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A NOTE ON TWO COMMUTATORS

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ABSTRACT: We show that two known conditions which naturally arose in commutator theory and in the theory of internal crossed modules coincide: *every star-multiplicative graph is multiplicative* if and only if *every two effective equivalence relations commute as soon as so do their normalisations*. This answers a question asked by George Janelidze.

KEYWORDS: Commutator, internal reflexive graph, star-multiplication, groupoid, protomodular category, semi-abelian category.
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Introduction

The purpose of this work is proving that for a semi-abelian category, the following conditions are equivalent:

(SM) every star-multiplicative graph is an internal groupoid;
(SH) two equivalence relations commute if and only if their normalisations commute.

The first condition comes from the study of internal crossed modules. In a semi-abelian category $\mathcal{A}$, the internal crossed modules introduced by Janelidze [Jan03] form a category which is equivalent to the category of internal groupoids in $\mathcal{A}$. To define a crossed module of groups, however, less structure is needed: already a reflexive graph equipped with a star-multiplication determines a crossed module. Nevertheless, there exist examples of semi-abelian categories where this is not true. Thus the question arose under which conditions on $\mathcal{A}$, the star-multiplicative graphs in $\mathcal{A}$ are internal groupoids.

The second condition is classical in the categorical approach to commutator theory. On one hand, there is the commutator of internal (effective) equivalence relations which was introduced by Smith [Smi76] in the context of Mal'tsev varieties and made categorical by Pedicchio [Ped95]. On the other hand, in the article [Huq68], Huq introduced a commutator for normal

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subobjects in a context which is roughly equivalent to that of semi-abelian categories. This definition was further studied by Borceux and Bourn [BB04]. Since, in any semi-abelian category, there is a bijective correspondence between the normal subobjects of an object and the effective equivalence relations on it, it is natural to ask how the two concepts of commutator correspond to each other. The answer is that in general, these concepts are not equivalent—not even in a variety of Ω-groups, as the counterexample of digroups shows [BB04]. On the other hand, in a category which is, for instance, strongly semi-abelian, two equivalence relations commute if and only if their normalisations commute.

It is already known that the second condition (SH) implies the first (SM): this was shown by Mantovani and Metere [MM09]. We shall prove that the other implication holds also. We do this in two steps: in the first section we work towards Theorem 1.4 which essentially states that Condition (SH) may be restricted to a special class of effective equivalence relations: those pairs of effective equivalence relations which are the kernel pairs of the domain and codomain morphisms of a reflexive graph. Then, in Section 2, we prove that a reflexive graph carries a star-multiplication if and only if it is a Peiffer graph if and only if the kernels of its domain and codomain morphisms commute (Proposition 2.11). This is enough to obtain our main result, Theorem 2.12, which states that (SM) is equivalent to (SH).

1. The *Smith is Huq* condition

We show that for a pointed protomodular category, the following two conditions are equivalent:

(SH) every two effective equivalence relations commute as soon as so do their normalisations;

(SH') every reflexive graph of which the kernels of the domain and the codomain morphisms commute is a groupoid.

Condition (SH) is the *Smith is Huq* condition in the title of this section; condition (SH') is well-known to hold, for instance, in the case of groups: recall the analysis of crossed modules given in the final chapter of Mac Lane's [Mac98].

1.1. The context. In this section we shall work in *pointed protomodular* categories. Recall that a category is **pointed** when it has a **zero object**, i.e., an initial object that is also terminal. A pointed category is **Bourn protomodular** [Bou91] when it has pullbacks along split epimorphisms and the **Split Short Five Lemma** holds: given a diagram

$$\begin{tikzcd} K[f] \arrow[r, "\mathsf{Ker}\, f"] \arrow[d, "k"'] & A \arrow[r, shift right, "f"'] \arrow[d, "a"] & B \arrow[l, shift right, "s"'] \arrow[d, "b"] \\ K[f'] \arrow[r, "\mathsf{Ker}\, f'"'] & A' \arrow[r, shift right, "f'"'] & B' \arrow[l, shift right, "s'"'] \end{tikzcd} \tag{A}$$

such that $fs = 1_B$ and $f's' = 1_{B'}$, the morphisms k and b being isomorphisms implies that a is an isomorphism.

In particular, a pointed protomodular category has binary products and kernels of split epimorphisms. Moreover, given a split epimorphism and its kernel as in

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "s"'] \end{tikzcd}$$

the kernel k and the section s are jointly strongly epic [BB04, Lemma 3.1.22]. Hence k and s are jointly epic [BG04, Lemma 2.2].

1.2. Commuting kernels. A coterminal pair of morphisms

$$X \xrightarrow{k} A \xleftarrow{l} Y$$

commutes (in the sense of Huq) [BB04, Huq68] when there is a (necessarily unique) morphism φ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\langle 1_X, 0\rangle"'] \arrow[dr, "k"] & \\ X \times Y \arrow[rr, dotted, "\varphi"] & & A \\ & Y \arrow[ul, "\langle 0, 1_Y\rangle"] \arrow[ur, "l"'] & \end{tikzcd}$$

is commutative.

We shall only consider the case where k and l are kernels. Of particular interest to us is the situation where they are the kernels of the domain and codomain morphisms of a reflexive graph $C = (C_1, C_0, d, c, e)$:

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}, \qquad de = ce = 1_{C_0}$$

and $k = \mathsf{Ker}\, d\colon X \to C_1$, $l = \mathsf{Ker}\, c\colon Y \to C_1$.

One usually views the elements of C_1 as arrows between the elements of C_0, so that the morphism $\varphi\colon X \times Y \to C_1$ is nothing but a partial composition on C_1 which sends a couple of arrows

$$\cdot \xleftarrow{\alpha} 0 \xleftarrow{\beta} \cdot$$

to its composite $\varphi(\alpha, \beta)$. The central question studied in this paper is, under which conditions such a partial composition extends to a composition on the entire graph. To answer it, we shall need the concept of commuting effective equivalence relations and its connection with commuting kernels.

1.3. Commuting effective equivalence relations. Consider a pair of reflexive graphs (R, S) on a common object A

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_0"] \arrow[r, shift right=2, "r_1"'] & A \arrow[l, "\Delta_R"' description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "s_0"] \end{tikzcd}$$

and consider the induced pullback of r_1 and s_0.

$$\begin{tikzcd} R \times_A S \arrow[r, "\pi_S"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\lrcorner" very near start] & S \arrow[d, "s_0"] \\ R \arrow[r, "r_1"'] & A \end{tikzcd}$$

The pair (R, S) **commutes (in the sense of Smith)** [Smi76, Ped95, BB04] when there is a (necessarily unique) morphism θ such that the diagram

$$\begin{tikzcd} & R \arrow[dl, "\langle 1_R, \Delta_S r_1 \rangle"'] \arrow[dr, "r_0"] & \\ R \times_A S \arrow[rr, dotted, "\theta"] & & A \\ & S \arrow[ul, "\langle \Delta_R s_0, 1_S \rangle"] \arrow[ur, "s_1"'] & \end{tikzcd}$$

is commutative.

We shall only consider the case where R and S are effective equivalence relations (i.e., kernel pairs). It is well-known that when for a span

$$\begin{tikzcd} & C_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ C_0 & & C_0' \end{tikzcd} \qquad \textbf{(B)}$$

the kernel pairs $R[d]$ and $R[c]$ commute, this means that (d, c) carries an internal pregroupoid structure [JP01]; briefly, any zigzag

$$\cdot \xleftarrow{\alpha} \cdot \xrightarrow{\beta} \cdot \xleftarrow{\gamma} \cdot$$

in C_1 may be composed to a single arrow $\theta(\alpha, \beta, \gamma)$, in such a way that $\theta(\alpha, \beta, \beta) = \alpha$ and $\theta(\beta, \beta, \gamma) = \gamma$. In particular, a reflexive graph $C = (C_1, C_0, d, c, e)$ is an internal groupoid if and only if $R[d]$ and $R[c]$ commute: then $\theta(\alpha, \beta, \gamma) = \alpha \circ \beta^{-1} \circ \gamma$.

It is also well-known that when a pair (R, S) of effective equivalence relations commutes, then so do their normalisations

$$X = K[r_0] \xrightarrow{k = r_1 \mathsf{Ker}\, r_0} A \xleftarrow{l = s_1 \mathsf{Ker}\, s_0} K[s_0] = Y :$$

see [BB04, Proposition 2.7.7]. In particular, for any internal groupoid C the composition on C restricts in such a way that the kernels of its domain and codomain morphisms commute. The converse is not true: in general, it is not possible to extend the partial composition on a reflexive graph which is given by its commuting kernels to a composition on the entire graph which makes it into a groupoid. This is explained by the next result (inspired by Lemma 2.1 in [Joh91]), together with the fact that a pair of effective equivalence relations of which the normalisations commute need not commute itself [BB04].

Theorem 1.4. *For a pointed protomodular category, the following conditions are equivalent:*

- (SH) *every two effective equivalence relations commute as soon as so do their normalisations;*
- (SH') *every reflexive graph of which the kernels of the domain and the codomain morphisms commute is a groupoid.*

Proof: It is clear that (SH') is just (SH) in the special case where the effective equivalence relations considered are the kernel pairs of the domain and the codomain maps of a reflexive graph. This special case implies the general case. Let indeed $R = R[d]$ and $S = R[c]$ be the effective equivalence relations induced by a span ($\mathbf{B}$) and assume that the kernels $k = \mathsf{Ker}\, d$ and $l = \mathsf{Ker}\, c$ commute in the sense of Huq. We have to prove that R and S commute in the sense of Smith, i.e., the span (d, c) is a pregroupoid.

If one thinks of the "elements" of the object C_1 as arrows $d(\alpha)\xrightarrow{\alpha}c(\alpha)$ then R and S consist of couples

$$\cdot\xleftarrow{\beta}\cdot\xrightarrow{\alpha}\cdot \qquad \text{and} \qquad \cdot\xrightarrow{\gamma}\cdot\xleftarrow{\delta}\cdot,$$

respectively. Forming the pullback of r_0 and s_0 we obtain a reflexive graph

$$\begin{tikzcd} R\times_0 S \arrow[r, shift left=2, "\mathrm{dom}=r_1\pi_R"] \arrow[r, shift right=2, "\mathrm{cod}=s_1\pi_S"'] & C_1 \arrow[l, "\langle\Delta_R,\Delta_S\rangle" description] \end{tikzcd}. \tag{$\mathbf{C}$}$$

Note that there is an isomorphism $R\times_{C_1}S\cong R\times_0 S$ under which an element $((\alpha,\beta),(\beta,\gamma))$ corresponds to $((\beta,\alpha),(\beta,\gamma))$—in both cases, a triple (α,β,γ). Thus an arrow in this reflexive graph is a triple

$$\cdot\xleftarrow{\alpha}\cdot\xrightarrow{\beta}\cdot\xleftarrow{\gamma}\cdot$$

considered as an arrow β with domain $\alpha=\mathrm{dom}(\alpha,\beta,\gamma)=r_1\pi_R(\alpha,\beta,\gamma)$ and codomain $\gamma=\mathrm{cod}(\alpha,\beta,\gamma)=s_1\pi_S(\alpha,\beta,\gamma)$. The kernels of dom and cod commute because so do k and l. The needed morphism

$$K[\mathrm{dom}]\times K[\mathrm{cod}]\to R\times_0 S$$

takes a couple

$$(\cdot\xleftarrow{0}\cdot\xrightarrow{\beta}\cdot\xleftarrow{\gamma}\cdot,\ \cdot\xleftarrow{\delta}\cdot\xrightarrow{\epsilon}\cdot\xleftarrow{0}\cdot)$$

in the product $K[\mathrm{dom}]\times K[\mathrm{cod}]$ and maps it to the element

$$\cdot\xleftarrow{\delta}\cdot\xrightarrow{\varphi(\beta,\epsilon)}\cdot\xleftarrow{\gamma}\cdot$$

of $R\times_0 S$. The hypothesis that (SH') holds now implies that this reflexive graph is a groupoid. This, in turn, establishes a pregroupoid structure on the span (d,c): the required morphism $\theta\colon R\times_{C_1}S\cong R\times_0 S\to C_1$ is determined by

$$(\cdot\xleftarrow{\gamma}\cdot\xrightarrow{\theta(\alpha,\beta,\gamma)}\cdot\xleftarrow{\alpha}\cdot)=(\cdot\xleftarrow{\beta}\cdot\xrightarrow{\alpha}\cdot\xleftarrow{\alpha}\cdot)\circ(\cdot\xleftarrow{\gamma}\cdot\xrightarrow{\gamma}\cdot\xleftarrow{\beta}\cdot)$$

where the composition takes place in the groupoid ($\mathbf{C}$). ■

Condition (SH) is usually called the **Smith is Huq** condition. It is known to hold in quite diverse situations: in pointed and *strongly protomodular* categories (by Theorem 6.6.1 in [BB04]; see also [Bou04]) and in pointed and *action accessible* categories (as explained in [MM09]; see also [BJ]). This condition is also weaker than the *reflected admissibility* condition studied in [MF09].

2. Star-multiplication

In this section we show that, in a semi-abelian category, three types of (uniquely determined) structure on a reflexive graph $C = (C_1, C_0, d, c, e)$ coincide: a reflexive graph C is *star-multiplicative* if and only if it is *Peiffer* if and only if the kernels of d and c commute (Proposition 2.11). This allows us to prove Theorem 2.12 which states that a semi-abelian category has the *Smith is Huq* property if and only if every star-multiplicative graph is a groupoid.

2.1. The context. A category is **semi-abelian** [JMT02] when it is pointed, Bourn protomodular and Barr exact with binary coproducts. **Barr exact** means that every internal equivalence relation is **effective** (i.e., it is the kernel pair of its coequaliser) and the category is **regular**: finitely complete with pullback-stable regular epimorphisms. A **homological** category is pointed, regular and protomodular [BB04].

In a homological category regular epimorphisms (coequalisers), strong epimorphisms and normal epimorphisms (cokernels) coincide, and every morphism $f\colon A \to B$ may be factored as a regular epimorphism $A \to I[f]$ followed by a monomorphism $\mathsf{Im}\, f\colon I[f] \to B$. The monomorphism $\mathsf{Im}\, f$ is the **image** of f. A morphism f is **proper** when it has a normal image, i.e., $\mathsf{Im}\, f$ is a kernel. In a semi-abelian category, the **direct image** $\mathsf{Im}\,(pm)$ of a normal monomorphism m along a regular epimorphism p is always a normal monomorphism (condition (SA*6) in [JMT02]).

We need the following lemma; see [Bou01, Proposition 7] or [BB04].

Lemma 2.2. *In a homological category, given a commutative diagram*

$$\begin{tikzcd} K[f] \arrow[r, "\mathsf{Ker}\, f"] \arrow[d, "k"'] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\ K[f'] \arrow[r, "\mathsf{Ker}\, f'"'] & A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

where f and f' are regular epimorphisms, the morphism k is an isomorphism if and only if the right hand side square $bf = f'a$ is a pullback. ∎

2.3. Commuting kernels implies isomorphic kernels. Using this lemma we may show that when the kernels k and l of the morphisms d and c in a reflexive graph $C = (C_1, C_0, d, c, e)$ commute, their domains are isomorphic.

Lemma 2.4. *Let k and l be induced by a reflexive graph C in a homological category as above. If k and l commute then the following commutative squares are pullbacks.*

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[d, "\varphi"'] & X \arrow[d, "h=ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\varphi"'] & Y \arrow[d, "dl"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd}$$

This makes X and Y isomorphic in a strong sense: there exist morphisms $i\colon X \to Y$ and $j\colon Y \to X$ such that

$$ji = 1_X, \qquad ij = 1_Y, \qquad ckj = dl \qquad \textit{and} \qquad ck = dli.$$

Proof: The left hand side diagram commutes because $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly (strongly) epimorphic and moreover $c\varphi\langle 1_X, 0\rangle = ck = ck\pi_X\langle 1_X, 0\rangle$ and $c\varphi\langle 0, 1_Y\rangle = cl = 0 = ck\pi_X\langle 0, 1_Y\rangle$. It is a pullback by Lemma 2.2 since the induced morphism between the kernels of π_X and c is 1_Y. Similarly the right hand side square is a pullback.

The morphism $i\colon X \to Y$ is obtained through the universal property of the first pullback as follows. The equality $ceck = ck = h1_X$ gives rise to a morphism $\iota\colon X \to X \times Y$ such that $\varphi\iota = eck$ and $\pi_X\iota = 1_X$; considering $X \times Y$ as a product now, this ι is a couple $\langle 1_X, i\rangle\colon X \to X \times Y$. Clearly,

$$dli = dl\pi_Y\langle 1_X, i\rangle = d\varphi\langle 1_X, i\rangle = deck = ck.$$

Using the second pullback one obtains a morphism $j\colon Y \to X$ that satisfies $\varphi\langle j, 1_Y\rangle = edl$, so that $ckj = dl$.

Now we only have to prove that i and j are each other's inverse. This again follows from the universal properties of the pullbacks. Indeed, the maps $\langle j, ij\rangle\colon Y \to X \times Y$ and $\langle j, 1_Y\rangle\colon Y \to X \times Y$ are both universally induced by the equality $cedl = ckj = hj$, hence they are equal. Likewise, $\langle 1_X, i\rangle$ is equal to $\langle ji, i\rangle$ so that $ji = 1_X$. ∎

2.5. Star-multiplicative graphs. A reflexive graph $C = (C_1, C_0, d, c, e)$ is **star-multiplicative** [Jan03] when there is a (necessarily unique) morphism

$$\varsigma\colon C_1 \times_{C_0} X \to X$$

such that $\varsigma\langle k, 0\rangle = 1_X$ and $\varsigma\langle eck, 1_X\rangle = 1_X$. Here the square

$$\begin{tikzcd} C_1 \times_{C_0} X \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & X \arrow[d, "h=ck"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd}$$

is a pullback. A star-multiplication takes a composable pair of arrows

$$\cdot \xleftarrow{\alpha} \cdot \xleftarrow{\beta} 0$$

and sends it to their composite $\varsigma(\alpha, \beta)$.

2.6. Peiffer graphs. A reflexive graph $C = (C_1, C_0, d, c, e)$ is **Peiffer** [MM09] when there is a (necessarily unique) morphism

$$\omega\colon X \times X \to C_1$$

such that $\omega\langle 1_X, 0\rangle = k$ and $\omega\langle 1_X, 1_X\rangle = eck$. The structure ω sends a composable pair of arrows

$$\cdot \xleftarrow{\alpha} 0 \xrightarrow{\beta} \cdot$$

to the composite $\omega(\alpha, \beta)$—which should be considered as $\alpha \circ \beta^{-1}$.

In [MM09] these two structures are shown to be equivalent; we recall the argument.

Proposition 2.7. *A reflexive graph $C = (C_1, C_0, d, c, e)$ in a pointed protomodular category is star-multiplicative if and only if it is Peiffer.*

Proof: Given $\varsigma\colon C_1 \times_{C_0} X \to X$ put $\omega = \pi_0\langle \varsigma, \pi_1\rangle^{-1}$; given $\omega\colon X \times X \to C_1$ put $\varsigma = \pi_0\langle \omega, \pi_1\rangle^{-1}$. Notations are as above. The inverse morphisms exist by the Split Short Five Lemma. ∎

Now we work towards an equivalence with reflexive graphs of which the kernel of the domain map commutes with the kernel of the codomain map. In Lemma 2.10 we need the surrounding category to be semi-abelian.

Lemma 2.8. *Let $g\colon X \times X \to A$ be a morphism with $g\langle 0, 1_X\rangle = 0$ and write $g_0 = g\langle 1_X, 0\rangle$. Then $g = g_0\pi_0$, so that $g\langle 1_X, 1_X\rangle = g_0$.*

Proof: The morphism g is uniquely determined by the equalities $g\langle 0, 1_X\rangle = 0$ and $g\langle 1_X, 0\rangle = g_0$. Since also $g_0\pi_0\langle 0, 1_X\rangle = 0$ and $g_0\pi_0\langle 1_X, 0\rangle = g_0$ we have that $g = g_0\pi_0$. ∎

Lemma 2.9. *For any Peiffer graph C, the morphism c is the cokernel of the composite $\omega\langle 0, 1_X\rangle$.*

Proof: Consider $f\colon C_1 \to A$ such that $f\omega\langle 0, 1_X\rangle = 0$; we claim that the morphism $fe\colon C_0 \to A$ satisfies $fec = f$. Indeed, by Lemma 2.8 the equalities $f\omega\langle 0, 1_X\rangle = 0$ and $f\omega\langle 1_X, 0\rangle = fk$ imply $f\omega\langle 1_X, 1_X\rangle = fk$, so that $feck = fk$. Since also $fece = fe$ and k and e are jointly epic we may conclude that $fec = f$. ■

Lemma 2.10. *For any Peiffer graph C in a semi-abelian category the induced commutative squares*

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] \arrow[dr, phantom, "\texttt{(i)}"] & X \arrow[d, "h=ck"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "\pi_0"] \arrow[d, "\omega"'] \arrow[dr, phantom, "\texttt{(ii)}"] & X \arrow[d, "h=ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

are pullbacks.

Proof: Both squares commute because the morphisms $\langle 1_X, 0\rangle$ and $\langle 1_X, 1_X\rangle$ are jointly epic and

$$\begin{aligned} d\omega\langle 1_X, 0\rangle &= dk = 0 = h\pi_1\langle 1_X, 0\rangle, \\ d\omega\langle 1_X, 1_X\rangle &= deck = ck = h\pi_1\langle 1_X, 1_X\rangle, \\ c\omega\langle 1_X, 0\rangle &= ck = 0 = ck\pi_0\langle 1_X, 0\rangle \end{aligned}$$

and $c\omega\langle 1_X, 1_X\rangle = ceck = ck = h\pi_0\langle 1_X, 1_X\rangle$. Taking kernels horizontally in `(i)` induces the identity morphism 1_X; hence the square is a pullback by Lemma 2.2. The rest of the proof is devoted to showing that `(ii)` is also a pullback.

Taking kernels vertically gives rise to the reflexive graph

$$\begin{tikzcd} K[\omega] \arrow[r, shift left=2, "\pi'_0"] \arrow[r, shift right=2, "\pi'_1"'] & K[h] \arrow[l, "\Delta" description] \end{tikzcd};$$

Since `(i)` is a pullback, the morphism π'_1, and hence also π'_0, is an isomorphism. It follows by Lemma 2.2 that the top square in the vertical regular epi-mono factorisation

$$\begin{tikzcd}
X \arrow[r, dotted, "\langle 0,1_X\rangle"] \arrow[d, dotted, no head, equal] & X \times X \arrow[r, "\pi_0"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads] \\
X \arrow[r, dotted, "\mathsf{Ker}\,\overline{c}"] \arrow[d, dotted, "i"'] \arrow[dr, phantom, "\texttt{(iii)}"] & I[\omega] \arrow[r, "\overline{c}"] \arrow[d, tail, "\mathsf{Im}\,\omega"] & I[h] \arrow[d, tail, "\mathsf{Im}\,h"] \\
Y \arrow[r, dotted, "\mathsf{Ker}\,c"'] & C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

of `(ii)` is a pullback. Taking kernels to the left induces morphisms as indicated. We have to show that i is an isomorphism.

Being a composite $h = ck$ of a kernel with a regular epimorphism, the morphism h is proper, i.e., its image $\mathsf{Im}\,h$ is a kernel. Since the square `(i)` is a pullback, ω is also proper, so that $\mathsf{Im}\,\omega$ is a kernel. The morphism $\mathsf{Im}\,h$ being mono implies that the square `(iii)` is a pullback. Since both $\mathsf{Im}\,\omega$ and $\mathsf{Ker}\,c$ are kernels, this implies that the diagonal of `(iii)`—the morphism $\omega\langle 0, 1_X\rangle$—is also a kernel. Lemma 2.9 tells us that c is its cokernel, so that $\omega\langle 0, 1_X\rangle$ is the kernel of c. This means that i is an isomorphism, and the square `(ii)` is a pullback by Lemma 2.2. ■

Proposition 2.11. *For a reflexive graph $C = (C_1, C_0, d, c, e)$ in a semi-abelian category, the following three conditions are equivalent:*

1. *C is star-multiplicative;*
2. *C is Peiffer;*
3. $\mathsf{Ker}\,d$ *and* $\mathsf{Ker}\,c$ *commute.*

Proof: Conditions (1) and (2) are equivalent by Proposition 2.7. If C is Peiffer then $\mathsf{Ker}\,d$ and $\mathsf{Ker}\,c$ commute. Indeed, by Lemma 2.10 we can put $\varphi = \omega$ since $\omega\langle 0, 1_X\rangle$ is the kernel l of c. Conversely, if Condition (3) holds then by Lemma 2.4 we have

$$\iota = \langle 1_X, i\rangle \colon X \to X \times Y$$

such that $\varphi\iota = eck$. Now $\omega = \varphi(1 \times i)\colon X \times X \to C_1$ is a Peiffer structure on C because $\omega\langle 1_X, 0\rangle = \varphi(1_X \times i)\langle 1_X, 0\rangle = \varphi\langle 1_X, 0\rangle = k$ and $\omega\langle 1_X, 1_X\rangle = \varphi(1_X \times i)\langle 1_X, 1_X\rangle = \varphi\iota = eck$. ■

Theorem 2.12. *For a semi-abelian category, the following conditions are equivalent:*

(SM) *every star-multiplicative graph is multiplicative;*

(SH) *two (effective) equivalence relations commute if and only if their normalisations commute.*

Proof: We already explained above that one implication of (SH) always holds by [BB04, Proposition 2.7.7]. Hence by Theorem 1.4 we may replace the second condition with

(SH') *every reflexive graph of which the kernels of the domain and the codomain morphisms commute is a groupoid.*

The result now follows from Proposition 2.11 and the fact that in a semi-abelian category, multiplicative graphs (i.e., categories) and groupoids coincide. ∎

Note that only in Lemma 2.10 we use that the underlying category is semi-abelian rather than homological. In the homological context, this suggests a modification of the concept of Peiffer graph, where the pullback property of square `(ii)` in Lemma 2.10 becomes an axiom. (Or, equivalently, the morphism $\omega\langle 0, 1_X\rangle$ is demanded to be a kernel.) The concept of star-multiplicative graph allows a similar modification, where now one asks that the morphism of reflexive graphs

$$\begin{tikzcd} C_1 \times_{C_0} X \arrow[r, "\pi_0"] \arrow[d, shift right=2, "\pi_1"'] \arrow[d, shift left=2, "\varsigma"] & C_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ X \arrow[u] \arrow[r, "h"'] & C_0 \arrow[u] \end{tikzcd}$$

is not just a discrete cofibration (i.e., the square $h\pi_1 = d\pi_0$ is a pullback) but also a discrete fibration ($h\varsigma = c\pi_0$ is a pullback). These definitions extend Theorem 2.12 to homological categories.

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