Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 09–48

THE CHANGE IN EIGENVALUE MULTIPLICITY ASSOCIATED WITH PERTURBATION OF A DIAGONAL ENTRY OF THE MATRIX

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ABSTRACT: Here we investigate the relation between perturbing the i-th diagonal entry of $A \in \mathcal{M}_n(\mathbb{F})$ and extracting the principal submatrix $A(i)$ from A with respect to the possible changes in multiplicity of a given eigenvalue. A complete description is given and used to both generalize and improve prior work about Hermitian matrices whose graph is a given tree.

KEYWORDS: Eigenvalue multiplicity, submatrices, Hermitian matrices, graph.

In a series of papers investigating the possible multiplicity lists for the eigenvalues among Hermitian matrices whose graph is a given tree, considerable information relating the structure of the tree to the change in the multiplicity of an eigenvalue when passing to a principal submatrix has been generated ([13], [14], [3], [5], [8], [6], [9], [11], [10]). Our purpose here is several-fold: (1) to focus on a single multiple eigenvalue, but in the very general setting of square matrices over a field; (2) to understand the relationship between change in multiplicity with change in a diagonal entry and change in multiplicity when passing to a principal submatrix; and (3) to apply these observations to get further, or different, insights in the Hermitian/tree case. Our basic results generalize certain aspects of the Hermitian case. Beyond what we do here, we suspect that these observations may be powerful tools for further generalization. A priori, there are substantial differences in the general matrix case. For example, when passing to a principal submatrix of size one smaller a multiplicity may change by more than one, in contrast to the Hermitian case; the same distinction occurs when there is a change in the value of a diagonal entry. This alone makes a substantial difference in the strength of the statements that can be made. Nonetheless, remarkable parts of the Hermitian/tree case carry over.

All our results will be about algebraic multiplicity, and we use polynomial methods. We begin with a simple observation about polynomials that will be used repeatedly. We note two important aspects of the way we present

Received November 26, 2009.

all the results. First, we imagine that the polynomial roots or eigenvalues in question lie in the given field from which the coefficients or entries are chosen. This emphasizes that if a value of a diagonal entry is to be chosen to achieve a certain change, the value can be chosen from the same field, but there is no loss of generality, as the results may be applied to an extension field in which the eigenvalue lies and the matrix entries still lay. Second, when a value is not, by hypothesis, confined to the ground field, it may be chosen completely freely (in any field extension) and the statement is still valid.

We employ the standard matrix notation in which $A(i) \in \mathcal{M}_{n-1}(\mathbb{F})$ denotes the principal submatrix of $A \in \mathcal{M}_n(\mathbb{F})$ after deletion of the i-th row and column, while $\mathbb{F}$ denotes a general field; $E_{ii} \in \mathcal{M}_n(\mathbb{F})$ denotes the n-by-n matrix with a 1 in the (i, i) position and zeros elsewhere. We also use $p_A(x)$ for the characteristic polynomial, $\det(xI - A)$, of the matrix A and $\mathrm{m}_A(\lambda)$ or $\mathrm{m}_p(\lambda)$ to indicate the multiplicity of λ as an eigenvalue of $A \in \mathcal{M}_n(\mathbb{F})$ or a root of the polynomial $p(x) \in \mathbb{F}[x]$.

Leading to one of our main results, we present three lemmata, one dealing with each of three cases in which $\mathrm{m}_{A(i)}(\lambda)\ <,\ =,\ >\ \mathrm{m}_A(\lambda)$. Note that all three can occur, $|\mathrm{m}_A(\lambda) - \mathrm{m}_{A(i)}(\lambda)|$ may be any nonnegative integer and that we allow that either $\mathrm{m}_A(\lambda)$ or $\mathrm{m}_{A(i)}(\lambda)$ be 0 or 1. We note that to support a proof of Theorem 5 to come, only a much weaker version of each of Lemmas 2, 3, 4 is necessary (essentially just the forward implication of each), but we give the more complete version of each for clarity.

Lemma 1. *Let* $0 \not\equiv p_1, p_2 \in \mathbb{F}[x]$, *with* $\mathbb{F}$ *a field. If* $\lambda \in \mathbb{F}$, $\mathrm{m}_{p_1}(\lambda) = m \geqslant 0$, $\mathrm{m}_{p_2}(\lambda) \geqslant m$ *and* $p_a(x) = (x - a)p_1(x) + p_2(x)$, *then* $\mathrm{m}_{p_a}(\lambda) = m$ *for all* a, *except for a unique* $a_0 \in \mathbb{F}$ *for which* $\mathrm{m}_{p_{a_0}}(\lambda) > m$.

Proof. Obviously, by a divisibility argument, we have $\mathrm{m}_{p_a}(\lambda) \geqslant m$, for all $a \in \mathbb{F}$. Let $p_i(x) = (x - \lambda)^m q_i(x)$, $i = 1, 2, a$. Then

$$q_a(x) = (x - a)q_1(x) + q_2(x)\,.$$

Now $\mathrm{m}_{p_a}(\lambda) > m$ if and only if $q_a(\lambda) = 0$, that is, if and only if

$$a = \lambda + \frac{q_2(\lambda)}{q_1(\lambda)}$$

(which is an element of $\mathbb{F}$), and so this is the only element a_0 for which $\mathrm{m}_{p_{a_0}}(\lambda) > m$. ∎

We note that $\mathrm{m}_{p_a}(\lambda)-m$ may be arbitrarily large, as with the remark after Lemma 4, which also shows that the decrease in multiplicity in Lemma 3 may be arbitrarily large.

First, we consider the case in which $\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$.

Lemma 2. *Suppose that $A \in \mathcal{M}_n(\mathbb{F})$, that i, $1 \leqslant i \leqslant n$, is an index and that $\lambda \in \mathbb{F}$. Then,*

$$\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$$

if and only if

$$\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_A(\lambda)$$

for all t.

Proof. Using the Laplace expansion of the characteristic polynomial of A, we have

$$p_A(x) = (x - a_{ii})p_{A(i)}(x) + q(x), \tag{1}$$

in which $q(x)$ is a certain polynomial of degree $n-2$. Using the same expansion for the characteristic polynomial of $A + tE_{ii}$, $t \in \mathbb{F}$, we obtain

$$p_{A+tE_{ii}}(x) = (x - a_{ii} - t)p_{A(i)}(x) + q(x) = p_A(x) - tp_{A(i)}(x). \tag{2}$$

Suppose that $\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$. The hypothesis and a divisibility argument applied to equation (2), $\mathrm{m}_{A+tE_{ii}}(\lambda) \geqslant \mathrm{m}_A(\lambda)$. But, if $\mathrm{m}_{A+tE_{ii}}(\lambda) > \mathrm{m}_A(\lambda)$, then $\mathrm{m}_A(\lambda) \geqslant \min\left\{\mathrm{m}_{A+tE_{ii}}(\lambda), \mathrm{m}_{A(i)}(\lambda)\right\} > \mathrm{m}_A(\lambda)$, a contradiction. So $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_A(\lambda)$, which proves the necessity.

Consider sufficiency. Since $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_A(\lambda)$, a divisibility argument applied to (2) implies that λ is also a root of $p_{A(i)}(x)$ with multiplicity at least $\mathrm{m}_A(\lambda)$, i.e., $\mathrm{m}_{A(i)}(\lambda) \geqslant \mathrm{m}_A(\lambda)$. Suppose that $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)$. By a divisibility argument applied to (1) we have that $\mathrm{m}_q(\lambda) \geqslant \mathrm{m}_A(\lambda)$. In that case, by Lemma 1, there would exist a (unique) $t_0 \in \mathbb{F}$ such that $\mathrm{m}_{A+t_0E_{ii}}(\lambda) > \mathrm{m}_A(\lambda)$, a contradiction. Therefore we have $\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$. ■

We note that indices of the sort described in Lemma 2 are known to always exist in case A is Hermitian, its graph is a tree and $\mathrm{m}_A(\lambda) > 1$ [5, 13, 14]. Otherwise, there need not be such indices [4], but, of course, they may occur, in situations beyond the Hermitian/tree case.

Next we consider the case in which the multiplicity declines when passing to the principal submatrix.

Lemma 3. *Suppose that $A \in \mathcal{M}_n(\mathbb{F})$ that i, $1 \leqslant i \leqslant n$, is an index and that $\lambda \in \mathbb{F}$. Then,*

$$\mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$$

if and only if

$$\mathrm{m}_{A+tE_{ii}}(\lambda) < \mathrm{m}_A(\lambda)$$

for all $t \neq 0$.

Moreover, in this event, $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_{A(i)}(\lambda)$ $(< \mathrm{m}_A(\lambda))$ for any $t \neq 0$, so that for any $t \neq 0$, the decline in multiplicity when passing from A to $A + tE_{ii}$ is the same.

Proof. We show first that, for $t \neq 0$, both: (a) $\mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$ implies $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_{A(i)}(\lambda)$ and (b) $\mathrm{m}_{A+tE_{ii}}(\lambda) < \mathrm{m}_A(\lambda)$ implies $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_{A(i)}(\lambda)$, from which the entire statement of the lemma follows.

For (a), a divisibility argument applied to (1) gives $\mathrm{m}_q(\lambda) = \mathrm{m}_{A(i)}(\lambda)$. Then, Lemma 1, applied to (2), shows that, for any $t \neq 0$ ($t = 0$ being the only exception in Lemma 1), $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_{A(i)}(\lambda)$.

For (b), Lemma 1, applied to (2), shows that $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_{A(i)}(\lambda)$, with the lone exception of $t = 0$. ■

Finally, we consider the case in which the multiplicity does not change when passing to the principal submatrix.

Lemma 4. *Suppose that $A \in \mathcal{M}_n(\mathbb{F})$, that i, $1 \leqslant i \leqslant n$, is an index, and $\lambda \in \mathbb{F}$. Then,*

$$\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)$$

if and only if

$$\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_A(\lambda)$$

for all t, except for a unique $t_0 \in \mathbb{F}$ for which $\mathrm{m}_{A+t_0E_{ii}}(\lambda) > \mathrm{m}_A(\lambda)$.

Proof. Suppose that $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)$. A divisibility argument applied to (1) shows that $\mathrm{m}_q(\lambda) \geqslant \mathrm{m}_A(\lambda)$. By Lemma 1, applied to equation (2), there is a unique $t_0 \in \mathbb{F}$, obviously $t_0 \neq 0$, for which $\mathrm{m}_{A+t_0E_{ii}}(\lambda) > \mathrm{m}_A(\lambda)$.

To prove sufficiency, we show that neither $\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$ nor $\mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$ can occur. If we suppose that $\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$, by Lemma 2, we would have, for any $t \in \mathbb{F}$, $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_A(\lambda)$, which is a contradiction. If we suppose that $\mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$, by Lemma 3, we would have, for any $0 \neq t \in \mathbb{F}$, $\mathrm{m}_{A+tE_{ii}}(\lambda) = \mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$ which is again a contradiction. Therefore, we have $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)$, completing the proof. ■

An analysis of the proofs of Lemmas 2 to 4 shows that if $\mathbb{F}_1$ is a subfield of $\mathbb{F}$ and $p_A(x)$, $p_{A(i)}(x) \in \mathbb{F}_1[x]$, $\lambda \in \mathbb{F}_1$ then the reverse implication of each of the lemmas is still true with t restricted to $\mathbb{F}_1$ (because we still can apply Lemma 1); in this case we have $t_0 \in \mathbb{F}_1$ in Lemma 4. This situation occurs for Hermitian matrices over $\mathbb{C}$, with $\mathbb{F}_1 = \mathbb{R}$. We note also that, in each of the Lemmas 2 to 4, the parameter t may be taken to be unrestricted in the forward implication and need only to be assumed in $\mathbb{F}$ (or $\mathbb{F}_1$) for the reverse implication.

Each of Lemmas 2, 3, 4 has a natural and interesting specialization to the case in which $A \in \mathcal{M}_n(\mathbb{C})$ is Hermitian. Then,

$$\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$$

if and only if

$$\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1,$$

and

$$\mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$$

if and only if

$$\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) - 1\,,$$

because of the interlacing [1, Chap 4] ($\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)$ may still occur). We do not state these, but they may be referred to as Lemma 2(Hermitian), Lemma 3(Hermitian) and Lemma 4(Hermitian). The statements should be clear, except that we note that in Lemma 4(Hermitian), the very last part would be $\mathrm{m}_{A+t_0E_{ii}}(\lambda) = \mathrm{m}_A(\lambda) + 1$, and $t_0 \in \mathbb{R}$.

By analogy with the Hermitian case, we give names for each of the above three cases. In the general setting, we say that an index i, $1 \leqslant i \leqslant n$, is *Parter* for λ in A if $\mathrm{m}_{A(i)}(\lambda) > \mathrm{m}_A(\lambda)$ (as in Lemma 2). Similarly, i is *neutral* (respectively, a *downer*) for λ in A if $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)$ (respectively, $\mathrm{m}_{A(i)}(\lambda) < \mathrm{m}_A(\lambda)$). If the amount of change is of interest, we say, for example, that i is Parter for λ in A of order k if $\mathrm{m}_{A(i)}(\lambda) - \mathrm{m}_A(\lambda) = k$. Of course, if A is Hermitian, "Parter" implies "Parter of order 1".

The increase mentioned in Lemma 4 could be bigger than 1 in the general case. Since this is the only situation in which multiplicity may increase (or increase by more than 1) when passing from A to $A+tE_{ii}$, we give an example of what may occur.

Example. The increase in multiplicity due to the unique change in one diagonal entry corresponding to a neutral index can be arbitrarily large in general. An n-by-n matrix of the form

$$A = \begin{bmatrix} w-t & y^\top \\ z & D \end{bmatrix}$$

with D diagonal, with distinct nonzero diagonal entries, and $w = \operatorname{tr} D$ and such that $A + tE_{11}$ is nilpotent gives an example such that $\mathrm{m}_A(0) = 0$ but $\mathrm{m}_{A+tE_{11}}(0) = n$. According to [12], y and z may be chosen to produce such an A for any such D.

The three Lemmas 2, 3 and 4 may be combined into a single summary statement, a main result. Of course, there is also a Hermitian specialization.

Theorem 5. *Suppose that $A \in \mathcal{M}_n(\mathbb{F})$, that i, $1 \leqslant i \leqslant n$, is an index and $\lambda \in \mathbb{F}$. Then, i is Parter (respectively, downer, neutral) for λ in A if and only if*

$$\mathrm{m}_{A+tE_{ii}}(\lambda) = (\textit{ respectively } <, =)\ \mathrm{m}_A(\lambda)$$

for all t (respectively, all t except $t = 0$, for all t except for a unique $t_0 \in \mathbb{F}$ for which we have $>$).

Now, to make some applications of Theorem 5, we give another lemma that may be of independent interest.

Lemma 6. *If $A \in \mathcal{M}_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$ is an eigenvalue of A, then there is a downer index for λ in A.*

Proof. Since $p'_A(x) = \sum_{i=1}^n p_{A(i)}(x)$ [1, p. 43] and $\mathrm{m}_{p'_A}(\lambda) = \mathrm{m}_A(\lambda) - 1$, $(x-\lambda)^{\mathrm{m}_A(\lambda)}$ cannot divide $\sum p_{A(i)}(x)$. But, if each index is either neutral or Parter, $(x-\lambda)^{\mathrm{m}_A(\lambda)}$ would divide each $p_{A(i)}(x)$ and thus the sum. This contradiction means that there must be at least one downer. ■

Remarks. Another (longer) proof of Lemma 6 may be given by applying ideas about compound matrices [1] to the characteristic matrix $xI - A$. In case A is Hermitian with graph a tree, there are at least two downer indices for any eigenvalue. In general, for an eigenvalue of multiplicity one, there may be only one, as in the case of a diagonal matrices. When $\mathrm{m}_A(\lambda) > 1$, must there be at least two downer indices (or more under certain circumstances)? In general, the answer is "no", as is shown by the example in $\mathcal{M}_3$:

$$A = \begin{bmatrix} a & a & -a \\ -a & -a & a \\ -a & -a & a \end{bmatrix}, \quad a \neq 0,$$

in which $\mathrm{m}_A(0) = 2$ since $\operatorname{rank} A = 1$ and $\operatorname{tr} A = a$. Since $\mathrm{m}_{A(1)}(0) = \mathrm{m}_{A(3)}(0) = 2$, index 2 is the only downer ($\mathrm{m}_{A(2)}(0) = 1$).

For the Hermitian case we have the following result.

Theorem 7. *If $A \in \mathcal{M}_n(\mathbb{C})$ ($n > 1$) is Hermitian and $\lambda \in \mathbb{R}$ and either $\mathrm{m}_A(\lambda) > 1$ or $\mathrm{m}_A(\lambda) = 1$ and A is irreducible, then there are at least two downer indices.*

Proof. It follows from [2, Corollary 2] that if there were zero or one downer index then any eigenvector of A associated with λ will have, respectively, all components zero (which is impossible) or all but one component zero; in this last event A will be reducible and if $\mathrm{m}_A(\lambda) > 1$ then one the direct summands of A will have λ as an eigenvalues; this will allows us to find another downer index in this summand which will be also a downer index of the original matrix A contradicting the supposition that there were just one such index. ■

One of the uses of Lemma 6 is to show that if λ is a highly multiple eigenvalue of A, then there are matrices A' that agree with A off the diagonal and have λ as an eigenvalue with lower multiplicity. Typically, only a few diagonal entries need be changed. If A has a downer index of order 1, then a change in one diagonal entry to produce A' will lower the multiplicity by 1. If there is another such index, we may lower the multiplicity by 1, etc., so that all lower multiplicities will be achieved. This will be the case if A is Hermitian, as the only possible order for a downer index is 1. This gives the following result.

Theorem 8. *If $A \in \mathcal{M}_n(\mathbb{C})$ is Hermitian, $\lambda \in \mathbb{R}$ is an eigenvalue of A and $\mathrm{m}_A(\lambda) = m > 1$, then for any $0 \leqslant l \leqslant m$, there exist $k = m - l$ diagonal entries that may be changed to produce A' for which $\mathrm{m}_{A'}(\lambda) = l$.*

We note that, according to Theorem 5, if $A \in \mathcal{M}_n(\mathbb{F})$ has a Parter index for $\lambda \in \sigma(A)$, then any change in the diagonal entry corresponding to that index gives a new matrix for which the same index is Parter for λ. If a diagonal entry associated with a neutral index is changed so as to increase the multiplicity (a unique change does this), then that index becomes a downer. (A change other than the unique one will leave the index neutral.) Similarly, any change in a downer makes the index become neutral. It is more subtle what changes in a diagonal entry can do to the status of other indices.

Since, for a tree T, diagonal entries only can be changed to produce maximum multiplicity among Hermitian matrices with graph T, we have a stronger result.

Let G be an undirected graph on n vertices and denote by $\mathcal{H}(G)$ the collection of Hermitian matrices whose graph is G. Further, let $M(G)$ be the maximum multiplicity of an eigenvalue among matrices in $\mathcal{H}(G)$. When the graph is a tree T, $M(T)$ has been characterized as the path cover number in [3].

Corollary 9. *Let T be a tree, $B \in \mathcal{H}(T)$ and $M = M(T)$. Then for any $0 \leqslant m \leqslant M$ and any $\lambda \in \mathbb{R}$, there is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ such that $\mathrm{m}_{B+D}(\lambda) = m$.*

Proof. To apply Theorem 8, we need only show that there is a real diagonal perturbation A of B for which $\mathrm{m}_A(\lambda) = M(T)$. This may be done using an idea of [3]. If q vertices are removed from T to leave p paths so that $p - q = M(T)$ and for the submatrix corresponding to each path, diagonal entries are changed (e.g. via translation), if necessary, to place λ on the path. Then, reconstructing the tree will produce a new A with $\mathrm{m}_A(\lambda) = M(T)$. ■

Another proof of Corollary 8 may be given via a translation and diagonal congruence/ rank argument. It is an open conjecture that any *possible* (unordered) multiplicity list for a tree may be attained for any off-diagonal entries.

Bringing multiplicities down via diagonal perturbation is always possible because of the existence of downer indices. The reverse is, not surprisingly, not always possible, as there may not be neutral indices. The nonexistence of neutral vertices is the condition for a certain sort of local maximum in multiplicity. (Of course, neutral indices may, or may not occur.)

The following result is a direct consequence of Theorem 5, as the only way that a single diagonal perturbation can increase multiplicity is if neutral indices exist.

Corollary 10. *If $A \in \mathcal{M}_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$, then $\mathrm{m}_A(\lambda)$ can be increased by change of a single diagonal entry of A if and only if there is an index that is neutral for λ in A.*

Corollary 10 allows us to give a very different proof of a generalization of a result of [7].

Corollary 11. *Let G be any graph and let $A \in \mathcal{H}(G)$ and $\lambda \in \sigma(A)$ be such that* $\mathrm{m}_A(\lambda) = M(G)$. *Then no vertex of G is neutral in G for λ and A.*

Proof. If there were a neutral vertex for λ in A, then $\mathrm{m}_A(\lambda)$ could be increased, which is not possible if it is already a maximum. ■

We close by noting that there are other circumstances in which there can be no neutral vertices. For example, if there is an eigenvalue λ such that $\mathrm{m}_A(\lambda) = M(G) = m_1$ and another eigenvalue μ such that $\mathrm{m}_A(\mu) = m_2$, which is a maximum given m_1, then if multiplicities $m_1 - 1$ and $m_2 + 1$ cannot occur (there are examples, see [8, § 5]), there can be no neutral vertices for μ in T given A. Other such situations can be identified. It would be of interest to characterize all situations for a given graph in which neutral vertices cannot occur.

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