Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 09–51

ON AN INDEX TWO SUBGROUP OF PUZZLE AND LITTLEWOOD-RICHARDSON TABLEAU $\mathbb{Z}_2 \times \mathbb{S}_3$-SYMMETRIES (EXTENDED ABSTRACT)

OLGA AZENHAS, ALESSANDRO CONFLITTI AND RICARDO MAMEDE

Abstract: We consider an action of the dihedral group $\mathbb{Z}_2 \times \mathbb{S}_3$ on Littlewood-Richardson tableaux which carries a linear time action of a subgroup of index two. This index two subgroup action on Knutson-Tao-Woodward puzzles is the group generated by the puzzle mirror reflections with label swapping. One shows that, as happens in puzzles, half of the twelve symmetries of Littlewood-Richardson coefficients may also be exhibited on Littlewood-Richardson tableaux by surprisingly easy maps. The other hidden half symmetries are given by a remaining generator which enables to reduce those symmetries to the Schützenberger involution. Purbhoo mosaics are used to map the action of the subgroup of index two on Littlewood-Richardson tableaux into the group generated by the puzzle mirror reflections with label swapping. After Pak and Vallejo one knows that Berenstein-Zelevinsky triangles, Knutson-Tao hives and Littlewood-Richardson tableaux may be put in correspondence by linear algebraic maps. We conclude that, regarding the symmetries, the behaviour of the various combinatorial models for Littlewood-Richardson coefficients is similar, and the bijections exhibiting them are in a certain sense unique.

Keywords: Litlewood-Richardson coefficients, mosaics, puzzles, tableaux, action of the dihedral group of cardinality twelve.

1. Introduction

The positive integers $0 < d < n$ are fixed throughout, and the partitions are those whose Young diagrams fit inside a $d \times (n-d)$ rectangle of height d and width $n-d$. We mainly think of a partition in terms of its Young diagram. The French convention is adopted. The complement of a partition λ is the partition $\lambda^\vee$ whose Young diagram is the set complement of λ in the $d\times(n-d)$ rectangle, followed by a rotation of 180 degrees. Clearly, $\lambda^{\vee\vee} = \lambda$. As our partitions fit inside a $d \times (n-d)$ rectangle, the boundary data of a Littlewood-Richardson (LR)-tableau of shape $\lambda^\vee/\mu$ and content ν is defined to be (μ, ν, λ).

Received December 28, 2009.
The three authors are supported by CMUC - Centre for Mathematics, University of Coimbra. The second author is also supported by FCT (Fundação para a Ciência e a Tecnologia) Portuguese Foundation of Science and Technology.

Littlewood-Richardson numbers $c_{\mu\,\nu\,\lambda}$ are non negative integers, depending on three partitions μ, ν, and λ. As one knows they appear in various mathematical contexts as multiplicity numbers, for example, in the decomposition of tensor products of irreducible representations of general linear groups, in the linear expansion of the product of two Schur functions, or in the linear expansion of the cup product of Schubert classes. Simultaneously, they are counted by a quantity of combinatorial models with boundary data (μ, ν, λ) as, for example, LR tableaux [LiRi], Berenstein-Zelevinsky (BZ) triangles [BZ], Knutson-Tao hives [KT], Knutson-Tao-Woodward puzzles [KTW] or Purbhoo mosaics [Pu]. They are invariant under the action of the group $\mathbb{Z}_2 \times \mathbb{S}_3$ where the nonidentity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν, λ, and $\mathbb{S}_3$ permutes μ, ν, λ [BZ, BSS]. As already pointed out by Pak and Vallejo in [PV1, PV2], for the case of the $\mathbb{S}_3$–symmetries, these numbers are quite intriguing since a subgroup of index two of the $\mathbb{Z}_2 \times \mathbb{S}_3$-symmetries may be given by surprising easily maps in either combinatorial model, and, therefore, the symmetries outside of this group are linear time reducible either to the commutativity or to the transposition symmetry, given by a remaining generator. In the case of $\mathbb{S}_3$-symmetries, the subgroup of index two may be given by linear cost maps, and since the commutative symmetry is a remaining generator, the symmetries outside of the index two subgroup are given by maps which are linear time reducible to the map exhibiting the commutative symmetry. This observation [PV1, PV2] has been made on the basis that if one operates with LR tableaux, one may pass through a linear map into BZ triangles, perform the symmetry there, and return back to LR tableaux. As a follow-up we study the $\mathbb{Z}_2 \times \mathbb{S}_3$-symmetries of Littlewood-Richardson coefficients and a particular index two subgroup which contains the index two subgroup of $\mathbb{S}_3$-symmetries. However, in our approach we exhibit explicitly, both in the LR-tableau and in the puzzle model, the group of symmetries, *i.e.* the group of bijections, and we show that they coincide using Purbhoo mosaics [Pu].

We denote by τ the non–identity element of $\mathbb{Z}_2$, by σ_1 and σ_2 the simple transpositions of $\mathbb{S}_3$, and think of $\mathbb{Z}_2 \times S_3$ as the dihedral group $\langle \tau, \tau\sigma_1, \tau\sigma_2 \rangle$ generated by the involutions τ, $\tau\sigma_1$, $\tau\sigma_2$ where τ commutes with both σ_1 and σ_2. The action of the index two subgroup $H = \langle \tau\sigma_1, \tau\sigma_2 \rangle$ will be the main object of our study. The group H acts on puzzles through the mirror reflections and label swapping. We shall see that the group H also acts on

LR tableaux by surprisingly easy maps. This shows that the symmetries

$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu} = c_{\nu^t\,\mu^t\,\lambda^t} = c_{\mu^t\,\lambda^t\,\nu^t} = c_{\lambda^t\,\nu^t\,\mu^t}$$

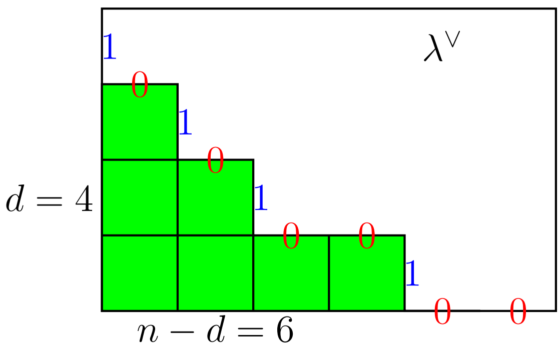
are intrinsically easy to exhibit in either model, whereas both the conjugation symmetry and the commutativity, *i.e.* the equalities

$$c_{\mu\,\nu\,\lambda} = c_{\nu\,\mu\,\lambda} = c_{\mu^t\,\nu^t\,\lambda^t} = c_{\mu\,\lambda\,\nu} = c_{\lambda^t\,\mu^t\,\nu^t} = c_{\lambda\,\nu\,\mu} = c_{\nu^t\,\lambda^t\,\mu^t},$$

are very difficult. Since H is a normal subgroup, and the index of H in $\mathbb{Z}_2 \times S_3$ is a prime number, we have that the quotient group $\mathbb{Z}_2 \times S_3/H$ is cyclic and every element different from the identity is a generator. Therefore the latter symmetries are linear time equivalent and may be reduced to the Schützenberger involution [ACM]. The paper is divided in four sections. In the next, we give the necessary background on tableaux and puzzles. In Section 3, the $\mathbb{Z}_2 \times S_3$-symmetries on LR tableaux and on puzzles are studied with a special focus on the index two subgroup H. The last section is devoted to Purbhoo mosaics and their use on showing how the symmetry involutions on LR tableaux and on puzzles commute. Several illustrations are provided.

2. Background on tableaux and puzzles

2.1. Partitions, Young diagrams, skew-shapes and 01-words. A partition $\lambda = (\lambda_1, \dots, \lambda_d)$ is a decreasing sequence of non negative integers, and we mainly think of it in terms of a Young diagram. We assume that our partitions have at most d parts, and $\lambda_1 \leq n - d$ so that they fit, according to the French convention, a $d \times (n-d)$ rectangle of height d and width $n - d$. In particular, we may regard this rectangle as a Young diagram, and hence all our Young diagrams are subsets of the Young diagram $d \times (n-d)$ rectangle. Thus our partitions are in bijection with 01-words of length n, comprising $n - d$ 0's and d 1's as follows: the positions of the zeroes and ones in a 01-word are respectively the positions of the horizontal and vertical steps along the boundary of the corresponding Young diagram, starting in the right lower corner of the rectangle, as shown in the example below, with $d = 4$, $n = 10$, and $\lambda = (4, 2, 1, 0, 0)$. The partition λ is identified with the 01-word 0010010101.



A partition μ is said to be contained in a partition λ if the Young diagram of μ is contained in the Young diagram of λ. In this case, one defines the skew shape λ/μ to be the set of boxes in the Young diagram of λ that remains after one removes those boxes corresponding to μ. Considering the Young diagram $d \times n-d$ rectangle, one obtains the skew shape $d \times (n-d)/\lambda$, also called anti-normal shape. Thus rotating the skew shape $d \times (n-d)/\lambda$, one obtains the partition $\lambda^\vee = (6,5,4,2) = 1010100100$ the reverse of the 01-word of λ. Transposing the Young diagram $d \times (n-d)$ rectangle, we obtain $\lambda^t = 0101011011$, the transpose of λ, where the 01-word is the one obtained from λ by reversing and swapping zeroes and ones. T. Finally, rotating and transposing gives $\lambda^{\vee\, t} = 1101101010$, where the 01-word is the one obtained from λ by swapping zeroes and ones.

2.2. Tableaux and Littlewood-Richardson tableaux. For any skew shape λ/μ, a (semistandard) tableau T of shape λ/μ and content $\nu = (\nu_1, \ldots, \nu_d)$, is a filling f which assigns to each box of λ/μ a positive integer (or a symbol in a finite alphabet, that is, a finite totally ordered set) such that:

(i) the entries are weakly increasing in each row from west to east;

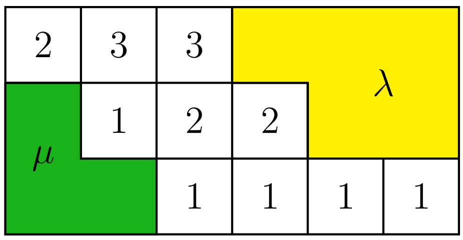
(ii) the entries are strictly increasing in each column from south to north.

The word of T is the list of labels when one reads the labeled boxes in reverse reading order, that is, from right to left across rows taken in turn from bottom to top, where ν_k is precisely the number of k's which occurs in the filling.

A semistandard tableau T of shape λ/μ and content ν is called a Littlewood-Richardson tableau (or LR-tableau) if additionally the word satisfies:

(iii) in any prefix the number of i's is at least as large as the number of $(i+1)$'s, *i.e.*, $\#1's \geq \#2's \ldots$. This condition is called the lattice permutation property of the word which lists the labels. The boundary data of a tableau of shape λ/μ and content ν is defined to be $(\mu, \nu, \lambda^\vee)$. The number of LR tableaux with boundary data (μ, ν, λ) is the Littlewood-Richardson number $c_{\mu\,\nu\,\lambda}$.

Below it is shown a LR tableau, with $n = 9$, $d = 3$, of boundary data (μ, ν, λ), where $\nu = (5, 3, 2, 0)$, $\mu = (2, 1, 0, 0)$, $\lambda = (3, 2, 0, 0)$, and word 1111221332.



The canonical LR tableau $Y(\mu)$ of shape μ, also called Yamanouchi tableau of shape μ, is the unique tableau of shape and content μ, that is, a tableau of shape μ where each row i is filled with μ_i i's, for every i. Applying reverse *jeu de taquin* slides to $Y(\mu)$ in the smallest rectangle containing it, we obtain the tableau of anti-normal shape of $Y(\mu)$, denoted by $Y(\mu)^{\mathrm{a}}$. As $Y(\mu)^{\mathrm{a}}$ fits the upper right corner of the $n \times (n-d)$ rectangle, $Y(\mu)^{\mathrm{a}}$ may be regarded as the LR tableau of shape $d \times (n-d)/\mu^\vee$ and content μ. The rotation of the shape of $Y(\mu)^{\mathrm{a}}$ is the shape μ.

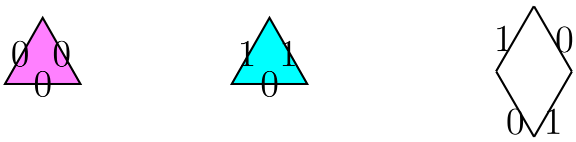
Given a decomposition of the $d \times (n-d)$ rectangle into shapes μ, $\lambda^\vee/\mu$, and $d \times (n-d)/\lambda^\vee$, the triple (U_1, U_2, U_3) is said to be a three-fold multitableau of outershape the $d \times (n-d)$ rectangle, if U_1 is a filling of the shape μ, U_2 is a filling of $\lambda^\vee/\mu$, and U_3 is a filling of $d \times (n-d)/\lambda^\vee$. A three-fold LR multitableau of boundary data (μ, ν, λ) is a three-fold multitableau of outer shape the $d \times (n-d)$ rectangle, where the inner tableau is the Yamanouchi tableau $Y(\mu)$, the middle one is the LR tableau of shape $\lambda^\vee/\mu$ and content ν, and the outer tableau is $Y(\lambda)^{\mathrm{a}}$.

If T is a tableau of shape $\lambda^\vee/\mu$ and content $\nu = (\nu_1, \ldots, \nu_d)$, $T^\bullet$ is the tableau of shape $\mu^\vee/\lambda$ and content $(\nu_d, \ldots, \nu_1)$ obtained by rotation of 180^0 of T and replacing the entry i with $d - i + 1$, for every i. Indeed $T^{\bullet\bullet} = T$.

The standard order on a semistandard Young tableau is the numerical ordering of the labels with priority, in the case of equality, given by rule northwest=smaller, southeast=larger. The standardization of a semistandard tableau T of content ν, denoted by $\widehat{T}$, is the enumeration of the labeled boxes according to the standard order of T. The transposition of a standard tableau T is still a tableau denoted by T^t.

2.3. Knutson-Tao-Woodward puzzles. A puzzle of size n [KTW] is a tiling of an equilateral triangle of side length n with three kind of puzzle pieces

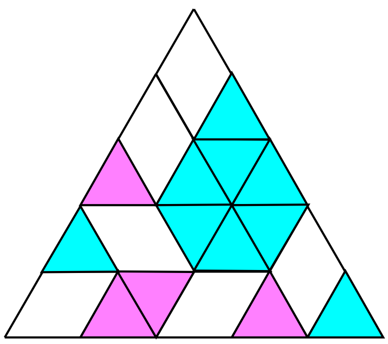
1. unit equilateral triangles with all edges labeled 1 (here also represented in blue colour);
2. unit equilateral triangles with all edges labeled 0 (here also represented in pink colour); and
3. unit rhombi (two equilateral triangles joined together) with the two edges clockwise of acute vertices labeled 0, and the other two labeled 1,



such that wherever two pieces share an edge, the numbers on the edge must agree. Puzzle pieces may be rotated in any orientation but rhombi can not be reflected.

The boundary data of the puzzle is (μ, ν, λ) if the partitions μ, ν and λ appear clockwise, starting in the left corner, as 01-words. The partitions μ,ν and λ as 01-words have exactly d 1's and $n-d$ 0's. This means that the blue unitary triangles constitute a triangle of size d, and the pink unitary triangles a triangle of size $n-d$.

A puzzle with $n = 5$, $d = 3$ and boundary $\mu = 01011$; $\nu = 01101$ and $\lambda = 10101$, read clockwise starting in the left corner,



Theorem 2.1. *(Knutson-Tao-Woodward)* [KTW] *The number of puzzles with μ, ν and λ appearing clockwise as 01-strings along the boundary is equal to $c_{\mu\,\nu\,\lambda}$.*

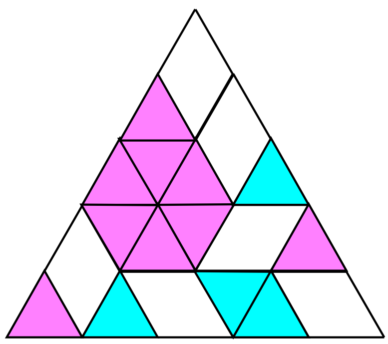
We recall that puzzles are in bijection with LR tableaux [KTW]. We shall use Tao's bijection without words which appears in [V] and also in [Pu].

3. The $\mathbb{Z}_2 \times \mathbb{S}_3$-symmetries

3.1. The puzzle H-subgroup of symmetries. The group of symmetries of a puzzle generated by the action of H is the group generated by the mirror reflections with 0 and 1's swapped which comprises the rotational symmetries. The action of $\tau\sigma_i$, $i = 1, 2$, and $\tau\sigma_1\sigma_2\sigma_1$ on a puzzle is the reflection of that puzzle such that the boundary data (μ, ν, λ) is respectively permuted according to σ_i, $i = 1, 2$, $\sigma_1\sigma_2\sigma_1$, and the 0 and 1's labels are swapped. They are the unique maps which swap the pink (label 0) and blue (label 1) colours in a puzzle such that the resulting tiled triangle is still a puzzle, equivalently, the blue triangle of size d becomes a pink triangle, and the pink triangle of size $n-d$ becomes a blue triangle. We shall denote them respectively by $\spadesuit$, $\clubsuit$ and $\blacklozenge$, and we may write

$$H = \langle \spadesuit, \clubsuit \rangle = \{\mathbf{1}, \spadesuit, \clubsuit, \spadesuit\clubsuit, \clubsuit\spadesuit, \clubsuit\spadesuit\clubsuit = \spadesuit\clubsuit\spadesuit\} \simeq \mathbb{S}_3.$$

For example, the action of $\tau\sigma_1$ on the puzzle above results in the puzzle



whose boundary data is $\nu^t = 01001$; $\mu^t = 00101$; $\lambda^t = 01010$. This illustrates $\spadesuit$. The size of the blue triangle is now two and the pink one has size three.

3.2. The LR tableau H-subgroup of symmetries. As LR tableaux are in bijection with puzzles, the group H also acts on LR tableaux. We analyse the group of symmetries on LR tableaux generated by this action. Intentionally we use for LR tableau H-symmetries the same notation as for the H-symmetries on puzzles. In the last section, we shall see that the symmetries defined in this section are exactly those we obtain under the bijection between puzzles and LR tableaux.

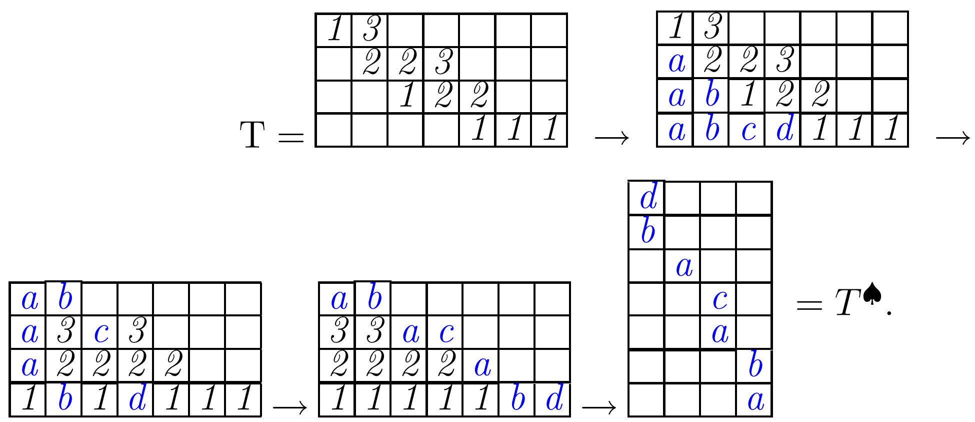
Let $LR(\mu, \nu, \lambda)$ denote the set of LR tableaux with boundary data (μ, ν, λ). We define the involution $\spadesuit : LR(\mu, \nu, \lambda) \longrightarrow LR(\nu^t, \mu^t, \lambda^t)$ as a five step procedure.

Definition 3.1 (Map $\spadesuit$). *Let $T \in LR(\mu, \nu, \lambda)$.*

1. *Fill the inner shape μ, using an alphabet different from the "numerical" filling of T, so that its transpose is $Y(\mu^t)$.*
2. *Slide down vertically the 1's to the first row, the 2's to the second row, the 3's to the third row and so on.*
3. *Slide horizontally the numbers to the left.*
4. *Transpose the resulting tableau.*
5. *Forget about the numerical filling.*

The example illustrates this procedure.

Example 3.2. *Let $d = 4$, $n = 11$, $\mu = (4, 2, 1, 0)$, $\nu = (5, 4, 3, 0)$ and $\lambda = (5, 3, 2, 0)$*



The procedure is clearly reversible. Next we check that it yields the desired tableau.

Let ξ_i, $i = 1, 2$, [BSS] denote the tableau-switching operation on the LR-multitableau of boundary data (μ, ν, λ) which switches the first two LR tableaux and the last two respectively.

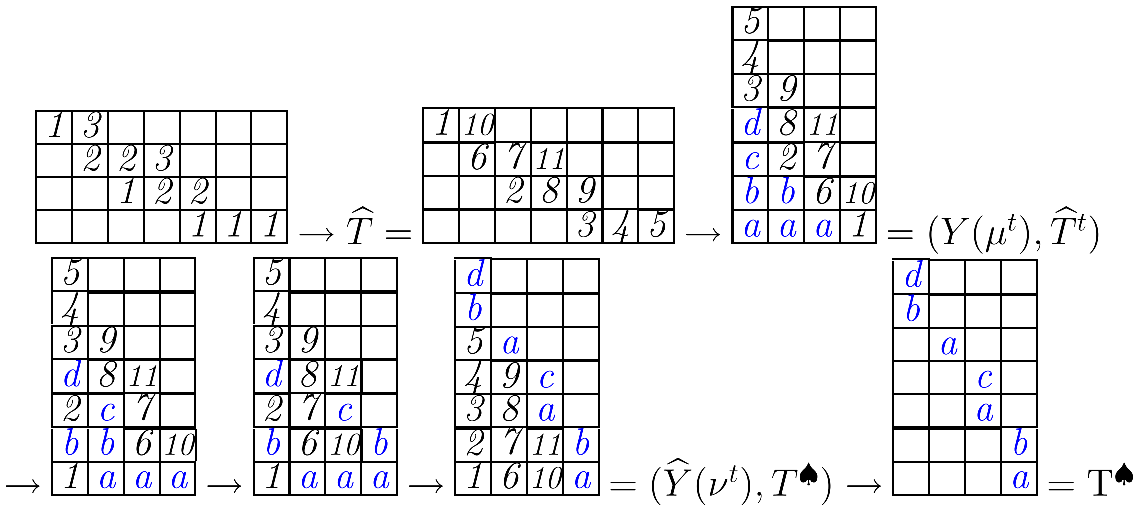
Proposition 3.1. *The map $\spadesuit$ is such that*

$$T \longrightarrow \widehat{T} \longrightarrow \widehat{T}^t \longrightarrow (Y(\mu^t), \widehat{T}^t, Y(\lambda^t)^{\mathrm{a}}) \longrightarrow \xi_1(Y(\mu^t), \widehat{T}^t, Y(\lambda^t)^{\mathrm{a}}) =$$
$$= (\widehat{Y}(\nu^t), T^{\spadesuit}, Y(\lambda^t)^{\mathrm{a}}).$$

Proof: The second and third steps of the definition of the map ♠ correspond to the action of the operation ξ_1 on $(Y(\mu^t), \widehat{T}^t)$. ■

The example illustrates the proposition, where for simplification we omit the $Y(\lambda)^{\mathrm{a}}$ filling.

Example 3.3. *Let* $d = 4$, $n = 11$, $\mu = (4, 2, 1, 0)$ $\nu = (5, 4, 3, 0)$ *and* $\lambda = (5, 3, 2, 0)$



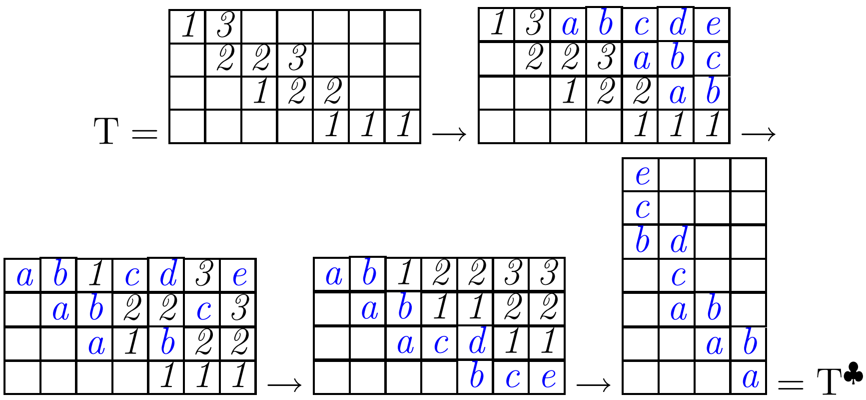
The involution $LR(\mu, \nu, \lambda) \xrightarrow{\clubsuit} LR(\mu^t, \lambda^t, \nu^t)$ is defined similarly as a five step procedure

Definition 3.4 (Map ♣). *Let* $T \in LR(\mu, \nu, \lambda)$.

(1) *Fill the outer shape* λ*, using an alphabet different from the "numerical" filling* T*, so that its transpose is* $Y(\lambda^t)^{\mathrm{a}}$.
(2) *Slide horizontally the numbers, in reverse numerical order, that is, . . . 3's, 2's and 1', to the right so that the semistandard condition on columns is not broken.*
(3) *Slide up vertically, in reverse numerical order, the numbers.*
(4) *Transpose the resulting tableau.*
(5) *Forget about the numerical filling.*

The example illustrates the procedure where we omit the $Y(\mu)$ filling.

Example 3.5.



The procedure is clearly reversible and as before we check that it yields the desired tableau.

Proposition 3.2. *The map* $\clubsuit$ *is such that*

$$T \longrightarrow \widehat{T} \longrightarrow \widehat{T}^t \longrightarrow (Y(\mu^t), \widehat{T}^t, Y(\lambda^t)^{\mathrm{a}}) \longrightarrow \xi_2(Y(\mu^t), \widehat{T}^t, Y(\lambda^t)^{\mathrm{a}}) =$$

$$= (Y(\mu^t), T^{\clubsuit}, \widehat{Y}(\nu^t)^{\mathrm{a}}).$$

Proof: The second and third steps of the definition of the map $\clubsuit$ correspond exactly to the action of the operation ξ_2 on $(\widehat{T}^t, Y(\lambda^t)^{\mathrm{a}})$. ■

We now briefly sketch some notions of Relative (Computational) Complexity which are needed. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. The map δ is said to have linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$, where $\langle A \rangle$ is the bit–size of A. A tableau A is encoded through its recording matrix $(c_{i,j})$, where $c_{i,j}$ is the number of j's in the ith row of A. A function f reduces linearly to g, if it is possible to compute f in time linear in the time it takes to compute g; f and g are linearly equivalent if f reduces linearly to g and vice versa. This defines an equivalence relation on functions. For further details, see for instance [ACM, CLRS, PV2].

Theorem 3.6. *The maps* $\spadesuit$ *and* $\clubsuit$ *have linear cost.*

Proof: Looking at the definitions of the maps $\spadesuit$ and $\clubsuit$, and examples 3.2, 3.5, it is clear that they can be performed using a fixed number, which does not depend on the size of the starting tableau or the starting filling, of scans of the input tableau. ■

In the next section we show that the H action on puzzles corresponds on LR-tableaux to the H action defined by the group

$$\langle \clubsuit, \spadesuit \rangle = \{\mathbf{1}, \spadesuit, \clubsuit, \spadesuit\clubsuit, \clubsuit\spadesuit, \clubsuit\spadesuit\clubsuit = \spadesuit\clubsuit\spadesuit\} \simeq S_3,$$

with $\spadesuit$ and $\clubsuit$ as described respectively in definitions 3.1 and 3.4. From the previous theorem, we may say that H is a linear time subgroup of index 2 of $\mathbb{Z}_2 \times S_3$. In particular, the $\mathbb{S}_3$ index two subgroup of symmetries is given by $\langle \spadesuit\clubsuit, \clubsuit\spadesuit \rangle$. We have not proved directly the braid factorisation of $\clubsuit\spadesuit\clubsuit = \spadesuit\clubsuit\spadesuit$. This follows from the bijection between puzzles and LR tableaux. It was observed in [BSS] that the tableau-switching maps ξ_1 and ξ_2 when applied to general semistandard multitableaux do not satisfy the braid factorisation. As a consequence, from propositions 3.1 and 3.2, we may conclude that, in the case of three-fold LR multitableaux, with shape factorisation as mentioned, we have $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$.

We make now explicit the involution $\blacklozenge := \clubsuit\spadesuit\clubsuit$ in a concise manner. Let Q denotes the Q-symbol or recording tableau of the Robinson-Schensted insertion procedure of a word. The involution $\blacklozenge : LR(\mu, \nu, \lambda) \longrightarrow LR(\lambda^t, \nu^t, \mu^t)$ is such that $T^\blacklozenge$ is the unique tableau in $LR(\lambda^t, \nu^t, \mu^t)$ with $Q(T^\blacklozenge) = Q(T)^t$. Moreover $T^\blacklozenge$ can be determined as in the following procedure. (For details see [ACM].)

Definition 3.7 (Map $\blacklozenge$). *Let $T \in LR(\mu, \nu, \lambda)$.*

(1) *Rotate T by 180 degrees.*
(2) *Replace the ν_i i's with $1, 2, \ldots, \nu_i$ according to the standard order on the boxes.*
(3) *Transpose.*

Example 3.8.

$$\mathrm{T} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 3 & & & & & \\ \hline & 2 & 2 & 3 & & & \\ \hline & & 1 & 2 & 2 & & \\ \hline & & & & 1 & 1 & 1 \\ \hline \end{array} \xrightarrow{\blacklozenge} \mathrm{T}^\blacklozenge = \begin{array}{|c|c|c|c|}\hline 5 & & & \\ \hline 2 & 4 & & \\ \hline & 3 & 4 & \\ \hline & 1 & 2 & \\ \hline & & 1 & 3 \\ \hline & & & 2 \\ \hline & & & 1 \\ \hline \end{array}$$

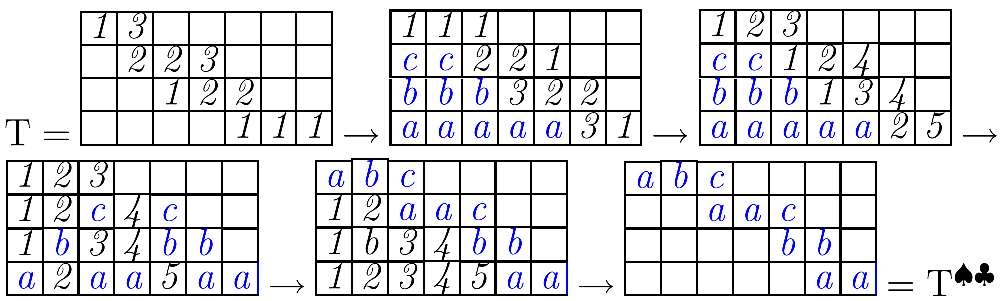
The rotation symmetries on LR tableaux can be easily performed noting that $\spadesuit\blacklozenge = \clubsuit\spadesuit$ and $\clubsuit\blacklozenge = \spadesuit\clubsuit$.

Definition 3.9 (Rotation symmetry map $\spadesuit\blacklozenge$). *Let $T \in LR(\mu, \nu, \lambda)$.*

(1) *Rotate T by* 180 *degrees.*
(2) *Fill the inner shape λ with the Yamanouchi tableau $Y(\lambda)$ using an alphabet different from the "numerical" filling of T.*
(3) *Replace the ν_i i's with $1, 2, \ldots, \nu_i$ according to the standard order on the boxes.*
(4) *Slide horizontally the* 1*'s to the first column, the* 2*'s to the second column, the* 3*'s to the third column and so on.*
(5) *Slide down vertically the numbers.*
(6) *Forget about the numerical filling.*

The example illustrates the procedure from which we see that it is reversible.

Example 3.10.



The rotation symmetry ♣♦ can be performed in a similar way, considering $Y(\mu)^{\mathrm{a}}$ as the filling of the outer shape μ after rotating T 180 degrees.

We may now observe that, in these symmetries, the tableau-switching involution is always performed with very simple slides, therefore, in order to get maps of linear cost, the operation to be avoided is the standardization which we have done.

3.3. The symmetries outside of H. Consider two equivalence relations on a pair of tableaux. Two tableaux are Knuth equivalent [K] if one can be obtained from the other by a sequence of (reverse) *jeu de taquin* slides, and, therefore, both have the same rectified tableau. They are dual Knuth equivalent [H] if such a (any) sequence results in tableaux of the same shape. In particular, both have the same shape and the same rectification shape. Tableaux of the same (anti) normal shape are dual equivalent. A pair of tableaux that are both Knuth and dual Knuth equivalent must be equal. If $\mathcal{D}$ is a dual Knuth equivalence class and $\mathcal{K}$ is a Knuth equivalence class, both corresponding to the same rectification shape, then there is a unique tableau in $\mathcal{D} \cap \mathcal{K}$.

In [A2, A3, ACM, F, PV2, DK], it has been shown that the known LR conjugation symmetry maps [HS, A1, A2, BSS] as well as the commutativity symmetry maps [A3, HK, BSS, PV2] are identical. In fact they can be uniquely characterized by Knuth, dual Knuth equivalence and the complementary property of Horn inequalities. The involutions $\spadesuit$, $\clubsuit$ and $\blacklozenge$ are better explained geometrically on puzzles: they are the unique maps which swap the pink (label 0) and blue (label 1) colours in a puzzle such that the resulting tiled triangle is still a puzzle. Let $\varrho : LR(\mu,\nu,\lambda) \longrightarrow LR(\mu^t,\nu^t,\lambda^t)$ be the conjugation symmetry involution defined by

$$\varrho(T) = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK} \qquad \text{see [BSS]},$$

where the brackets $[\ ]_k$ and $[\ ]_{dK}$ denote respectively the Knuth and dual Knuth equivalence class. That is, $\varrho(T)$ is the unique LR tableau of boundary data (μ^t,ν^t,λ^t) dual Knuth equivalent to $\widehat{T}^t$. Let $\rho : LR(\nu,\mu,\lambda) \longrightarrow LR(\lambda,\nu,\mu)$ be the commutativity symmetry involution such that

$$\rho(T) = [Y(\nu)]_K \cap [T^\bullet]_{dK}.$$

In other words, $\rho(T)$ is the unique LR tableau of boundary data (λ,ν,μ) dual Knuth equivalent to $T^\bullet$ [BSS, ACM]. In [BSS], it was defined T^e, the reverse of T, as the unique tableau Knuth equivalent to $T^\bullet$ and dual equivalent to T. The map $T \longrightarrow T^e = [Y(\nu)^\bullet]_K \cap [T]_{dK}$ is an involution called reversal. Therefore,

$$\rho = e\ \bullet .$$

We define the bijections $\rho_1 : LR(\mu,\nu,\lambda) \longrightarrow LR(\nu,\mu,\lambda)$ and
$\rho_2 : LR(\mu,\nu,\lambda) \longrightarrow LR(\mu,\lambda,\nu)$ as follows

$$\varrho = \spadesuit\,\rho_1 = \blacklozenge\,\rho = \clubsuit\rho_2.$$

Equivalently, $\rho_1 = \spadesuit\blacklozenge\, e \bullet = \clubsuit\spadesuit\, e \bullet$, $\varrho = \blacklozenge\, e \bullet$, $\rho_2 = \clubsuit\blacklozenge\, e \bullet = \spadesuit\clubsuit\, e \bullet$, $\rho = e \bullet$. The two remaining symmetry involutions, exhibiting $c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\lambda^t\,\mu^t} = c_{\lambda^t\,\mu^t\,\nu^t}$, are given by $\clubsuit\spadesuit\varrho = \spadesuit\, e \bullet$ and $\spadesuit\clubsuit\varrho = \clubsuit\, e \bullet$ respectively. The symmetries outside of H are therefore linearly reducible to each other, and, in particular, to the reversal involution e. On its turn, as shown in [ACM], the reversal map e is linearly reducible to the Schützenberger involution E,

$$\begin{tikzcd} T \arrow[r, leftrightarrow, "e"] \arrow[d, leftrightarrow, "\tau"'] & T^e \arrow[r, leftrightarrow, "\bullet"] \arrow[d, leftrightarrow, "\tau"'] & T^{e\bullet} \arrow[d, leftrightarrow, "\tau"'] \\ P \arrow[r, leftrightarrow, "\mathrm{a}"'] & P^{\mathrm{a}} \arrow[r, leftrightarrow, "\bullet"] & P^E. \end{tikzcd}$$

The transposition of the recording matrix of a LR tableau is the recording matrix of a tableau of straight shape. The linear map τ defines a bijection between tableaux of normal (straight) shape and LR tableaux [Lee1, Lee2, PV2, O]. Hence, we may write

$$\mathbb{Z}_2 \times S_3 = \langle \clubsuit, \blacklozenge, \rho : \rho^2 = \clubsuit^2 = \blacklozenge^2 = (\clubsuit\blacklozenge)^3 = (\clubsuit\rho)^2 = (\blacklozenge\rho)^2 = 1 \rangle.$$

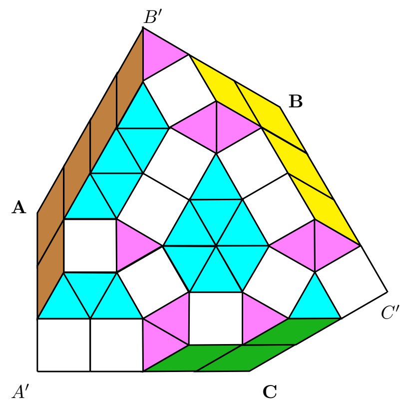
In [Pu2] are given geometrical interpretations for most of the combinatorial operations on tableaux as *jeu de taquin*, and also Knuth equivalence and dual Knuth equivalence. We wonder whether such an interpretation will give further links on the symmetries of LR coefficients.

4. Purbhoo mosaics: a rhombus-square-triangle model

In this section we follow closely [Pu]. Consider a puzzle of side length n and replace the rhombi by unitary squares. The puzzle will be distorted and a convex diagram can be recovered by adding thin rhombi with angles of 150 and 30 degrees to the three distorted edges of the puzzle. If one forgets the labels on the puzzle pieces, the resulting diagram, called mosaic, is a tiled hexagon by the following three shapes:

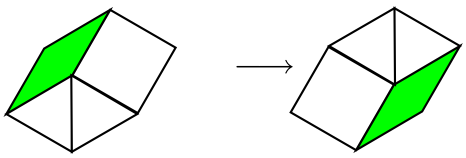
(a) the equilateral triangle with side length 1;
(b) the square with side length 1;
(c) the rhombus with side length 1 and angles 30 and 150 degrees,

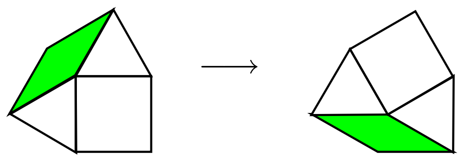
in a way that all rhombi are packed into the three nests A, B and C of the hexagon. See the picture below where the colours should be looked at this point as decoration. The mosaic has side lengths $A'A = B'B = C'C = n-d$ and $AB' = BC' = CA' = d$. The collections of rhombi in the nests A, B and C, denoted respectively by α, β, and γ, define the boundary data (α, β, γ) of the mosaic.



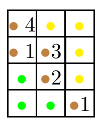
From this construction we see a natural bijection between mosaics and puzzles. Walking clockwise from A', this extra rhombi can be regarded as the three Young diagrams encoded by the 01-words on the boundary of the puzzle. By removing the extra rhombi and straightening, we can go from a mosaic to a puzzle: walking from A' to B', the shape left by removing α turns into the string of 0's and 1's, 0 for each unit step west, and a 1 for each unit step north; and straightening the squares they will become 30/60 degrees rhombi. This will determine the remain labels of the puzzle pieces. (Similarly, walking anticlockwise from C' to B', we get the dual puzzle, that is, the one obtained by mirror reflection and label swapping.) In the standard orientation, that is, read clockwise, a mosaic of boundary (α, β, γ) can be identified with the corresponding puzzle of boundary data (μ, ν, λ), where α is identified with the Young diagram of μ, β with ν, and γ with λ. The number of mosaics with boundary data (α, β, γ) is equal to the number of puzzles of boundary (μ, ν, λ).

One of the advantages of mosaics over puzzles is that we can give different orientations to the nests A, B, and C. This allows us to relate the symmetry bijections on puzzles and on LR tableaux. Define unit vectors E_A, N_A, E_B, N_B, E_C, N_C in the directions of AA', AB', BB', BC', CC', CA' respectively, and fix orientations (E_A, N_A), (E_B, N_B), (E_C, N_C) on the nests at A, B, and C respectively. The letters E, N, $-E$, $-N$ are thought as compass directions east, north, west and south, respectively. Thus the orientations (E, N) and (N, E) in a nest means respectively the standard or clockwise orientation, and the counter clockwise orientation. Flocks are (skew) tableau-like structures, defined on the thin rhombi in a mosaic, packed into one of the nests A, B or C. Four orientations can be given to a nest. Each orientation uniquely determines the flock as an LR tableau. Fix a nest, say A, the rhombi α under the orientation (E, N) becomes the Yamanouchi tableau $Y(\mu)$; under (N, E) becomes $Y(\mu^t)$; under $(-E, -N)$ becomes $Y(\mu)^{\mathrm{a}}$; and under $(-N, -E)$ becomes $Y(\mu^t)^{\mathrm{a}}$. Migration is an invertible operation that takes a flock to a new nest. The rhombi must move in the standard order in a tableau. Choose the target nest. Rhombi move in the chosen direction of migration, inside a smallest hexagon in which the thin rhombus $\diamondsuit$ is contained:

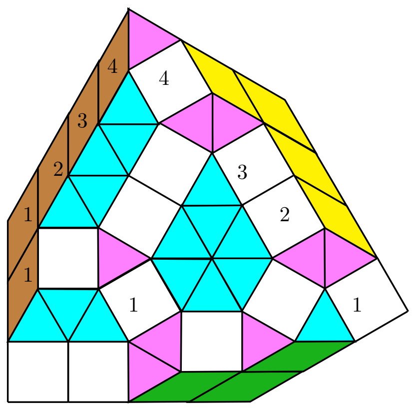




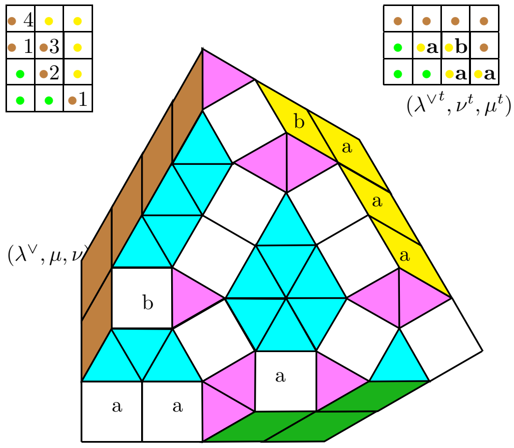
Migration from the left to the right side of the mosaic, of the flock α with standard orientation (E, N), gives a bijection between mosaics of boundary (α, β, γ) and LR tableaux of boundary $(\lambda^\vee, \mu, \nu)$ where $\nu = \beta$, $\lambda^\vee = \gamma$ and $\mu = \alpha$. This bijection coincides with the Tao's bijection without words [V], between LR tableaux of boundary data $(\lambda^\vee, \mu, \nu)$ and puzzles of boundary (μ, ν, λ). An illustration of the Tao's bijection which coincides with the migration from the left to the right side of the mosaic:



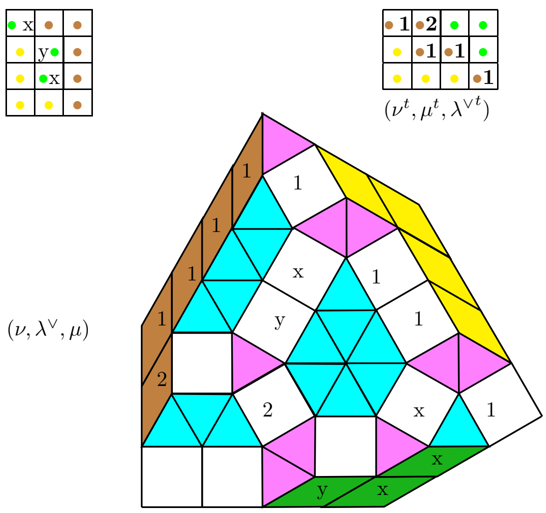
$(\lambda^\vee, \mu, \nu)$



Next we see that the bijections $\clubsuit, \spadesuit, \blacklozenge$ defined by the action of the group H on puzzles and on LR tableaux are exactly what we get in mosaics. In this discussion it is better to consider the presentation $H =< \clubsuit, \clubsuit\blacklozenge >$. Migration from the right to the left side of the mosaic, of the flock β with orientation (N, E) (or read counterclockwise), coincides with the Tao's bijection on the back side of the mosaic, that is, reflects the mosaic along the vertical axis and swap the colours in the puzzle. This defines simultaneously the $\clubsuit$ involution on puzzles and on LR tableaux, as illustrated below.

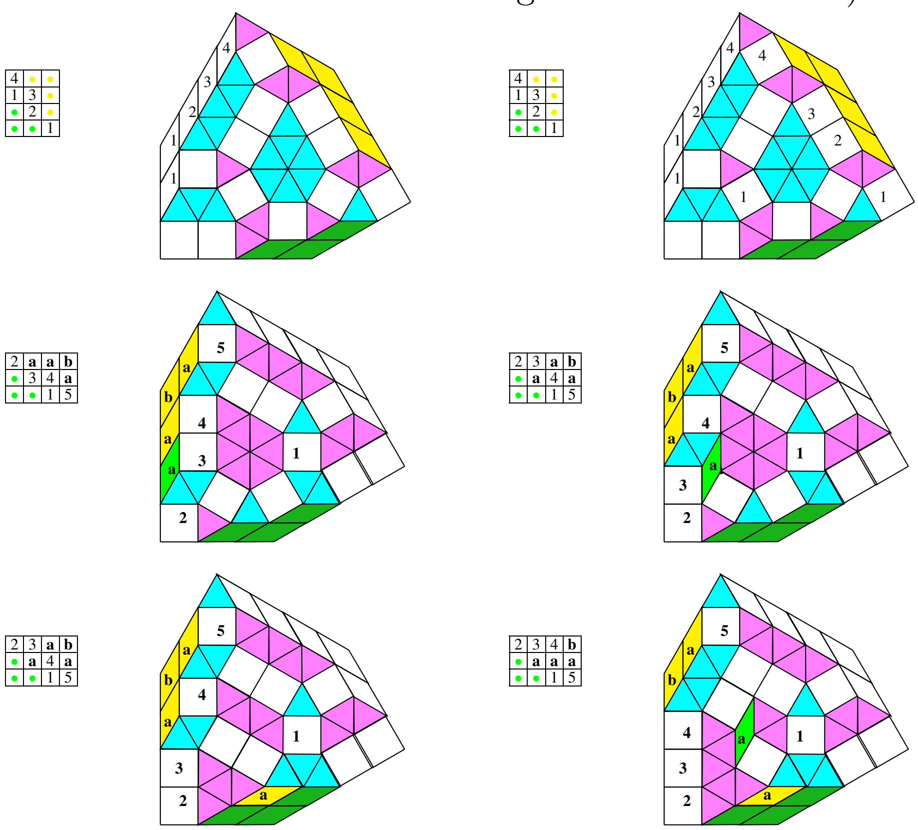


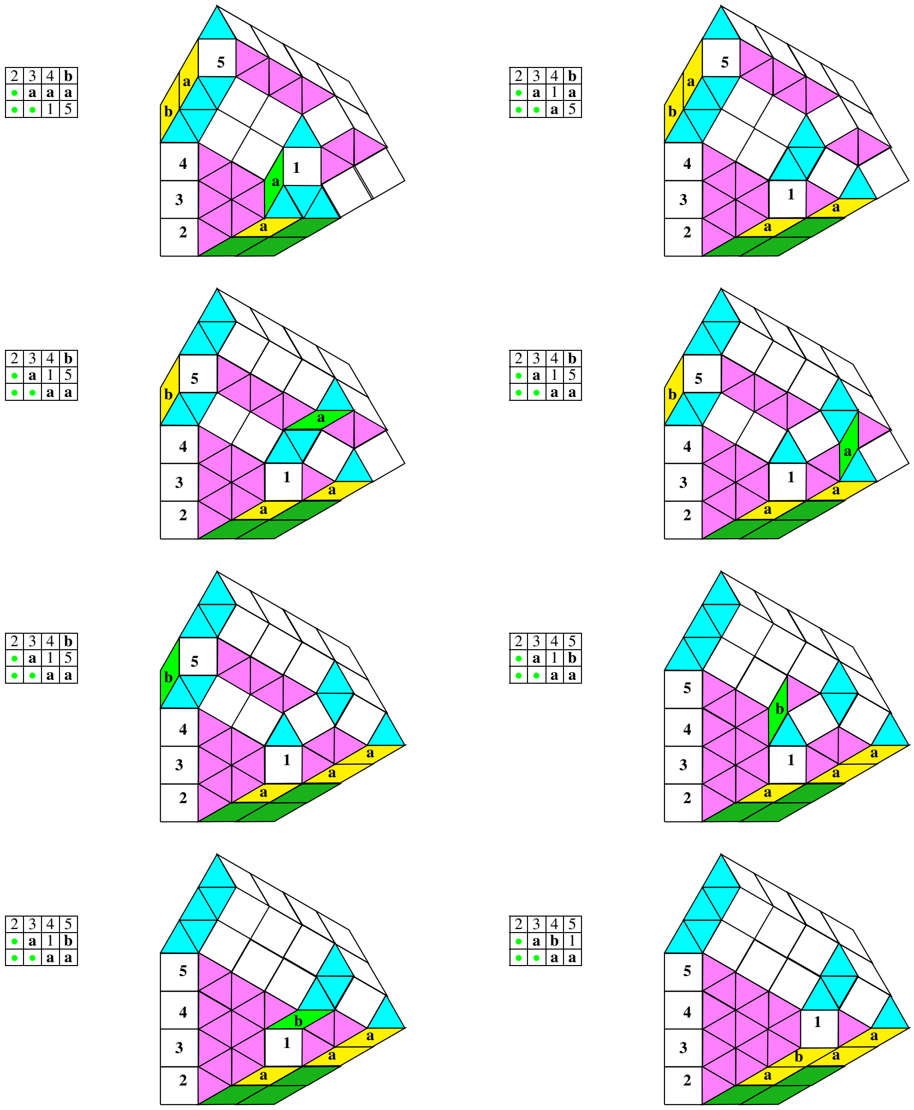
Migration from the left to the bottom side of the mosaic, of the flock α with orientation (N, E) (or read anticlockwise), coincides with Tao's bijection on the back side of the mosaic after rotating it 120 degrees clockwise. This defines ♦ involution on puzzles and on LR tableaux. Migration from the bottom to the left side of the mosaic, of the flock γ with standard orientation (E, N), corresponds to a rotation of 120 degrees clockwise in the mosaic, and this defines the involution ♣♦. We illustrate both involutions at the same time.



Similarly, migration of the flock γ with orientation (N, E) (read anticlockwise), from the bottom to the right side of the mosaic, defines the involution ♠ = ♣(♦♣). It coincides with Tao's bijection on the back side of the mosaic after rotating it 120 degrees counter clockwise (or 240 degrees clockwise).

In [Pu] it is discussed how the migration of a single rhombus in a mosaic is related with *jeu de taquin* slides on LR tableaux. This explains the correspondence between the action of $\mathbb{Z}_2 \times \mathbb{S}_3$ on puzzles and on LR tableaux. We have seen that ♣ can be described as the migration of the flock β with orientation (N, E), thus identified with $Y(\nu^t)$, from the left to the right side of the mosaic. However ♣ can also be described as the migration of the flock β with orientation $(-N, -E)$, thus identified with $Y(\nu^t)^{\mathrm{a}}$, from the right to the bottom side of the mosaic. Equivalently, on the back of the mosaic, it is the migration from the left to the bottom side, of the corresponding flock ν with orientation $(-E, -N)$, thus identified with $Y(\nu)^{\mathrm{a}}$. Combining the Tao's bijection, which represents the mosaic of boundary (α, β, γ) by a puzzle of boundary (μ, ν, λ) and by a LR tableau of boundary $(\lambda^\vee, \mu, \nu)$, with this later description of ♣, we get Proposition 3.2. This is illustrated below. (Note that some of the moves in the migration are omitted.)





If one forgets the numerical filling, the last mosaic gives the tableau already obtained with Tao's bijection above.

References

[A1] Olga Azenhas. *The admissible interval for the invariant factors of a product of matrices*, Linear and Multilinear Algebra 46 (1999), 51–99.

[A2] Olga Azenhas. *Littlewood–Richardson fillings and their symmetries*, Matrices and group representations (Coimbra, 1998), Textos Mat. Sér. B, 19, Univ. Coimbra, Coimbra, 1999, 81–92.

[A3] Olga Azenhas. *A variation on tableau switching and a Pak-Vallejo's Conjecture*, Discrete Mathematics & Theoretical Computer Science Proceedings, AJ, (2008) 529–542. Slides from the presentation available at `http://inst-mat.utalca.cl/fpsac2008/talks/Azenhas.pdf`

[ACM] Olga Azenhas, Alessandro Conflitti, Ricardo Mamede.*Linear time equivalent Littlewood-Richardson coefficient maps*, Discrete Mathematics & Theoretical Computer Science Proceedings, AK, (2009) 127–144.

[BSS] Georgia Benkart, Frank Sottile, Jeffrey Stroomer. *Tableau switching: algorithms and applications*, J. Combin. Theory Ser. A 76 (1996), 11–34.

[BZ] Arkady Berenstein, Andrei Zelevinsky. *Triple multiplicities for* $\mathrm{sl}(r+1)$ *and the spectrum of the exterior algebra of the adjoint representation.* J. Algebraic Combin. 1 (1992), 7–22.

[CLRS] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*, The MIT Press, 2nd edition, 2001.

[DK] Vladimir I. Danilov and Gleb A. Koshevoĭ. *Arrays and the octahedron recurrence*, available at `http://arxiv.org/abs/math/0504299`

[F] William Fulton. Young Tableaux: With Applications to Representation Theory and Geometry, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, USA, 1997.

[H] Mark Haiman. *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math. 99 (1992), 79-113.

[HS] Philip Hanlon, Sheila Sundaram. *On a bijection between Littlewood–Richardson fillings of conjugate shape*, J. Combin. Theory Ser. A 60 (1992), 1–18.

[HK] André Henriques, Joel Kamnitzer. *The octahedron recurrence and* $\mathfrak{gl}_\mathfrak{n}$ *crystals*, Adv. Math. 206 (2006), 211–249.

[K] Donald E. Knuth. *Permutations, matrices, and generalized Young tableaux*, Pacific Journal of Mathematics, 34 (1970), 709–727.

[KT] A. Knutson, T. Tao. *The honeycomb model of GLn(C) tensor products I: proof of the saturation conjecture*, J. of Amer. Math. Soc, 12 (1999), no. 4, 1055–1090.

[KTW] A. Knutson, T. Tao and C. Woodward. *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. II: Puzzles determine facets of the Littlewood–Richardson cone*, J. Amer. Math. Soc. 17 (2004), 19–48.

[LiRi] Dudley E. Littlewood, Archibald R. Richardson. *Group characters and algebra*, Philos. Trans. Roy. Soc. London Ser. A 233 (1934), 99-142.

[Lee1] Mark A. A. van Leeuwen. *Tableau algorithms defined naturally for pictures*, Discrete Math. 157 (1996), 321–362.

[Lee2] Mark A. A. van Leeuwen. *The Littlewood–Richardson rule and related combinatorics*, in Interaction of combinatorics and representation theory, 95–145, MSJ Mem., 11, Math. Soc. Japan, Tokyo, 2001.

[O] Erik Ouchterlony. *Commutation for Young tableau involutions*, Linköping University, Phd Thesis, 2005.Available at `http://arxiv.org/abs/0908.0255`

[PV1] Igor Pak, Ernesto Vallejo. *Combinatorics and geometry of Littlewood-Richardson cones*, European J. Combin. 26 (2005), 995–1008.

[PV2] Igor Pak, Ernesto Vallejo. *Reductions of Young tableau bijections*, to appear in SIAM J. Discrete Math. Available at `http://www.arXiv:math/0408171`

[Pu] Kevin Purbhoo. *Puzzles, tableaux, and mosaics*,J. Algebraic Combin., 28 (2008), 461–480.

[Pu2] Kevin Purbhoo.*Jeu de taquin and a monodromy problem for Wronskians of polynomials.* Available at `http://arXiv:0902.1321v3`

[V] Ravi Vakil. *A geommetric Littlewood-Richardson rule*, Ann. Math. 164 (2006), 371–422.

Olga Azenhas
CMUC, Centre for Mathematics, University of Coimbra, Apartado 3008, 3001–454 Coimbra, Portugal.

E-mail address: `oazenhas@mat.uc.pt`

Alessandro Conflitti

CMUC, Centre for Mathematics, University of Coimbra, Apartado 3008, 3001–454 Coimbra, Portugal.

E-mail address: conflitt@mat.uc.pt

Ricardo Mamede
CMUC, Centre for Mathematics, University of Coimbra, Apartado 3008, 3001–454 Coimbra, Portugal.

E-mail address: mamede@mat.uc.pt