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ON REGULAR AND HOMOLOGICAL CLOSURE OPERATORS

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

ABSTRACT: Observing that weak heredity of regular closure operators in Top and of homological closure operators in homological categories identifies torsion theories, we study these closure operators in parallel, showing that regular closure operators play the same role in topology as homological closure operators do algebraically.

KEYWORDS: regular closure operator, homological closure operator, maximal closure operator, torsion theory.
AMS SUBJECT CLASSIFICATION (2000): 18E10, 18E40, 54B30, 18B30, 18A32.

0. Introduction

Homological categories were introduced recently by Borceux and Bourn [2], and since then studied by several authors, as the right non-abelian setting to study homology. As shown by Bourn and Gran [6], these categories are also a suitable setting to study torsion theories. In [6] the authors introduce torsion theories in homological categories and show that they are identifiable by weak heredity of their homological closure operators. This result resembles the characterization of disconnectednesses of topological spaces via weak heredity of their regular closure operators, and encompasses the characterization of torsion-free subcategories of abelian categories via weak heredity of their regular closure operators obtained in [7] (see also [13]). Having as starting point this common property, we establish parallel properties of regular and homological closure operators, in topological spaces and in homological categories, respectively. Since regular closure operators are exactly the homological ones in abelian categories, this study raises the question of knowing in which cases these closure operators coincide in homological categories. We solve this question showing that they are the same exactly when they are induced by a regular-epireflective subcategory of abelian objects.

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In Section 1 we describe briefly regular and homological closures, defining also maximal closure operators, which include homological closure operators in homological categories. In Section 2 we establish parallel results for regular and homological closures, based in the results obtained by Bourn and Gran in [6]. In Theorem 2.5 we show the validity of the corresponding topological version of the characterization of hereditary torsion theories via hereditary homological closure operators. Next we investigate openness and closedness of regular epimorphisms, with respect to the regular closure, showing in Propositions 2.7 and 2.9 that these properties are unlikely topological. Finally we characterize in Corollary 2.12 the regular-epireflective subcategories of homological categories for which the regular and the homological closures coincide, generalising the result of [14] stating that in abelian categories regular and homological closure operators coincide.

Further analysis of a common setting to study these closure operators will be included in a forthcoming paper [9].

1. Regular and homological closure operators.

Throughout $\mathbf{C}$ is a finitely complete category with cokernel pairs, and $\mathcal{M}$ is a left-cancellable and pullback-stable class of monomorphisms of $\mathbf{C}$; that is,

- if $n \in \mathcal{M}$ and $n \cdot a \in \mathcal{M}$, also $a \in \mathcal{M}$;
- if $n : N \to Y \in \mathcal{M}$, then its pullback $f^{-1}(n) : f^{-1}(N) \to X$ along $f : X \to Y$ belongs to $\mathcal{M}$.

For any object X of $\mathbf{C}$, we consider in the class of morphisms of $\mathcal{M}$ with codomain X the preorder $\leq$ defined by $m \leq n$ if m factors through n; that is, there exists a (necessarily in $\mathcal{M}$) morphism j such that

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "j"'] & & X \\ & N \arrow[ur, "n"'] & \end{tikzcd}$$

we write $m \cong n$ if $m \leq n$ and $n \leq m$.

A *closure operator* c on $\mathbf{C}$ with respect to $\mathcal{M}$ assigns to each $m : M \to X$ in $\mathcal{M}$ a morphism $c_X(m) : c_X(M) \to X$ in $\mathcal{M}$ such that, for every object X,

(C1) c_X is *extensive*: $m \leq c_X(m)$ for every $m : M \to X$ in $\mathcal{M}$;

(C2) c_X is *monotone*: $m \leq m' \;\Rightarrow\; c_X(m) \leq c_X(m')$, for every $m : M \to X$, $m' : M' \to X$ in $\mathcal{M}$;

(C3) morphisms are *c-continuous*: $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for every morphism $f : X \to Y$ and every $n : N \to Y$ in $\mathcal{M}$.

Extensivity of c says that every $m : M \to X \in \mathcal{M}$ factors as

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "j_m"'] & & X \\ & c_X(M) \arrow[ur, "c_X(m)"'] & \end{tikzcd}$$

The morphism $m : M \to X$ is *c-closed* if $c_X(m) \cong m$, and *c-dense* if $c_X(m) \cong 1_X$. A closure operator c is said to be

- *idempotent* if $c_X(m)$ is c-closed for every $m : M \to X \in \mathcal{M}$;
- *weakly hereditary* if j_m is c-dense for every $m \in \mathcal{M}$;
- *hereditary* if, for $m : M \to X$, $l : X \to Y$ and $l \cdot m : M \to Y$ in $\mathcal{M}$,

$$c_X(m) \cong l^{-1}(c_Y(l \cdot m)).$$

It is immediate that every hereditary closure operator is in particular weakly hereditary.

Closure operators with respect to $\mathcal{M}$ can be preordered by

$$c \leq d \;:\Leftrightarrow\; \forall m : M \to X \in \mathcal{M} \; c_X(m) \leq d_X(m).$$

For any such class $\mathcal{M}$ of monomorphisms containing the regular monomorphisms, every reflective subcategory $\mathbf{A}$ of $\mathbf{C}$ induces a *regular closure operator* $\mathrm{reg}^{\mathbf{A}}$ on $\mathbf{C}$ with respect to $\mathcal{M}$, assigning to each $m : M \to X$ in $\mathcal{M}$ the equaliser of the following diagram

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, "\rho_Y"] & RY \end{tikzcd} ,$$

where (u, v) is the cokernel pair of m and ρ_Y is the $\mathbf{A}$-reflection of Y; that is,

$$\mathrm{reg}^{\mathbf{A}}_X(m) = \mathrm{eq}(\rho_Y \cdot u, \rho_Y \cdot v).$$

Regular closure operators are idempotent but not weakly hereditary in general.

When the category $\mathbf{C}$ is pointed, replacing equalisers by kernels in the construction above gives rise to another interesting closure operator. Let $\mathcal{M}$ be a left-cancellable and pullback-stable class of monomorphisms containing the kernels, and let $\mathbf{A}$ be a reflective subcategory of $\mathbf{C}$. The *homological* *closure operator* $\mathrm{h}^{\mathbf{A}}$ *induced by* $\mathbf{A}$ in $\mathcal{M}$ assigns to each $m : M \to X$ the kernel of the following composition of morphisms

$$X \xrightarrow{\pi_M} Y \xrightarrow{\rho_Y} RY\ ,$$

where π_M is the cokernel of m and ρ_Y is the $\mathbf{A}$-reflection of Y; that is,

$$\mathrm{h}^{\mathbf{A}}_X(m) = \ker(\rho_Y \cdot \pi_M).$$

Homological closure operators are idempotent but not weakly hereditary in general.

While regular closure operators were introduced by Salbany [16] more than 30 years ago, and widely studied since then, homological closure operators were recently introduced by Bourn and Gran [6] in the context of homological categories.

A pointed category $\mathbf{C}$ is *homological* if it is

(1) *(Barr-)regular*, that is if it is finitely complete and (regular epimorphisms, monomorphisms) is a pullback-stable factorization system in $\mathbf{C}$, and
(2) *protomodular*, that is given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & B \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "2"] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

where the dotted vertical arrow is a regular epimorphism, if $\boxed{1}$ and the whole rectangle are pullbacks, then $\boxed{2}$ is a pullback as well.

In a homological category the homological closure operator induced by a regular-epireflective subcategory $\mathbf{A}$ can be equivalently described as the maximal closure operator induced by $\mathbf{A}$ we describe next. For any reflective subcategory $\mathbf{A}$ of a pointed category $\mathbf{C}$ there is a maximal closure operator c with $0_A : 0 \to A$ c-closed for every object A of $\mathbf{A}$. It is called the *maximal closure operator induced by* $\mathbf{A}$ and denoted by $\max^{\mathbf{A}}$.

For comprehensive accounts on closure operators and homological categories we refer the Reader to [13] and [2, 15] respectively.

2. How close are regular and homological closure operators.

The study of weak heredity of regular closure operators presented in [7] encompasses the following topological and algebraic results.

Theorem 2.1 ([7]). *For a regular-epireflective subcategory* $\mathbf{A}$ *of* **Top**, *the following assertions are equivalent:*

(i) $\mathrm{reg}^{\mathbf{A}}$ *is weakly hereditary;*
(ii) $\mathbf{A}$ *is a disconnectedness.*

Theorem 2.2 ([7]). *For a (regular-)epireflective subcategory* $\mathbf{A}$ *of an abelian category* $\mathbf{C}$, *the following conditions are equivalent:*

(i) $\mathrm{reg}^{\mathbf{A}}$ *is weakly hereditary;*
(ii) $\mathbf{A}$ *is a torsion-free subcategory.*

Disconnectedness in topological spaces, as defined by Arhangelskii [1], and torsion-free subcategories in abelian categories, as defined by Dickson [11], are particular cases of right-constant subcategories (see [10] for details), hence the two theorems above are instances of a more general result. Moreover, as shown in [14], if $\mathbf{C}$ is an abelian category, then the regular closure operator induced by an epireflective subcategory $\mathbf{A}$ coincides with the maximal closure operator induced by $\mathbf{A}$. This shows, moreover, that Theorem 2.2 is a particular case of the following result, due to Bourn and Gran [6].

Theorem 2.3 ([6]). *For a regular-epireflective subcategory* $\mathbf{A}$ *of a homological category* $\mathbf{C}$, *the following conditions are equivalent:*

(i) $\mathrm{max}^{\mathbf{A}}$ *is weakly hereditary;*
(ii) $\mathbf{A}$ *is a torsion-free subcategory.*

The two authors show also that heredity of $\mathrm{max}^{\mathbf{A}}$ identifies hereditary torsion theories, that is those torsion theories with hereditary torsion part.

Theorem 2.4 ([6]). *For a regular-epireflective subcategory* $\mathbf{A}$ *of a homological category* $\mathbf{C}$, *the following conditions are equivalent:*

(i) $\mathrm{max}^{\mathbf{A}}$ *is hereditary;*
(ii) $\mathbf{A}$ *is an hereditary torsion-free subcategory.*

As for weak heredity there is a corresponding result in topology.

Theorem 2.5. *For a regular-epireflective subcategory* $\mathbf{A}$ *of* **Top**, *the following conditions are equivalent:*

(i) $\mathrm{reg}^{\mathbf{A}}$ *is hereditary;*

(ii) $\mathbf{A}$ *is an hereditary disconnectedness (that is, its connectedness counterpart* $\mathrm{l}(\mathbf{A})$ *is hereditary);*

(iii) $\mathbf{A}$ *is either* **Top** *or the category* $\mathbf{Top}_0$ *of* T_0*-spaces or the category* **Sgl** *consisting of singletons and the empty set.*

Proof. First we prove that (ii)$\Leftrightarrow$(iii). If $\mathbf{A} = \mathbf{Top}$ then $\mathrm{l}(\mathbf{A}) = \mathbf{Sgl}$ is hereditary, and if $\mathbf{A} = \mathbf{Top}_0$ then $\mathrm{l}(\mathbf{A})$ is the category of indiscrete spaces, which is hereditary as well, while $\mathrm{l}(\mathbf{Sgl}) = \mathbf{Top}$ is trivially hereditary. To prove the converse let $\mathbf{A} \subset \mathbf{Top}_0$, which means $\mathbf{A} \subseteq \mathbf{Top}_1$ since any $\mathbf{A} \subseteq \mathbf{Top}_0$ containing a space which is not T_1 has the Sierpinski space $S = \{0, 1\}$, with $\{0\}$ the only non-trivial open subset, as a subspace and therefore contains every T_0-space. Hence the Sierpinski space S and its square $S \times S$ belong to $\mathrm{l}(\mathbf{A})$. However, the two points discrete space $D = \{(0, 1), (1, 0)\}$ is a subspace of $S \times S$, hence $D \in \mathrm{l}(\mathbf{A})$ if $\mathrm{l}(\mathbf{A})$ is hereditary, which reduces to $\mathbf{A} = \mathbf{Sgl}$.

(iii)$\Leftrightarrow$(i): If $\mathbf{A} = \mathbf{Top}$, then $\mathrm{reg}^{\mathbf{A}}$ is the discrete closure, which is trivially hereditary. If $\mathbf{A} = \mathbf{Top}_0$, then $\mathrm{reg}^{\mathbf{A}}$ is the b-closure, with, for $A \subseteq X$,

$$\mathrm{b}_X(A) = \{x \in X \,|\, \text{for every neighbourhood } U \text{ of } x, \overline{\{x\}} \cap U \cap A \neq \emptyset\},$$

which is known to be hereditary (see for instance [13]). If $\mathbf{A} = \mathbf{Sgl}$, then $\mathrm{reg}^{\mathbf{A}}$ is the indiscrete closure, that is

$$\mathrm{reg}_X^{\mathbf{Sgl}}(A) = X \text{ for every } \emptyset \neq A \subseteq X \text{ and } \mathrm{reg}_X^{\mathbf{Sgl}}(\emptyset) = \emptyset,$$

which is hereditary. Conversely, assume that $\mathbf{A}$ is none of these three subcategories. We use again $S \times S$ and the embedding $D = \{(0, 1), (1, 0)\} \to S \times S$. While the $\mathbf{A}$-reflection of $S \times S$ is a singleton, $D \in \mathbf{A}$. Hence $\mathrm{reg}_D^{\mathbf{A}}(0, 1) = (0, 1)$ while $\mathrm{reg}_{S\times S}^{\mathbf{A}}(0, 1) = S \times S$, and therefore $\mathrm{reg}^{\mathbf{A}}$ is not hereditary. $\square$

Another interesting feature of homological closure operators pointed out by Bourn and Gran [6] is to make regular epimorphisms open. Recall that, given a closure operator c, a morphism $f : X \to Y$ is c-*open* if, for every $n : N \to Y \in \mathcal{M}$,

$$c_X(f^{-1}(n)) \cong f^{-1}(c_Y(n));$$

that is, the inequality in the c-continuity condition (C3) becomes an isomorphism.

Proposition 2.6 ([8])**.** *For an idempotent closure operator c in a homological category* $\mathbf{C}$ *the following conditions are equivalent:*

(i) $c = \max^{\mathbf{A}}$ *for some regular-epireflective subcategory* $\mathbf{A}$*;*
(ii) *regular epimorphisms in* $\mathbf{C}$ *are c-open.*

It is easy to check that in general this is not a common property of regular closure operators in **Top**.

Proposition 2.7. *For a closure operator c in* **Top** *the following conditions are equivalent:*

(i) *c is a regular closure operator making regular epimorphisms c-open;*
(ii) *c is either the discrete or the indiscrete closure operator.*

Proof. (ii)⇒(i) is trivial.

(i)⇒(ii): Let c be a regular closure operator induced by a regular-epireflective subcategory $\mathbf{A} \neq \mathbf{Top}$. Then either $\mathbf{A} = \mathbf{Top}_0$ or $\mathbf{A} \subseteq \mathbf{Top}_1$. If $\mathbf{A} = \mathbf{Top}_0$, then $\mathrm{reg}^{\mathbf{A}}$ is the b-closure, which does not satisfy (i): take $X = \{0,1,2,3\} \to Y = \{0,1,2\}$ with $f(i) = i$ if $i \leq 2$ and $f(3) = 2$, where the only non-trivial open subset of X is $\{1,2\}$, hence the quotient topology is indiscrete; then $f^{-1}(\mathrm{b}(0)) = X$ and $\mathrm{b}(f^{-1}(0)) = \{0,3\}$. If $A \subseteq \mathbf{Top}_1$, then $\mathrm{reg}^{\mathbf{A}}$ is indiscrete in the Sierpinski space. Hence, for every Kuratowski-closed, non-open, subset C of a space Z, since $\chi_C : Z \to S$ is a quotient, hence $\mathrm{reg}^{\mathbf{A}}$-open, one has $\mathrm{reg}^{\mathbf{A}}_Z(C) = \chi_C^{-1}(\mathrm{reg}^{\mathbf{A}}_S(1)) = Z$. Therefore, if Z is T_1 and non-discrete, it has a non-open point z, and so $\mathrm{reg}^{\mathbf{A}}_Z(z) = Z$, which implies that $Z \notin \mathbf{A}$. This means then that $\mathbf{A}$ has only discrete spaces, hence $\mathbf{A} = \mathbf{Sgl}$.

□

Closed morphisms with respect to a closure operator are defined analogously to open morphisms, replacing inverse images by direct images. These are easily defined in the categories we are considering using their natural factorization systems: the (regular epi, mono)-factorization in the case of regular categories and the (epi, regular mono)-factorization in **Top**. If $(\mathcal{E},\mathcal{M})$ is a factorization system in $\mathbf{C}$, $m : M \to X \in \mathcal{M}$ and $f : X \to Y$ any morphism, $f(m)$ is the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of $f \cdot m$. A morphism $f : X \to Y$ is said to be *c-closed* if, for every $m \in \mathcal{M}$,

$$f(c_X(m)) \cong c_Y(f(m)).$$

(We remark that the inequality $f(c_X(m)) \leq c_Y(f(m))$ is equivalent to c-continuity of f.)

We recall that an epireflective subcategory is said to be *Birkhoff* if it is closed under quotients.

Next we analyse the topological counterpart of the following result.

Proposition 2.8 ([6]). *If* $\mathbf{C}$ *is a semi-abelian category, and* $\mathbf{A}$ *is a regular-epireflective subcategory of* $\mathbf{C}$*, the following assertions are equivalent:*

(i) *regular epimorphisms are* $\mathrm{max}^{\mathbf{A}}$*-closed;*
(ii) $\mathbf{A}$ *is a Birkhoff subcategory.*

Proposition 2.9. *For a regular-epireflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *the following conditions are equivalent:*

(i) *regular epimorphisms are* $\mathrm{reg}^{\mathbf{A}}$*-closed;*
(ii) $\mathbf{A}$ *is a Birkhoff subcategory;*
(iii) $\mathbf{A} = \mathbf{Top}$ *or* $\mathbf{A} = \mathbf{Sgl}$.

Proof. Trivially (iii)⇒(ii). To show that (ii)⇒(iii), first note that $\mathbf{Top_0}$ is not closed under quotients, hence it is not a Birkhoff subcategory. Now, if $\mathbf{A} \subseteq \mathbf{Top_1}$ and $\mathbf{A}$ contains a non-discrete space Z, hence with a closed non-open subset C, then $\chi_C : Z \to S$ is a quotient although the Sierpinski space S does not belong to $\mathbf{A}$. Hence every object of $\mathbf{A}$ is discrete, which implies that $\mathbf{A} = \mathbf{Sgl}$.

(iii)⇒(i) is clear, since $\mathrm{reg}^{\mathbf{Top}}$ is the discrete closure and $\mathrm{reg}^{\mathbf{Sgl}}$ is the indiscrete closure, both making regular epimorphisms c-closed.

(i)⇒(iii): If $\mathbf{A} = \mathbf{Top_0}$, $\mathrm{reg}^{\mathbf{A}}$ is the b-closure. The quotient $X \to Y$ used in the proof of Proposition 2.7 is not b-closed since

$$f(\mathrm{b}(0)) = f(\{0,3\}) = \{0,2\} \text{ and } \mathrm{b}(f(0)) = \mathrm{b}(0) = \{0,1,2\}.$$

If $\mathbf{A} \subseteq \mathbf{Top_1}$ and C is a closed, non-open, subset of $Z \in \mathbf{A}$, then $\chi_C : Z \to S$ is a quotient. Moreover, $\mathrm{reg}^{\mathbf{A}}$ is indiscrete in S, because the $\mathbf{A}$-reflection of S is a singleton, and every point in Z is $\mathrm{reg}^{\mathbf{A}}$-closed, since every object in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-separated. For any $z \in C$ one has

$$\chi_C(\mathrm{reg}^{\mathbf{A}}_Z(z)) = \chi_C(z) = 1 \neq \mathrm{reg}^{\mathbf{A}}_S(\chi_C(z)) = \mathrm{reg}^{\mathbf{A}}_S(1) = S.$$

Therefore every object of $\mathbf{A}$ is discrete, and so $\mathbf{A} = \mathbf{Sgl}$. □

Finally, it is natural to ask in which pointed regular categories regular and maximal closure operators coincide. Until the end of this section, we will assume that these closure operators are defined in the class of monomorphisms of $\mathbf{C}$. To compare them it is useful the following alternative description of the regular closure operator.

Lemma 2.10 ([12]). *If $(\mathcal{E}, \mathcal{M})$ is a factorization system in $\mathbf{C}$, then the regular closure $\mathrm{reg}^{\mathbf{A}}$ defined in $\mathcal{M}$ by the reflective subcategory $\mathbf{A}$ of $\mathbf{C}$ is described by*

$$\mathrm{reg}^{\mathbf{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m))).$$

Theorem 2.11. *Let $\mathbf{A}$ be a regular-epireflective subcategory of a pointed and regular category with cokernels. The following conditions are equivalent:*

(i) *when restricted to $\mathbf{A}$, $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide;*
(ii) $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*;*
(iii) *in $\mathbf{A}$ every equaliser is a kernel;*
(iv) *for every object A of $\mathbf{A}$, the diagonal $\delta_A : A \to A \times A$ is a kernel in $\mathbf{A}$.*

Proof. (i)⇒(ii): On one hand, since the maximal closure $\max^{\mathbf{A}}$ is the largest closure c with $0_A : 0 \to A$ c-closed for any $A \in \mathbf{A}$ and $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide in $\mathbf{A}$, $\mathrm{reg}^{\mathbf{A}} \leq \max^{\mathbf{A}}$. On the other hand, $\mathrm{reg}^{\mathbf{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m)))$ is $\max^{\mathbf{A}}$-closed since $\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m)) \cong \max^{\mathbf{A}}_{RX}(\rho_X(m))$, hence $\mathrm{reg}^{\mathbf{A}}_X \geq \max^{\mathbf{A}}_X$.

(ii)⇒(iii): Since every equaliser $m : M \to A$ in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-closed, hence $\max^{\mathbf{A}}$-closed by (ii), and the $\max^{\mathbf{A}}$-closure of m in $\mathbf{A}$ is the kernel of

$$A \xrightarrow{\pi_M} Y \xrightarrow{\rho_Y} RY \ \in \mathbf{A},$$

$m \cong \ker(\rho_Y \cdot \pi_M)$ is a kernel in $\mathbf{A}$ as claimed.

(iii)⇒(iv) is obvious, while (iv)⇒(iii) follows from the fact that the equaliser of $f, g : A \to B$ is the pullback of δ_B along $< f, g >: A \to B \times B$.

(iii)⇒(i): A monomorphism in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-closed (resp. $\max^{\mathbf{A}}$-closed) if, and only if, it is an equaliser in $\mathbf{A}$ (resp. a kernel in $\mathbf{A}$). If equalisers are kernels, then, as idempotent closure operators, necessarily $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide in $\mathbf{A}$. □

If $\mathbf{A}$ is a regular-epireflective subcategory of a homological category, then $\mathbf{A}$ is homological as well (see [4]), and so in $\mathbf{A}$ every coequaliser is a cokernel. In the theorem above the dual property is required for $\mathbf{A}$ so that its homological and regular closure operators coincide. Indeed this condition leads us again to an abelian-like condition, as we show next.

Corollary 2.12. *If* $\mathbf{A}$ *is a regular-epireflective subcategory of a homological category* $\mathbf{C}$*, then the following conditions are equivalent:*

(i) $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*;*
(ii) *every object in* $\mathbf{A}$ *is abelian.*

Proof. First we remark that both $\mathbf{C}$ and $\mathbf{A}$ are homological, and that in a homological category

X is abelian $\Leftrightarrow$ δ_X is a kernel $\Leftrightarrow$ X has an internal abelian group structure

(see [5]).

If $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$, then $\delta_A : A \to A \times A$ is a kernel, for every $A \in \mathbf{A}$. Hence, A is abelian. Conversely, if A is abelian then it has an internal abelian group structure in $\mathbf{C}$, hence also in $\mathbf{A}$, and so δ_A must be a kernel in $\mathbf{A}$. □

We point out that there are non-abelian homological categories where every equaliser is a kernel. In fact such categories are necessarily additive but may fail to be exact. (We recall that an exact and additive category is abelian: see [15].) This is the case, for instance, of the category of topological abelian groups, which is regular and protomodular but not exact (see [3] for details.)

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