} = \left(J_-\right)_{11}$. Hence, comparing the right hand sides of (35) and (36) we arrive to (23) (we recall that C_0 is an invertible matrix).

Now, we prove (f)⇒ (g). Since (5) and Lemma 4,

$$\mathcal{F}_J(\zeta) = \sum_{n\geq 0} \frac{\mathcal{U}(z^n \mathcal{P}_0)}{\zeta^{n+1}}, \quad |\zeta| > \|J\| . \tag{37}$$

Then, from (23),

$$\begin{aligned}
\dot{\mathcal{F}}_J(\zeta) &= \sum_{n\geq 0} \frac{\mathcal{U}(z^{n+1}\mathcal{P}_0) - \mathcal{U}(z^n\mathcal{P}_0)\mathcal{U}(z\mathcal{P}_0)}{\zeta^{n+1}} \\
&= \zeta \sum_{n\geq 0} \frac{\mathcal{U}(z^{n+1}\mathcal{P}_0)}{\zeta^{n+2}} - \sum_{n\geq 0} \frac{\mathcal{U}(z^n\mathcal{P}_0)}{\zeta^{n+1}} \mathcal{U}(z\mathcal{P}_0) \\
&= \zeta \left(\mathcal{F}_J(\zeta) - \frac{1}{\zeta} I_2 \right) - \mathcal{F}_J(\zeta)\mathcal{U}(z\mathcal{P}_0) .
\end{aligned}$$

This is, (24) is verify.

Given (24), we are going to obtain the derivative of the vector of functionals $\mathcal{U}$. For doing this, we use the linearity of $\mathcal{U}$ and the convergence of the series,

$$\mathcal{F}_J(\zeta) = \mathcal{U}\left(\sum_{n\geq 0} \frac{z^n}{\zeta^{n+1}} \mathcal{P}_0 \right) = \mathcal{U}\left(\frac{1}{\zeta - z} \mathcal{P}_0 \right) , \quad |\zeta| > \|J\| . \tag{38}$$

(Here and in the next expressions, as usual, $\mathcal{U} = \mathcal{U}_z$ is the vector of functionals defined on the closure of the space $\mathcal{P}^2$ of vectorial polynomials $(q_1, q_2)^T$ in the variable z.)

From (38) and (24),

$$\begin{aligned}
\frac{d}{dt}\mathcal{U}\left(\frac{1}{\zeta - z}\mathcal{P}_0 \right) &= \mathcal{U}\left(\frac{1}{\zeta - z}\mathcal{P}_0 \right)\left(\zeta I_2 - \mathcal{U}\left(z\mathcal{P}_0\right)\right) - I_2 \\
&= \mathcal{U}\left(\left(1 + \frac{z}{\zeta - z} \right) \mathcal{P}_0 \right) - \mathcal{U}\left(\frac{1}{\zeta - z}\mathcal{P}_0 \right) \mathcal{U}\left(z\mathcal{P}_0\right) - I_2
\end{aligned}$$

and so

$$\frac{d}{dt}\mathcal{U}\left(\frac{1}{\zeta - z}\mathcal{P}_0 \right) = \mathcal{U}\left(\frac{z}{\zeta - z}\mathcal{P}_0 \right) - \mathcal{U}\left(\frac{1}{\zeta - z}\mathcal{P}_0 \right) \mathcal{U}\left(z\mathcal{P}_0\right) . \tag{39}$$

Define the vectors of functionals $\mathcal{U}_1 \,,\, \mathcal{U}_2 : \mathcal{P}^2 \longrightarrow \mathcal{M}_{2\times 2}$ such that

$$\left.\begin{array}{rcl} \mathcal{U}_1(\mathcal{B}) & = & \mathcal{U}(z\mathcal{B}) \\ \mathcal{U}_2(\mathcal{B}) & = & \mathcal{U}(\mathcal{B})\,\mathcal{U}(z\mathcal{P}_0) \end{array}\right\} \tag{40}$$

for each $\mathcal{B} \in \mathcal{P}^2$. We remark that $\frac{1}{\zeta - z}\mathcal{P}_0$ do not depend on $t \in \mathbb{R}$. In (39), denoting $\dot{\mathcal{U}} = \frac{d}{dt}\mathcal{U}$, we have $\dot{\mathcal{U}} = \mathcal{U}_1 - \mathcal{U}_2$ over $\frac{1}{\zeta - z}\mathcal{P}_0$, being

$$\mathcal{U}\left(\frac{1}{\zeta - z}\mathcal{P}_0\right) = \frac{1}{\zeta}\mathcal{U}\left(\mathcal{P}_0\right) + \frac{1}{\zeta^2}\mathcal{U}\left(z\mathcal{P}_0\right) + \cdots \,, \quad |\zeta| > \|J\| \,.$$

Hence, we have $\dot{\mathcal{U}} = \mathcal{U}_1 - \mathcal{U}_2$ over $\mathcal{P}^2$, this is, we have (25).

For proving (h)$\Rightarrow$ (i), as in (30), $\dot{\mathcal{B}}_n$ can be written in terms of the sequence $\{\mathcal{B}_n\}$,

$$\dot{\mathcal{B}}_n(z) = D_0^{(n)}\,\mathcal{B}_0(z) + D_1^{(n)}\,\mathcal{B}_1(z) + \cdots + D_n^{(n)}\,\mathcal{B}_n(z) \tag{41}$$

If $n = 0, 1$, the above expression is

$$\dot{\mathcal{B}}_n(z) = D_{n-1}^{(n)}\,\mathcal{B}_{n-1}(z) + D_n^{(n)}\,\mathcal{B}_n(z) \tag{42}$$

Let $n \geq 2$ be fixed. We are going to show that (42) holds, also, for n. Due to the orthogonality, from (41), $\mathcal{U}(\dot{\mathcal{B}}_n) = D_0^{(n)}\,\mathcal{U}\left(\mathcal{B}_0\right)$. In fact, using (25),

$$\begin{array}{rcl} \mathcal{O}_2 = \dfrac{d}{dt}\left(\mathcal{U}(\mathcal{B}_n)\right) & = & \dot{\mathcal{U}}(\mathcal{B}_n) + \mathcal{U}(\dot{\mathcal{B}}_n) \\ & = & \mathcal{U}(z\mathcal{B}_n) - \mathcal{U}(\mathcal{B}_n)\mathcal{U}(z\mathcal{P}_0) + D_0^{(n)}\,\mathcal{U}\left(\mathcal{B}_0\right) \,. \end{array}$$

Thus, $D_0^{(n)} = \mathcal{O}_2$. We proceed by induction on n, assuming

$$D_0^{(n)} = \cdots = D_{j-1}^{(n)} = \mathcal{O}_2$$

for a fixed $j < n - 1$. Using (41) and, again, (25) and (19),

$$\begin{array}{rcl} \mathcal{O}_2 = \dfrac{d}{dt}\mathcal{U}(z^j\mathcal{B}_n) & = & \left(\dfrac{d}{dt}\mathcal{U}\right)(z^j\mathcal{B}_n) + \mathcal{U}(z^j\dot{\mathcal{B}}_n) \\ & = & \mathcal{U}(z^{j+1}\mathcal{B}_n) - \mathcal{U}(z^j\mathcal{B}_n)\mathcal{U}(z\mathcal{P}_0) + D_j^{(n)}\,\mathcal{U}\left(z^j\mathcal{B}_j\right) \\ & = & D_j^{(n)}\,\mathcal{U}\left(z^j\mathcal{B}_j\right) \,, \end{array}$$

where $\mathcal{U}\left(z^j\mathcal{B}_j\right)$ is an invertible matrix. Thus, $D_j^{(n)} = \mathcal{O}_2$ and (42) is verified for any $n \in \mathbb{N}$.

Our next purpose is to determine $D_j^{(n)}$, $j = n-1, n$. From (42),

$$\mathcal{U}(z^{n-1}\dot{\mathcal{B}}_n) = D_{n-1}^{(n)}\,\mathcal{U}\left(z^{n-1}\mathcal{B}_{n-1}\right) \,.$$

Then, because of (25) and (19),

$$\mathcal{O}_2 = \frac{d}{dt}\mathcal{U}\left(z^{n-1}\mathcal{B}_n\right) = \mathcal{U}\left(z^{n}\mathcal{B}_n\right) - \mathcal{U}\left(z^{n-1}\mathcal{B}_n\right)\mathcal{U}\left(z\mathcal{P}_0\right) + D_{n-1}^{(n)}\,\mathcal{U}\left(z^{n-1}\mathcal{B}_{n-1}\right)$$

and, therefore, $D_{n-1}^{(n)} = -C_n$.

On the other hand, writing

$$\mathcal{B}_n(z) = \sum_{i=0}^{n} F_i^{(n)}\mathcal{P}_i(z) \tag{43}$$

and comparing the coefficient of z^{2n} and z^{2n+1} in both sides of (43), we obtain

$$F_n^{(n)} = \begin{pmatrix} 1 & 0 \\ f_n & 1 \end{pmatrix} .$$

Moreover, taking derivatives in (43) and comparing with (42), we see $D_n^{(n)} = \dot{F}_n^{(n)}$ or, what is the same,

$$D_n^{(n)} = \begin{pmatrix} 0 & 0 \\ d_n & 0 \end{pmatrix} ,$$

where we need to determine d_n. From (25) and (42),

$$\begin{aligned} D_n^{(n)}\mathcal{U}\left(z^{n}\mathcal{B}_n\right) = \left(\frac{d}{dt}\mathcal{U}\left(z^{n}\mathcal{B}_n\right) - \mathcal{U}\left(z^{n+1}\mathcal{B}_n\right) + \mathcal{U}\left(z^{n}\mathcal{B}_n\right)\mathcal{U}\left(z\mathcal{P}_0\right)\right) \\ + C_n\,\mathcal{U}\left(z^{n}\mathcal{B}_{n-1}\right) \,. \end{aligned}$$

Then, using (19) and (7),

$$\begin{aligned} D_n^{(n)} = \left(\frac{d}{dt}\left(C_nC_{n-1}\cdots C_0\right)\right)\left(C_nC_{n-1}\cdots C_0\right)^{-1} \\ - B_{n+1} + \left(C_nC_{n-1}\cdots C_0\right)\mathcal{U}\left(z\mathcal{P}_0\right)\left(C_nC_{n-1}\cdots C_0\right)^{-1} , \end{aligned}$$

thus

$$\begin{aligned} D_n^{(n)} + B_{n+1} = \left(\frac{d}{dt}\left(C_nC_{n-1}\cdots C_1\right)\right)\left(C_nC_{n-1}\cdots C_1\right)^{-1} \\ + \left(C_nC_{n-1}\cdots C_1\right)\left(\dot{C}_0\,C_0^{-1} + J_{11}\right)\left(C_nC_{n-1}\cdots C_1\right)^{-1} . \end{aligned} \tag{44}$$

The matrix $C_n C_{n-1} \cdots C_1$ is upper triangular. Moreover, because of $\dot{a}_1 = b_1$ also $\dot{C}_0 \, C_0^{-1} + J_{11}$ is an upper triangular matrix and, then, the matrix in the left hand side of (44) is upper triangular and, consequently, $d_n = b_{2n+1}$.

Finally, we show (i) $\Rightarrow$ (e). Taking derivatives in (7),

$$\begin{aligned} &\dot{C}_n \, \mathcal{B}_{n-1}(z) + \dot{B}_{n+1} \, \mathcal{B}_n(z) \\ &\qquad + C_n \, \dot{\mathcal{B}}_{n-1}(z) + (B_{n+1} - zI_2)\dot{\mathcal{B}}_n(z) + A \, \dot{\mathcal{B}}_{n+1}(z) = 0 \,, \quad n = 0, 1, \ldots \end{aligned}$$

Using (12) and taking into account $A \, D_{n+1} = D_n \, A = \mathcal{O}_2$,

$$\begin{aligned} &\dot{C}_n \, \mathcal{B}_{n-1}(z) + \dot{B}_{n+1} \, \mathcal{B}_n(z) + C_n \, (-C_{n-1} \, \mathcal{B}_{n-2}(z) + D_{n-1} \, \mathcal{B}_{n-1}(z)) \\ &\qquad + (B_{n+1} - zI_2) \, (-C_n \, \mathcal{B}_{n-1}(z) + D_n \, \mathcal{B}_n(z)) - A \, C_{n+1} \, \mathcal{B}_n(z) = 0 \,. \end{aligned} \tag{45}$$

Using, again, (7) for eliminating the explicit expression in z,

$$\left. \begin{array}{rcl} z\mathcal{B}_n(z) & = & C_n \, \mathcal{B}_{n-1}(z) + B_{n+1} \, \mathcal{B}_n(z) + A \, \mathcal{B}_{n+1}(z) \\ z\mathcal{B}_{n-1}(z) & = & C_{n-1} \, \mathcal{B}_{n-2}(z) + B_n \, \mathcal{B}_{n-1}(z) + A \, \mathcal{B}_n(z) \,. \end{array} \right\}$$

Substituting in (45), and identifying with zero the coefficients of the vectorial polynomials in the obtained expression, we arrive to

$$\left. \begin{array}{rcl} \dot{B}_n & = & A \, C_n - C_{n-1} \, A + D_{n-1} \, B_n - B_n \, D_{n-1} \\ \dot{C}_n & = & D_n \, C_n - C_n \, D_{n-1} + B_{n+1} \, C_n - C_n \, B_n \end{array} \right\} n = 1, 2, \ldots \tag{46}$$

Taking into account that, with the above notation, $D_n = (J_-)_{n+1,n+1}$, we see that (46) is equivalent to (9) when we consider J as a blocked matrix. ■

4. Proof of Theorem 1 and Corollaries

4.1. Proof of Theorem 1. We start by establishing the equivalence between (10) and (11). The key idea is the convergence in the respective operator norm, for $|z| > \|J\|$, of the series given in the right hand side of (3) and (5). Starting by (10), to obtain $\dot{\mathcal{R}}_J(z)$ it is sufficient to take derivatives in (5) and to substitute $\dot{J}_{11}^n$ in

$$\dot{\mathcal{R}}_J(z) = \sum_{n \geq 0} \frac{\dot{J}_{11}^n}{z^{n+1}} \,, \quad |z| > \|J\| \,.$$

Reciprocally, if (11) holds, substituting $\mathcal{R}_J(z)$ and $\dot{\mathcal{R}}_J(z)$ by its Laurent expansion, and comparing their coefficients, we arrive to (10).

The rest of the proof is to show the equivalence between (a), (b) and (d).

(a)$\Rightarrow$ (b) is Lemma 5.

Now, we are going to prove (b) $\Rightarrow$ (d). We assume that (10) is verified. Taking $n = 1$ in this expression we immediately deduce $\dot{a}_1 = b_1$. Moreover, from (10) we arrive to (23) in the same way that in the proof of (e)$\Rightarrow$ (f) in Theorem 2. Then we are under the hypothesis of Theorem 2 and, therefore, we have (12).

Finally, (12) $\Rightarrow$ (9) was proved in Theorem 2. ■

4.2. Proof of Corollary 1. It is easy to see that C_0, $M(t,z)$ and $N(t)$ are, respectively, the solutions of the following Cauchy problems:

$$\left.\begin{array}{r} \dot{X} = -\left(J_-\right)_{11} X \\ X(0) = C_0(0) \end{array}\right\}, \quad \left.\begin{array}{r} \dot{X} = -\exp(-zt)\, C_0^{-1}\, N(t) \\ X(0) = \left(C_0(0)\right)^{-1} \mathcal{R}_0(z) \end{array}\right\},$$

$$\text{and} \quad \left.\begin{array}{r} \dot{X} = \left(J_{11} - \left(J_-\right)_{11}\right) X \\ X(0) = I_2 \end{array}\right\}.$$

Taking derivatives in the right hand side of (26), and checking the initial condition, we can prove that $\exp(zt)\, C_0\, M(t,z)\, \left(N(t)\right)^{-1}$ is a solution of the following Cauchy problem,

$$\left.\begin{array}{r} \dot{X} = X\left(zI_2 - J_{11}\right) - I_2 + \left[X, \left(J_-\right)_{11}\right] \\ X(0) = \mathcal{R}_0(z) \end{array}\right\}. \tag{47}$$

From [11], we know that (47) has a unique solution. On the other hand, since Theorem 1, $\mathcal{R}(z)$ is solution of (47). Then, we arrive to (26). ■

4.3. Proof of Corollary 2. Since (20), in the conditions of Corollary 2, necessarily (28) implies

$$M = \left[\left(e^{zt}\, \mathcal{U}_0\right)\left(\mathcal{P}_0\right)\right]^{-1}.$$

On the other hand, for proving that $\{a_n, b_n, c_n\}$, $n \in \mathbb{N}$, is a solution of (1), it is sufficient to show that (23) holds. Let a fixed $k \in \{0, 1, \ldots,\}$ be. Taking into account

$$\frac{d}{dt}\left(e^{zt}\, \mathcal{U}_0\right)\left(z^k \mathcal{P}_0\right) = \left(e^{zt}\, \mathcal{U}_0\right)\left(z^{k+1} \mathcal{P}_0\right)$$

and

$$\frac{d}{dt} M = -M\, \mathcal{U}(z\mathcal{P}_0),$$

and taking derivatives in

$$\mathcal{U}(z^k \mathcal{P}_0) = \left(e^{zt}\, \mathcal{U}_0\right)\left(z^k \mathcal{P}_0\right) M,$$

see (28), we arrive to (23). ■

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