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PERFECT CATEGORY-GRADED ALGEBRAS

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ABSTRACT: In a perfect category every object has a minimal projective resolution. We give a sufficient condition for the category of modules over a category-graded algebra to be perfect.

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In [6] the second author explored homological properties of algebras graded over a small category. Our interest in these algebras arose from our research on the homological properties of Schur algebras, but we believe that they play an important organizational role in representation theory in general.

Recall that an abelian category $\mathcal{C}$ is called *perfect* if every object of $\mathcal{C}$ has a *projective cover* (see Section 1). The existence of projective covers for every object guarantees the existence of minimal projective resolutions for every object in the category. The category $\mathcal{C}$ is called *semi-perfect* if every finitely generated object has a projective cover. We say that a category-graded algebra A is (semi)-perfect if the category of A-modules is (semi)-perfect. In [6] it was given a criterion for category-graded algebras to be semi-perfect. This criterion is sufficient to ensure that all category-graded algebras which appear in [5] are semi-perfect. But this is not enough to prove the existence of a minimal projective resolution for some of them, as the kernel of a projective cover may not be finitely generated. In this article we fill this gap by giving a criterion for a category-graded algebra to be perfect.

Now we introduce the notions related with category-graded algebras that will be needed and explain the main result in more detail. Recall that, given a small category C, a C-*graded algebra* (see [6]) is a collection of vector spaces A_α parametrised by the arrows α of C, with preferred elements $e_s \in A_{1_s}$ for every object s of C and a collection of maps $\mu_{\alpha,\beta}\colon A_\alpha \otimes A_\beta \to A_{\alpha\beta}$ for every composable pair of morphisms α, β of C. For $a \in A_\alpha$ and $b \in A_\beta$ we shall write ab for $\mu_{\alpha,\beta}(a \otimes b)$. For every composable triple α, β, and γ of arrows

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in C and $a \in A_\alpha$, $b \in A_\beta$, and $c \in A_\gamma$ we require associativity

$$a(bc) = (ab)c.$$

Suppose also that $\alpha \colon s \to t$. Then we require

$$e_t a = a = a e_s.$$

A C-*graded module* M over a C-graded algebra A is a collection of vector spaces M_γ parametrised by the arrows γ of C with maps $r_{\alpha,\beta} \colon A_\alpha \otimes M_\beta \to M_{\alpha\beta}$ for every composable pair of morphisms in C. We shall write am instead of $r_{\alpha,\beta}(a \otimes m)$ for $a \in A_\alpha$ and $m \in M_\beta$.

As always we will assume the usual module axioms:

$$a(bm) = (ab)m \text{ for } a \in A_\alpha,\ b \in A_\beta,\ m \in M_\gamma,$$

where α, β, and γ are composable; and

$$e_t m = m,$$

where $\gamma \colon s \to t$ and $m \in M_\gamma$.

An A-*homomorphism* between two C-graded A-modules M and N is a collection of linear homomorphisms $f_\gamma \colon M_\gamma \to N_\gamma$ such that for every composable pair of morphisms α, $\beta \in$ C

$$f_{\alpha\beta}(am) = a f_\beta(m).$$

We denote the category of all C-graded A-modules by A-mod.

Given a morphism $\gamma \colon s \to t$ of C, *the left stabiliser* St^l_γ of γ is the submonoid of $\mathrm{C}(t,t)$

$$St^l_\gamma = \{\alpha \in \mathrm{C}(t,t) | \alpha\gamma = \gamma\}.$$

For every C-graded algebra A the multiplication maps $\mu_{\alpha,\beta}$ induce an (ungraded) algebra structure on the vector space

$$A^l(\gamma) := \bigoplus_{\alpha \in St^l_\gamma} A_\alpha$$

with unity e_t.

The main result of this paper is

Theorem 0.1. *Let* C *be a small category such that every sequence* β_1, β_2, . . . *of morphisms in* C, *where* β_{k+1} *is a right divisor of* β_k, *stabilizes. Suppose* A *is a* C*-graded algebra such that* $A^l(\gamma)$ *is (left) perfect for all* γ. *Then* A*-mod is a perfect category.*

In particular we will recover a well-known result that for every positively graded algebra $A = \oplus_{n\geq 0} A_n$ with $A_0 \cong \mathbb{K}$ the category of graded A-modules is perfect.

The main idea of the proof of this theorem is to apply the general criterion of perfectness obtained in [3]. Therefore we start in Section 1 with a result on the radical of an abelian category and a recollection of notions used in that work. Section 2 is devoted to Harada's criterion and the study of perfectness of a class of abelian categories, which will be useful in the sequel. In Section 3 we prove the main result and in Section 4 we give examples.

For undefined notation the reader is referred to [6].

1. Preliminaries

The notion of radical for general additive categories was introduced in [4]. Let $\mathcal{C}$ be an additive category. An *ideal* I of $\mathcal{C}$ is a collection of subgroups $I(A,B)$ of $\mathcal{C}(A,B)$ for each A, $B \in \operatorname{Ob}\mathcal{C}$, such that

$$I(B,C)\mathcal{C}(A,B) \subset I(A,C)$$
$$\mathcal{C}(B,C)I(A,B) \subset I(A,C).$$

Definition 1.1 ([4]). *A* radical *of an additive category $\mathcal{C}$ is an ideal I of $\mathcal{C}$ such that for every object A of $\mathcal{C}$ we have $I(A,A) = J(\mathcal{C}(A,A))$, where J denotes the Jacobson radical of the ring.*

It can be shown that for an abelian category there is a unique radical. For convenience of the reader we provide a proof of this fact in the appendix.

Let $\mathcal{C}$ be an abelian category and A, B objects of $\mathcal{C}$. Denote by $\pi_A\colon A\oplus B \to A$, $\pi_B\colon A\oplus B \to B$, $i_A\colon A \to A\oplus B$, and $i_B\colon B \to A\oplus B$ the canonical projections and inclusions associated with the definition of the direct sum $A\oplus B$. Using these maps we can identify the ring $\mathcal{C}(A\oplus B, A\oplus B)$ with the matrix ring

$$\begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(B,A) \\ \mathcal{C}(A,B) & \mathcal{C}(B,B) \end{pmatrix}.$$

It will be shown in Proposition 5.1 that the radical J of $\mathcal{C}$ is given by

$$J\left(A,B\right) = \left\{ f \,\middle|\, \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in J\begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(B,A) \\ \mathcal{C}(A,B) & \mathcal{C}(B,B) \end{pmatrix} \right\}.$$

We will also need the following technical property of the radical of $\mathcal{C}$.

Proposition 1.1. *Let $\mathcal{C}$ be an abelian category and A, B objects of $\mathcal{C}$. Suppose that $A = A' \oplus A''$ and $B = B' \oplus B''$. Then $J(A', B') = \pi_{B'} J(A, B) i_{A'}$.*

Proof: We can identify $\mathcal{C}(A \oplus B, A \oplus B)$ with the matrix ring

$$R := \begin{pmatrix} \mathcal{C}(A', A') & \mathcal{C}(A'', A') & \mathcal{C}(B', A') & \mathcal{C}(B'', A') \\ \mathcal{C}(A', A'') & \mathcal{C}(A'', A'') & \mathcal{C}(B', A'') & \mathcal{C}(B'', A'') \\ \mathcal{C}(A', B') & \mathcal{C}(A'', B') & \mathcal{C}(B', B') & \mathcal{C}(B'', B') \\ \mathcal{C}(A', B'') & \mathcal{C}(A'', B'') & \mathcal{C}(B', B'') & \mathcal{C}(B'', B'') \end{pmatrix}.$$

Then if we repeat the considerations in the proof of Proposition 5.1 we see that $J(A', B')$ is the set of maps $f\colon A' \to B'$ such that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is an element of $J(R)$. Now the elements of $J(A, B)$ can be identified with the elements of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \end{pmatrix}$$

in $J(R)$. Thus $f\colon A' \to B'$ is an element of $J(A', B')$ if and only if $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ is an element of $J(A, B)$. Note that in matrix notation $\pi_{B'} = (1_{B'}, 0)$ and $i_{A'} = \begin{pmatrix} 1_{A'} \\ 0 \end{pmatrix}$. This shows that $J(A', B') \subset \pi_{B'} J(A, B) i_{A'}$. Now suppose that $f_{11} \in \pi_{B'} J(A, B) i_{A'}$. Then there is a matrix

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

that lies in $J(A, B)$. Since J is an ideal of $\mathcal{C}$ we get that

$$\begin{pmatrix} f_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_{B'} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 1_{A'} & 0 \\ 0 & 0 \end{pmatrix} \in J(A, B),$$

which shows that $f_{11} \in J(A', B')$. Thus $\pi_{B'} J(A, B) i_{A'} \subset J(A', B')$. ■

Next we introduce some standard notation which will be used in the following sections.

We say that $X \subset Y$ is *a small subobject* of Y if for any $S \subset Y$ such that $X + S = Y$ we have $S = Y$. An epimorphism $\pi\colon P \twoheadrightarrow Y$, where P is projective, is called a *projective cover* of Y whenever $\operatorname{Ker}\pi$ is a small subobject of P.

Note that in a perfect abelian category every object has a (unique up to isomorphism) *minimal projective resolution*. By definition a minimal projective resolution of an object X is an exact complex $(P_\bullet, d_\bullet)$ with a map $\varepsilon\colon P_0 \twoheadrightarrow X$, such that the maps $d_k\colon P_{k+1} \to \operatorname{Ker}(d_{k-1})$ and ε are projective covers. The existence of minimal projective resolutions in a perfect category can be shown by induction.

2. Harada criterion

In this section we give a sufficient condition for a Grothendieck category $\mathcal{C}$ to be perfect. This is based on Harada's criterion of perfectness, Corollary 1 p.338 of [3]. The crucial ingredient of this criterion is the notion of locally right T-nilpotent system with respect to an ideal I of $\mathcal{C}$.

Definition 2.1. *A set of objects $\{\, M_i \,|\, i \in I\}$ in an additive category is called* locally right T-nilpotent *with respect to an ideal I if for any sequence of maps $f_k \in I\left(M_{i_k}, M_{i_{k+1}}\right)$, $k = 1, 2, \dots,$ and every small subobject X of M_1 there is a natural number m such that $f_m f_{m-1} \dots f_1(X) = 0$.*

Since the only ideal we are interested in is the radical J of an abelian category $\mathcal{C}$, we abbreviate the term "locally right T-nilpotent system with respect to J" to "T-nilpotent system".

Definition 2.2. *Let $\mathcal{C}$ be an abelian category. We say that an object $B \in \mathcal{C}$ is* semi-perfect *(*completely indecomposable*) if the ring $\mathcal{C}(B, B)$ is semi-perfect (local).*

Note that our definition of semi-perfect object is different from the definition given in [3] on p. 330, but this does not interfere with the work.

Let $\{\, P_\alpha \,|\, \alpha \in I\}$ be a generating set of semi-perfect objects of an abelian category $\mathcal{C}$. Then each ring $\mathcal{C}(P_\alpha, P_\alpha)$ is semi-perfect. By Theorem 27.6 of [1] for each α the ring $\mathcal{C}(P_\alpha, P_\alpha)$ has a complete orthogonal set of idempotents $e_{\alpha,1}$, $e_{\alpha,2}$, $\dots$, e_{α,n_α} and for every $\alpha \in I$ and every $1 \le i \le n_\alpha$ the ring $e_{\alpha,i}\mathcal{C}(P_\alpha, P_\alpha)e_{\alpha,i}$ is local. We denote by $P_{\alpha,i}$ the direct summand of P_α that corresponds to $e_{\alpha,i}$. We also write $\pi_{\alpha,j}$ for the canonical projection of P_α on $P_{\alpha,j}$ and $i_{\alpha,j}$ for the canonical embedding of $P_{\alpha,j}$ in P_α.

Proposition 2.1. *The objects $P_{\alpha,i}$ are completely indecomposable.*

Proof: Since $e_{\alpha,1}, \ldots, e_{\alpha,n_\alpha}$ is a complete orthogonal set of idempotents the ring $\mathcal{C}(P_{\alpha,i}, P_{\alpha,i}) \cong e_{\alpha,i}\mathcal{C}(P_\alpha, P_\alpha)e_{\alpha,i}$ is local. Thus $P_{\alpha,i}$ is completely indecomposable. ∎

Proposition 2.2. *Let $\mathcal{C}$ be a Grothendieck category with a generating set of finitely generated objects. Suppose $\mathcal{C}$ has a generating set $\{ P_\alpha \mid \alpha \in I\}$ of semi-perfect projective objects. If $\{ P_\alpha \mid \alpha \in I\}$ is a right T-nilpotent system then $\mathcal{C}$ is perfect.*

Proof: In this proof we are going to apply Corollary 1 on p.338 of [3]. This claims that if $\mathcal{C}$ has a generating set of finitely generated objects and $\{ Q_\beta \mid \beta \in K\}$ is a T-nilpotent generating set of completely indecomposable projective objects, then $\mathcal{C}$ is perfect. Thus we have to construct a T-nilpotent generating set of completely indecomposable projective objects.

If we apply the construction described above to $\{ P_\alpha \mid \alpha \in I\}$, we get a generating set $\mathcal{G} = \{ P_{\alpha,i} \mid \alpha \in I,\ i = 1, \ldots, n_\alpha\}$. Every object $P_{\alpha,i}$ is a direct summand of P_α and so $P_{\alpha,i}$ is projective. The object $P_{\alpha,i}$ is also completely indecomposable by Proposition 2.1.

Now we will show that $\mathcal{G}$ is T-nilpotent. Let $P_{\alpha_1,i_1}, P_{\alpha_2,i_2}, \ldots$ be a sequence of objects in $\mathcal{G}$ and $f_k \in J(P_{\alpha_k,i_k}, P_{\alpha_{k+1},i_{k+1}})$. From Proposition 1.1 it follows that $J(P_{\alpha_k,i_k}, P_{\alpha_{k+1},i_{k+1}}) = \pi_{\alpha_{k+1},i_{k+1}} J(P_{\alpha_k}, P_{\alpha_{k+1}}) i_{\alpha_k,i_k}$. Thus there is $\widetilde{f}_k \in J(P_{\alpha_k}, P_{\alpha_{k+1}})$ such that $f_k = \pi_{\alpha_{k+1},i_{k+1}} \widetilde{f}_k i_{\alpha_k,i_k}$. Denote by g_k the element $i_{\alpha_{k+1},i_{k+1}} \pi_{\alpha_{k+1},i_{k+1}} \widetilde{f}_k$ of $J(P_{\alpha_k}, P_{\alpha_{k+1}})$. Then we have

$$f_r \ldots f_1 = \pi_{\alpha_{r+1},i_{r+1}} g_r \ldots g_1 i_{\alpha_1,i_1}.$$

Let X be a small subobject of P_{α_1,i_1}. Then $i_{\alpha_1,i_1}(X)$ is a small subobject of P_α. Since $\{ P_\alpha \mid \alpha \in I\}$ is T-nilpotent there is some n such that

$$g_n \ldots g_1 i_{\alpha_1,i_1}(X) = 0.$$

But then $f_n \ldots f_1(X) = 0$. ∎

3. The main result

Let C be a small category. We define a C-graded vector space V over a field $\mathbb{K}$ to be a collection of $\mathbb{K}$-vector spaces V_γ parametrized by the arrows of C. A map of a C-graded vector space V to a C-graded vector space W is a collection of linear homomorphisms $f_\gamma\colon V_\gamma \to W_\gamma$. The category of C-graded vector spaces is denoted by $\mathcal{V}$. We denote by $\mathbb{K}[\gamma]$ the C-graded vector space with all components equal to zero except the one with index γ and $(\mathbb{K}[\gamma])_\gamma = \mathbb{K}$.

Let A be a $\mathcal{C}$-graded algebra. In Section 2 of [6] it was defined a functor

$$F_A\colon \mathcal{V} \to A\text{-mod}$$

given on objects by the formula

$$F_A\,(V)_\gamma = \bigoplus_{\gamma=\alpha\beta} A_\alpha \otimes V_\beta.$$

We will call the objects in the image of F_A free. Note that the functor F_A is a left adjoint to the obvious forgetful functor A-mod $\to \mathcal{V}$.

Recall from the introduction that given an arrow $\gamma\colon s \to t$ of the small category C we denote by $A^l(\gamma)$ the algebra

$$\bigoplus_{\alpha\in St^l_\gamma} A_\alpha,$$

where $St^l(\gamma)$ is the left stabilizer of γ.

Theorem 3.1. *Let* C *be a small category such that every sequence* β_1, β_2, ... *of morphisms in* C*, where* β_{k+1} *is a right divisor of* β_k*, stabilizes. Suppose* A *is a* C*-graded algebra such that* $A^l(\gamma)$ *is (left) perfect for all* γ*. Then* A*-mod is a perfect category.*

Proof: We will use several results of [6] and Proposition 2.2. It is proved in Proposition 4.3 [6] that A-mod is a Grothendieck category. By Proposition 4.1 [6] the set

$$\{\, F_A\,(\mathbb{K}[\gamma])\,|\,\gamma \text{ an arrow in C}\}$$

is a generating set of A-mod. Every A-module $F_A\,(\mathbb{K}[\gamma])$ is projective by Proposition 5.1 [6] and finitely generated by Corollary 6.1 and Proposition 6.1 of [6].

It is proved in Theorem 8.1 of [6] that there is an isomorphism of rings $A\text{-mod}(F_A(\mathbb{K}[\gamma]), F_A\,(\mathbb{K}[\gamma])) \cong A^l(\gamma)^{op}$. Note that every left or right perfect ring is semi-perfect. Thus, since $A^l(\gamma)^{op}$ is a right perfect ring, we get that $A\text{-mod}(F_A(\mathbb{K}[\gamma],)F_A\,(\mathbb{K}[\gamma]))$ is a semi-perfect ring. Hence $F_A(\mathbb{K}[\gamma])$ is a semi-perfect object.

We will show now that $\{ F_A(\mathbb{K}[\gamma]) \mid \gamma \text{ an arrow in C}\}$ is a T-nilpotent system. Let $f_k\colon F_A(\mathbb{K}[\beta_k]) \to F_A(\mathbb{K}[\beta_{k+1}])$ be a sequence of maps such that

$$f_k \in J\left(F_A\left(\mathbb{K}[\beta_k]\right), F_A\left(\mathbb{K}[\beta_{k+1}]\right)\right).$$

From the adjunction isomorphism referred above we have an isomorphism of C-graded vector spaces

$$\begin{aligned} A\text{-mod}\left(F_A\left(\mathbb{K}[\beta_k]\right), F_A\left(\mathbb{K}[\beta_{k+1}]\right)\right) &\cong \mathcal{V}\left(\mathbb{K}[\beta_k], F_A\left(\mathbb{K}[\beta_{k+1}]\right)\right) \\ &\cong \left(F_A\left(\mathbb{K}[\beta_{k+1}]\right)\right)_{\beta_k} \cong \bigoplus_{\alpha\beta_{k+1}=\beta_k} A_\alpha. \end{aligned}$$

Therefore if β_{k+1} is not a right divisor of β_k then necessary $f_k = 0$. Now by the theorem hypothesis there are two possibilities:

1) there is some n such that β_{n+1} is not a right divisor of β_n. Then $f_n = 0$ and $f_n\left(f_{n-1}\dots f_1\left(X\right)\right) = 0$ for any small subobject X of $F_A\left(\mathbb{K}[\beta_1]\right)$.

2) There is some N such that for all $n \geq N$ we have $\beta_n = \beta_N$. Then

$$f_n \in J\left(A\text{-mod}\left(F_A\left(\mathbb{K}[\beta_n]\right)\right), F_A\left(\mathbb{K}[\beta_{n+1}]\right)\right) \cong J\left(A^l(\beta_N)^{op}\right).$$

Denote by $\widetilde{f_n}$ the element of $J\left(A^l\left(\beta_N\right)^{op}\right)$ that corresponds to f_n under this isomorphism. Then $f_n \dots f_N$ corresponds to $\widetilde{f_n}\dots\widetilde{f_N}$ in $J(A^l(\beta_N)^{op}) \subset A^l(\beta_N)^{op}$. This last product corresponds under duality to the product $\widetilde{f}_N^{op}\dots\widetilde{f}_r^{op}$ in $J\left(A^l\left(\beta_N\right)\right)$. Now the algebra $A^l(\beta_N)$ is left perfect by the hypothesis of the theorem. Therefore by Theorem 28.4(b) [1] the ideal $J\left(A^l\left(\beta_n\right)\right)$ is left T-nilpotent. Thus there is $r \geq N$ such that $\widetilde{f}_N^{op}\dots\widetilde{f}_r^{op}$ in $J\left(A^l\left(\beta_N\right)\right) = 0$. Hence $f_r \dots f_N = 0$. ■

4. Examples

In this section we apply the main theorem to some classes of interesting algebras.

4.1. Algebras graded over a monoid. Let Γ be a monoid with unit e. We denote by $(*, \Gamma)$ the category with one object $*$ and the set of morphisms given by Γ. Recall that a Γ-graded algebra is an algebra A with a fixed direct sum decomposition into subspaces A_γ, $\gamma \in \Gamma$ such that $e_A \in A_e$ and $A_\alpha A_\beta \subset A_{\alpha\beta}$. Analogously, a Γ-graded module M over a Γ-graded algebra A is defined as an A-module with a direct sum decomposition $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ such that $A_\alpha M_\beta \subset M_{\alpha\beta}$. Homomorphisms of Γ-graded algebras (A-modules) are homomorphisms of algebras (A-modules) that preserve the components of the direct sum decomposition.

It immediately follows that purely syntactical replacement of the sign $\bigoplus_{\gamma\in\Gamma}$ by the brackets $(\ )_{\gamma\in\Gamma}$ gives an equivalence between the category of Γ-graded algebras and the category $(*,\Gamma)$-graded algebras. By the same argument, if $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is an Γ-graded algebra then the category of Γ-graded A-modules is equivalent to the category of A'-modules, where A' is the $(*,\Gamma)$-graded algebra that corresponds to A.

Recall that a poset $(S,\leq)$ is called *artinian* if every descending sequence $s_1 \geq s_2 \geq \dots$ of elements in S stabilizes.

Proposition 4.1. *Let Γ be an artinian ordered monoid such that e is the least element. Suppose A is a Γ-graded algebra such that A_e is left perfect. Then the category of left Γ-graded A-modules is perfect.*

Proof: Let A' be the $(*,\Gamma)$-graded algebra that corresponds to A under the equivalence described above. It is sufficient to show that the category A'-mod is perfect. We apply Theorem 3.1 to A'. Let $\gamma\in\Gamma$. Then $St^l(\gamma)=\{e\}$. In fact, suppose $\alpha\gamma=\gamma$ and $\alpha\neq e$. Since e is the least element of Γ we have $\alpha>e$, and, since Γ is an ordered monoid it follows that $\alpha\gamma>e\gamma=\gamma$. Contradiction. Therefore for all $\gamma\in\Gamma$ the algebra $(A')^l(\gamma)=A'_e=A_e$ is left perfect.

Suppose $\gamma_1, \gamma_2, \dots$ is a sequence of elements in Γ such that γ_{k+1} is a right divisor of γ_k. Since e is the least element of Γ we get that $\gamma_k>\gamma_{k+1}$. Therefore $\gamma_1, \gamma_2, \dots$ is a descending sequence and must stabilize as Γ is artinian. ∎

An example of a graded algebra in the conditions just described is the Kostant form of the universal enveloping algebra of the complex Lie algebra of strictly upper triangular matrices. In our work on Schur algebras [5], we were led to the construction of a minimal projective resolution of the trivial module of this Kostant form. Although this module is obviously finitely generated the same can not to be said of the kernels of the projective covers which appear in the resolution. It was this example that motivated the present paper.

Now we give an example which shows that the condition “Γ is artinian” in Proposition 4.1 is essential.

Let $\Gamma = (\mathbb{Z}, +)$ and denote $(*, \Gamma)$ by C. Define a C-graded algebra A by

$$A_k := \begin{cases} \mathbb{K}a_k & k \geq 0 \\ 0 & \text{otherwise} \end{cases}, \ k \in \mathbb{Z}$$

and multiplication $a_k a_l = a_{k+l}$. In fact, A is just the polynomial algebra in one variable considered as a C-graded algebra. We define a $\mathcal{C}$-graded A-module X by

$$X_k := \mathbb{K}x_k,\ k \in \mathbb{Z}$$

and the action of A on X is given by $a_k x_l = x_{k+l}$.

Proposition 4.2. *The module X has no projective cover in A-mod.*

Proof: Suppose P is a projective cover of X. We show first that P is a free module. Since every object in A-mod can be covered by a free module (Proposition 4.1 [6]) and every epimorphism to P is splitable, we know that P is a direct summand of a free module $F_A(V)$, where V is a C-graded vector space. Let $e\colon F_A(V) \to F_A(V)$ be the idempotent that corresponds to P. Note that for every k we have

$$F_A(V)_k = \bigoplus_{l \geq k} a_{l-k} \otimes V_l.$$

We claim that the following maps are idempotent:

$$e_k\colon V_k \to (F_A(V))_k \xrightarrow{e} (F_A(V))_k \to V_k,$$

where the first map is the inclusion induced by the unity of A and the last map is the natural projection π_k with respect to above direct decomposition. Let $v \in V_k$, then $e(a_0 \otimes v) = \sum_{l \geq 0} \sum_{i \in I_l} \lambda_l a_l v_l$, where $v_l \in V_{k-l}$. Now all elements of $e\left(\sum_{l>0} \sum_{i \in I_l} \lambda_l a_l v_l\right)$ lie in the direct sum $\bigoplus_{l>k} a_{l-k} \otimes V_l$. Therefore

$$\pi_k e\left(\sum_{l>0} \sum_{i \in I_l} \lambda_l a_l v_l\right) = 0.$$

This show that

$$\pi_k e^2(v) = \pi_k e(\pi_k e(v)).$$

Since $e^2 = e$ we get $e_k(v) = e_k^2(v)$. Now for every k we choose a basis $\{\, v_{k,i} \mid i \in I_k \} \cup \{\, v_{k,j} \mid j \in J_k \}$ such that $e_k(v_{k,i}) = v_{k,i}$ for all $i \in I_k$ and $e_k(v_{k,j}) = 0$ for all $j \in J_k$. Then $e(v_{k,i}) = v_{k,i} + \sum_{l>k} \sum_{j \in I_l} \lambda_{l,j} a_{l-k} v_{l,j}$ for $i \in I_k$, and $e(v_{k,j}) = \sum_{l>k} \sum_{j \in I_l} \lambda_{l,j} a_{l-k} v_l$ for $j \in J_k$. Let W be a

C-graded vector space with basis $\{\, w_{k,i} \,|\, i \in I_k \}$, $w_{k,i} \in W_k$. Define a map $f\colon F_A(W) \to F_A(V)$ by

$$f\left(a_s \otimes w_{k,i}\right) = a_s \otimes v_{k,i} + \sum_{l>k} \sum_{j\in I_l} \lambda_{l,j} a_{s+l-k} v_{l,j}.$$

This is obviously an injective map of A-modules. Moreover, from the above computations it follows that all elements of $e\left(F_A\left(V\right)\right)$ belong to the image of f. Thus f provides an isomorphism between $F_A\left(W\right)$ and P.

Suppose $\phi\colon F_A(W) \to X$ is a projective cover. By the adjunction property of F_A the map ϕ is given by its values on $w_{k,i}$, $i \in I_k$. Note first that $\phi(w_{k,i}) \neq 0$, since otherwise the direct summand $Aw_{k,i}$ of $F_A\left(W\right)$ would lie in the kernel of ϕ, which contradicts the fact that $\operatorname{Ker}\phi$ is small. Analogously, we see that all the elements $\phi\left(w_{k,i}\right)$, $i \in I_k$, should be linearly independent. Since the X_k are one dimensional, every set I_k contains no more than one element. We shall denote this unique element, if it exists, by w_k. After scalar multiplication we can assume without loss of generality that $\phi\left(w_k\right) = x_k$. Let us fix k such that I_k is non-empty. Since x_l for $l > k$ are not in the image of Aw_k there is $r > k$ such that I_r is non-empty. Denote by W' the C-graded vector space spanned by $\{\, w_l \,|\, I_l$ is non-empty and $l > k \}$. Let ϕ' be the restriction of ϕ on $F_A(W')$. Then ϕ can be factorized via ϕ' by $\phi = \phi' \circ \theta$, where $\theta\left(w_s\right) = a_{r-s} w_r$ for all $s \geq k$ and $\theta\left(w_s\right) = w_s$ for $s < k$. Since θ is surjective we see that $F_A(W')$ has a direct complement P in $F_A(W)$. As $\theta(P) = 0$, it follows that $P \subset \operatorname{Ker}\phi$ which contradicts the fact that $\operatorname{Ker}\phi$ is small. ∎

4.2. Poset-graded algebras. Let $(\Lambda, \leq)$ be a poset. Denote by $\widetilde{\Lambda}$ the category with the set of objects Λ and exactly one morphism $\overline{\mu\lambda}$ from λ to μ for $\mu \geq \lambda$.

Let A be a $\mathbb{K}$-algebra with an orthogonal unit decomposition

$$e = \sum_{\lambda\in\Lambda} e_\lambda,\ e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda$$

such that $e_\mu A e_\lambda = 0$ if $\mu \ngeq \lambda$. Define the $\widetilde{\Lambda}$-graded algebra $\widetilde{A}$ by

$$\widetilde{A}_{\overline{\mu\lambda}} := \{\, [\![a]\!] \,|\, a \in e_\mu A e_\lambda \}$$

with the vector space structure inhereted from $e_\mu A e_\lambda$ via the bijection $[\![a]\!] \mapsto a$. We define multiplication on $\widetilde{A}$ by

$$[\![x]\!]\,[\![y]\!] := [\![xy]\!]\,,\ x \in e_\nu A e_\mu,\ y \in e_\mu A e_\lambda.$$

Then $[\![e_\lambda]\!] \in \widetilde{A}_{\lambda\lambda}$ are local units in $\widetilde{A}$.

Proposition 4.3. *Let Λ, A and $\widetilde{A}$ be as above. Suppose that for every λ, $\mu \in \Lambda$, $\mu \geq \lambda$ the interval $[\lambda, \mu]$ is an artinian poset and that for every $\lambda \in \Lambda$ the algebra $e_\lambda A e_\lambda$ is left perfect. Then the category $\widetilde{A}$-mod is perfect.*

Proof: We apply Theorem 3.1. Note first that $St^l(\overline{\mu\lambda}) = \overline{\mu\mu}$ for every $\mu \geq \lambda$. Therefore $\widetilde{A}^l(\overline{\mu\lambda}) = \widetilde{A}_{\overline{\mu\mu}} = e_\mu A e_\mu$ are left perfect algebras for all maps in $\widetilde{\Lambda}$.

Suppose α_1, α_2, …is a sequence of maps in $\widetilde{\Lambda}$ such that α_{k+1} is a right divisor of α_k. Then there are λ, μ_1, μ_2, …in Λ such that $\alpha_k = \overline{\mu_k\lambda}$ and $\mu_{k+1} \leq \mu_k$. Since $[\lambda, \mu_1]$ is artinian and every μ_k lies in this interval, we get that $\mu_1 > \mu_2 > \dots$ stabilizes. Therefore α_1, α_2, …stabilizes as well. ∎

5. Appendix

As we mentioned before, in this appendix we prove that the radical of an abelian category is unique and characterize it.

Proposition 5.1. *If $\mathcal{C}$ is an abelian category then there is a unique radical in $\mathcal{C}$.*

Proof: We use the notation introduced immediately after Definition 1.1. Let I be a radical of $\mathcal{C}$. Then

$$i_B I(A, B)\pi_A \subset I(A \oplus B, A \oplus B)$$

and

$$\pi_B I(A \oplus B, A \oplus B) i_A \subset I(A, B).$$

Therefore

$$I(A, B) = \pi_B i_B I(A, B)\pi_A i_A \subset \pi_B I(A \oplus B, A \oplus B) i_A \subset I(A, B)$$

and, as a consequence,

$$I(A, B) = \pi_B I(A \oplus B, A \oplus B) i_A = \pi_B J\left(\mathcal{C}(A \oplus B, A \oplus B)\right) i_A.$$

This shows that the radical is unique if it exists.

Next we show the existence of a radical in $\mathcal{C}$. Define $J(A,B)$ by

$$J(A,B) = \left\{ f \,\middle|\, \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(B,A) \\ \mathcal{C}(A,B) & \mathcal{C}(B,B) \end{pmatrix} \right\}.$$

First we show that J is an ideal in $\mathcal{C}$. Let C be an object in $\mathcal{C}$. Denote by E the idempotent

$$\begin{pmatrix} 1_A & 0 & 0 \\ 0 & 1_B & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $\mathcal{C}(A\oplus B\oplus C, A\oplus B\oplus C)$. By Proposition 5.13 [2] we have an isomorphism

$$J(\mathcal{C}(A\oplus B, A\oplus B)) \cong EJ(\mathcal{C}(A\oplus B\oplus C, A\oplus B\oplus C))E.$$

Therefore

$$J(A,B) = \left\{ f\colon A\to B \,\middle|\, \begin{pmatrix} 0 & 0 & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(B,A) & \mathcal{C}(C,A) \\ \mathcal{C}(A,B) & \mathcal{C}(B,B) & \mathcal{C}(C,B) \\ \mathcal{C}(A,C) & \mathcal{C}(B,C) & \mathcal{C}(C,C) \end{pmatrix} \right\}.$$

Let $g\colon B\to C$. Then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ gf & 0 & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(B,A) & \mathcal{C}(C,A) \\ \mathcal{C}(A,B) & \mathcal{C}(B,B) & \mathcal{C}(C,B) \\ \mathcal{C}(A,C) & \mathcal{C}(B,C) & \mathcal{C}(C,C) \end{pmatrix}.$$

Switching the roles of B and C in the above considerations we obtain

$$J(A,C) = \left\{ h\colon A\to C \,\middle|\, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ h & 0 & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(B,A) & \mathcal{C}(C,A) \\ \mathcal{C}(A,B) & \mathcal{C}(B,B) & \mathcal{C}(C,B) \\ \mathcal{C}(A,C) & \mathcal{C}(B,C) & \mathcal{C}(C,C) \end{pmatrix} \right\}$$

and therefore $gf \in J(A,C)$. This shows that J is a left ideal of $\mathcal{C}$. That J is a right ideal can be shown analogously.

Now we have to check that $J(A,A) = J(\mathcal{C}(A,A))$. By definition we have

$$J(A,A) = \left\{ f\colon A\to A \,\middle|\, \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(A,A) \\ \mathcal{C}(A,A) & \mathcal{C}(A,A) \end{pmatrix} \right\}.$$

Let $f \in J(A,A)$. Then

$$\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(A,A) \\ \mathcal{C}(A,A) & \mathcal{C}(A,A) \end{pmatrix}$$

since $J\left(\mathcal{C}(A \oplus A, A \oplus A)\right)$ is an ideal of $\mathcal{C}(A \oplus A, A \oplus A)$. As

$$\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = e \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} e,$$

where $e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$, we obtain by Proposition 5.13 [2]

$$\begin{aligned} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in eJ \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(A,A) \\ \mathcal{C}(A,A) & \mathcal{C}(A,A) \end{pmatrix} e &= J\left(e \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(A,A) \\ \mathcal{C}(A,A) & \mathcal{C}(A,A) \end{pmatrix} e \right) \\ &= J \begin{pmatrix} \mathcal{C}(A,A) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore $f \in J(\mathcal{C}(A,A))$ and $J(A,A) \subset J\left(\mathcal{C}(A,A)\right)$.

Now suppose that $f \in J(\mathcal{C}(A,A))$. Then

$$\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(A,A) \\ \mathcal{C}(A,A) & \mathcal{C}(A,A) \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1_A & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in J \begin{pmatrix} \mathcal{C}(A,A) & \mathcal{C}(A,A) \\ \mathcal{C}(A,A) & \mathcal{C}(A,A) \end{pmatrix}$$

since $J\left(\mathcal{C}(A \oplus A, A \oplus A)\right)$ is an ideal of $\mathcal{C}(A \oplus A, A \oplus A)$. Thus $f \in J(A,A)$ and $J\left(\mathcal{C}(A,A)\right) \subset J(A,A)$.

■

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