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PRESENTATIONS OF SCHÜTZENBERGER GROUPS OF MINIMAL SUBSHIFTS

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ABSTRACT: In previous work, the first author established a natural bijection between minimal subshifts and maximal regular $\mathcal{J}$-classes of free profinite semigroups. In this paper, the Schützenberger groups of such $\mathcal{J}$-classes are investigated in particular in respect to a conjecture proposed by the first author concerning their profinite presentation. The conjecture is established for several types of minimal subshifts associated with substitutions. The Schützenberger subgroup of the $\mathcal{J}$-class corresponding to the Prouhet-Thue-Morse subshift is shown to admit a somewhat simpler presentation, from which it follows that it satisfies the conjecture, that it has rank three, and that it is non-free relatively to any pseudovariety of groups.

KEYWORDS: Profinite group presentation, free profinite semigroup, subshift, Prouhet-Thue-Morse substitution.
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1. Introduction

In recent years, several results on closed subgroups of free profinite semigroups have appeared in the literature [2, 3, 4, 15, 18]. The first author explored a link between symbolic dynamics and free profinite semigroups that allowed him to show, for several classes of maximal subgroups of free profinite semigroups, all associated with minimal subshifts [2, 3], that they are free profinite groups. Rhodes and Steinberg [15] proved that the closed subgroups of free profinite semigroups are precisely the projective profinite groups. Without using ideas from symbolic dynamics, Steinberg proved that the Schützenberger group of the minimal ideal of the free profinite semigroup over a finite alphabet with at least two letters is a free profinite group with infinite countable rank [18]. The same result holds for the Schützenberger group of the regular $\mathcal{J}$-class associated to a non-periodic irreducible sofic

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subshift [7]; the proof is based on the techniques of [18] and on the conjugacy invariance of the group for arbitrary subshifts [6].

In this paper, we investigate the minimal subshift associated with the iteration of a substitution φ over a finite alphabet A and the Schützenberger group $G(\varphi)$ of the corresponding $\mathcal{J}$-class of the free profinite semigroup on A. A minimal subshift can be naturally associated with the substitution φ if and only if it is *weakly primitive* [2, Theorem 3.7]. Such a substitution always admits a so-called *connection*, which is a special two-letter block ba of the subshift. Provided φ is an encoding of bounded delay, from the set X of return words for ba, which constitute a finite set, one can then obtain a generating set for $G(\varphi)$ by cancelling the prefix b, adding the same letter as a suffix, and applying the idempotent (profinite) iterate φ^ω. In a lecture given at the *Fields Workshop on Profinite Groups and Applications* (Carleton University, August 2005), the first author proposed, as a problem, a natural profinite presentation for $G(\varphi)$, namely

$$\langle X \mid \Phi^\omega(x) = x \ (x \in X)\rangle, \tag{1.1}$$

where Φ is the continuous endomorphism of the free profinite group on X which encodes the action of a power of φ which acts on the semigroup freely generated by X.

The main results of this paper establish that indeed $G(\varphi)$ admits the presentation (1.1) in the following cases:

(1) the mapping φ is the Prouhet-Thue-Morse substitution τ, given by $\tau(a) = ab$ and $\tau(b) = ba$ (Corollary 5.2);
(2) the substitution φ is ultimately group invertible, in the sense that it induces an automorphism of the free profinite group on the letters that do not eventually disappear in the iteration of φ (Theorem 5.3);
(3) all words $\varphi(a)$ ($a \in A$) start with the same letter and end with the same letter (Theorem 5.8).

This adds evidence to the conjecture that the group $G(\varphi)$ always admits the above presentation (1.1).

Every finitely generated projective profinite group has a finite presentation [13] (as a profinite group). Hence, by the previously mentioned result obtained by Rhodes and Steinberg, every finitely generated closed subgroup of a free profinite semigroup has some finite presentation. Yet, the conjecture is finer since it entails the decidability of whether a finite group is a continuous homomorphic image of $G(\varphi)$ (Corollary 6.3). On the other hand, in case φ induces an automorphism of $G(\varphi)$, we may always replace Φ in (1.1) by a continuous endomorphism of the free profinite group on X to obtain a presentation of $G(\varphi)$ (Proposition 3.5).

The Prouhet-Thue-Morse infinite word and the corresponding subshift are among the most studied in the literature [8]. To establish case (1) of the above conjecture, we first prove that the profinite group $G(\tau)$ admits a related profinite presentation with three generators and three relations (Theorem 4.3). Using just the fact that the relations are satisfied, we show that $G(\tau)$ cannot be relatively free with respect to any pseudovariety of groups (Theorem 6.6). This answers in a very strong sense the question raised by the first author as to whether this profinite group is free [2]. In the same paper there is already an argument to reduce the proof of this fact to showing that the Schützenberger group $G(\tau)$ has rank three. From the same simpler presentation, we do prove that this group has rank three (Theorem 6.7).

We also consider the only other type of example in the literature of a non-free Schützenberger group $G(\varphi)$ of a subshift defined by a substitution, illustrated by the substitution $\varphi(a) = ab$, $\varphi(b) = a^3b$ [2, Example 7.2], which fits in the framework of the above case (3). For this group, again we prove that it is not free relatively to any pseudovariety of groups (Theorem 6.4).

2. Preliminaries

We indicate [3, 16] as supporting references on free profinite semigroups, and [11, 8] for symbolic dynamics.

Let A be a finite alphabet. We denote by A^+ the free semigroup on A. A *code* is a nonempty subset of A^+ that generates a free subsemigroup. An equality $u_1 \cdots u_m = v_1 \cdots v_n$ with $u_i, v_j \in A^+$ is said to be *reducible* if there are indices r and s such that $2 < r + s \leq m + n$ and $u_r \cdots u_m = v_s \cdots v_n$. A subset C of A^+ is *of bounded delay with respect to* a language $L \subseteq A^+$ if there is an integer N such that every equality of one of the forms $uc_1 \cdots c_m v = c'_1 \cdots c'_n$ or $uc_1 \cdots c_m = c'_1 \cdots c'_n v$ is reducible whenever the two products belong to L, $c_i, c'_j \in C$, u is a suffix and v a prefix of some word in the elements of C, and $m + n > N$. A homomorphism $\varphi : A^+ \to B^+$ is an *encoding* if it is injective and *of bounded delay with respect to* $L \subseteq B^+$ if so is the set $\varphi(A)$.

We denote by $\overline{\Omega}_A\mathsf{S}$ (respectively $\overline{\Omega}_A\mathsf{G}$) the free profinite semigroup (respectively free profinite group) freely generated by A [3]. A *symbolic dynamical system* $\mathcal{X}$ of $A^{\mathbb{Z}}$, also called *subshift* or *shift space* of $A^{\mathbb{Z}}$, is a nonempty closed subset of $A^{\mathbb{Z}}$ invariant under the shift operation and its inverse [11]. We denote by $L(\mathcal{X})$ the set of *finite blocks* of elements of $\mathcal{X}$.

A subshift $\mathcal{X}$ is *minimal* if it does not contain proper subshifts. There is another useful characterization of minimal subshifts, with a combinatorial flavor. Given a subshift $\mathcal{X}$ and $u \in L(\mathcal{X})$, say that a nonempty word v is a *return word of* u *in* $\mathcal{X}$ if $vu \in L(\mathcal{X})$, u is a prefix of vu and u occurs in vu only as a prefix and a suffix. See [5] for a recent account on return words. A subshift is minimal if and only if each of its blocks has a finite set of return words.

If the subshift $\mathcal{X}$ is *minimal*, then the topological closure of $L(\mathcal{X})$ in $\overline{\Omega}_A\mathsf{S}$ is the disjoint union of $L(\mathcal{X})$ and a $\mathcal{J}$-class $J(\mathcal{X})$ of maximal regular elements of $\overline{\Omega}_A\mathsf{S}$. The correspondence $\mathcal{X} \mapsto J(\mathcal{X})$ is a bijection between the set of minimal subshifts of $A^{\mathbb{Z}}$ and the set of maximal regular $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{S}$ [2, Section 2]. It is natural to ask what is the structure of the Schützenberger group (that is, of any maximal subgroup) of $J(\mathcal{X})$, denoted $G(\mathcal{X})$ and called the *Schützenberger group of* $\mathcal{X}$. For instance, it is proved in [2] that, if $\mathcal{X}$ is an Arnoux-Rauzy subshift of degree k, of which the case $k = 2$ is that of the extensively studied Sturmian subshifts [12, 8], then $G(\mathcal{X})$ is a free profinite group of rank k. An example of a minimal subshift $\mathcal{X}$ such that $G(\mathcal{X})$ is not freely generated, with rank two, is also given in the same paper [2, Example 7.2].

A *substitution over a finite alphabet* A is an endomorphism of the free semigroup A^+. It is well known that to each *primitive substitution* φ over a finite alphabet A, we can associate a minimal subshift $\mathcal{X}_\varphi$. Actually, it suffices to assume that φ is *weakly primitive*, which means that there exists some positive integer n such that all $\varphi^n(a)$ $(a \in A)$ have length at least two and the same factors of length two. The language $L(\mathcal{X}_\varphi)$ is the set of factors of words of the form $\varphi^k(a)$, where a is an arbitrary element of A, and $k \geq 1$ is sufficiently large. See [2] for consequences of this definition and for the following technical notions. We shall denote $J(\mathcal{X}_\varphi)$ and $G(\mathcal{X}_\varphi)$ respectively by $J(\varphi)$ and $G(\varphi)$: this notation is more synthetic and emphasizes the exclusive dependence of these structures on φ, which in turn is a mathematical object completely determined by a finite amount of data, namely the images of letters by φ.

The unique continuous endomorphism of $\overline{\Omega}_A\mathsf{S}$ extending φ will also be denoted by φ. We shall use the fact that the monoid $\operatorname{End} S$ of continuous endomorphisms of a finitely generated profinite semigroup S is profinite for the pointwise topology, which coincides with the compact-open topology [3, Section 4.3]. In particular, this justifies the consideration of the idempotent continuous endomorphism φ^ω.

A *connection for* φ is a word ba, with $b, a \in A$, such that $ba \in L(\mathcal{X}_\varphi)$, the first letter of $\varphi^\omega(a)$ is a, and the last letter of $\varphi^\omega(b)$ is b. Every weakly primitive substitution has a connection. For a connection ba, the intersection of the $\mathcal{R}$-class containing $\varphi^\omega(a)$ with the $\mathcal{L}$-class containing $\varphi^\omega(b)$ is a maximal subgroup of $J(\varphi)$; we also use the notation $G(\varphi)$ to refer to this specific maximal subgroup. If, additionally, φ is an encoding of bounded delay with respect to the factors of $J(\varphi)$, then the group $G(\varphi)$ is generated by the set $\varphi^\omega(X_\varphi(a,b))$, where $X_\varphi(a,b) = b^{-1}R(ba)b$ and $R(ba)$ is the set of return words of ba in $\mathcal{X}_\varphi$.

We now introduce the main example considered in this paper. Let A be the two-letter alphabet $\{a, b\}$. The *Prouhet-Thue-Morse substitution* is the substitution τ over A given by $\tau(a) = ab$ and $\tau(b) = ba$ [8]. It is a primitive substitution, and thus we can consider the corresponding minimal subshift $\mathcal{X}_\tau$ of $A^{\mathbb{Z}}$. It is also an encoding of bounded delay with respect to the factors of $J(\tau)$.

3. Presentations of profinite groups

Consider a profinite group T and an onto continuous homomorphism π from $\overline{\Omega}_X\mathsf{G}$ onto T, where X is a finite set. Let φ be a continuous endomorphism of T. Since $\overline{\Omega}_X\mathsf{G}$ is a projective profinite group, there is at least one continuous endomorphism $\Phi : \overline{\Omega}_X\mathsf{G} \to \overline{\Omega}_X\mathsf{G}$ such that $\pi \circ \Phi = \varphi \circ \pi$, that is, such that Diagram (3.1) commutes. Call such an endomorphism a *lifting of* φ *via* π.

$$\begin{tikzcd} \overline{\Omega}_X\mathsf{G} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \overline{\Omega}_X\mathsf{G} \arrow[d, "\pi"] \\ T \arrow[r, "\varphi"] & T \end{tikzcd} \tag{3.1}$$

Remark 3.1. *If φ is an automorphism of T then $\pi \circ \Phi^\omega = \pi$.*

Proof: The facts that Diagram (3.1) commutes and π is continuous entail the equality $\pi \circ \Phi^\omega = \varphi^\omega \circ \pi$. On the other hand, φ^ω is the identity on T because φ is an automorphism of T. ∎

From hereon we assume that φ is an automorphism. Put $R = \{\Phi^\omega(x)x^{-1} : x \in X\}$ and let N be the topological closure of the normal closure of R.

From Remark 3.1 it follows that $R \subseteq \operatorname{Ker}\pi$, thus $N \subseteq \operatorname{Ker}\pi$. If $N = \operatorname{Ker}\pi$, then $\langle X \mid \Phi^\omega(x) = x,\ (x \in X)\rangle$ is a presentation of T (as a profinite group).

Lemma 3.2. *We have $N = \operatorname{Ker}\pi$ if and only if $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\Phi^\omega$, if and only if $\operatorname{Ker}\pi = \operatorname{Ker}\Phi^\omega$.*

For proving Lemma 3.2, the following lemma will be useful. It is part of the proof of [13, Proposition 1.1], which states that every finitely generated projective profinite group admits a finite presentation.

Lemma 3.3. *If f is an onto continuous homomorphism from $\overline{\Omega}_X\mathsf{G}$ onto a profinite group H and if g is a continuous homomorphism from H into $\overline{\Omega}_X\mathsf{G}$ such that $f \circ g$ is the identity, then $\operatorname{Ker} f$ is the topological closure of the normal closure of the set $\{g(f(x)) \cdot x^{-1} \mid x \in X\}$.*

Proof of Lemma 3.2: By Remark 3.1, we have $\operatorname{Ker}\pi \supseteq \operatorname{Ker}\Phi^\omega$, and so it only remains to prove the first equivalence. Clearly, $R \subseteq \operatorname{Ker}\Phi^\omega$ since Φ^ω is idempotent, thus $N \subseteq \operatorname{Ker}\Phi^\omega$. This proves the “only if” part. Conversely, if $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\Phi^\omega$ then there is a unique continuous homomorphism $\phi : T \to \overline{\Omega}_X\mathsf{G}$ such that $\phi \circ \pi = \Phi^\omega$. The composition $\pi \circ \phi$ is the identity map in T: indeed, every element of T is of the form $\pi(u)$, and $\pi \circ \phi \circ \pi = \pi \circ \Phi^\omega = \pi$ by Remark 3.1. We then apply Lemma 3.3 with $f = \pi$ and $g = \phi$. ■

Denote by $L_\varphi(\pi)$ the set of liftings Φ of φ via π such that $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\Phi^\omega$. This motivates the following proposition.

Proposition 3.4. *If T is a projective profinite group then the set $L_\varphi(\pi)$ is nonempty.*

Proof: Since T is projective, there is a homomorphism $\zeta : T \to \overline{\Omega}_X\mathsf{G}$ such that $\pi \circ \zeta = \varphi$. Putting $\Phi = \zeta \circ \pi$, we obtain $\pi \circ \Phi = \varphi \circ \pi$ and $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\Phi \subseteq \operatorname{Ker}\Phi^\omega$. ■

The existence of a finite presentation of T defined by an element of $L_\varphi(\pi)$ is thus guaranteed by Lemma 3.2 and Proposition 3.4, but the latter is not constructive. The following result summarizes the above discussion.

Proposition 3.5. *Let $\pi : \overline{\Omega}_X\mathsf{G} \to T$ be a continuous homomorphism onto a projective profinite group and let φ be a continuous automorphism of T. Then T admits a presentation of the form*

$$\langle X \mid \Phi^\omega(x) = x\ (x \in X)\rangle \tag{3.2}$$

where Φ is a continuous endomorphism of $\overline{\Omega}_X\mathsf{G}$ such that $\varphi \circ \pi = \pi \circ \Phi$. ■

Sections 4 and 5, exhibit constructive examples of such presentations.

By a *retraction* of a profinite group G we mean a continuous idempotent endomorphism of G. The image of a retraction of G is called a *retract* of G.

The following is a simple characterization of the groups that admit presentations of the form (3.2).

Corollary 3.6. *The following are equivalent for a profinite group G:*

(1) G admits a presentation of the form (3.2) for some continuous endomorphism Φ of $\overline{\Omega}_X\mathsf{G}$;
(2) G is projective and X-generated;
(3) G is a retract of $\overline{\Omega}_X\mathsf{G}$.

Proof: The implication (2) $\Rightarrow$ (1) is given by Proposition 3.5 while (3) $\Rightarrow$ (2) follows from the well-known fact that the closed subgroups of a free profinite group are projective (cf. [17, Lemma 7.6.3]). It remains to prove the implication (1) $\Rightarrow$ (3). Let Φ be a continuous endomorphism of $\overline{\Omega}_X\mathsf{G}$ and denote by H the image of the retraction Φ^ω. It suffices to establish that H admits the presentation (3.2). For this purpose, we apply the general setting of this section to the following commutative diagram:

$$\begin{tikzcd} \overline{\Omega}_X\mathsf{G} \arrow[r, "\Phi^\omega"] \arrow[d, "\Phi^\omega"'] & \overline{\Omega}_X\mathsf{G} \arrow[d, "\Phi^\omega"] \\ H \arrow[r, "\mathrm{id}"] & H \end{tikzcd}$$

From Lemma 3.2 we deduce that indeed H admits the presentation (3.2). ■

Note that the implication (2) $\Rightarrow$ (3) is well known.

4. A finite presentation of $G(\tau)$

We now focus our attention on the Prouhet-Thue-Morse substitution τ introduced in Section 2. The word aa is a connection for τ. The four elements of $X_\tau(a,a)$ are $abba$, $ababba$, $abbaba$ and $ababbaba$, cf. [5, Section 3.2.]. While the list of elements of $X_\tau(a,a)$ given in [2, Example 7.3] is incorrect, it is correctly observed that τ is an encoding of bounded delay with respect to the factors of $J(\tau)$, and that, consequently, the maximal subgroup $G(\tau)$ is generated by $\tau^\omega(X_\tau(a,a))$. Let $\alpha = \tau^\omega(abba)$, $\beta = \tau^\omega(ababba)$, $\gamma = \tau^\omega(abbaba)$ and $\delta = \tau^\omega(ababbaba)$.

Remark 4.1. *Let ζ be a continuous semigroup homomorphism from $\overline{\Omega}_A\mathsf{S}$ into a profinite semigroup S. Suppose that $\zeta(abba)$ belongs to a subgroup of S. Then $\zeta(ababba)\cdot\zeta(abba)^{\omega-1}\cdot\zeta(abbaba)=\zeta(ababbaba)$.*

Proof: We have $\zeta(ababba)\cdot\zeta(abba)^{\omega-1}\cdot\zeta(abbaba)=\zeta(ab)\cdot\zeta(abba)^{\omega+1}\cdot\zeta(ba)$, and $\zeta(abba)^{\omega+1}=\zeta(abba)$, because $\zeta(abba)$ is a group element of S. ■

Applying Remark 4.1 to the continuous homomorphism τ^ω, we conclude that $\beta\alpha^{-1}\gamma=\delta$ in $G(\tau)$, so that the profinite group $G(\tau)$ is generated by $\{\alpha,\beta,\gamma\}$.

As argued in [2, Example 7.3], the restriction ψ of τ^2 to $G(\tau)$ is a continuous automorphism of $G(\tau)$.

Remark 4.2. *The images of the generators α, β and γ of $G(\tau)$ by ψ are given by $\psi(\alpha)=\gamma\alpha\beta$, $\psi(\beta)=\gamma\beta\alpha^{-1}\cdot\gamma\alpha\beta$, $\psi(\gamma)=\gamma\alpha\beta\cdot\alpha^{-1}\gamma\beta$.*

Proof: We have the following routine computations:

$$\begin{aligned}
\psi(\alpha)&=\tau^\omega(\tau^2(abba))=\tau^\omega(abbaba\cdot abba\cdot ababba)=\gamma\alpha\beta\\
\psi(\beta)&=\tau^\omega(\tau^2(ababba))=\tau^\omega(abbaba\cdot ababbaba\cdot abba\cdot ababba)=\gamma\delta\alpha\beta\\
\psi(\gamma)&=\tau^\omega(\tau^2(abbaba))=\tau^\omega(abbaba\cdot abba\cdot ababbaba\cdot ababba)=\gamma\alpha\delta\beta\,.
\end{aligned}$$

Finally, we use the already mentioned equality $\delta=\beta\alpha^{-1}\gamma$. ■

In this section we give an explicit presentation of $G(\tau)$ as a profinite group. To apply the general approach introduced in Section 3, consider $T=G(\tau)$, $X=\{\alpha,\beta,\gamma\}$, the above automorphism ψ of $G(\tau)$, the surjection π given by the unique homomorphism $\overline{\Omega}_X\mathsf{G}\to T$ whose restriction to X is the identity, and the lifting of ψ via π which is the unique continuous endomorphism Ψ of $\overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G}$ such that $\Psi(\alpha)=\gamma\alpha\beta$, $\Psi(\beta)=\gamma\beta\alpha^{-1}\cdot\gamma\alpha\beta$, $\Psi(\gamma)=\gamma\alpha\beta\cdot\alpha^{-1}\gamma\beta$.

By Lemma 3.2, every element of $L_\psi(\pi)$ defines a presentation of $G(\tau)$. The existence of such an element follows from Proposition 3.4 since every closed subgroup of a free profinite semigroup is projective [15]. But the proof of Proposition 3.4 does not construct any elements of $L_\psi(\pi)$. The following result provides such a constructive element.

Theorem 4.3. *The endomorphism Ψ belongs to $L_\psi(\pi)$, and hence the profinite group $G(\tau)$ admits the following presentation:*

$$\langle\alpha,\beta,\gamma\mid\Psi^\omega(\alpha)=\alpha,\Psi^\omega(\beta)=\beta,\Psi^\omega(\gamma)=\gamma\rangle.$$

Proof: Let u be an arbitrary element of $\overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G}$ not belonging to $\operatorname{Ker}\Psi^\omega$. If we show that $\pi(u)\neq 1$, then the theorem is proved, thanks to Lemma 3.2.

Since $\Psi^\omega(u)\neq 1$, there is a finite group H and an onto continuous homomorphism $\lambda:\overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G}\to H$ such that $\lambda(\Psi^\omega(u))\neq 1$. Note that $\lambda_0=\lambda\circ\Psi^\omega$ entails $\lambda_0(\Psi^\omega(u))=\lambda(\Psi^\omega(u))$. Hence we may replace λ by λ_0, that is, we may assume that $\lambda=\lambda\circ\Psi^\omega$.

Suppose that $\mu:\overline{\Omega}_{\{a,b\}}\mathsf{S}\to S$ is a continuous homomorphism into a finite semigroup S such that H is (isomorphic to) a subgroup of S and such that the following diagram commutes:

$$\begin{tikzcd}
\overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G} \arrow[r, "\lambda"] \arrow[d, "\pi"'] & H \arrow[r, hook] & S \\
G(\tau) \arrow[rr, hook] & & \overline{\Omega}_{\{a,b\}}\mathsf{S} \arrow[u, "\mu"']
\end{tikzcd}$$

Then we have $\mu(\pi(u))=\lambda(u)=\lambda(\Psi^\omega(u))\neq 1$ and so $\pi(u)\neq 1$. The proof of the theorem is thus reduced to the existence of the homomorphism μ.

For each integer ℓ, let $g_\ell=\lambda(\Psi^{\omega+\ell}(\alpha))$, $h_\ell=\lambda(\Psi^{\omega+\ell}(\beta))$ and $k_\ell=\lambda(\Psi^{\omega+\ell}(\gamma))$. Let $w=\alpha\beta^{-1}\alpha\gamma^{-1}\alpha$ and $p=\lambda(w)$. A straightforward computation shows that $\Psi(w)=w$. Therefore, the equality $\Psi^{\omega+\ell}(w)=\Psi^\omega(w)$ holds for every integer ℓ. Since $\lambda\circ\Psi^\omega=\lambda$, we then have $p=\lambda(\Psi^{\omega+\ell}(w))$ and

$$p=g_\ell h_\ell^{-1}g_\ell k_\ell^{-1}g_\ell \tag{4.1}$$

for every integer ℓ.

Let S be the Rees matrix semigroup $\mathcal{M}(\{1,2\},H,\{1,2\};P)$ with sandwich matrix $P=\left(\begin{smallmatrix}1&1\\1&p\end{smallmatrix}\right)$. For each $(x,y)\in H\times H$, let $\mu_{x,y}$ be the unique continuous semigroup homomorphism $\overline{\Omega}_{\{a,b\}}\mathsf{S}\to S$ such that $\mu_{x,y}(a)=(1,x,1)$ and $\mu_{x,y}(b)=(2,y,2)$.

For each pair of integers (n,ℓ) such that $n\geq 0$, let $\mathcal{S}_{(n,\ell)}$ be the set of elements (x,y) of $H\times H$ such that

$$\begin{cases}\mu_{x,y}(\tau^{2n}(abba))=(1,g_\ell,1)\\ \mu_{x,y}(\tau^{2n}(ababba))=(1,h_\ell,1)\\ \mu_{x,y}(\tau^{2n}(abbaba))=(1,k_\ell,1).\end{cases}$$

Claim 4.4. *$\mathcal{S}_{(n,\ell)}\subseteq\mathcal{S}_{(n+1,\ell+1)}$ and $\mathcal{S}_{(n,\ell)}\neq\emptyset$.*

Proof of the claim: It will be convenient to denote by $\mu'_{x,y}$ the composition of $\mu_{x,y}$ with the projection $(i,z,j)\mapsto z$. Note that $\mu_{x,y}(u)\in\{i\}\times H\times\{j\}$

if and only if u starts with the i-th letter in $\{a, b\}$ (where a is the first letter and b the second) and ends with the j-th letter. Since τ^m preserves the first and last letters when m is even, by the choice of the matrix P the element $\mu'_{x,y}(\tau^{2n}(u_1u_2))$ is equal to $\mu'_{x,y}(\tau^{2n}(u_1))\cdot p\cdot\mu'_{x,y}(\tau^{2n}(u_2))$ if u_1 ends with b and u_2 starts with b, and is equal to $\mu'_{x,y}(\tau^{2n}(u_1))\cdot\mu'_{x,y}(\tau^{2n}(u_2))$ otherwise. These facts allow us to make the following computations:

$$
\begin{aligned}
(x,y)\in\mathcal{S}_{(n,\ell)} \iff & \begin{cases} \mu'_{x,y}(\tau^{2n}(abba)) = g_\ell \\ \mu'_{x,y}(\tau^{2n}(ab))\cdot\mu'_{x,y}(\tau^{2n}(abba)) = h_\ell, \\ \mu'_{x,y}(\tau^{2n}(abba))\cdot\mu'_{x,y}(\tau^{2n}(ba)) = k_\ell. \end{cases} \\
\iff & \begin{cases} \mu'_{x,y}(\tau^{2n}(abba)) = g_\ell \\ \mu'_{x,y}(\tau^{2n}(ab))\cdot g_\ell = h_\ell, \\ g_\ell\cdot\mu'_{x,y}(\tau^{2n}(ba)) = k_\ell. \end{cases} \\
\iff & \begin{cases} \mu'_{x,y}(\tau^{2n}(ab))\cdot p\cdot\mu'_{x,y}(\tau^{2n}(ba)) = g_\ell \\ \mu'_{x,y}(\tau^{2n}(ab)) = h_\ell g_\ell^{-1}, \\ \mu'_{x,y}(\tau^{2n}(ba)) = g_\ell^{-1}k_\ell. \end{cases}
\end{aligned}
\tag{4.2}
$$

By (4.1), the first equality in (4.2) is redundant. Hence the equivalence

$$
(x,y)\in\mathcal{S}_{(n,\ell)} \iff \begin{cases} \mu'_{x,y}(\tau^{2n}(ab)) = h_\ell g_\ell^{-1}, \\ \mu'_{x,y}(\tau^{2n}(ba)) = g_\ell^{-1}k_\ell \end{cases}
\tag{4.3}
$$

holds for all integers n,ℓ with $n\geq 0$. Suppose now that $(x,y)\in\mathcal{S}_{(n,\ell)}$. Then we may perform the following calculation:

$$
\begin{aligned}
\mu'_{x,y}(\tau^{2(n+1)}(ab)) &= \mu'_{x,y}(\tau^{2n}(abbabaab)) \\
&= \mu'_{x,y}(\tau^{2n}(abbaba))\cdot\mu'_{x,y}(\tau^{2n}(ab)) \\
&= k_\ell\cdot h_\ell g_\ell^{-1} \\
&= \lambda(\Psi^{\omega+\ell}(\gamma\beta\alpha^{-1})) \\
&= \lambda(\Psi^{\omega+\ell}(\gamma\beta\alpha^{-1}\cdot\Psi(\alpha)\cdot\Psi(\alpha^{-1})))
\end{aligned}
$$

Since $\Psi(\beta) = \gamma\beta\alpha^{-1}\cdot\Psi(\alpha)$, we deduce that

$$
\mu'_{x,y}(\tau^{2(n+1)}(ab)) = \lambda(\Psi^{\omega+\ell+1}(\beta\alpha^{-1})) = h_{\ell+1}g_{\ell+1}^{-1}.
\tag{4.4}
$$

Similarly, we have $\mu'_{x,y}(\tau^{2(n+1)}(ba)) = g_{\ell+1}^{-1}k_{\ell+1}$. From this equality, (4.4) and (4.3), we obtain $(x,y) \in \mathcal{S}_{(n+1,\ell+1)}$. Hence $\mathcal{S}_{(n,\ell)}$ is contained in $\mathcal{S}_{(n+1,\ell+1)}$ for all integers n, ℓ, with $n \geq 0$.

We may then conclude that $\mathcal{S}_{(0,\ell-n)} \subseteq \mathcal{S}_{(n,\ell)}$, and so to prove that $\mathcal{S}_{(n,\ell)} \neq \emptyset$ for all integers n, ℓ with $n \geq 0$, it suffices to prove that $\mathcal{S}_{(0,\ell)} \neq \emptyset$ for all ℓ. Thanks to (4.3),

$$(x,y) \in \mathcal{S}_{(0,\ell)} \iff \begin{cases} \mu'_{x,y}(ab) = h_\ell g_\ell^{-1} \\ \mu'_{x,y}(ba) = g_\ell^{-1} k_\ell \end{cases} \iff \begin{cases} xy = h_\ell g_\ell^{-1} \\ yx = g_\ell^{-1} k_\ell. \end{cases}$$

Hence $\mathcal{S}_{(0,\ell)} \neq \emptyset$ if and only if $h_\ell g_\ell^{-1}$ and $g_\ell^{-1}k_\ell$ are conjugate elements of H. Straightforward computations show that $\Psi(\alpha^{-1}\gamma) = \alpha^{-1} \cdot \Psi(\beta\alpha^{-1}) \cdot \alpha$. Applying $\lambda \circ \Psi^{\omega+\ell-1}$ to both members of this equality we obtain $g_\ell^{-1}k_\ell = g_{\ell-1}^{-1} \cdot h_\ell g_\ell^{-1} \cdot g_{\ell-1}$, whence $\mathcal{S}_{(0,\ell)} \neq \emptyset$. ■

We proceed with the proof of the theorem. For each $n \geq 2$, let (x_n, y_n) be an element of the nonempty set $\mathcal{S}_{(\frac{n!}{2},0)}$. Since $H \times H$ is finite, the sequence $(x_n, y_n)_n$ has some subsequence $(x_{n_k}, y_{n_k})_k$ with constant value (x,y). Then

$$\mu_{x,y}(\alpha) = \mu_{x,y}(\tau^\omega(abba)) = \lim_{k\to\infty} \mu_{x,y}(\tau^{n_k!}(abba))$$

Since $(x,y) = (x_{n_k}, y_{n_k}) \in \mathcal{S}_{(\frac{n_k!}{2},0)}$, we have $\mu_{x,y}(\tau^{n_k!}(abba)) = (1, g_0, 1)$, whence $\mu_{x,y}(\alpha) = (1, g_0, 1)$. Identifying $\{1\} \times H \times \{1\}$ with H via the isomorphism $(1,z,1) \mapsto z$, and because $\lambda \circ \Psi^\omega = \lambda$, we have therefore $\mu_{x,y}(\alpha) = \lambda(\alpha)$. Similarly, we obtain the equalities $\mu_{x,y}(\beta) = \lambda(\beta)$ and $\mu_{x,y}(\gamma) = \lambda(\gamma)$. Hence the maps $\mu_{x,y} \circ \pi$ and λ coincide on the set $\{\alpha, \beta, \gamma\}$. This set (freely) generates $\overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G}$, thus $\mu_{x,y} \circ \pi = \lambda$. According to the remarks at the beginning of the proof, this establishes the theorem. ■

5. The general problem

We now consider the general framework introduced in [2]. Let φ be a weakly primitive substitution over a finite alphabet A, with connection ba. There is a finite power $\tilde{\varphi}$ of φ such that the first letter of $\tilde{\varphi}(a)$ is a and the last letter of $\tilde{\varphi}(b)$ is b. This implies that the restriction of $\tilde{\varphi}$ to $G(\varphi)$ is an endomorphism of $G(\varphi)$. If, additionally, φ is an encoding of bounded delay with respect to the factors of $J(\varphi)$, then this restriction is an automorphism of $G(\varphi)$. We have already mentioned in Section 2 that the profinite group $G(\varphi)$ is generated by $\varphi^\omega(X_\varphi(a,b))$. Note that $\varphi^\omega = \tilde{\varphi}^\omega$, since $\tilde{\varphi}$ is a power of φ.

To avoid overloaded notation, $X_\varphi(a,b)$ will be denoted by X. The set $R(ba)$ is easily recognized to be a code and so is $X = b^{-1}R(ba)b$. Let i be the unique homomorphism from the semigroup freely generated by X into the semigroup freely generated by A such that $i(x) = x$ for all $x \in X$. Then i is injective, because X is a code. If $x \in X$ then $\tilde\varphi(x)$ belongs to the subsemigroup of A^+ generated by X. Therefore, we can consider the word $w_x = i^{-1}(\tilde\varphi(x))$, the unique decomposition of $\tilde\varphi(x)$ in the elements of X. The homomorphism i has a unique extension to a continuous homomorphism $\overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$, which we also denote by i. Let q be the canonical projection $\overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_X\mathsf{G}$, namely the unique continuous homomorphism from $\overline{\Omega}_X\mathsf{S}$ into $\overline{\Omega}_X\mathsf{G}$ that is the identity on the generators. Then there are unique continuous endomorphisms $\tilde\varphi_X$ and $\tilde\varphi_{X,\mathsf{G}}$ such that Diagram (5.1) commutes. More explicitly, for each $x \in X$ we have $\tilde\varphi_X(x) = w_x$ and $\tilde\varphi_{X,\mathsf{G}}(x) = w_x$, where we regard w_x as a semigroup and a group word, respectively.

$$\begin{tikzcd}
\overline{\Omega}_A\mathsf{S} \arrow[d, "\tilde\varphi"'] & \overline{\Omega}_X\mathsf{S} \arrow[l, "i"'] \arrow[r, "q"] \arrow[d, "\tilde\varphi_X"] & \overline{\Omega}_X\mathsf{G} \arrow[d, "\tilde\varphi_{X,\mathsf{G}}"] \\
\overline{\Omega}_A\mathsf{S} & \overline{\Omega}_X\mathsf{S} \arrow[l, "i"'] \arrow[r, "q"] & \overline{\Omega}_X\mathsf{G}
\end{tikzcd} \tag{5.1}$$

The following conjecture was proposed as a problem by the first author in a lecture given at the *Fields Workshop on Profinite Groups and Applications* (Carleton University, August 2005).

Conjecture 5.1. *Under the above hypotheses, the profinite group $G(\varphi)$ admits the presentation*

$$\langle X \mid (\tilde\varphi_{X,\mathsf{G}})^\omega(x) = x \ (x \in X)\rangle.$$

Note that, by Remark 3.1, for each $x \in X$ the relation $(\tilde\varphi_{X,\mathsf{G}})^\omega(x) = x$ on the generators of $G(\varphi)$ is valid, since $\tilde\varphi_{X,\mathsf{G}}$ is a lifting of $\tilde\varphi$ via the unique continuous homomorphism $\pi : \overline{\Omega}_X\mathsf{G} \to G(\varphi)$ such that $\pi(x) = \varphi^\omega(x)$ for all $x \in X$.

Corollary 5.2. *Conjecture 5.1 holds for the Prouhet-Thue-Morse substitution: denoting by $\bar\psi$ the homomorphism $\tau^2_{X_\tau(a,a),\mathsf{G}}$, the profinite group $G(\tau)$ has the following presentation:*

$$\langle X_\tau(a,a) \mid \bar\psi^\omega(x) = x \ (x \in X_\tau(a,a))\rangle.$$

Proof: Note that indeed we can take $\tilde\tau = \tau^2$. Let $\alpha' = abba$, $\beta' = ababba$, $\gamma' = abbaba$ and $\delta' = ababbaba$. Recall that $X_\tau(a,a) = \{\alpha', \beta', \gamma', \delta'\}$ and, for each $x \in \{\alpha, \beta, \gamma, \delta\}$, one has $x = \tau^\omega(x')$ by the definition of x. We have $w_\alpha = \gamma'\alpha'\beta'$, $w_\beta = \gamma'\delta'\alpha'\beta'$, $w_{\gamma'} = \gamma'\alpha'\delta'\beta'$ and $w_{\delta'} = \gamma'\delta'\alpha'\delta'\beta'$. In Diagram (5.2), the continuous homomorphisms ι and ξ are respectively defined by $\iota(x') = x$ for each $x \in \{\alpha, \beta, \gamma, \delta\}$, and by $\xi(x') = x$ for each $x \in \{\alpha, \beta, \gamma\}$ and $\xi(\delta') = \beta\alpha^{-1}\gamma$. Then ι is an isomorphism and Diagram (5.2) commutes.

$$\begin{tikzcd} \overline{\Omega}_{X_\tau(a,a)}\mathsf{G} \arrow[r, "\iota"] \arrow[d, "\bar\psi"'] & \overline{\Omega}_{\{\alpha,\beta,\gamma,\delta\}}\mathsf{G} \arrow[r, "\xi"] \arrow[d, "\iota\circ\bar\psi\circ\iota^{-1}"'] & \overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G} \arrow[r, "\pi"] \arrow[d, "\Psi"] & G(\tau) \arrow[d, "\psi"] \\ \overline{\Omega}_{X_\tau(a,a)}\mathsf{G} \arrow[r, "\iota"] & \overline{\Omega}_{\{\alpha,\beta,\gamma,\delta\}}\mathsf{G} \arrow[r, "\xi"] & \overline{\Omega}_{\{\alpha,\beta,\gamma\}}\mathsf{G} \arrow[r, "\pi"] & G(\tau) \end{tikzcd} \tag{5.2}$$

Our claim then follows from Theorem 4.3. ■

Another situation in which Conjecture 5.1 holds is the main setting considered in [2]. For this discussion and for later parts of the paper, the following notation is convenient: given a pseudovariety of semigroups V containing G, and an endomorphism φ of $\overline{\Omega}_A\mathsf{V}$, let φ_G be the unique continuous endomorphism of $\overline{\Omega}_A\mathsf{G}$ such that $\varphi_\mathsf{G} \circ p = p \circ \varphi$, where $p : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{G}$ is the canonical projection. In particular, if $\varphi \in \operatorname{End}\overline{\Omega}_A\mathsf{G}$, then $\varphi_\mathsf{G} = \varphi$. We say that a weakly primitive substitution φ over the alphabet A is *ultimately group invertible* if, for the alphabet B of all letters that occur in any $\varphi^\omega(a)$ with $a \in A$, the restriction of φ_G to $\overline{\Omega}_B\mathsf{G}$ is an automorphism of $\overline{\Omega}_B\mathsf{G}$.

Theorem 5.3. *Let φ be an ultimately group invertible weakly primitive substitution over the finite alphabet A, with connection ba. Then, in the notation of the beginning of this section, Conjecture 5.1 holds.*

Proof: Let H be the subgroup obtained by intersection of the $\mathcal{R}$-class of $\varphi^\omega(a)$ with the $\mathcal{L}$-class of $\varphi^\omega(b)$. Combining [2, Theorem 5.3, Corollaries 5.6 and 5.8], we know that the projection $p : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{G}$ sends H isomorphically to the closed subgroup generated by X, which is a free profinite group. As in [2], denote by Z the unique basis of this free profinite group which is contained in A^+. It is obtained by successively applying the operations $(xy, x) \mapsto (x, y)$ and $(yx, x) \mapsto (x, y)$, while $y \in A^+$, starting with the set X.

Besides the associated diagram (5.1), we also have the following commutative diagram, where q_Z is the canonical projection, and where we also take into account that Z is a code (so that we may consider the coding homomorphism i_Z) such that $\tilde{\varphi}(A) \subseteq Z^+$ (yielding the homomorphism $\tilde{\varphi}_Z$):

$$\begin{tikzcd}
\overline{\Omega}_A\mathsf{S} \arrow[rrrr, "p"] \arrow[ddd, "\tilde{\varphi}"'] & & & & \overline{\Omega}_A\mathsf{G} \arrow[ddd, "\tilde{\varphi}_{\mathsf{G}}"] \\
& \overline{\Omega}_Z\mathsf{S} \arrow[ul, "i_Z"'] \arrow[r, "q_Z"] \arrow[d, "\tilde{\varphi}_Z"'] & \overline{\Omega}_Z\mathsf{G} \arrow[r, hook] \arrow[d, "\tilde{\varphi}_{Z,\mathsf{G}}"] & \overline{\Omega}_B\mathsf{G} \arrow[ur, hook] \arrow[d, "(\tilde{\varphi}_{\mathsf{G}})|_{\overline{\Omega}_B\mathsf{G}}"] & \\
& \overline{\Omega}_Z\mathsf{S} \arrow[dl, "i_Z"] \arrow[r, "q_Z"] & \overline{\Omega}_Z\mathsf{G} \arrow[r, hook] & \overline{\Omega}_B\mathsf{G} \arrow[dr, hook] & \\
\overline{\Omega}_A\mathsf{S} \arrow[rrrr, "p"] & & & & \overline{\Omega}_A\mathsf{G}
\end{tikzcd} \tag{5.3}$$

The restriction of φ_{G} to $\overline{\Omega}_B\mathsf{G}$ is a continuous automorphism, whence, since $\tilde{\varphi}$ is a finite power of φ, the restriction of $\tilde{\varphi}_{\mathsf{G}}$ to $\overline{\Omega}_B\mathsf{G}$ is also a continuous automorphism. Therefore, $\tilde{\varphi}_{Z,\mathsf{G}}$ is a continuous automorphism of $\overline{\Omega}_Z\mathsf{G}$. Hence $(\tilde{\varphi}_{Z,\mathsf{G}})^\omega$ is the identity mapping of $\overline{\Omega}_Z\mathsf{G}$ and, since H is isomorphic to $\overline{\Omega}_Z\mathsf{G}$, it follows that H admits the profinite group presentation

$$\langle Z \mid (\tilde{\varphi}_{Z,\mathsf{G}})^\omega(z) = z \ (z \in Z)\rangle. \tag{5.4}$$

Thus, it remains to show that, without changing the group, one can modify the presentation (5.4) to the presentation of Conjecture 5.1, namely

$$\langle X \mid (\tilde{\varphi}_{X,\mathsf{G}})^\omega(x) = x \ (x \in X)\rangle.$$

To establish such a claim, taking into account how Z is constructed from X, it is better to analyse how such presentations are affected by the transformation $(x, y) \mapsto (x, xy)$, which is considered in Lemma 5.4 below, and its dual. From the lemma and its dual, a simple induction argument completes the proof of the theorem. ∎

Let B be a finite alphabet and let ψ be a continuous automorphism of $\overline{\Omega}_B\mathsf{G}$. Suppose that $C \subseteq B^+$ is a finite code such that the closed subgroup generated by C is stable under the action of ψ. Let i_C be the induced continuous homomorphism $\overline{\Omega}_C\mathsf{G} \to \overline{\Omega}_B\mathsf{G}$ that sends each element of C to itself and let Ψ_C be the unique continuous endomorphism of $\overline{\Omega}_C\mathsf{G}$ such that $i_C \circ \Psi_C = \psi \circ i_C$. Given distinct $c, d \in C$, note that $D = (C \setminus \{d\}) \cup \{cd\}$ is again a code. The continuous homomorphism $\overline{\Omega}_C\mathsf{G} \to \overline{\Omega}_D\mathsf{G}$ that sends the generator d to the group word $c^{-1} \cdot cd$ and fixes every other element of C is denoted $\varepsilon_{c,d}$. Then we have the following diagram

$$\begin{tikzcd}
\overline{\Omega}_C\mathsf{G} \arrow[rrr, "\Psi_C"] \arrow[dr, "\varepsilon_{c,d}"] \arrow[dd, "i_C"'] & & & \overline{\Omega}_C\mathsf{G} \arrow[dl, "\varepsilon_{c,d}"'] \arrow[dd, "i_C"] \\
& \overline{\Omega}_D\mathsf{G} \arrow[r, "\Psi_D"] \arrow[dl, "i_D"] & \overline{\Omega}_D\mathsf{G} \arrow[dr, "i_D"'] & \\
\overline{\Omega}_B\mathsf{G} \arrow[rrr, "\psi"] & & & \overline{\Omega}_B\mathsf{G}
\end{tikzcd} \tag{5.5}$$

where $\operatorname{Im} i_C = \operatorname{Im} i_D$. It is immediate to check that the diagram commutes.

Lemma 5.4. *In the above setting,* $\langle C \mid \Psi_C^\omega(t) = t \ (t \in C)\rangle$ *is a profinite presentation of the group* $\operatorname{Im} i_C$ *if and only if so is* $\langle D \mid \Psi_D^\omega(t) = t \ (t \in D)\rangle$*.*

Proof: In view of Lemma 3.2, we need to show that $\operatorname{Ker} i_C \subseteq \operatorname{Ker} \Psi_C^\omega$ if and only if $\operatorname{Ker} i_D \subseteq \operatorname{Ker} \Psi_D^\omega$. Since $\varepsilon_{c,d}$ is an isomorphism, the commutativity of Diagram (5.5) implies that $\operatorname{Ker} i_D = \varepsilon_{c,d}(\operatorname{Ker} i_C)$ and $\operatorname{Ker} \Psi_D^\omega = \varepsilon_{c,d}(\operatorname{Ker} \Psi_C^\omega)$. The desired equivalence is now immediate. ∎

The remaining part of this section is dedicated to a special case in which the Schützenberger group of the subshift is realized as a retract of the free profinite semigroup under the ω-power of a weakly primitive substitution. The section closes with a result (Theorem 5.8) that gives further evidence towards Conjecture 5.1. Before that, we prove a related result that was also announced in the same lecture where the problem was proposed; its proof appears here for the first time.

Theorem 5.5. *Let* φ *be a weakly primitive substitution over a finite alphabet* A *such that:*

(1) all $\varphi(a)$ $(a \in A)$ *start with the same letter and end with the same letter;*
(2) φ *is an encoding of bounded delay with respect to factors of* $J(\varphi)$*.*

Then we have the following profinite group presentation:

$$G(\varphi) = \langle A \mid \varphi_{\mathsf{G}}^\omega(a) = a \ (a \in A)\rangle.$$

Proof: Let $H = G(\varphi)$ be the maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ containing all elements of the form $\varphi^\omega(a)$, where $a \in A$. By [2, Theorem 4.13], the image of φ^ω is H. Since $\overline{\Omega}_A\mathsf{G}$ is a free profinite group, φ^ω factorizes through p as $\varphi^\omega = \nu \circ p$, where $\nu : \overline{\Omega}_A\mathsf{G} \to H$ is an onto continuous homomorphism. By Lemma 3.2, it suffices to prove that $\operatorname{Ker} \nu \subseteq \operatorname{Ker} \varphi_{\mathsf{G}}^\omega$. Let $u \in \operatorname{Ker} \nu$. There is $w \in \overline{\Omega}_A\mathsf{S}$ such that $u = p(w)$. Then $\varphi^\omega(w) = \nu(p(w)) = \nu(u) = 1$, and so $\varphi^\omega_{\mathsf{G}}(u) = \varphi^\omega_{\mathsf{G}}(p(w)) = p(\varphi^\omega(w)) = 1$. ∎

Here is a couple of examples to illustrate Theorem 5.5.

Example 5.6. Let $A = \{a, b\}$ and define a substitution φ by $\varphi(a) = ab$ and $\varphi(b) = a^3b$. Then φ is primitive and it is easy to check that it satisfies the hypotheses of Theorem 5.5 so that

$$G(\varphi) = \langle a, b \mid \varphi^\omega_{\mathsf{G}}(a) = a,\ \varphi^\omega_{\mathsf{G}}(b) = b\rangle.$$

It is shown in [2, Example 7.2] that $G(\varphi)$ is not a free profinite group. This result is improved in the next section (Theorem 6.4).

Example 5.7. Let $A = \{a, b, c\}$ and consider the substitution defined by $\varphi(a) = ac$, $\varphi(b) = aca^2c$, and $\varphi(c) = ac^2ac$. Again, we may apply Theorem 5.5 to φ to deduce that

$$G(\varphi) = \langle a, b, c \mid \varphi^\omega_{\mathsf{G}}(a) = a,\ \varphi^\omega_{\mathsf{G}}(b) = b,\ \varphi^\omega_{\mathsf{G}}(c) = c\rangle.$$

Note that, since b does not occur in $\varphi^\omega(b)$, the second relation allows us to drop the generator b, so that a simpler presentation is given by

$$G(\varphi) = \langle a, c \mid \varphi^\omega_{\mathsf{G}}(a) = a,\ \varphi^\omega_{\mathsf{G}}(c) = c\rangle.$$

Moreover, since in the free group on the set A the subgroup generated by $\{ac, aca^2c, ac^2ac\}$ is also generated by $\{a, c\}$, the substitution φ is ultimately group invertible and [2, Theorem 5.3] yields that the two relations in the simpler presentation are trivial. Hence $G(\varphi)$ is actually a free profinite group of rank two.

Let φ be a substitution as in Theorem 5.5. Let a and b be the letters such that all elements of the image of φ start with a and end with b. Then ba is the unique connection for φ. Note that in this case we can take $\tilde{\varphi} = \varphi$.

Theorem 5.8. *With the above notation,*

$$\langle X \mid (\tilde{\varphi}_{X,\mathsf{G}})^\omega(x) = x\ (x \in X)\rangle$$

is a presentation of $G(\varphi)$.

Proof: We adopt the notation in the proof of Theorem 5.5. We also retain all homomorphisms in Diagram (5.1). We define some additional homomorphisms, included in Diagram (5.6).

$$
\begin{tikzcd}
\overline{\Omega}_X\mathsf{S} \arrow[rrrrr, "\varphi_X"] \arrow[dr, "q"] \arrow[dddd, shift left, "i"] & & & & & \overline{\Omega}_X\mathsf{S} \arrow[dl, "q"'] \arrow[dddd, shift left, "i"] \\
& \overline{\Omega}_X\mathsf{G} \arrow[rrr, dashed, "\varphi_{X,\mathsf{G}}"] \arrow[dr, dashed, "\pi"] \arrow[dd, shift left, "i_{\mathsf{G}}"] & & & \overline{\Omega}_X\mathsf{G} \arrow[dl, dashed, "\pi"'] \arrow[dd, shift left, "i_{\mathsf{G}}"] & \\
& & H \arrow[r, dashed, "\varphi|_H"] & H & & \\
& \overline{\Omega}_A\mathsf{G} \arrow[rrr, "\varphi_{\mathsf{G}}"] \arrow[ur, "\nu"'] \arrow[uu, shift left, "j_{\mathsf{G}}"] & & & \overline{\Omega}_A\mathsf{G} \arrow[ul, "\nu"] \arrow[uu, shift left, "j_{\mathsf{G}}"] & \\
\overline{\Omega}_A\mathsf{S} \arrow[rrrrr, "\varphi"] \arrow[ur, "p"'] \arrow[uuuu, shift left, "j"] & & & & & \overline{\Omega}_A\mathsf{S} \arrow[ul, "p"] \arrow[uuuu, shift left, "j"]
\end{tikzcd}
\tag{5.6}
$$

The extension of an injective homomorphism of finitely generated free semigroups to a homomorphism of absolutely free profinite semigroups is also injective [14, Proposition 2.1]. Hence, the extension $i : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ is injective.

For every $c \in A$, the word $\varphi(c)$ is in the image of i, whence $\operatorname{Im}\varphi \subseteq \operatorname{Im} i$. Therefore, since i is injective, there is a unique continuous homomorphism $j : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_X\mathsf{S}$ such that $j(c) = i^{-1}(\varphi^\omega(c))$ for all $c \in A$.

Let i_{G} and j_{G} be the unique continuous homomorphisms such that $i_{\mathsf{G}} \circ q = p \circ i$ and $j_{\mathsf{G}} \circ p = q \circ j$. Finally, let $\pi = \nu \circ i_{\mathsf{G}}$, with which the definition of all homomorphisms in Diagram (5.6) is concluded.

We shall prove that the dashed trapezoid in the diagram is commutative. Since q is onto, that amounts to proving the equality $\pi \circ \varphi_{X,\mathsf{G}} \circ q = \varphi \circ \pi \circ q$. We first note that

$$\pi \circ \varphi_{X,\mathsf{G}} \circ q = \nu \circ i_{\mathsf{G}} \circ \varphi_{X,\mathsf{G}} \circ q = \nu \circ i_{\mathsf{G}} \circ q \circ \varphi_X = \nu \circ p \circ i \circ \varphi_X.$$

Since $\nu \circ p = \varphi^\omega$ and $i \circ \varphi_X = \varphi \circ i$, we deduce that $\pi \circ \varphi_{X,\mathsf{G}} \circ q = \varphi^{\omega+1} \circ i$. On the other hand, we also have

$$\varphi \circ \pi \circ q = \varphi \circ \nu \circ i_{\mathsf{G}} \circ q = \varphi \circ \nu \circ p \circ i = \varphi \circ \varphi^\omega \circ i = \varphi^{\omega+1} \circ i,$$

which concludes the proof of the commutativity of the dashed trapezoid in Diagram (5.6). Therefore, $\varphi_{X,\mathsf{G}}$ is a lifting of $\varphi|_H$ via π.

By Lemma 3.2, to prove the theorem it suffices to prove that $\operatorname{Ker}\pi \subseteq \operatorname{Ker}(\varphi_{X,\mathsf{G}})^\omega$. Let $u \in \overline{\Omega}_X\mathsf{S}$ be such that $\pi(q(u)) = 1$. Then $i_{\mathsf{G}}(q(u)) \in$

$\operatorname{Ker}\nu$. Theorem 5.5 was obtained by proving that $\operatorname{Ker}\nu \subseteq \operatorname{Ker}\varphi_{\mathsf{G}}^{\omega}$. Therefore, $\varphi_{\mathsf{G}}^{\omega} \circ i_{\mathsf{G}} \circ q(u) = 1$, which justifies the following equality:

$$j_{\mathsf{G}} \circ \varphi_{\mathsf{G}}^{\omega} \circ i_{\mathsf{G}} \circ q(u) = 1. \tag{5.7}$$

Since $i_{\mathsf{G}} \circ q = p \circ i$, $\varphi_{\mathsf{G}} \circ p = p \circ \varphi$ and $j_{\mathsf{G}} \circ p = q \circ j$, we conclude that the composite $j_{\mathsf{G}} \circ \varphi_{\mathsf{G}}^{\omega} \circ i_{\mathsf{G}} \circ q$ is equal to $q \circ j \circ \varphi^{\omega} \circ i$. The homomorphism φ^{ω} is idempotent, whence by the definition of j we have $j \circ \varphi^{\omega} = j$, and so $j_{\mathsf{G}} \circ \varphi_{\mathsf{G}}^{\omega} \circ i_{\mathsf{G}} \circ q = q \circ j \circ i$. Note also that $i \circ j = \varphi^{\omega}$. Thus we have the equalities $i \circ j \circ i = \varphi^{\omega} \circ i = i \circ \varphi_X^{\omega}$. Because i is injective, it follows that $j \circ i = \varphi_X^{\omega}$. This yields the equality $j_{\mathsf{G}} \circ \varphi_{\mathsf{G}}^{\omega} \circ i_{\mathsf{G}} \circ q = q \circ \varphi_X^{\omega}$. Since $q \circ \varphi_X = \varphi_{X,\mathsf{G}} \circ q$, this means that $j_{\mathsf{G}} \circ \varphi_{\mathsf{G}}^{\omega} \circ i_{\mathsf{G}} \circ q = (\varphi_{X,\mathsf{G}})^{\omega} \circ q$. Then, from (5.7) we conclude that $q(u)$ belongs to the kernel of $(\varphi_{X,\mathsf{G}})^{\omega}$. This establishes the inclusion $\operatorname{Ker}\pi \subseteq \operatorname{Ker}(\varphi_{X,\mathsf{G}})^{\omega}$ and concludes the proof of the theorem. ∎

6. Applications

For a set X, denote by $\mathcal{T}(X)$ the semigroup of all full transformations of X. The following technical lemma will be useful. Its proof uses the methods introduced in [1].

Lemma 6.1. *Let* V *be a pseudovariety of semigroups,* $\varphi \in \operatorname{End}\overline{\Omega}_A\mathsf{V}$*, and* S *a semigroup from* V*. Consider the transformation* $\overline{\varphi} \in \mathcal{T}(S^A)$ *defined by* $\overline{\varphi}(f) = \hat{f} \circ \varphi|_A$*, where* $\hat{f}$ *is the unique extension of* $f \in S^A$ *to a continuous homomorphism* $\overline{\Omega}_A\mathsf{V} \to S$*. Then the correspondence*

$$\begin{aligned} \operatorname{End}\overline{\Omega}_A\mathsf{V} &\to \mathcal{T}(S^A) \\ \varphi &\mapsto \overline{\varphi} \end{aligned}$$

is a continuous anti-homomorphism. In particular, we have $\overline{\varphi^{\omega}} = \overline{\varphi}^{\omega}$*.*

Proof: Let $\varphi, \psi \in \operatorname{End}\overline{\Omega}_A\mathsf{V}$ and $f \in S^A$. Since $\widehat{\hat{f} \circ \varphi|_A} = \hat{f} \circ \varphi$, we obtain the following chain of equalities:

$$\overline{\varphi \circ \psi}(f) = \hat{f} \circ \varphi \circ \psi|_A = \widehat{\hat{f} \circ \varphi|_A} \circ \psi|_A = \bar{\psi}(\hat{f} \circ \varphi|_A) = \bar{\psi} \circ \bar{\varphi}(f),$$

which proves that our mapping is an anti-homomorphism. To prove that it is continuous, consider a net limit $\varphi = \lim \varphi_i$ in $\operatorname{End}\overline{\Omega}_A\mathsf{V}$. Then, for every $f \in \mathcal{T}(S^A)$ and every $a \in A$, we may perform the following computation:

$$\begin{aligned} \overline{\varphi}(f)(a) &= \hat{f}(\varphi(a)) = \hat{f}\big((\lim \varphi_i)(a)\big) = \hat{f}(\lim \varphi_i(a)) \\ &= \lim \hat{f}(\varphi_i(a)) = \lim \overline{\varphi_i}(f)(a), \end{aligned}$$

which yields the desired equality $\overline{\varphi} = \lim \overline{\varphi_i}$. ∎

For a semigroup S, we say that a mapping $f \in S^A$ is a *generating mapping* if $f(A)$ generates S.

The following result will be useful to draw structural and computational information about our presentations.

Proposition 6.2. *Let* V *be a pseudovariety of semigroups containing* G*,* A *a finite alphabet, and* φ *a continuous endomorphism of* $\overline{\Omega}_A\mathsf{V}$*. The following are equivalent:*

(1) S *is a continuous homomorphic image of the group presented by* $\langle A \mid \varphi_{\mathsf{G}}^{\omega}(a) = a \ (a \in A)\rangle$*;*
(2) there is some generating mapping $f : A \to S$ *and some integer* n *such that* $1 \leq n \leq |S^A|$ *and* $\hat{f} \circ \varphi^n|_A = f$*;*
(3) there is some generating mapping $f : A \to S$ *and some integer* n *such that* $\hat{f} \circ \varphi^n|_A = f$*.*

Proof: Let H be the profinite group defined by the presentation of the statement and consider the natural homomorphisms $p : \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_A\mathsf{G}$ and $\pi : \overline{\Omega}_A\mathsf{G} \to H$.

We begin by proving (1) ⇒ (2). Suppose that $\theta : H \to S$ is an onto continuous homomorphism. Consider the mapping $f = \theta \circ \pi \circ p|_A \in S^A$, whose unique continuous homomorphic extension $\hat{f} : \overline{\Omega}_A\mathsf{V} \to S$ is the mapping $\theta \circ \pi \circ p$. Since $\varphi_{\mathsf{G}} \circ p = p \circ \varphi$, we deduce that $\hat{f} \circ \varphi^k = \theta \circ \pi \circ \varphi_{\mathsf{G}}^k \circ p$ for every $k \geq 0$, so that, for every $a \in A$, the following equalities hold: $\hat{f} \circ \varphi^{\omega}(a) = \theta \circ \pi \circ \varphi_{\mathsf{G}}^{\omega}(a) = \theta \circ \pi \circ \varphi_{\mathsf{G}}^{0}(a) = \hat{f} \circ \varphi^0(a) = f(a)$. We have thus proved that $\hat{f} \circ \varphi^{\omega}|_A = f$. In terms of the anti-homomorphism of Lemma 6.1, this means $\overline{\varphi}^{\omega}(f) = f$. As $\overline{\varphi}$ is a transformation of the set S^A, the successive iterates $f, \overline{\varphi}(f), \overline{\varphi}^2(f), \ldots, \overline{\varphi}^{|S^A|}(f)$ cannot all be distinct and $\overline{\varphi}^{\omega}(f)$ must be found in the sequence on or after the first repeated point. Hence the equality $\overline{\varphi}^{\omega}(f) = f$ implies that $\overline{\varphi}^{n}(f) = f$ for some integer n such that $1 \leq n \leq |S^A|$. This proves the implication since $\overline{\varphi}^{n}(f) = f$ yields $\hat{f} \circ \varphi^n|_A = f$ by Lemma 6.1.

The implication (2) ⇒ (3) being trivial, it remains to prove the implication (3) ⇒ (1). It suffices to show that $\hat{f}$ factors through $\pi \circ p$. Since $S \in \mathsf{G}$, $\hat{f}$ factors through p, and we have the following commutative diagram, where the existence of the dashed arrow θ is yet to be established:

$$\begin{tikzcd} \overline{\Omega}_A\mathsf{V} \arrow[r, "\hat{f}"] \arrow[d, "p"'] & S \\ \overline{\Omega}_A\mathsf{G} \arrow[r, "\pi"] \arrow[ur, "\eta"] & H \arrow[u, dashed, "\theta"'] \end{tikzcd}$$

Thus, it is enough to verify that, for every $a \in A$, $\eta(\varphi_{\mathsf{G}}^{\omega}(a)) = \eta(a)$. Taking into account the definition of p and the commutativity of the diagram, the desired equality is equivalent to $\eta(\varphi_{\mathsf{G}}^{\omega}(p(a))) = \eta(p(a))$. In view of $\varphi_{\mathsf{G}} \circ p = p \circ \varphi$ and $\eta \circ p = \hat{f}$, this translates into the equality $\hat{f}(\varphi^{\omega}(a)) = \hat{f}(a)$. We proceed to prove that $\hat{f} \circ \varphi^{\omega}|_A = f$.

Lemma 6.1 yields $\hat{f} \circ \varphi^{\omega}|_A = \overline{\varphi}^{\omega}(f)$. Since, also by the same lemma and by hypothesis, $\overline{\varphi}^{n}(f) = \hat{f} \circ \varphi^{n}|_A = f$ and as $\overline{\varphi}^{\omega} = (\overline{\varphi}^{n})^{\omega}$, we conclude that $\hat{f} \circ \varphi^{\omega}|_A = f$, as was claimed. ■

We say that a profinite semigroup S is *decidable* if there is an algorithm to determine, for a given finite semigroup T, whether there is a continuous homomorphism from S onto T. For instance, if V is a pseudovariety of semigroups and A is a finite set, then $\overline{\Omega}_A\mathsf{V}$, the pro-$\mathsf{V}$ semigroup freely generated by A, is decidable if and only if it is decidable whether a finite A-generated semigroup belongs to V. Thus, the pseudovariety V has a decidable membership problem if and only if all finitely generated free pro-V semigroups are decidable.

The following immediate application of Proposition 6.2 could be stated, and essentially proved in the same way, for much more general presentations. To avoid introducing further notation, we stick here to the type of presentations that we have been considering.

Corollary 6.3. *Let φ be an endomorphism of the free group $FG(A)$ on a finite set A and let $\hat{\varphi}$ be its unique extension to a continuous endomorphism of $\overline{\Omega}_A\mathsf{G}$. Then the profinite group presented by $G = \langle A \mid \hat{\varphi}^{\omega}(a) = a \ (a \in A)\rangle$ is decidable.* ■

For the profinite non-free Schützenberger group of the subshift of Example 5.6, we can prove the stronger property that it is not relatively free, although in fact we do not know whether the pseudovariety generated by all its finite continuous homomorphic images is a proper subclass of G.

Theorem 6.4. *Let φ be the substitution given by $\varphi(a) = ab$ and $\varphi(b) = a^3b$. Then $G(\varphi)$ is not a relatively free profinite group.*

Proof: As has been observed in [2, Example 7.2], the Schützenberger group $H = G(\varphi)$ is the image of the continuous endomorphism φ^ω of $\overline{\Omega}_{\{a,b\}}\mathsf{S}$. In particular, H is the closed subsemigroup generated by $\{\varphi^\omega(a), \varphi^\omega(b)\}$. The argument in [2, Example 7.2] shows that H cannot be relatively free with respect to any pseudovariety containing the two-element group. Hence, it suffices to show that the pseudovariety generated by the finite continuous homomorphic images of H contains the two-element group, i.e., that H has a continuous homomorphic image of finite even order. We claim, more specifically, that the alternating group A_5 is a continuous homomorphic image of H.

Let $A = \{a, b\}$ and let $\theta : \overline{\Omega}_A\mathsf{S} \to A_5$ be the unique continuous homomorphism such that $\theta(a) = (1\,2\,3)$ and $\theta(b) = (3\,4\,5)$. Note that θ is onto. To establish the claim, in view of Proposition 6.2 it is enough to check that $\theta \circ \varphi^{12}|_A = \theta_A$. Although the length of the word $\varphi^n(a)$ depends exponentially on n, the verification can be carried out easily by using Lemma 6.1 since $\theta \circ \varphi^{12}|_A = \overline{\varphi}^{12}(\theta|_A)$. The computation of $\overline{\varphi}^{12}(\theta|_A)$ can be easily done either by hand or by using a computer algebra system like GAP [9] and it confirms that indeed $\overline{\varphi}^{12}$ fixes $\theta|_A$, thereby proving the theorem. ∎

For a profinite group G and a pseudovariety of groups V, denote by G_V the largest factor group of G that belongs to V. For a prime p, let Ab_p denote the pseudovariety of all elementary Abelian p-groups. The following result is well known (cf. [17, Proposition 3.4.2]).

Lemma 6.5. *Let V be a pseudovariety of groups. Suppose that G is a relatively free finitely generated profinite group. Then G_V is a free pro-V group and the two groups have the same rank.*

In [2, Example 7.3] it was proved that the profinite group $G(\tau)$ is not free on three generators: although the computation is wrong, as it starts from an incorrect set of return words, the same argument goes through with the correct set. We may now show the following improvement. The proof follows an approach different from that in [2, Example 7.3].

Theorem 6.6. *The profinite group $G(\tau)$ is not relatively free.*

Proof: In view of Lemma 6.5 it suffices to establish that, for the prime $p = 2$ the factor group $H_p = G(\tau)_{\mathsf{Ab}_p}$ has rank one while $G(\tau)$ has rank at least two.

We first note that, since, in $G(\tau)$, the relations $\Psi^\omega(x) = x$ $(x \in \{a, \beta, \gamma\})$ hold (which follows from Theorem 4.3 but is much weaker than it, as it suffices to apply Remark 3.1), H_p is a quotient of the finite group K_p with the following presentation:

$$\langle \alpha, \beta, \gamma \mid \Psi^\omega(\alpha) = \alpha,\ \Psi^\omega(\beta) = \beta,\ \Psi^\omega(\gamma) = \gamma,$$
$$\alpha\beta = \beta\alpha,\ \beta\gamma = \gamma\beta,\ \gamma\alpha = \alpha\gamma,\ \alpha^p = \beta^p = \gamma^p = 1\rangle.$$

In the group K_p, we have $\Psi(\beta) = \beta^2\gamma^2 = \Psi(\gamma)$, from which we deduce that $\beta = \Psi^\omega(\beta) = \Psi^\omega(\gamma) = \gamma$. Moreover, identifying each function f from $X = \{\alpha, \beta, \gamma\}$ to K_p with the triple $(f(\alpha), f(\beta), f(\gamma))$ and applying iteratively the transformation $\overline{\Psi} \in \mathcal{T}(K_p^X)$, one obtains inductively $\overline{\Psi}^n(\alpha, \beta, \beta) = (\alpha\beta^{2(4^n-1)/3}, \beta^{4^n}, \beta^{4^n})$. By Lemma 6.1, it follows that, in K_p and for $n = m!$ sufficiently large, the equalities $\alpha\beta^{2(4^n-1)/3} = \alpha$ and $\beta^{4^n} = \beta$ hold. In particular, for $p = 2$, we get $\beta = 1$, which shows that K_2, whence also H_2, is indeed cyclic.

For a prime $p > 3$, the above calculations show that K_p has rank two. This would suffice to complete the proof by invoking Theorem 4.3. We prefer to give an additional argument which is independent of that theorem to prove that $G(\tau)$ has rank at least two, which is quite similar to the one given for the proof of Theorem 6.4. Let $A = \{a, b\}$ and consider the transformation $\overline{\tau} \in A_5^A$ associated with the Prouhet-Thue-Morse substitution according to Lemma 6.1. Identifying here $f \in A_5^A$ with the pair $(f(a), f(b))$, we have $\overline{\tau}(x, y) = (xy, yx)$. Again, a straightfoward calculation shows that $\overline{\tau}^6$ fixes the pair of 3-cycles $((1\,2\,3), (3\,4\,5))$. Hence, for the continuous homomorphism $\eta : \overline{\Omega}_A\mathsf{S} \to A_5$ given by $\eta(a) = (1\,2\,3)$ and $\eta(b) = (3\,4\,5)$, by Lemma 6.1 we obtain the equalities $\eta(\tau^\omega(a)) = (1\,2\,3)$ and $\eta(\tau^\omega(a)) = (3\,4\,5)$, from which it follows that $\eta(\tau^\omega(abba)) = (1\,3\,2\,5\,4)$ and $\eta(\tau^\omega(ababba)) = (1\,5\,2)$. Since A_5 is generated by the latter two cycles, while $\tau^\omega(abba)$ and $\tau^\omega(ababba)$ belong to $G(\tau)$, we conclude that the restriction of η to $G(\tau)$ is onto, thereby showing that $G(\tau)$ has rank at least two. ■

The proof of Theorem 6.6 shows that the rank of $G(\tau)$ is either two or three. The following result settles the precise value of the rank. It is an application of the presentation of $G(\tau)$ given by Theorem 4.3.

Theorem 6.7. *The group $G(\tau)$ has a group of order 18 of rank three as a continuous homomorphic image. Hence $G(\tau)$ has rank three.*

Proof: Set $X = \{\alpha, \beta, \gamma\}$. Let H be the group given by the following presentation

$$\langle a, b, c \mid a^2 = b^3 = c^3 = 1,\ bc = cb,\ aba = b^2,\ aca = c^2 \rangle.$$

Note that H is the semidirect product of the subgroup $\langle b, c \rangle$, which is the direct product of two three-element groups, by the two-element subgroup $\langle a \rangle$. Let $\theta : \overline{\Omega}_X\mathsf{G} \to H$ be the continuous homomorphism that sends α, β, γ respectively to a, b, c.

We first verify that $\theta(\Psi^2(x)) = \theta(x)$ for all $x \in X$. Since the calculations are quite similar, we treat only the case where $x = \beta$, leaving the other two cases for the reader to check:

$$\begin{aligned} \theta(\Psi^2(\beta)) &= caba^{-1}cb \cdot cba^{-1}cab \cdot (cab)^{-1} \cdot caba^{-1}cb \cdot cab \cdot cba^{-1}cab \\ &= c \cdot aba \cdot cbcbac \cdot aba \cdot cbcabcb \cdot aca \cdot b \\ &= c \cdot b^2 \cdot cbcbac \cdot b^2 \cdot cbcabcb \cdot c^2 \cdot b = b = \theta(\beta). \end{aligned}$$

From Proposition 6.2, it follows that H is a continuous homomorphic image of $G(\tau)$. Since it is easily checked that H has rank three, it follows that so does $G(\tau)$. ■

The following result adds further information about the presentation of Theorem 4.3.

Proposition 6.8. *For each $x \in \{\alpha, \beta, \gamma\}$, the pseudoidentity $\Psi^\omega(x) = x$ fails in the two-element group C_2. Hence, for each $x \in \{\alpha, \beta, \gamma\}$, the relation $\Psi^\omega(x) = x$, which holds in $G(\tau)$, is nontrivial.*

Proof: Let a be the nonidentity element of C_2 and let $X = \{\alpha, \beta, \gamma\}$. Then, in the notation of Lemma 6.1, one verifies that the transformation $\overline{\Psi}$ is idempotent. Moreover, if we identify each function $h \in C_2^X$ with the triple $(h(\alpha), h(\beta), h(\gamma))$, then $\overline{\Psi}(h_1, h_2, h_3) = (h_3h_1h_2, 1, 1)$. In particular, we obtain $\overline{\Psi}^\omega(a, a, 1) = \overline{\Psi}^\omega(a, 1, a) = (1, 1, 1)$ for all $n \geq 1$. Hence none of the pseudoidentities $\Psi^\omega(x) = x$, with $x \in X$, is satisfied by C_2. ■

Remark 6.9. Note that the proof of Proposition 6.8 also shows that the Schützenberger group $G(\tau)$ cannot be free relatively to any pseudovariety containing some group of even order. Since, among the continuous homomorphic images of $G(\tau)$ is a group of even order by Theorem 6.7 or by the proof of Theorem 6.6, this gives another proof that $G(\tau)$ is not relatively free. Actually, every finite cyclic group is a continuous homomorphic image of $G(\tau)$. Indeed, if we add the relations $\beta = \gamma = 1$ to the presentation of $G(\tau)$ given by Theorem 4.3, the relations $\Psi^\omega(x) = x$ ($x \in \{\alpha, \beta, \gamma\}$) become trivial and so the free procyclic group $\langle\alpha\rangle$ is a continuous homomorphic image of $G(\tau)$.

We end with a couple of open problems.

Let φ be a weakly primitive substitution over a finite alphabet. If Conjecture 5.1 holds, it follows from Corollary 6.3 that, in case φ is an encoding of bounded delay with respect to the factors of $J(\varphi)$, then the associated Schützenberger group $G(\varphi)$ is decidable. This raises the following questions.

Problem 6.10. *(1) Is the Schützenberger group $G(\varphi)$ always decidable for a weakly primitive substitution φ over a finite alphabet?*

(2) More generally, for which (minimal) subshifts $\mathcal{X}$, is the associated Schützenberger group $G(\mathcal{X})$ decidable?

(3) In particular, is there any such group which is undecidable?

It is well known that a free profinite group relatively to an extension-closed pseudovariety is projective as profinite group (cf. [10] and [19, Corollary 11.2.3]) and so, in view of the results of [2] or [15], finitely generated such groups certainly appear as closed subgroups of free profinite semigroups on two generators. P. Zalesskiĭ asked in the *Fields Workshop on Profinite Groups and Applications* (Carleton University, August 2005) and also in the Meeting of the *ESI Programme on Profinite Groups* (Vienna, December 2008) whether in particular free pro-p groups can appear as Schützenberger groups of free profinite semigroups. In our setting, and in view of the results of this section, this suggests the following questions.

Problem 6.11. *Let Φ be a continuous endomorphism of $\overline{\Omega}_X\mathsf{G}$ and let G be the profinite group presented by $\langle X \mid \Phi^\omega(x) = x \ (x \in X)\rangle$.*

(1) Under what assumptions on Φ is G a relatively free profinite group?

(2) Is there any such group G that is relatively free and not absolutely free?

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