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CLASSES OF NON-HERMITIAN OPERATORS WITH REAL EIGENVALUES

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ABSTRACT: Classes of non-Hermitian operators that have only real eigenvalues are presented. The investigated operators appear in quantum mechanics and are expressed in terms of the generators of the Weyl-Heisenberg algebra. For each non-Hermitian operator A, it is found a Hermitian involutive operator $\hat{J}$ such that A is $\hat{J}$-Hermitian, that is, $\hat{J}A = A^*\hat{J}$. Moreover, we construct a positive definite Hermitian Q such that A is Q-Hermitian, allowing for the standard probabilistic interpretation of quantum mechanics. Finally, it is shown that the considered matrices are similar to Hermitian matrices.

KEYWORDS: Infinite matrices, pseudo-Hermitian matrices, creation and annihilation operators, Krein spaces.

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1. Motivation

In [4] Bhatia studies families of matrices that have only real eigenvalues, mentioning that the behaviour of the eigenvalues of those matrices has similarities to those of Hermitian matrices. Such matrices appear in the investigation of vectorial hyperbolical differential equations. The purpose of this note is to present classes of infinite matrices [5] which are non-Hermitian but have real eigenvalues. This investigation is motivated by certain operators that appear in quantum mechanics and our approach to determine their eigenvalues and eigenvectors is inspired by Dirac's method.

The interest in the study of non-Hermitian Hamiltonians in physics has been related, in the past, with the interpretation of some properties, such as transfer phenomena, typical of open systems. At present, it is also associated with new kinds of quantum theories characterized by non-Hermitian Hamiltonians with $\mathcal{PT}$-symmetry (the product of parity and time reversal) and real spectra, the recent developments being motivated by field-theoretic models, such as the Lee model ([3] and the references therein). The results

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obtained originated a consistent extension of the standard quantum mechanics. The notion of $\mathcal{PT}$-symmetry can be placed in a general mathematical context known as *pseudo -Hermiticity*, a concept studied in the Krein space framework.

For the sake of completeness we introduce some terminology. As usual, we will refer to operators and matrices interchangeably. The *states* of a quantum system are described by vectors of an appropriate Hilbert space H of finite or infinite dimension, where the inner product of two vectors u, v is denoted by $\langle u, v\rangle$. The state of the physical system at the instant t is given by

$$u(t) = \mathrm{e}^{-itH}u(0),$$

or equivalently by

$$i\frac{\mathrm{d}u}{dt} = Hu,$$

where the linear operator $H : H \to H$ is called the *Hamiltonian* of the system assumed to be defined in a dense domain Dom$H$. The *energy* of the state u is given by the *Rayleigh quotient* $R(u) = \langle Hu, u\rangle/\langle u, u\rangle$, $u \neq 0$. In non-relativistic quantum mechanics, the Hamiltonian operator is Hermitian, that is, $H = H^*$ where $\langle Hx, y\rangle := \langle x, H^*y\rangle$, for any $x, y \in H$ in the domain of H. Throughout, we use indistinctly the terms Hermitian and self-adjoint. However, some relativistic extensions of quantum mechanics, such as the Klein-Gordon theory [7], lead to Hamiltonian operators H which are non-Hermitian. In such theories an involutive selfadjoint operator P (i.e., P^2 is the identity) arises which renders the Hamiltonian *P-Hermitian* (also said *P-selfadjoint* or *pseudo-Hermitian*), that is, $PH = H^*P$. This operator endows H, assumed for simplicity finite dimensional, with a Krein space structure $\langle u, v\rangle_P := v^*Pu$ (cf. e.g., [1, 2] for the theory of indefinite inner product spaces). In indefinite inner product spaces, there exist nonvanishing vectors such that $\langle u, u\rangle_P = 0$ (*isotropic vectors*) and the selfadjoint operators between these spaces may have nonreal eigenvalues. This mathematical structure is the natural context to formulate nonrelativistic quantum mechanics. A Hermitian involutive operator P with *signature* $(r, n-r)$ (i.e., with r positive eigenvalues and $n-r$ negative eigenvalues) allows for the definition of an indefinite norm

$$||u||_P^2 := \langle u, u\rangle_P = u_1^2 + \cdots + u_r^2 - u_{r+1}^2 + \cdots - u_n^2.$$

However, this norm is not suitable for the standard probabilistic quantum mechanical interpretation. So the problem arises of finding a positive definite Hermitian operator Q which can play the role of P and moreover is compatible with the conventional interpretation of quantum mechanics.

The main aim of this paper is the investigation of two classes of non-Hermitian infinite matrices that have only real eigenvalues and complete systems of eigenvectors (so they are diagonalizable under a similarity transformation). For each matrix, we construct a Hermitian involutive operator $\hat{J}$ which renders it $\hat{J}$-Hermitian. It is also proved that this allows for the unambiguous definition of a positive definite norm operator suitable for the conventional quantum mechanical interpretation. These results are obtained in Section 3. In Section 2 some notation is fixed and auxiliary results are given. Finally, in Section 4 open questions are formulated.

2. Prerequisites

As usual, $L^2(\mathbb{R})$ denotes the Hilbert space of square-integrable complex valued functions of a real variable x endowed with the usual inner product

$$\langle\psi,\phi\rangle = \int_{-\infty}^{+\infty} \bar{\phi}(x)\,\psi(x)\,\mathrm{d}x.$$

We denote by l^2 the Hilbert space of complex valued sequences $\{x_n\}_0^{+\infty}$, such that the series $\sum_{n=0}^{+\infty}|x_n|^2$ converges, endowed with the usual inner product

$$\langle x,y\rangle = \sum_{k=0}^{+\infty} \bar{y}_k x_k.$$

Throughout the paper, operations which are well-defined on l^2 for bounded operators (such as commutation) are used in the context of unbounded operators [5], a short explanation of their applicability being given.

For the sake of completeness, we briefly describe Dirac's approach to determine the eigenvalues and eigenvectors of the *harmonic oscillator Hamiltonian* $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined on a dense domain as follows,

$$H := \frac{1}{2}(\hat{p}^2 + \hat{x}^2).$$

In Physics, $\hat{p} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the differential operator $f(x) \to -i(\mathrm{d}f/\mathrm{d}x)$ with dense domain $\mathrm{Dom}\hat{p}$ and $\hat{x} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the multiplicative operator $f(x) \to xf(x)$, with dense domain $\mathrm{Dom}\hat{x}$. For an arbitrary $f \in$

$\mathrm{Dom}\hat{p} \cap \mathrm{Dom}\hat{x} \subset L^2(\mathbb{R})$, let the commutator $[\hat{p}, \hat{x}]$ be defined as follows

$$[\hat{p}, \hat{x}]f := (\hat{p}\hat{x} - \hat{x}\hat{p})f.$$

It can be easily seen that the above operators satisfy the *quantum condition*

$$[\hat{p}, \hat{x}] = -i\hat{1}\,,$$

where $\hat{1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the identity operator. The following relations also hold

$$[\hat{p}, [\hat{p}, \hat{x}]] = [\hat{x}, [\hat{p}, \hat{x}]] = 0.$$

The *Weyl-Heisenberg algebra* is the algebra generated by $\hat{p}, \hat{x}, \hat{1}$. This algebra is equivalently generated by the *creation* and *annihilation operators* $\hat{a}^*$ and $\hat{a}$, respectively, defined by the linear combinations

$$\hat{a} := \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \quad \hat{a}^* := \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}). \tag{2.1}$$

As the notation conveys, $\hat{a}^*$ is the adjoint operator of $\hat{a}$ and these operators satisfy the commutation relations

$$[\hat{a}, \hat{a}^*] = \hat{1}, \quad [\hat{a}, \hat{1}] = 0, \quad [\hat{a}^*, \hat{1}] = 0, \tag{2.2}$$

being the commutation defined as previously. The harmonic oscillator Hamiltonian is Hermitian and so its eigenvalues are real. Following Dirac's approach, we determine them with the help of the Weyl-Heisenberg algebra as shown above. One easily finds that H may be written in terms of the creation and annihilation operators as follows:

$$H = \hat{a}^*\hat{a} + \frac{1}{2}\hat{1}.$$

We determine the vector ϕ_0 such that $\hat{a}\phi_0 = 0$ by solving the differential equation

$$\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\phi_0(x) = 0.$$

We easily obtain $\phi_0 = K_0 \mathrm{e}^{-\frac{1}{2}x^2}$, with K_0 a real constant, and so $H\phi_0 = \frac{1}{2}\phi_0$. For $K_0 = \pi^{-\frac{1}{4}}$, we get $\langle \phi_0, \phi_0 \rangle = 1$. It can be easily verified that if $H\psi = \lambda\psi$, $\psi \neq 0$, then

$$H\hat{a}\psi = (\lambda - 1)\hat{a}\psi, \quad \hat{a}\psi \neq 0,$$

$$H\hat{a}^*\psi = (\lambda + 1)\hat{a}^*\psi, \ \psi \neq 0.$$

On the other hand, the functions

$$\phi_n(x) = K_n \left(x - \frac{\mathrm{d}}{\mathrm{d}x} \right)^n \mathrm{e}^{-\frac{1}{2}x^2}, \tag{2.3}$$

where $K_n = \pi^{-\frac{1}{4}}(n!2^n)^{-\frac{1}{2}}$ is a real normalization constant, are eigenvectors associated with distinct eigenvalues. Indeed,

$$H\phi_n = \left(n + \frac{1}{2} \right) \phi_n, \quad n = 0, 1, 2, \cdots .$$

For $K_n = \pi^{-\frac{1}{4}}(n!2^n)^{-\frac{1}{2}}$, the eigenvectors ϕ_n are orthonormal, that is, $\langle \phi_n, \phi_m \rangle = \delta_{nm}$, $n, m = 0, 1, 2, \cdots$ (δ_{ij} denotes the Kronecker symbol, i.e., δ_{ij}=1 for $i = j$ and δ_{ij}=0 for $i \neq j$). It is a well known fact that the eigenvectors form a complete system [6].

3. Non-Hermitian operators with real eigenvalues

3.1. The extended harmonic oscillator. Let $H_\beta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be an operator with a dense domain DomH_β, defined by

$$H_\beta := \frac{\beta}{2}(\hat{p}^2 + \hat{x}^2) + i\sqrt{2}\,\hat{p} = \beta \hat{a}^* \hat{a} + (\hat{a} - \hat{a}^*) + \frac{\beta}{2}\hat{1}, \quad \beta > 0.$$

This operator is the so called *extended harmonic oscillator.* Although H_β is non-Hermitian, it is nevertheless P-Hermitian, i.e., $PH_\beta = H_\beta^* P$, where $P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the *parity operator* defined by $Pf(x) = f(-x)$. This operator performs spatial reflections, so that it has the effect $p \to -p$ and $x \to -x$, and is Hermitian and involutive.

Proposition 3.1. *The matrix representation of the annihilation operator $\hat{a}$ in (2.1) in the basis of the harmonic oscillator (2.3) is the infinite matrix*

$$\breve{a} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.1}$$

Proof: By straightforward computations, we may conclude that

$$[\hat{a}, \hat{a}^*] = \hat{1}, \quad [\hat{a}, \hat{a}^{*n}] = [\hat{a}, \hat{a}^*]\hat{a}^{*n-1} + \hat{a}^*[\hat{a}, \hat{a}^*]\hat{a}^{*n-2} + \cdots + \hat{a}^{*n-1}[\hat{a}, \hat{a}^*] = n\hat{a}^{*n-1}.$$

Having in mind (2.3) we find

$$\begin{aligned}
&\hat{a}\phi_0 = 0,\\
&\hat{a}\phi_n = K_n\hat{a}\ \hat{a}^{*n}\mathrm{e}^{-\frac{1}{2}x^2} = K_n[\hat{a},\hat{a}^{*n}]\mathrm{e}^{-\frac{1}{2}x^2} = nK_n\hat{a}^{*(n-1)}\mathrm{e}^{-\frac{1}{2}x^2} =\\
&\quad = \sqrt{n}\phi_{n-1}, \quad n \geq 1,
\end{aligned}$$

and the result clearly follows. ■

Proposition 3.2. *The matrix representation of the creation operator $\hat{a}^*$ in (2.1) in the basis of the harmonic oscillator (2.3) is the infinite matrix $\breve{a}^*$.*

Proof: We have

$$\hat{a}^*\phi_n = K_n\hat{a}^*\ \hat{a}^{*n}\mathrm{e}^{-\frac{1}{2}x^2} = nK_n\hat{a}^{*(n+1)}\mathrm{e}^{-\frac{1}{2}x^2} = \sqrt{n+1}\ \phi_{n+1}.$$

■

We find that, in the basis (2.3), H_β is represented by the matrix

$$M_\beta = \beta\breve{a}^*\breve{a} + (\breve{a} - \breve{a}^*) + \frac{\beta}{2}I, \tag{3.2}$$

where I denotes the infinite diagonal matrix $I = \operatorname{diag}(1,1,1,\cdots)$, that is,

$$M_\beta = \begin{bmatrix} \frac{1}{2}\beta & \sqrt{1} & 0 & 0 & \cdots \\ -\sqrt{1} & \frac{3}{2}\beta & \sqrt{2} & 0 & \cdots \\ 0 & -\sqrt{2} & \frac{5}{2}\beta & \sqrt{3} & \cdots \\ 0 & 0 & -\sqrt{3} & \frac{7}{2}\beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.3}$$

The collection of matrices

$$\mathcal{S} = \{\alpha(\breve{a} - \breve{a}^*) + \beta(\breve{a}^*\breve{a} + I/2),\ \alpha \in \mathbb{R},\ \beta > 0\}$$

with the infinite zero matrix form a convex cone of the real vector space generated by $\breve{a} - \breve{a}^*$ and $\breve{a}^*\breve{a} + I/2$. Obviously, an analogous situation occurs if $\beta < 0$. We show that the matrices of $\mathcal{S}$ have real eigenvalues, complete systems of eigenvectors and moreover they are Q-Hermitian, for Q a positive definite Hermitian operator. This result is proved in the next theorem, where we consider a matrix of $\mathcal{S}$ with $\alpha = 1$ and $\beta = 2$ for simplicity.

Theorem 3.3. *Consider the infinite matrix $M = 2\breve{a}\breve{a}^* + (\breve{a} - \breve{a}^*) + I$. Then:*

(i) *The point spectrum of the matrix M is the set*

$$\sigma(M) = \left\{ \frac{3}{2} + 2j : j = 0, 1, 2, \cdots \right\},$$

and the vector

$$v_j = K_j \left(\breve{a}^* + \frac{I}{2} \right)^j \left(1, \frac{1}{2\sqrt{1}}, \frac{1}{2^2\sqrt{1 \cdot 2}}, \frac{1}{2^3\sqrt{1 \cdot 2 \cdot 3}}, \cdots \right)^T \in l^2,$$

K_j being a normalization constant, is a right eigenvector associated with the eigenvalue $\frac{3}{2} + 2j$, for $j = 0, 1, 2, \cdots$. This eigensystem is complete.

(ii) *We have $M = \mathrm{e}^S(2a^*\breve{a} + \frac{3}{2}I)\mathrm{e}^{-S}$, where $S = \frac{1}{2}(\breve{a} + \breve{a}^*)$. Moreover, M is Q-Hermitian for Q the positive Hermitian operator $Q = \mathrm{e}^{-(\breve{a}+\breve{a}^*)}$.*

Proof: Consider the matrix $M = 2\breve{a}^*\breve{a} + \breve{a} - \breve{a}^* + I$. Let

$$\breve{c}^\ddagger = \breve{a}^* + \frac{1}{2}I, \quad \breve{c} = \breve{a} - \frac{1}{2}I. \tag{3.4}$$

Then we obtain

$$M = 2\breve{c}^\ddagger\breve{c} + \frac{3}{2}I.$$

Let $[\breve{c}, \breve{c}^\ddagger] := \breve{c}\ \breve{c}^\ddagger - \breve{c}^\ddagger\ \breve{c}$. It may be easily seen that $[\breve{c}, \breve{c}^\ddagger]f = f$ for any $f \in L_2(\mathbb{R})$ and so the following holds

$$[\breve{c}, \breve{c}^\ddagger] = I. \tag{3.5}$$

Similarly, it may be shown that

$$[M, \breve{c}^\ddagger] = 2\breve{c}^\ddagger, \quad [M, \breve{c}] = -2\breve{c}. \tag{3.6}$$

If $v \in l^2$ and $\lambda \in \mathbb{R}$ are such that $Mv = \lambda v$, then by (3.6) we get

$$M\breve{c}^\ddagger v = [M, \breve{c}^\ddagger]v + \breve{c}^\ddagger M v = (\lambda + 2)\breve{c}^\ddagger v, \quad M\breve{c}v = (\lambda - 2)\breve{c}v.$$

If $\breve{c}v \neq 0$, then $\breve{c}v$ is an eigenvalue of M associated to the eigenvalue $\lambda - 2$. We observe that $\breve{c}^\ddagger v$ is different from 0 and so is an eigenvector of M associated to the eigenvalue $\lambda + 2$.

We *claim* that there exists a nonnegative integer j such that $\breve{c}^j v = 0$. We will accept this for the moment and the claim will be proved in (ii). It may be easily checked that, for $v_0 = \left(1, \frac{1}{2\sqrt{1}}, \frac{1}{2^2\sqrt{1\cdot 2}}, \frac{1}{2^3\sqrt{1\cdot 2\cdot 3}}, \cdots\right)^T \in l^2$, we have $\breve{c}v_0 = 0$. Thus,

$$M(\breve{c}^\ddagger)^j v_0 = \left(2j + \frac{3}{2}\right)(\breve{c}^\ddagger)^j v_0, \quad j = 0, 1, 2, 3, \cdots ,$$

and so $v_j = (\breve{c}^\ddagger)^j v_0$. Taking into account (3.4), we obtain the asserted expression for v_j in (i).

The eigenvectors v_j, $j = 0, 1, 2, 3, \cdots$, of M have been expressed in terms of the basis elements e_j, $j = 0, 1, 2, 3, \cdots$, with $e_0 = (1, 0, 0, \cdots)^T$, $e_1 = (0, 1, 0, 0, \cdots)^T$, $e_2 = (0, 0, 1, 0, \cdots)^T, \cdots$, being

$$\breve{a}\, e_0 = 0,\; e_1 = \breve{a}^* \sqrt{1!} e_0,\; e_2 = \breve{a}^{*2}\sqrt{2!} e_0, \cdots .$$

By inverting the transformation (3.4), we obtain

$$\breve{a}^* = \breve{c}^\ddagger - \frac{1}{2} I, \quad \breve{a} = \breve{c} + \frac{1}{2} I.$$

Thus, we are able to express the vectors e_j in terms of the eigenvectors v_j. This proves that these eigenvectors form a complete basis and (i) follows.

Now we prove (ii). Let $S = \frac{1}{2}(\breve{a} + \breve{a}^*)$, where $\breve{a}$ is given in (3.1). By definition $\mathrm{e}^S = I + S + \frac{S^2}{2!} + \cdots$ and $\mathrm{e}^S \breve{a} \mathrm{e}^{-S}$ is given by the formal series

$$\mathrm{e}^S \breve{a} \mathrm{e}^{-S} = \breve{a} + [S, \breve{a}] + \frac{1}{2}[S, [S, \breve{a}]] + \cdots .$$

This series has only a finite number of nonvanishing terms, because $[\breve{a}, \breve{a}^*] = \hat{1}$. We notice that since the series expansion have a finite number of nonvanishing terms, the domain of the unbounded operator in the left side is equal to the domain of the operator in the right side, namely the subspace of l^2 constituted by the vectors $(x_1, x_2, \cdots)^T$ such that the series $\sum_{j=1}^{\infty} j\ |x_j|^2$ converges.

Having in mind (2.2), we find

$$\mathrm{e}^S \breve{a} \mathrm{e}^{-S} = \breve{a} - \frac{1}{2} I.$$

Analogously,

$$\mathrm{e}^S \breve{a}^* \mathrm{e}^{-S} = \breve{a}^* + \frac{1}{2} I.$$

Thus,

$$\mathrm{e}^{S}\left(2\breve{a}^*\breve{a}+\frac{3}{2}I\right)\mathrm{e}^{-S}=2\breve{a}^*\breve{a}+\breve{a}-\breve{a}^*+I=M.$$

We have shown that for $M_0=2\breve{a}^*\breve{a}+\frac{3}{2}I$, we obtain $\mathrm{e}^{S}M_0\mathrm{e}^{-S}=M$. We clearly have $\mathrm{e}^{-S}M_0\mathrm{e}^{S}=M^*$, or, equivalently,

$$M_0=\mathrm{e}^{-S}M\mathrm{e}^{S}=\mathrm{e}^{S}M^*\mathrm{e}^{-S}.$$

Hence

$$\mathrm{e}^{-2S}M=M^*\mathrm{e}^{-2S}.$$

Finally, we prove the *claim*. In fact, since M is similar to the harmonic oscillator Hamiltonian $M_0=2\breve{a}^*\breve{a}+\frac{3}{2}I$, M and M_0 have the same eigenvalues. The point spectrum of M_0 is bounded from below (cf. [7]), so the claim follows. ∎

Remark 3.4. A left eigenvector of M satisfies

$$w_j^*M=(2j+\frac{3}{2})w_j^*,\quad j=0,1,2,\cdots,\quad w_0^*\breve{c}^{\ddagger}=0,\quad w_j^*\breve{c}=\kappa_j w_{j+1}^*,$$

for some $\kappa_j>0$, and is such that

$$w_j=K_j\left(\breve{a}^*-\frac{I}{2}\right)^j\left(1,-\frac{1}{2\sqrt{1}},\frac{1}{2^2\sqrt{1\cdot 2}},-\frac{1}{2^3\sqrt{1\cdot 2\cdot 3}},\cdots\right)^T\in l^2.$$

Moreover, for $K_j=\sqrt{\frac{\mathrm{e}}{j!}}$, we have $w_j^*v_l=\delta_{kl}$, i.e., the left and right eigenvectors are orthonormal.

3.2. The Swanson-Jones Hamiltonian. Next, we consider an operator of a type investigated by Swanson [9] and Jones [8], namely the non-Hermitian Hamiltonian $H_\theta: L^2(\mathbb{R})\to L^2(\mathbb{R})$

$$H_\theta:=\frac{1}{2}(\hat{p}^2+\hat{x}^2)-\frac{i}{2}\tan 2\theta(\hat{p}^2-\hat{x}^2),$$

where θ is a real constant, $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$, and whose domain is assumed to be dense. In terms of creation and annihilation operators $\hat{a}^*,\hat{a}$ (cf. eq. (2.1)), we easily find

$$H_\theta=\hat{a}^*\hat{a}+i\frac{\tan 2\theta}{2}\left(\hat{a}^{*2}+\hat{a}^2\right)+\frac{1}{2}\hat{1}.$$

In the basis of the mentioned eigenvectors of the harmonic oscillator, this operator has the following matrix representation:

$$M_\theta = \begin{bmatrix} \frac{1}{2} & 0 & i\kappa\sqrt{1\cdot 2} & 0 & 0 & \cdots \\ 0 & \frac{3}{2} & 0 & i\kappa\sqrt{2\cdot 3} & 0 & \cdots \\ i\kappa\sqrt{1\cdot 2} & 0 & \frac{5}{2} & 0 & i\kappa\sqrt{3\cdot 4} & \cdots \\ 0 & i\kappa\sqrt{2\cdot 3} & 0 & \frac{7}{2} & 0 & \cdots \\ 0 & 0 & i\kappa\sqrt{3\cdot 4} & 0 & \frac{9}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.7}$$

with $\kappa = \frac{1}{2}\tan 2\theta$. Consider the collection of matrices

$$\mathcal{R} = \{i\alpha(\hat{a}^{*2} + \hat{a}^2) + \beta(\hat{a}^*\hat{a} + I/2),\ \alpha \in \mathbb{R},\ \beta > 0\}.$$

The union of the set $\mathcal{R}$ with the infinite zero matrix forms a convex cone of the real vector space generated by $\hat{a}^*\hat{a} + I/2$ and $\hat{a}^{*2} + \hat{a}^2$. Obviously, an analogous situation occurs if $\beta < 0$. We show that the matrices of $\mathcal{R}$ have similar properties to those of $\mathcal{S}$. Indeed, they have real eigenvalues, complete systems of eigenvectors and they are Q-Hermitian, for Q a positive Hermitian operator. This result is proved in the next theorem, where we consider a matrix in $\mathcal{R}$ with $\alpha = 2$ and $\beta = 2$ for simplicity.

Theorem 3.5. *Consider the infinite matrix $M = 2M_{\frac{\pi}{8}}$, where $M_{\frac{\pi}{8}}$ is of the form (3.7). Then:*

(i) *The point spectrum of M is $\sigma(M) = \left\{\sqrt{2}(1+2j) : j = 0, 1, 2, \cdots\right\}$, and the vector*

$$v_j = K_j\,(\breve{a}^* + i\gamma\ \breve{a})^j \left(1, 0, -i\gamma\sqrt{\frac{1}{2}}, 0, (-i\gamma)^2\sqrt{\frac{1\cdot 3}{2\cdot 4}}, 0, (-i\gamma)^3\sqrt{\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}} \cdots\right)^T \in l^2,$$

with $\gamma = \tan\frac{\pi}{8}$ and K_j This eigensystem is complete.

(ii) *We have $M = \mathrm{e}^S\sqrt{2}(2\breve{a}^*\breve{a} + I)\mathrm{e}^{-S}$, where $S = \frac{\pi}{16}i(\breve{a}^2 - \breve{a}^{*2})$. Moreover, M is Q-Hermitian for Q the positive Hermitian operator $Q = \mathrm{e}^{-\frac{\pi}{8}i(\breve{a}^2 - \breve{a}^{*2})}$.*

Proof. Consider the matrix $M = 2\breve{a}^*\breve{a} + i(\breve{a}^2 + \breve{a}^{*2}) + I$. Let

$$\breve{c}^\ddagger = \cos\frac{\pi}{8}\ \breve{a}^* + i\sin\frac{\pi}{8}\ \breve{a}, \quad \breve{c} = \cos\frac{\pi}{8}\ \breve{a} + i\sin\frac{\pi}{8}\ \breve{a}^*,$$

$$M = \sqrt{2}(2\breve{c}^\ddagger\breve{c} + I),$$

and the following commutation relations are valid

$$[\breve{c}, \breve{c}^{\ddagger}] = I, \quad [M, \breve{c}^{\ddagger}] = 2\sqrt{2}\breve{c}^{\ddagger}, \quad [M, \breve{c}] = -2\sqrt{2}\breve{c}.$$

If $v \in l^2, \lambda \in \mathbb{R}$ are such that $Mv = \lambda v$, then

$$M\breve{c}^{\ddagger}v = [M, \breve{c}^{\ddagger}]v + \breve{c}^{\ddagger}Mv = (\lambda + 2\sqrt{2})\breve{c}^{\ddagger}v, \quad M\breve{c}v = (\lambda - 2\sqrt{2})\breve{c}v.$$

As in the previous theorem, we can show that there exists a nonnegative integer j such that $\breve{c}^j v = 0$. It may be easily checked that, for

$$v_0 = \left(1, 0, -i\gamma\sqrt{\frac{1}{2}}, 0, (-i\gamma)^2\sqrt{\frac{1\cdot 3}{2\cdot 4}}, 0, (-i\gamma)^3\sqrt{\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}}, \cdots\right)^T \in l^2,$$

$\gamma = \tan\frac{\pi}{8}$, we have $\breve{c}v_0 = 0$. Thus,

$$M(\breve{c}^{\ddagger})^j v_0 = \sqrt{2}(2j+1)(\breve{c}^{\ddagger})^j v_0, \quad j = 0, 1, 2, 3, \cdots,$$

and this yields the expression for v_j in (i). Using analogous arguments to those in the proof of Theorem 3.1, it can be shown that this eigensystem is complete.

Now we prove (ii). Let $S = \frac{\pi}{16}i(\breve{a}^2 - \breve{a}^{*2})$. Similarly to the proof of Theorem 3.1 (ii), we easily find

$$[S, \breve{a}] = \frac{\pi}{8}i\breve{a}^*, \quad [S, \breve{a}^*] = \frac{\pi}{8}i\breve{a}.$$

Thus,

$$\mathrm{e}^S\breve{a}\mathrm{e}^{-S} = \cos\frac{\pi}{8}\,\breve{a} + i\sin\frac{\pi}{8}\,\breve{a}^*, \quad \mathrm{e}^S\breve{a}^*\mathrm{e}^{-S} = \cos\frac{\pi}{8}\,\breve{a}^* + i\sin\frac{\pi}{8}\,\breve{a}.$$

Hence, by similar arguments to those in the proof of Theorem 3.1 we get

$$\mathrm{e}^S\sqrt{2}(2\breve{a}^*\breve{a} + I)\mathrm{e}^{-S} = 2\breve{a}^*\breve{a} + i(\breve{a}^{*2} + \breve{a}^2) + I = M.$$

We have shown that for $M_0 = \sqrt{2}(2\breve{a}^*\breve{a} + I)$, we obtain $\mathrm{e}^S M_0\mathrm{e}^{-S} = M$. We also have $\mathrm{e}^{-S}M_0\mathrm{e}^S = M^*$, or equivalently,

$$M_0 = \mathrm{e}^{-S}M\mathrm{e}^S = \mathrm{e}^S M^*\mathrm{e}^{-S}.$$

Therefore

$$\mathrm{e}^{-2S}M = M^*\mathrm{e}^{-2S}. \ \square$$

Remark 3.6. A left eigenvector of M satisfies

$$w_j^* M = \sqrt{2}(1+2j)w_j^*, \quad j = 0, 1, 2, \cdots, \quad w_0^*\breve{c}^{\ddagger} = 0, \quad w_j^*\breve{c} = \kappa_j w_{j+1}^*,$$

for some $\kappa_j > 0$, and is such that

$$w_j = K_j\,(\breve{a}^* - i\gamma\;\breve{a})^j\left(1, 0, i\gamma\sqrt{\frac{1}{2}}, 0, (i\gamma)^2\sqrt{\frac{1\cdot 3}{2\cdot 4}}, 0, (i\gamma)^3\sqrt{\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}}\cdots\right)^T \in l^2,$$

$\gamma = \tan\frac{\pi}{8}$. Moreover, for an appropriate K_j, we have $w_j^* v_l = \delta_{kl}$, i.e., the left and right eigenvectors are orthonormal.

We recall that an $n\times n$ matrix U is called *J-unitary* if $UJU^*J = JU^*JU = I_n$, where I_n denotes the identity matrix of size n and J is a Hermitian involution with signature $(r, n-r)$. The J-unitary matrices form a locally compact group called the *J-unitary group*.

Theorem 3.7. *Let $J = I_r \oplus I_{n-r}$ and let A be an $n\times n$ J-Hermitian matrix diagonalizable under a J-unitary similarity. Assume that A has distinct eigenvalues. Then there exists a (non-unique) positive definite Hermitian operator Q such that QA is Hermitian.*

Proof: Under the hypothesis, the eigenvalues of A are real. Let $\alpha_1, \cdots, \alpha_n$ be (the distinct) eigenvalues of A. Denote by $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ the associated eigenvectors of A and A^*, respectively. That is,

$$Ax_j = \alpha_j x_j, \quad A^* y_j = \alpha_j y_j, \quad j = 1, \cdots, n.$$

By straightforward computation, it follows that

$$y_j^* x_k = 0, \qquad \text{for}\ \ j \neq k,\ j, k = 1, \cdots, n.$$

Assume the eigenvectors x_j are normalized according to $x_j^* J x_j = 1$ for $j = 1, \cdots, r$, and $x_j^* J x_j = -1$ for $j = r+1, \cdots, n$. We chose $y_j = Jx_j$ for $j = 1, \cdots, r$, and $y_j = -Jx_j$ for $j = r+1, \cdots, n$. Then, we have $y_j^* x_j = 1,\ j = 1, \cdots, n$. Consider the matrix Q defined by $y_j = Qx_j,\ j = 1, \cdots, n$. This matrix is Hermitian, because $x_j^* Q x_k = \delta_{jk}$ and for an arbitrary vector $z = \sum_{j_1}^n c_j x_j$ we find $z^* Q^* z = z^* Q z$. We also get $z^* Q z = \sum_{j=1}^n |c_j|^2 > 0$, and so Q is positive definite. Moreover, QA is Hermitian. In fact,

$$x_j^* Q A x_k = y_j^* A x_k = \alpha_j y_j^* x_k = \alpha_j \delta_{jk}.$$

Considering an arbitrary vector $z = \sum_{j=1}^n c_j x_j$, we find that $z^* Q A z = \sum_{j=1}^n \overline{c}_j c_j \alpha_j$ is real. So the result clearly follows. ∎

4. Final Remarks

The above theorem still holds under the assumption of the eigenvalues of the matrix not being distinct. This can be easily seen by perturbative and continuity arguments.

We observe that the matrix M_β in (3.3) is J-Hermitian for

$$J = I_1 \oplus (-I_1) \oplus I_1 \oplus (-I_1) \oplus \cdots ,$$

while the matrix M_θ in (3.7) is J-Hermitian for

$$J = I_2 \oplus (-I_2) \oplus I_2 \oplus (-I_2) \oplus \cdots .$$

The eigenvalues of J-Hermitian matrices are real numbers and pairs of complex conjugate numbers (see, e.g., [1, 2]). If we consider finite matrices of the types (3.3) or (3.7), then complex conjugate eigenvalues may appear.

It would be of interest to answer the following questions:

1) Does the last theorem still hold for infinite matrices of the type here considered?

2) What can be said about the eigenvalues and eigenvectors of the matrices in $\mathcal{R}$ when we take α nonreal?

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