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OPTIMAL CONTROL AND QUASI-VELOCITIES

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Abstract: In this paper we study optimal control problems for nonholonomic systems defined on Lie algebroids by using quasi-velocities. We consider both kinemactic, i.e. systems whose cost functional depends only on position and velocities, and dynamic optimal control problems, i.e. systems whose cost functional depends also on accelarations. Formulating the problem directly at the level of Lie algebroids turns out to be the correct framework to explain in detail similar results appeared recently [48]. We also provide several examples to illustrate our construction.

1. Introduction

The principles of analytical mechanics established by D'Alembert, Lagrange, Gauss and Hamilton can also be contemplated from additional mathematical xperspectives providing us methods for understanding Nature's law from new viewpoints which may be helpful in solving specific problems and clarifying the way in which Nature works. The traditional techniques were only applied to very simple models but current technology needs efficient algorithms in areas ranging from robotics to spacecraft design. Furthermore the computer development with the corresponding capability of computation suggests the convenience of analysing different formulations to yield the differential equations for multibody dynamics that involve a certain number of constraints.

There exist different techniques to deal with such constrained systems. The geometric framework of manifolds replacing Euclidean spaces allows us to give a formulation for systems with holonomic constraints in terms of generalised coordinates and free of Lagrange multipliers. However it is not clear how to choose generalised coordinates improving computational efficiency. Kane's method [1, 19, 21, 22, 27, 28, 51, 52] which eliminate constraint forces and is a projection method is also useful for multibody dynamics analysis and mobile manipulators (see e.g [33, 34] for a modern geometric approach). The

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Maggi equations [11, 31] formulation uses true coordinates and it is quite similar to Kane's method [4]. Another recent alternative formulation is given in [54, 55].

Nonholonomic constraints are very relevant and appear in many problems in physics and engineering, and in particular in control theory. Such nonholonomic constraints restrict possible virtual displacements and when taking into account such constraints d'Alembert-Lagrange principle leads to Boltzmann–Hamel equations [15, 16, 47, 48, 49].

The concept of quasi-velocity (or generalised velocity) [10, 35, 50] is of a great relevance in the study of mechanical systems, mainly in nonholonomic ones because the conditions of nonholonomic constraints can be expressed in a simpler form. Boltzmann-Hamel equations, Gibbs–Appell and Gauss principles, for instance, make use of quasi-coordinates (also called nonholonomic coordinates) and the Hamel symbols [30, 50]. The use of quasi-velocities in the dynamics of nonholonomic system with symmetry has recently been investigated under different approaches in [3, 15, 16, 17] and Hamel's equations have been recovered from this perspective.

It has been shown in recent papers [10, 17, 16] that the appropriate geometric framework for studying systems with linear nonholonomic constraints is the framework of Lie algebroids and actually these structures are receiving a lot of attention [13, 14] during the last years due to its capability for dealing with such constraints and its role in reduction processes for Lagrangian systems with symmetry. The geometric approach to mechanics uses tangent bundles in the Lagrangian formulation and tangent bundles are but particular instances of algebroids. The usual geometric approach to Lagrangian formalism was then developed in this extended framework of Lie algebroids [7, 39, 40, 56] the main advantage being that such structure arises in reduction processes from tangent bundles when the vertical endomorphism character is not projectable [8, 9]. Moreover this techniques are also extended to the study of discrete mechanical systems [37, 25, 56]

Many of the techniques of classical mechanics can be used in control theory [12, 15, 17, 41, 45]. The theory of Lie algebroids can be applied to deal with control problems and the application of such geometric tools is very useful for a better understanding of different control problems. This is our motivation for developing the theory of optimal control theory using the properties of Lie algebroid theory which is going to be the appropriate approach to Boltzmann–Hamel equations.

This article is organized in the following way. A brief introduction with the fundamental concepts of the theory of Lie algebroids and the modern application to the theory of Lagrangian mechanics in Lie algebroids is given in Section 2. The symplectic and Lagrangian mechanics in the framework of Lie algebroids are respectively summarized in Sections 3 and 4. The approach to nonholonomic constrained systems and the Euler–Lagrange equations are also introduced in Section 4 within this geometric formalism which in the standard case of mechanics on tangent bundles reduce to Lagrange–d'Alembert equations in terms of quasi-velocities. Section 5 to 7 are specifically devoted to optimal control theory. Section 5 deals with the optimal control theory and the Pontryagin maximum principle [41], kinematic optimal control is studied in Section 6 and dynamical aspects are the aim of Section 7. The theory is illustrated in Section 8 with several examples: the Heisenberg system, the vertical rolling disk, rotational motion of the free rigid body, and constrained systems with symmetry.

2. Preliminaries

Lie algebroids. A Lie algebroid structure on a vector bundle $\tau\colon E \to M$ is given by a vector bundle map $\rho\colon E \to TM$ over the identity in M, called the anchor, together with a Lie algebra structure on the $C^\infty(M)$-module of sections of E such that the compatibility condition $[\sigma, f\eta] = (\rho(\sigma)f)\eta + f[\sigma, \eta]$ is satisfied for every $f \in C^\infty(M)$ and every $\sigma, \eta \in Sec(E)$. See [5, 36] for more information on Lie algebroids.

In what concerns Mechanics, it is convenient to think of a Lie algebroid as a generalization of the tangent bundle of M. One regards an element a of E as a generalized velocity, and the actual velocity v is obtained when applying the anchor to a, i.e., $v = \rho(a)$. A curve $a\colon [t_0, t_1] \to E$ is said to be admissible or an E-path if $\dot{s}(t) = \rho(a(t))$, where $s(t) = \tau(a(t))$ is the base curve. We will denote by Adm(E) the space of admissible curves on E.

A local coordinate system (x^i) in the base manifold M and a local basis $\{e_\alpha\}$ of sections of E, determine a local coordinate system (x^i, y^α) on E. The anchor and the bracket are locally determined by the local functions ρ^i_α and $C^\alpha_{\beta\gamma}$ on M given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \text{and} \qquad [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\, e_\gamma.$$

The functions ρ^i_α and $C^\alpha_{\beta\gamma}$ satisfy some relations due to the compatibility condition and the Jacobi identity which are called the structure equations:

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta} \tag{1}$$

$$\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + \rho^i_\beta \frac{\partial C^\nu_{\gamma\alpha}}{\partial x^i} + \rho^i_\gamma \frac{\partial C^\nu_{\alpha\beta}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} + C^\mu_{\gamma\alpha} C^\nu_{\beta\mu} + C^\mu_{\alpha\beta} C^\nu_{\gamma\mu} = 0. \tag{2}$$

Cartan calculus. The Lie algebroid structure is equivalent to the existence of a exterior differential on E, $d\colon Sec(\wedge^k E^*) \to Sec(\wedge^{k+1} E^*)$, defined as follows

$$\begin{aligned} d\omega(\sigma_0, \ldots, \sigma_k) \qquad &= \textstyle\sum_{i=0}^k (-1)^i \rho(\sigma_i)(\omega(\sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \sigma_k)) + \\ &+ \textstyle\sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \widehat{\sigma}_j, \ldots, \sigma_k), \end{aligned}$$

for $\omega \in Sec(\wedge^k E^*)$ and $\sigma_0, \ldots, \sigma_k \in Sec(\tau)$. d is a cohomology operator, that is, $d^2 = 0$. In particular, if $f\colon M \to \mathbb{R}$ is a real smooth function then $df(\sigma) = \rho(\sigma) f$, for $\sigma \in Sec(\tau)$. Locally,

$$dx^i = \rho^i_\alpha e^\alpha \qquad \text{and} \qquad de^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$. The above mentioned structure equations are equivalent to the relations $d^2 x^i = 0$ and $d^2 e^\alpha = 0$. We may also define the Lie derivative with respect to a section σ of E as the operator $\mathcal{L}_\sigma\colon Sec(\wedge^k E^*) \to Sec(\wedge^k E^*)$ given by $\mathcal{L}_\sigma = i_\sigma \circ d + d \circ i_\sigma$. Along this paper, except otherwise stated, the symbol d stands for the exterior differential on a Lie algebroid.

Morphisms. Given a second Lie algebroid $\tau'\colon E' \to M'$, a vector bundle map $\Phi\colon E \to E'$ over $\varphi\colon M \to M'$ is said to be admissible if it maps admissible curves in E into admissible curves in E', or equivalently if $\rho' \circ \Phi = T\varphi \circ \rho$. The map Φ is said to be a morphism of Lie algebroids if $\Phi^\star d\theta = d' \Phi^\star \theta$ for every p-form $\theta \in Sec(\wedge^p E^*)$. Every morphism is an admissible map.

In coordinates, a vector bundle map $\Phi(x, y) = (\varphi^i(x), \Phi^\alpha_\beta(x) y^\beta)$ is admissible if and only if

$$\rho'^k_\beta \Phi^\beta_\alpha = \rho^i_\alpha \frac{\partial \varphi^k}{\partial x^i}. \tag{3}$$

Moreover, such a map is a morphism if in addition to the above equation it satisfies

$$\Phi^\alpha_\gamma C^\gamma_{\mu\nu} = \rho^i_\mu \frac{\partial \Phi^\alpha_\nu}{\partial x^i} - \rho^i_\nu \frac{\partial \Phi^\alpha_\mu}{\partial x^i} + C'^\alpha_{\beta\gamma} \Phi^\beta_\mu \Phi^\gamma_\nu. \tag{4}$$

Prolongation. In what respect to Mechanics, the tangent bundle to a Lie algebroid, to its dual or to a more general fibration does not have an appropriate Lie algebroid structure. Instead one should use the so called prolongation bundle which has in every case the appropriate geometrical structures [39, 40, 46].

Let $(E, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over a manifold M and $\nu : P \to M$ be a fibration. For every point $p \in P$ we consider the vector space

$$\mathcal{T}^E_p P = \{ (b, v) \in E_x \times T_p P \mid \rho(b) = T_p\nu(v) \} ,$$

where $T\nu : TP \to TM$ is the tangent map to ν and $\nu(p) = x$. The set $\mathcal{T}^E P = \cup_{p\in P} \mathcal{T}^E_p P$ has a natural vector bundle structure over P, the vector bundle projection τ^E_P being just the projection $\tau^E_P(b, v) = \tau_P(v)$. We will frequently use the redundant notation (p, b, v) to denote the element $(b, v) \in \mathcal{T}^E_p P$. In this way, the projection τ^E_P is just the projection onto the first factor.

The vector bundle $\tau^E_P : \mathcal{T}^E P \to P$ can be endowed with a Lie algebroid structure. The anchor map is the projection onto the third factor, that is, the map $\rho^1 : \mathcal{T}^E P \to TP$ is given by $\rho^1(p, b, v) = v$. To define the bracket on sections of $\mathcal{T}^E P$ we will consider some special sections. A section $Z \in Sec(\mathcal{T}^E P)$ is said to be projectable if there exists a section $\sigma \in Sec(E)$ such that $Z(p) = (p, \sigma(\nu(p)), U(p))$, for all $p \in P$. Now, the bracket of two projectable sections Z_1, Z_2 given by $Z_i(p) = (p, \sigma_i(\nu(p)), U_i(p))$, $i = 1, 2$, is given by

$$[\![Z_1, Z_2]\!](p) = (p, [\![\sigma_1, \sigma_2]\!](\nu(p)), [U_1, U_2](p)), \text{ with } p \in P.$$

Since any section of $\mathcal{T}^E P$ can be locally written as a $C^\infty(M)$-linear combination of projectable sections, the definition of the Lie bracket for arbitrary sections of $\mathcal{T}^E P$ follows.

The Lie algebroid $\mathcal{T}^E P$ is called the prolongation of $\nu : P \to M$ with respect to E or the E-tangent bundle to ν.

Given local coordinates (x^i, u^A) on P and a local basis $\{e_\alpha\}$ of sections of E, we can define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ of sections of $\mathcal{T}^E P$ by

$$\mathcal{X}_\alpha(p) = \Big(p, e_\alpha(\nu(p)), \rho^i_\alpha \frac{\partial}{\partial x^i}\Big|_p\Big) \qquad \text{and} \qquad \mathcal{V}_A(p) = \Big(p, 0, \frac{\partial}{\partial u^A}\Big|_p\Big).$$

If $z = (p, b, v)$ is an element of $\mathcal{T}^E P$, with $b = z^\alpha e_\alpha$, then v is of the form $v = \rho^i_\alpha z^\alpha \frac{\partial}{\partial x^i} + v^A \frac{\partial}{\partial u^A}$, and we can write

$$z = z^\alpha \mathcal{X}_\alpha(p) + v^A \mathcal{V}_A(p).$$

Vertical elements are linear combinations of $\{\mathcal{V}_A\}$.

The anchor map ρ^1 applied to a section Z of $\mathcal{T}^E P$ with local expression $Z = Z^\alpha \mathcal{X}_\alpha + V^A \mathcal{V}_A$ is the vector field on P whose coordinate expression is

$$\rho^1(Z) = \rho^i_\alpha Z^\alpha \frac{\partial}{\partial x^i} + V^A \frac{\partial}{\partial u^A}.$$

In particular, the Lie brackets of the elements of the basis are

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!] = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma, \qquad [\![\mathcal{X}_\alpha, \mathcal{V}_B]\!] = 0 \qquad \text{and} \qquad [\![\mathcal{V}_A, \mathcal{V}_B]\!] = 0,$$

and, therefore, the exterior differential is determined by

$$\begin{gathered} dx^i = \rho^i_\alpha \mathcal{X}^\alpha, du^A = \mathcal{V}^A, \\ d\mathcal{X}^\gamma = -\tfrac{1}{2} C^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, d\mathcal{V}^A = 0, \end{gathered}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ is the dual basis to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

Prolongation of maps. We consider now how to prolong maps between two fibrations $\nu\colon P \to M$ and $\nu'\colon P' \to M'$. Let $\Psi\colon P \to P'$ be a map fibered over $\varphi\colon M \to M'$. We consider two Lie algebroids $\tau\colon E \to M$ and $\tau'\colon E' \to M'$ and a map $\Phi\colon E \to E'$ fibered also over φ. If Φ is admissible, then we can define a vector bundle map $\mathcal{T}^\Phi\Psi\colon \mathcal{T}^E P \to \mathcal{T}^{E'} P'$ by means of

$$\mathcal{T}^\Phi\Psi(p, b, v) = (\Psi(p), \Phi(b), T\Psi(v)).$$

It follows that $\mathcal{T}^\Phi\Psi$ is also admissible. In [43] it was proved that $\mathcal{T}^\Phi\Psi$ is a morphism of Lie algebroids if and only if Φ is a morphism of Lie algebroids.

In particular, when $E = E'$ and $\Phi = \mathrm{id}$ we have that any map from P to P' fibered over the identity can be prolonged to a morphism $\mathcal{T}^{\mathrm{id}}\Psi$ which will be denoted simply by $\mathcal{T}\Psi$. We will also identify $\mathcal{T}^E M$ (the prolongation of the 'fibration' $\mathrm{id}\colon M \to M$ with respect to E) with E itself by means of $(m, b, \rho(b)) \equiv b$. With this convention, the projection onto the second factor of $\mathcal{T}^E P$ is just $\mathcal{T}\nu\colon \mathcal{T}^E P \to E$. It follows that $\mathcal{T}\nu$ is a morphism of Lie algebroids.

3. Symplectic Mechanics on Lie algebroids

As our goal is to use Hamiltonian techniques to study optimal control problems on Lie algebroids, we shall begin by presenting the Hamiltonian approach to the problem of defining dynamics on a Lie algebroid. Basically, it is just a generalization of the usual symplectic description of dynamical system defined on cotangent bundles.

By a symplectic section on a vector bundle $\pi\colon F \to M$ we mean a section ω of $\wedge^2 F^*$ which is regular at every point when it is considered as a bilinear form. By a symplectic section on a Lie algebroid E we mean a symplectic section ω of the vector bundle E which is moreover d-closed, that is $d\omega = 0$. A symplectic Lie algebroid is a pair (E, ω) where E is a Lie algebroid and ω is a symplectic section on it.

On a symplectic Lie algebroid (E, ω) we can define a dynamical system for every function on the base, as in the standard case of a tangent bundle. Given a function $H \in C^\infty(M)$ there is a unique section $\sigma_H \in Sec(\tau)$ such that

$$i_{\sigma_H}\omega = dH.$$

The section σ_H is said to be the Hamiltonian section defined by H and the vector field $X_H = \rho(\sigma_H)$ is said to be the Hamiltonian vector field defined by H. In this way we get the dynamical system $\dot{x} = X_H(x)$.

A symplectic structure ω on a Lie algebroid E defines a Poisson bracket $\{\ ,\ \}$ on the base manifold M as follows. Given two functions $F, G \in C^\infty(M)$ we define the bracket

$$\{F, G\} = \omega(\sigma_F, \sigma_G).$$

It is easy to see that the closure condition $d\omega = 0$ implies that $\{\ ,\ \}$ is a Poisson structure on M. In other words, if we denote by Λ the inverse of ω as bilinear form, then $\{F, G\} = \Lambda(dF, dG)$. The Hamiltonian dynamical system associated to H can be written in terms of the Poisson bracket as $\dot{x} = \{x, H\}$.

An important particular class of symplectic dynamical systems on Lie algebroids is to the following:

Hamiltonian Mechanics. [32, 40] On $\mathcal{T}^E E^*$, the E-tangent to the dual bundle $\pi\colon E^* \to M$, we have a canonical symplectic structure.

The Liouville section $\Theta \in Sec((\mathcal{T}^E E^*)^*)$ is the 1-form given by

$$\langle\, \Theta\,, (\mu, b, w)\,\rangle = \langle\, \mu\,, b\,\rangle. \tag{5}$$

The canonical symplectic section $\Omega \in Sec(\wedge^2(\mathcal{T}^E E^*)^*)$ is the differential of the Liouville section

$$\Omega = -d\Theta.$$

Taking coordinates (x^i, μ_α) on E^* and denoting by $\{\mathcal{X}_\alpha, \mathcal{P}^\beta\}$ the associated local basis of sections $\mathcal{T}^E E^*$, the Liouville and canonical symplectic sections are written as

$$\Theta = \mu_\alpha \mathcal{X}^\alpha \qquad \text{and} \qquad \Omega = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\gamma C^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta,$$

where $\{\mathcal{X}^\alpha, \mathcal{P}_\beta\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{P}^\beta\}$.

The Hamiltonian section defined by a function $H \in C^\infty(E^*)$ is given in coordinates by

$$\Gamma_H = \frac{\partial H}{\partial \mu_\alpha}\mathcal{X}_\alpha - \left(\rho^i_\alpha \frac{\partial H}{\partial x^i} + \mu_\gamma C^\gamma_{\alpha\beta}\frac{\partial H}{\partial \mu_\beta}\right)\mathcal{P}^\alpha,$$

and therefore, Hamilton equations are

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha} \qquad \text{and} \qquad \frac{d\mu_\alpha}{dt} = -\rho^i_\alpha \frac{\partial H}{\partial x^i} - \mu_\gamma C^\gamma_{\alpha\beta}\frac{\partial H}{\partial \mu_\beta}. \tag{6}$$

The Poisson bracket $\{\ ,\ \}$ defined by the canonical symplectic section Ω on E^* is but the canonical Poisson bracket, which is known to exists on the dual of a Lie algebroid [5] and the Hamilton equations coincide with those defined by Weinstein in [56].

4. Lagrangian Mechanics

The Lie algebroid approach to Lagrangian Mechanics builds on the geometrical structure of the prolongation of a Lie algebroid [39] (where one can develop a geometric symplectic treatment of Lagrangian systems parallel to J. Klein's formalism [29]). It is also possible to derive the equations within a variational framework [44] which be extended for classical field theories [42]. We will discuss that point later because it shall be useful when commenting on the case of nonholonomic constrained motion.

On the E-tangent $\mathcal{T}^E E$ to E itself we do not have a canonical symplectic structure. Instead, we have the following two canonical objects: the vertical endomorphism $S\colon \mathcal{T}^E E \to \mathcal{T}^E E$ which is defined by

$$S(a, b, v) = (a, 0, b^{\mathsf{V}}_a),$$

where b_a^{v} denotes the vertical lift to T_aE of the element $b \in E$, and the Liouville section, which is the vertical section corresponding to the Liouville vector field,

$$\Delta(a) = (a, 0, a_a^{\mathsf{v}}).$$

Given a Lagrangian function $L \in C^\infty(E)$ we define the Cartan 1-section $\theta_L \in Sec((\mathcal{T}^E E)^*)$ and the Cartan 2-section $\omega_L \in Sec(\wedge^2(\mathcal{T}^E E)^*)$ and the Lagrangian energy $E_L \in C^\infty(E)$ as

$$\theta_L = S^*(dL), \qquad \omega_L = -d\theta_L \qquad \text{and} \qquad E_L = \mathcal{L}_\Delta L - L. \tag{7}$$

If the Cartan 2-section is regular, then it is a symplectic form on the Lie algebroid $\mathcal{T}^E E$, and we say that the Lagrangian L is regular. The Hamiltonian section Γ_L corresponding to the energy is the Euler-Lagrange section and the equations for the integral curves of the associated vector field are the Euler-Lagrange equations.

If (x^i, y^α) are local fibered coordinates on E, $(\rho^i_\alpha, C^\gamma_{\alpha\beta})$ are the corresponding local structure functions on E and $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ is the corresponding local basis of sections of $\mathcal{T}^E E$ then $S\mathcal{X}_\alpha = \mathcal{V}_\alpha$ and $S\mathcal{V}_\alpha = 0$, and the Liouville section is $\Delta = y^\alpha \mathcal{V}_\alpha$. The energy has the expression $E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L$, and the Cartan 2-section is

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\rho^i_\alpha + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta}\right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \tag{8}$$

from where we deduce that L is regular if and only if the matrix $W_{\alpha\beta} = \dfrac{\partial^2 L}{\partial y^\alpha \partial y^\beta}$ is regular. In such case, the local expression of Γ_L is

$$\Gamma_L = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha,$$

where the functions f^α satisfy the linear equations

$$\frac{\partial^2 L}{\partial y^\beta \partial y^\alpha} f^\beta + \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta y^\beta + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} = 0. \tag{9}$$

Thus, the Euler-Lagrange equations for L are

$$\dot{x}^i = \rho^i_\alpha y^\alpha, \quad \frac{d}{dt}\Big(\frac{\partial L}{\partial y^\alpha}\Big) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} = 0. \tag{10}$$

Within a varitional formalism, these equations arise as the extremals of the action functional $\mathcal{S} = \int_\gamma L(\gamma(t)dt$, defined on the set of admissible curves $\gamma : \mathbb{R} \to E$ on the Lie algebroid E, with an appropriate manifold structure (see [44] for the details). From a geometric point of view variations are encoded in the Euler-Lagrange operator $\delta L : \mathrm{Adm(E)} \to \mathrm{E}^*$ locally defined by

$$\delta L = \left[\frac{d}{dt}\Big(\frac{\partial L}{\partial y^\alpha}\Big) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i}\right] e^\alpha,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$, then the Euler-Lagrange differential equations read

$$\delta L = 0.$$

Finally, we mention that, as in the standard case, the relation between the Lagrangian and the Hamiltonian formalism is provided by the Legendre transformation $\mathcal{F}_L \colon E \to E^*$ defined by

$$\langle\, \mathcal{F}_L(a)\, , b\,\rangle = \frac{d}{dt} L(a + tb)\Big|_{t=0}, \tag{11}$$

for $a, b \in E$ with $\tau(a) = \tau(b)$. Then it is easy to see that

$$\mathcal{T}\mathcal{F}_L{}^\star(\Theta) = \theta_L \quad \text{and} \quad \mathcal{T}\mathcal{F}_L{}^\star(\Omega) = \omega_L$$

and therefore, in the regular case, the corresponding Hamiltonian sections are related by $\Gamma_H \circ \mathcal{F}_L = \mathcal{T}\mathcal{F}_L \circ \Gamma_L$.

Nonholonomic systems and Lagrange-d'Alembert equations [15, 16]**.** Consider now the case of a nonholonomic Lagrangian system defined by a Lagrangian $L \in C^\infty(E)$ and linear constraints given by a subbundle $\mathcal{D} \subset E$. D'Alembert principle affirms that the evolution of the system is given by the admissible curves a taking values in $\mathcal{D}$ and such that the work along virtual displacements vanishes. In other words, the evolution is given by curves a satisfying

$$a(t) \in \mathcal{D} \qquad \text{and} \qquad \delta L(a(t)) \in \mathcal{D}^\circ.$$

A nonholonomic system will be denoted by the triple $(E, L, \mathcal{D})$, or just by $(L, \mathcal{D})$ whenever E is understood.

Solution of Lagrange-d'Alembert equations. Let us now perform a precise global analysis of the existence and uniqueness of the solution of Lagrange-d'Alembert equations.

Definition 1. *A constrained Lagrangian system $(L, \mathcal{D})$ is said to be regular if the Lagrange-d'Alembert equations have a unique solution.*

Let G^L be the fundamental tensor associated with the Lagrangian function L, which locally reads

$$G^L = W_{\alpha\beta} e^\alpha \otimes e^\beta.$$

In order to characterize geometrically those nonholonomic systems which are regular, we define the tensor $G^{L,\mathcal{D}}$ as the restriction of G^L to $\mathcal{D}$, that is, $G^{L,\mathcal{D}}_a(b,c) = G^L_a(b,c)$ for every $a \in \mathcal{D}$ and every $b, c \in \mathcal{D}_{\tau(a)}$. The following theorem was proved in [16].

Theorem 1. *The following properties are equivalent:*

(1) *The constrained Lagrangian system* $(L, \mathcal{D})$ *is regular,*
(2) $\operatorname{Ker} G^{L,\mathcal{D}} = \{0\}$,
(3) $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap (\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp = \{0\}$.
(4) $\mathcal{T}^E E|_D = \mathcal{T}^{\mathcal{D}}\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp$.

Projection to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$. The regularity condition for the constrained system $(L, \mathcal{D})$ can be equivalently expressed by requiring that the subbundle $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$. It follows that, for every $a \in D$, we have a direct sum decomposition

$$\mathcal{T}^E E[a] = \mathcal{T}^{\mathcal{D}}_a\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}_a\mathcal{D})^\perp.$$

Let us denote by $\bar{P}$ and $\bar{Q}$ the complementary projectors defined by this decomposition, that is,

$$\bar{P}_a \colon \mathcal{T}^E E[a] \to \mathcal{T}^{\mathcal{D}}_a\mathcal{D} \qquad \text{and} \qquad \bar{Q}_a \colon \mathcal{T}^E E[a] \to (\mathcal{T}^{\mathcal{D}}_a\mathcal{D})^\perp, \quad \text{for all } a \in \mathcal{D}.$$

Then, we have the following result:

Theorem 2. *Let* $(L, \mathcal{D})$ *be a regular constrained Lagrangian system and let* Γ_L *be the solution of the free dynamics, i.e.,* $i_{\Gamma_L}\omega_L = dE_L$. *Then the solution of the constrained dynamics is the* SODE $\Gamma_{(L,\mathcal{D})}$ *obtained by projection* $\Gamma_{(L,\mathcal{D})} = \bar{P}(\Gamma_L|_{\mathcal{D}})$.

Lagrange-d'Alembert equations in local coordinates. Let us analyze the form of the Lagrange-d'Alembert equations in local coordinates. Let us choose a special coordinate system adapted to the structure of the problem as follows. We consider local coordinates (x^i) on an open set $\mathcal{U}$ of M and we take a basis $\{e_a\}$ of local sections of $\mathcal{D}$ and complete it to a basis $\{e_a, e_A\}$ of local sections of E (both defined on the open $\mathcal{U}$). In this way, we have coordinates (x^i, y^a, y^A) on E. In this set of coordinates, the constraints imposed by the submanifold $\mathcal{D} \subset E$ are simply $y^A = 0$. If $\{e^a, e^A\}$ is the dual basis of

$\{e_a, e_A\}$, then a basis for the annihilator $\mathcal{D}^\circ$ of $\mathcal{D}$ is $\{e^A\}$ and a basis for $\widetilde{\mathcal{D}^\circ}$ the pullback of $\mathcal{D}^\circ$ to $(\mathcal{T}^E E)^*$, is $\mathcal{X}^A$.

In this basis, the matrix elements of $G^{L,\mathcal{D}}$ are given by

$$\mathcal{C}_{ab}(x^i, y^c) = \frac{\partial^2 L}{\partial y^a \partial y^b}(x^i, y^c, 0) \tag{12}$$

and therefore, there exists a unique solution of the Lagrange-d'Alembert equations if and only if the matrix $[\mathcal{C}_{ab}]$ is regular.

The differential equations for the integral curves of the vector field $\rho^1(\Gamma)$ are the Lagrange-d'Alembert differential equations, which read

$$\begin{aligned} \dot{x}^i &= \rho^i_a y^a, \\ \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^a}\right) + \tfrac{\partial L}{\partial y^\gamma} C^\gamma_{ab} y^b - \rho^i_a \tfrac{\partial L}{\partial x^i} &= 0, \\ y^A &= 0. \end{aligned} \tag{13}$$

Finally, notice that the contraction with $\mathcal{X}_A$ just gives the components $\lambda_A = \langle\, i_\Gamma \omega_L - dE_L\,, \mathcal{X}_A \,\rangle|_{y^A=0}$ of the constraint forces $\lambda = \lambda_A e^A$.

Remark 1 (Equations in terms of the constrained Lagrangian)**.** In some occasions, it is useful to write the equations in the form

$$\begin{aligned} \dot{x}^i &= \rho^i_a y^a, \\ \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^a}\right) + \tfrac{\partial L}{\partial y^c} C^c_{ab} y^b - \rho^i_a \tfrac{\partial L}{\partial x^i} &= -\tfrac{\partial L}{\partial y^A} C^A_{ab} y^b, \\ y^A &= 0, \end{aligned} \tag{14}$$

where, on the left-hand side of the second equation, all the derivatives can be calculated from the value of the Lagrangian on the constraint submanifold $\mathcal{D}$. In other words, we can substitute L by the constrained Lagrangian L_c defined by $L_c(x^i, y^a) = L(x^i, y^a, 0)$.

Remark 2 (Lagrange-d'Alembert equations in quasi-coordinates)**.** A particular case of this construction is given by constrained systems defined in the standard Lie algebroid $\tau_M \colon TM \to M$. In this case, the equations (13) are the Lagrange-d'Alembert equations written in quasi-coordinates, where $C^\alpha_{\beta\gamma}$ are the so-called Hamel's transpositional symbols, which obviously are nothing but the structure coefficients (in the Cartan's sense) of the moving frame $\{e_\alpha\}$, see e.g., [20, 24].

5. Optimal control theory

As it is well known, optimal control theory is a generalization of classical mechanics. It is therefore natural to see whether our results can be extended to this more general context. The central result in the theory of optimal control systems is Pontryagin maximum principle. The reduction of optimal control problems can be performed within the framework of Lie algebroids, see [41]. This was done as in the case of classical mechanics, by introducing a general principle for any Lie algebroid and later studying the behavior under morphisms of Lie algebroids. See also [23] for a recent direct proof of Pontryagin principle in the context of general algebroids.

Pontryagin maximum principle [41]. By a control system on a Lie algebroid $\tau\colon E \to M$ with control space $\pi\colon B \to M$ we mean a section of E along π. A trajectory of the system σ is an integral curve of the vector field $\rho(\sigma)$ along π.

$$
\begin{tikzcd}
 & E \arrow[r, "\rho"] \arrow[d, "\tau"] & TM \arrow[dl, "\tau_M"] \\
B \arrow[ur, "\sigma"] \arrow[r, "\pi"'] & M &
\end{tikzcd}
$$

Given an index function $L \in C^\infty(B)$ we want to minimize the integral of L over some set of trajectories of the system which satisfies some boundary conditions. Then we define the Hamiltonian function $H \in C^\infty(E^* \times_M B)$ by $H(\mu, u) = \langle \mu, \sigma(u) \rangle - L(u)$ and the associated Hamiltonian control system σ_H (a section of $\mathcal{T}^E E^*$ along $\mathrm{pr}_1\colon E^* \times_M B \to E^*$) defined on a subset of the manifold $E^* \times_M B$, by means of the symplectic equation

$$
i_{\sigma_H} \Omega = dH. \tag{15}
$$

The integral curves of the vector field $\rho(\sigma_H)$ are said to be the critical trajectories.

In the above expression, the meaning of i_{σ_H} is as follows (see [6] for similar constructions in the context of mechanics). Let $\Phi\colon E \to E'$ be a morphism over a map $\varphi\colon M \to M'$ and let η be a section of E' along φ. If ω a section of $\bigwedge^p E'^*$ then $i_\eta \omega$ is the section of $\bigwedge^{p-1} E^*$ given by

$$
(i_\eta \omega)_m(a_1, \ldots, a_{p-1}) = \omega_{\varphi(n)}(\eta(m), \Phi(a_1), \ldots, \Phi(a_{p-1}))
$$

for every $m \in M$ and $a_1, \ldots, a_{p-1} \in E_m$. In our case, the map Φ is the prolongation $\mathcal{T}\mathrm{pr}_1\colon \mathcal{T}^E(E^* \times_M B) \to \mathcal{T}^E E^*$ of the map $\mathrm{pr}_1\colon E^* \times_M B \to E^*$

(this last map fibered over the identity in M), and σ_H is a section along pr_1. Therefore, $i_{\sigma_H}\Omega - dH$ is a section of the dual bundle to $\mathcal{T}^E(E^* \times_M B)$.

It is easy to see that the symplectic equation (15) has a unique solution defined on the following subset

$$S_H = \{\, (\mu, u) \in E^* \times_M B \mid \langle\, dH(\mu, u)\,, V\,\rangle = 0 \text{ for all } V \in \operatorname{Ker} \mathcal{T}\mathrm{pr}_1 \,\}\,.$$

Therefore, it is necessary to perform a stabilization constraint algorithm to find the integral curves of σ_H which are tangent to the constraint submanifold.

In local coordinates, the solution to the above symplectic equation is

$$\sigma_H = \frac{\partial H}{\partial \mu_\alpha}\mathcal{X}_\alpha - \left[\rho^i_\alpha \frac{\partial H}{\partial x^i} + \mu_\gamma C^\gamma_{\alpha\beta}\frac{\partial H}{\partial \mu_\beta}\right]\mathcal{P}^\alpha,$$

defined on the subset where

$$\frac{\partial H}{\partial u^A} = 0,$$

and therefore the critical trajectories are the solution of the differential-algebraic equations

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha \tfrac{\partial H}{\partial \mu_\alpha} \\ \dot{\mu}_\alpha &= -\left[\rho^i_\alpha \tfrac{\partial H}{\partial x^i} + \mu_\gamma C^\gamma_{\alpha\beta}\tfrac{\partial H}{\partial \mu_\beta}\right] \\ 0 &= \tfrac{\partial H}{\partial u^A}. \end{aligned} \tag{16}$$

Notice that $\frac{\partial H}{\partial \mu_\alpha} = \sigma^\alpha$.

One can easily see that whenever it is possible to write $\mu_\alpha = p_i\rho^i_\alpha$ then the above differential equations reduce to the critical equations for the control system $Y = \rho(\sigma)$ on TM and the index L. Nevertheless it is not warranted that μ is of that form. For instance in the case of a Lie algebra, the anchor vanishes, $\rho = 0$, so that the factorization $\mu_\alpha = p_i\rho^i_\alpha$ will not be possible in general.

6. Kinematic Optimal Control

Let $\tau : E \to M$ be a Lie algebroid and $\mathcal{D}$ a constraint distribution. Given a cost function $\boldsymbol{\kappa}\colon E \to \mathbb{R}$, we consider the following kinematic optimal control problem: we can control directly all the (constrained) velocities, and we want to minimize some index functional

$$I(a) = \int_\alpha^\beta \boldsymbol{\kappa}(a(t))\, dt,$$

for $a : [\alpha, \beta] \subset \mathbb{R} \to E$ over the set of admissible curves taking values in $\mathcal{D}$. We shall be using coordinates (x^i, y^a) to denote the elements of this bundle, where y^a will represent the coordinates with respect to some basis of section for $\mathcal{D}$, as in the last section.

Remark 3. *Whenever the cost function $\boldsymbol{\kappa}$ is a quadratic function defined on $\mathcal{D}$, the problem that we are considering is just the problem of sub-Riemannian geometry. In the case of a degree 1 homogeneous cost function this is sub-Finslerian geometry, and in the more general case this problem can be called sub-Lagrangian problem.*

Since we can control directly the velocities or pseudovelocities, the control bundle is $B = \mathcal{D}$ and the system map $\sigma : \mathcal{D} \to E$ is just the canonical inclusion $\sigma(a) = a$.

$$\begin{tikzcd}
 & E \arrow[r, "\rho"] \arrow[d, "\tau"'] & TM \arrow[dl, "\tau_M"] \\
\mathcal{D} \arrow[ur, "\sigma"] \arrow[r] & M &
\end{tikzcd}$$

Pontryagin Hamiltonian is a function $H \in C^\infty(E^* \times_M B)$ defined as $H(\mu, b) = \langle \mu, \sigma(b) \rangle - \boldsymbol{\kappa}(b)$, which in coordinates reads

$$H(x^i, \mu_a, \mu_A, u^a) = \mu_a u^a - \boldsymbol{\kappa}(x^i, u^a). \tag{17}$$

The Maximum principle corresponds to the choice of the control functions such that

$$\mu_a = \frac{\partial \boldsymbol{\kappa}}{\partial u^a} \qquad \left(\text{from } \frac{\partial \mathrm{H}}{\partial \mathrm{u}^{\mathrm{a}}} = 0 \right). \tag{18}$$

Under appropriate regularity conditions the set S_H of solutions of this equation is a submanifold of $E^* \times_M B$, which we call the critical submanifold. Frequently, this set is but the image of a section of $E^* \times_M B \to E^*$, given locally by

$$u^a = u^a(x, \mu). \tag{19}$$

Thus the set S_H is diffeomorphic to E^* and the optimal Hamiltonian, with local expression $H(x^i, \mu_\alpha, u^a(x, \mu))$, defines via the canonical symplectic form a Hamiltonian system on E^*.

The restriction of the Pontryagin-Hamilton equations to this submanifold provide us with the control system

$$\dot{x}^i = \rho^i_a u^a \tag{20}$$

and the dynamics of the momenta

$$\dot{\mu}_a = -\left[\rho^i_a \frac{\partial H}{\partial x^i} + \mu_c C^c_{ab} u^b + \mu_B C^B_{ab} u^b\right] \tag{21}$$

$$\dot{\mu}_A = -\left[\rho^i_A \frac{\partial H}{\partial x^i} + \mu_c C^c_{Ab} u^b + \mu_B C^B_{Ab} u^b\right]. \tag{22}$$

These are the equations to solve and use to determine, by using the mapping (19), the control functions defining the solution which optimizes the value of the cost function. By substitution of $\mu_a = \frac{\partial \boldsymbol{\kappa}}{\partial u^a}$ into these equations, and taking into account that $\frac{\partial H}{\partial x^i} = -\frac{\partial \boldsymbol{\kappa}}{\partial x^i}$, we get

$$\begin{aligned} \dot{x}^i &= \rho^i_a u^a \\ \frac{d}{dt}\left(\frac{\partial \boldsymbol{\kappa}}{\partial u^a}\right) - \rho^i_a \frac{\partial \boldsymbol{\kappa}}{\partial x^i} + \frac{\partial \boldsymbol{\kappa}}{\partial u^c} C^c_{ab} u^b + \mu_B C^B_{ab} u^b &= 0 \\ \dot{\mu}_A + \mu_B C^B_{Ab} u^b - \rho^i_A \frac{\partial \boldsymbol{\kappa}}{\partial x^i} + \frac{\partial \boldsymbol{\kappa}}{\partial u^c} C^c_{Ab} u^b &= 0. \end{aligned} \tag{23}$$

These equations are also obtained in [47, 48] for the case $E = TM$.

Remark 4. *For simplicity we are considering only normal extremals. For abnormal extremals we just have to consider the Hamiltonian function to be* $H = \mu_a u^a$ *and solve the same equations, i.e.*

$$\begin{aligned} \mu_a &= 0 \\ \dot{x}^i &= \rho^i_a u^a \\ \mu_B C^B_{ab} u^b &= 0 \\ \dot{\mu}_A + \mu_B C^B_{Ab} u^b &= 0. \end{aligned} \tag{24}$$

with $(\mu_A(t)) \neq (0)$ *for all* t.

Remark 5. *In the particular case when* $\mathcal{D} = E$ *and* $\sigma = \mathrm{id}_E$ *we recover the Euler-Lagrange equations on the Lie algebroid* E *for the Lagrangian* $L = \boldsymbol{\kappa}$. *Also when* $\mathcal{D} \subset E$ *we get the so-called vakonomic equations for the Lagrangian* $L = \boldsymbol{\kappa}$ *(see* [26, 38]*)*

7. Dynamic Optimal Control

In the dynamic problem, we can control directly the motion on a nonholonomic problem, with the exception of the constraint forces, of course. For instance, we can consider the equations of motion to be $\delta L(z)|_{\mathcal{D}} = u$, with

$u \in D^*$ the control variables representing the external (generalized) forces acting on the system. Another possibility would be to consider systems on which the accelerations are the control variables.

In both kind of problems the state space is the manifold $\mathcal{D}$ and the control bundle $\pi\colon B \to \mathcal{D}$ is

$$B = \left\{ (z, \nu) \in \mathcal{T}^{\mathcal{D}}\mathcal{D} \times \mathcal{D}^* \;\middle|\; z \in \mathrm{Adm}(E) \quad \text{and} \qquad \delta L(z)|_{\mathcal{D}} = \nu \right\}. \tag{25}$$

An element (z, ν) of B is of the form $z = (a, a, v)$ with $a \in \mathcal{D}$ and $v \in T_a\mathcal{D}$ and where ν is determined by the equations $\langle \nu, b\rangle = \langle \delta L(z), b\rangle$ for every $b \in \mathcal{D}$.

When we consider the forces as control variables, since we are assuming that the constrained Lagrangian system is regular, we can identify B with $\mathrm{pr}_1\colon \mathcal{D} \oplus \mathcal{D}^* \to \mathcal{D}$, via $(a, a, v; \nu) \equiv (a, \nu)$, because the vector v is determined by the point a and the equation $\delta L(z)|_{\mathcal{D}} = u$.

When we consider the accelerations as controls we can identify B with $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap \mathrm{Adm}(E) \to \mathcal{D}$, via $(a, a, v; \nu) \equiv (a, a, v)$ because ν is determined by $\nu = \delta L(z)|_{\mathcal{D}}$.

From a formal point of view both problems are equivalent, since the relation between them is one-to-one and thus it is possible to use the optimal solution written in terms of accelerations to determine the optimal forces and viceversa. In other words, they are related by a feedback transformation.

However, from the practical point of view the second problem produce simpler expressions. Therefore, we will identify B with $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap \mathrm{Adm}(E)$ and we can take coordinates (x^i, y^a, v^a) where v^a are the acceleration coordinates, i.e. our control variables.

On this set we also need to specify a control system where the optimization will be built. Such a system is specified by giving a section $\sigma : B \to \mathcal{T}^E\mathcal{D}$, i.e. the resulting system must always define an admissible velocity and acceleration. The Lie algebroid relevant for this case is the E-tangent to $\mathcal{D}$. An element of $\mathcal{T}^E\mathcal{D}$ is of the form $z = (a, b, w)$ with $a \in \mathcal{D}$, $b \in E$, with $\tau(a) = \tau(b)$ and $w \in T_a\mathcal{D}$ with $\rho(b) = T\tau(w)$. Taking a local basis $\{e_a\}$ for $\mathcal{D}$, and completing a local basis $\{e_a, e_A\}$ for E, we can write $a = y^a e_a$, $b = z^a e_a + z^A e_A$, and $w = (\rho^i_a z^a + \rho^i_A z^A)\frac{\partial}{\partial x^i} + w^a \frac{\partial}{\partial y^a}$. By taking coordinates (x^i) in the basis, we have coordinates $(x^i, y^a, z^a, z^A, w^a)$ on $\mathcal{T}^E\mathcal{D}$. A local basis of sections of $\mathcal{T}^E\mathcal{D} \to \mathcal{D}$ is $\{\mathcal{X}_a, \mathcal{X}_A, \mathcal{V}_a\}$ and the element z can be written $z = z^a\mathcal{X}_a(x, y) + z^A\mathcal{X}_A(a, y) + w^a\mathcal{V}_a(a, y)$, and

$\rho^1(z) = w = (\rho^i_a z^a + \rho^i_A z^A)\frac{\partial}{\partial x^i} + w^a \frac{\partial}{\partial y^a}$. The corresponding coordinates on the dual bundle $(\mathcal{T}^E\mathcal{D})^*$ will be denoted $(x^i, y^a, \mu_a, \mu_A, \pi_a)$.

If we choose to control the accelerations of the system $\{u^a\}$, the map $\sigma : B \to \mathcal{T}^E\mathcal{D}$ is given by the natural inclusion $\sigma(z) = z$,

$$\begin{tikzcd} & \mathcal{T}^E\mathcal{D} \arrow[r, "\rho^1"] \arrow[d, "\tau"] & T\mathcal{D} \arrow[dl, "\tau_{\mathcal{D}}"] \\ B \arrow[ur, "\sigma"] \arrow[r] & \mathcal{D} & \end{tikzcd}$$

which in coordinates corresponds to

$$\sigma(x^i, y^a, u^a) = (x^i, y^a, y^a, 0, u^a)$$

Given a cost function $\boldsymbol{\kappa} : B \to \mathbb{R}$ we take the Pontryagin Hamiltonian $H \in C^\infty((\mathcal{T}^E\mathcal{D})^* \times B)$, defined as $H(\mu, z) = \langle \mu, \sigma(z) \rangle - \boldsymbol{\kappa}(z)$ which in coordinates is

$$H(x^i, y^a, \mu_a, \mu_A, \pi_a, u^a) = \mu_a y^a + \pi_a u^a - \boldsymbol{\kappa}(x^i, y^a, u^a),$$

where the control functions are the accelerations u^a.

From $\frac{\partial H}{\partial u^a} = 0$, we get

$$\pi_a = \frac{\partial \boldsymbol{\kappa}}{\partial u^a}. \tag{26}$$

These equations determine the optimal submanifold S_H in $(\mathcal{T}^E\mathcal{D})^* \times B$, which in this case is a section of the projection $(\mathcal{T}^E\mathcal{D})^* \times B \to B$, given locally by the equations $u^a = u^a(x^i, y^a, \pi_a)$. On S_H, the equations of motion are the following. From $\dot{x} = \rho \frac{\partial H}{\partial \mu}$, since in this case the base variables are (x, y), we get the original control system

$$\dot{x}^i = \rho^i_a y^a + \rho^i_A 0 = \rho^i_a y^a \qquad \dot{y}^a = u^a. \tag{27}$$

Also, the equation of motion for π_a

$$\dot{\pi}_a = -\frac{\partial H}{\partial y^a} = \frac{\partial \boldsymbol{\kappa}}{\partial y^a} - \mu_a,$$

because all the structure functions involved vanish (i.e. $\mathcal{V}_a$ commute with all the others). Therefore we get

$$\mu_a = \frac{\partial \boldsymbol{\kappa}}{\partial y^a} - \frac{d}{dt}\left(\frac{\partial \boldsymbol{\kappa}}{\partial u^a}\right) = \frac{\partial \boldsymbol{\kappa}}{\partial y^a} - \dot{\pi}_a$$

which is the combination that appear in [47], equation (11), for the case $E = TM$, under the notation κ_J and without a justification.

The equation of motion for μ_a is

$$\begin{aligned} -\dot{\mu}_a &= \rho^i_a \frac{\partial H}{\partial x^i} + \mu_c C^c_{ab} y^b + \mu_C C^C_{ab} y^b \\ &= -\rho^i_a \frac{\partial \boldsymbol{\kappa}}{\partial x^i} + \mu_c C^c_{ab} y^b + \mu_C C^C_{ab} y^b, \end{aligned} \tag{28}$$

and the equation of motion for μ_A is

$$\begin{aligned} -\dot{\mu}_A &= \rho^i_A \frac{\partial H}{\partial x^i} + \mu_c C^c_{Ab} y^b + \mu_C C^C_{Ab} y^b \\ &= -\rho^i_A \frac{\partial \boldsymbol{\kappa}}{\partial x^i} + \mu_c C^c_{Ab} y^b + \mu_C C^C_{Ab} y^b. \end{aligned} \tag{29}$$

These two last equations corresponds to equations (18) and (19) obtained in [47, 47] for the case $E = TM$.

Remark 6. *As in the previous case, we have considered only normal extremals. For abnormal extremals we just have to take the cost function $\boldsymbol{\kappa} = 0$ and solve the same equations for $\boldsymbol{\kappa} = 0$, that is*

$$\mu_a = 0 \tag{30}$$

$$\pi_a = 0 \tag{31}$$

$$\dot{x}^i = \rho^i_a y^a \tag{32}$$

$$\mu_B C^B_{ab} y^b = 0 \tag{33}$$

$$\dot{\mu}_A + \mu_B C^B_{Ab} y^b = 0. \tag{34}$$

with $(\mu_A(t)) \neq (0)$ for all t. Interestingly, we get exactly the same equations (plus $\pi_a = 0$) as in the kinematic case.

8. Examples

First we discuss two of the examples discussed in [48] from the point of view of the theory of Lie algebroids. The result is naturally analogous to the results obtained previously, but within the new framework the treatement of dynamical control systems becomes much more natural.

8.1. Mechanical systems defined on tangent bundles.

8.1.1. *Kinematic optimal control of the Heisenberg system.* Consider the case of a system modelled on $M = \mathbb{R}^3$ using the set of coordinates $x = (x^1, x^2, x^3)$ and the nonholonomic constraint

$$\dot{x}^3 = x^2\dot{x}^1 - x^1\dot{x}^2.$$

Let us denote as $\mathcal{D}$ the subbundle of admissible velocities and consider a basis for it. The quasi-velocities of the system are thus the coordinates defined with respect to this new basis, and are written:

$$y^1 = x^2\dot{x}^1 - x^1\dot{x}^2 - \dot{x}^3, \quad y^2 = \dot{x}^1, \quad y^3 = \dot{x}^2.$$

Analogously we can write the velocities in terms of the quasi-velocities as:

$$\dot{x}^1 = y^2, \quad \dot{x}^2 = y^3, \quad \dot{x}^3 = -y^1 + x^2y^2 - x^1y^3.$$

The local basis of sections of TM determined by the quasi-velocities reads

$$e_1 = -\frac{\partial}{\partial x^3} \equiv (0, 0, -1), \tag{35}$$

$$e_2 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3} \equiv (1, 0, x^2), \tag{36}$$

$$e_3 = \frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial x^3} \equiv (0, 1, -x^1). \tag{37}$$

It is simple to compute the structure constants and the anchor mapping with respect to this basis $[e_2, e_3] = 2e_1 \Leftrightarrow C^1_{23} = -C^1_{32} = 2$ all other elements being zero. The corresponding anchor mapping reads $\rho(e_1) = -\frac{\partial}{\partial x^3}$, $\rho(e_2) = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}$ and $\rho(e_3) = \frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial x^3}$.

Consider now the problem of controlling the admissible velocities

$$y^2 = u^2 \qquad \text{and} \qquad y^3 = u^3,$$

i.e. the system

$$\begin{tikzcd} & T\mathbb{R}^3 \arrow[r, "id"] \arrow[d, "\tau"'] & T\mathbb{R}^3 \arrow[dl, "\tau"] \\ \mathcal{D} \arrow[ur, "inc"] \arrow[r] & \mathbb{R}^3 & \end{tikzcd},$$

in order to optimize the cost function $\kappa(x^i, u^a) = \frac{1}{2}[(u^2)^2 + (u^3)^2]$. The corresponding Pontryagin Hamiltonian $H \in C^\infty(T^*M \times_M \mathcal{D})$ is written as $H(x^i, \mu_a, u^a) = \mu_2u^2 + \mu_3u^3 - \frac{1}{2}[(u^2)^2 + (u^3)^2]$.

The Maximum principle implies thus - $\frac{\partial H}{\partial u^I} = 0$:

$$\frac{\partial H}{\partial u^2} = 0 \Leftrightarrow \mu_2 = u^2,$$
$$\frac{\partial H}{\partial u^3} = 0 \Leftrightarrow \mu_3 = u^3.$$

This defines the optimal submanifold W. On it, the dynamics reads

$$\begin{aligned}
\dot{x}^1 &= \mu_2,\\
\dot{x}^2 &= \mu_3,\\
\dot{x}^3 &= x^2\mu_2 - x^1\mu_3,\\
\dot{\mu}_1 &= 0 \text{ hence, } \mu_1 = c,\\
\dot{\mu}_2 &= -2c\mu_3,\\
\dot{\mu}_3 &= 2c\mu_2,
\end{aligned}$$

with c constant, or equivalently,

$$\begin{aligned}
\dot{x}^3 &= x^2\dot{x}^1 - x^1\dot{x}^2,\\
\ddot{x}^1 &= -2c\dot{x}^2,\\
\ddot{x}^2 &= 2c\dot{x}^1.
\end{aligned}$$

Remark 7. *We can also consider a case of abnormal extremals for this case. If we set the $\boldsymbol{\kappa} = 0$ (or analogously, $H = \mu_a u^a$), we find*

$$\dot{x}^1 = u^2, \qquad \dot{x}^2 = u^3, \qquad \dot{\mu}_1 = 0, \qquad \mu_1 u^2 = 0 \qquad \text{and} \qquad \mu_1 u^3 = 0,$$

which has solution of the form $x^1 = x_0, x^2 = y_0, x^3 = z_0, y^1 = 0, y^2 = 0,$ and $\mu_1 = \mu_1^0 \neq 0$. These are equilibrium points.

8.1.2. *Dynamic optimal control of the vertical rolling disc.* For such a system, the state space manifold corresponds to $M = \mathbb{R}^2 \times S^1 \times S^1$, and we will use the coordinates $x = (x^1, x^2, x^3, x^4)$, where $x^1 = x,\ x^2 = y,\ x^3 = \theta,$ and $x^4 = \phi$.

The rolling without slipping condition of the motion on the plane leads to a pair of nonholonomic constraints

$$\dot{x}^1 - \cos(x^4)\dot{x}^3 = 0 \ \text{ and } \ \dot{x}^2 - \sin(x^4)\dot{x}^3 = 0.$$

We can define then a set of coordinates adapted to these constraints and write a set of coordinates $\{y\}$ for the new velocities. Thus the quasi-velocities shall correspond to:

$$y^1 = \dot{x}^1 - \cos(x^4)\dot{x}^3, \quad y^2 = \dot{x}^2 - \sin(x^4)\dot{x}^3, \quad y^3 = \dot{x}^3, \quad y^4 = \dot{x}^4.$$

Analogously the inverse transformation allows us to write:

$$\dot{x}^1 = y^1 + \cos(x^4)y^3, \quad \dot{x}^2 = y^2 + \sin(x^4)y^3, \quad \dot{x}^3 = y^3, \quad \dot{x}^4 = y^4.$$

The local basis of sections of TM determined by the quasi-velocities turns out to be:

$$e_1 = \frac{\partial}{\partial x^1}, \quad e_2 = \frac{\partial}{\partial x^2}, \quad e_3 = \cos(x^4)\frac{\partial}{\partial x^1} + \sin(x^4)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \quad e_4 = \frac{\partial}{\partial x^4}.$$

The Lie algebroid structure is the usual one for the tangent bundle. But in the basis above, the anchor mapping is written as:

$$\rho(e_1) = \frac{\partial}{\partial x^1}, \quad \rho(e_2) = \frac{\partial}{\partial x^2}, \quad \rho(e_3) = \cos(x^4)\frac{\partial}{\partial x^1}+\sin(x^4)\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^3}, \quad \rho(e_4) = \frac{\partial}{\partial x^4}.$$

The Lie algebra structure is obtained also,

$$[e_3, e_4] = \sin(x^4)e_1 - \cos(x^4)e_2,$$

and we can read then the Hamel symbols $\gamma_{\alpha\beta}^{\epsilon}$ in [10, 48].

In what regards the control part, we are considering a situation where we control the external forces in the directions of the admissible velocities. Thus, as the velocities on the constrained system are of the form

$$y^1 = y^2 = 0, \;\; y^3 = \dot{x}^3 \qquad \text{and} \qquad y^4 = \dot{x}^4,$$

the natural coordinates are $\mathrm{a} = (x^1, x^2, x^3, x^4, y^3, y^4) = (x, y^a)$. The control bundle B becomes thus $\mathcal{D} \oplus \mathcal{D}^*$ and we take as coordinates $(x^i, y^3, y^4, u_3, u_4)$, where $u_3 = \frac{3}{2}\dot{y}^3$ and $u_4 = \frac{1}{4}\dot{y}^4$ for $u_a = (\delta L)_a$.

The cost function corresponds to $\kappa(\mathrm{a}, u) = \frac{1}{2}(u_3^2 + u_4^2)$ and the control system is defined as:

$$\dot{x}^1 = \cos(x^4)\dot{x}^3, \quad \dot{x}^2 = \sin(x^4)\dot{x}^3, \quad u_3 = \frac{3}{2}\ddot{x}^3, \quad u_4 = \frac{1}{4}\ddot{x}^4.$$

The Pontryagin Hamiltonian $H \in C^\infty((\mathcal{T}^E\mathcal{D})^* \times_{\mathcal{D}} B)$ corresponds now to

$$H(\mathrm{a}, p, u) = \langle p, \sigma_{\mathrm{a}}(u)\rangle - \kappa(\mathrm{a}, u) = \mu_I y^I + \pi_I \frac{u_I}{c_I} - \frac{1}{2}(u_3^2 + u_4^2),$$

with $c_3 = 3/2$ and $c_4 = 1/4$.

The Maximum principle is encoded as

$$\frac{\partial H}{\partial u_3} = 0 \;\Leftrightarrow u_3 = \tfrac{2}{3}\pi_3 \qquad \text{and} \qquad \frac{\partial H}{\partial u_4} = 0 \;\Leftrightarrow u_4 = 4\pi_4,$$

and the Pontryagin equations (optimal dynamical control equations) correspond to:

$$
\begin{array}{ll}
\dot{x}^1 = \cos(x^4)y^3 & \dot{x}^2 = \sin(x^4)y^3 \\
\dot{x}^3 = y^3 & \dot{x}^4 = y^4 \\
\dot{y}^3 = \frac{2}{3}u_3 & \dot{y}^4 = 4u_4 \\
\dot{\pi}_3 = -\mu_3 & \dot{\pi}_4 = -\mu_4 \\
\dot{\mu}_1 = 0 & \dot{\mu}_2 = 0 \\
\dot{\mu}_3 = [-\mu_1 \mathrm{sin}(x^4) + \mu_2 \mathrm{cos}(x^4)]y^4 & \dot{\mu}_4 = [\mu_1 \mathrm{sin}(x^4) - \mu_2 \mathrm{cos}(x^4)]y^3
\end{array}
$$

Since $y^3 = \dot{x}^3$, $y^4 = \dot{x}^4$ and $\dot{y}^I = u_I$, then $\mu^3 = -\dfrac{9}{4}\dddot{x}^3$ and $\mu^4 = -\dfrac{1}{16}\dddot{x}^4$. Thus, we can reduce the set of equations to:

$$
\begin{array}{ll}
\dot{x}^1 = \cos(x^4)\dot{x}^3, & \dot{x}^2 = \sin(x^4)\dot{x}^4, \\
\ddddot{x}^3 = \dfrac{4}{9}[\mu_1 \mathrm{sin}(x^4) - \mu_2 \mathrm{cos}(x^4)]\dot{x}^4, & \ddddot{x}^4 = 16[-\mu_1 \mathrm{sin}(x^4) + \mu_2 \mathrm{cos}(x^4)]\dot{x}^3,
\end{array}
$$

where μ_1, μ_2 are constants.

8.2. Optimal control problems of rotational motion of the free rigid body. Consider the problem of rotational motion of the free rigid body. As configuration manifold we take the Lie group $SO(3)$ and choose the type-I Euler angles (x^1, x^2, x^3) as local coordinate system. We shall consider the canonical Lie algebroid structure of the tangent bundle $TSO(3)$, whose anchor map is $\rho = id_{TSO(3)}$. Let $\{e_1, e_2, e_3\}$ be the set of sections for the bundle

$$
\begin{aligned}
e_1 &= \sec x^2 \mathrm{sin} x^3 \frac{\partial}{\partial x^1} + \mathrm{cos} x^3 \frac{\partial}{\partial x^2} + \mathrm{tan} x^2 \mathrm{sin} x^3 \frac{\partial}{\partial x^3} \\
e_2 &= \sec x^2 \mathrm{cos} x^3 \frac{\partial}{\partial x^1} - \mathrm{sin} x^3 \frac{\partial}{\partial x^2} + \mathrm{tan} x^2 \mathrm{cos} x^3 \frac{\partial}{\partial x^3} \\
e_3 &= \frac{\partial}{\partial x^3}
\end{aligned}
$$

whose Lie algebra structure is given by $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$. The anchor and the Lie bracket are locally determined by the functions

$$
\begin{array}{lll}
\rho_1^1 = \sec x^2 \mathrm{sin} x^3 & \rho_1^2 = \mathrm{cos} x^3 & \rho_1^3 = \mathrm{tan} x^2 \mathrm{sin} x^3 \\
\rho_2^1 = \sec x^2 \mathrm{cos} x^3 & \rho_2^2 = -\mathrm{sin} x^3 & \rho_2^3 = \mathrm{tan} x^2 \mathrm{cos} x^3 \\
\rho_3^1 = 0 & \rho_3^2 = 0 & \rho_3^3 = 1
\end{array}
$$

and

$$
C_{12}^3 = -C_{21}^3 = C_{23}^1 = -C_{32}^1 = C_{31}^2 = -C_{13}^2 = 1.
$$

We shall consider now two different situations, the rigid body without constraints and the same system subject to the constraint $\dot{x}^1 \mathrm{cos} x^2 \mathrm{sin} x^3 + \dot{x}^2 \mathrm{cos} x^3 =$

0. This implies that the constraint distribution D is the 2-dimensional subbundle of $TSO(3)$ generated by e_2 and e_3.

8.2.1. *Unconstrained kinematic problem.* Assume that the controls are all the components of the angular velocity, $y^1 = u^1, y^2 = u^2$ and $y^3 = u^3$. The control system is

$$\begin{array}{ll} \dot{x}^1 = \sec x^2 \sin x^3 u^1 + \sec x^2 \cos x^3 u^2 & \dot{x}^2 = \cos x^3 u^1 - \sin x^3 u^2 \\ \dot{x}^3 = \tan x^2 \sin x^3 u^1 + \tan x^2 \cos x^3 u^2 + u^3 & \end{array} \tag{38}$$

The cost function is the rotational kinetic energy given by

$$k(x^i, u^a) = \frac{1}{2}\left[I_1(u^1)^2 + I_2(u^2)^2 + I_3(u^3)^2\right],$$

where I_1, I_2 and I_3 represent the principal momenta of inertia of the body.

The optimality conditions (18) correspond now to $\mu_1 = I_1 u^1$, $\mu_2 = I_2 u^2$, $\mu_3 = I_3 u^3$. Thus, on that submanifold, the control system (38) and

$$\dot{\mu}_1 + \frac{I_3 - I_2}{I_2 I_3}\mu_2\mu_3 = 0; \quad \dot{\mu}_2 + \frac{I_1 - I_3}{I_1 I_3}\mu_1\mu_3 = 0; \quad \dot{\mu}_3 + \frac{I_2 - I_1}{I_1 I_2}\mu_1\mu_2 = 0 \tag{39}$$

are the Pontryagin-Hamilton equations for this case. Note that the equations (39) are the classical Euler equations of the rigid body.

8.2.2. *Unconstrained dynamic problem.* We study now the example considered in [48], Section 4.4. The controlled magnitudes are the components of the angular acceleration $v^1 = u^1$, $v^2 = u^2$ and $v^3 = u^3$. The control system is

$$\begin{array}{ll} \dot{x}^1 = \sec x^2 \sin x^3 y^1 + \sec x^2 \cos x^3 y^2 & \dot{x}^2 = \cos x^3 y^1 - \sin x^3 y^2 \\ \dot{x}^3 = \tan x^2 \sin x^3 y^1 + \tan x^2 \cos x^3 y^2 + y^3 & \dot{y}^1 = u^1,\ \dot{y}^2 = u^2,\ \dot{y}^3 = u^3 \end{array} \tag{40}$$

In this case the relevant Lie algebroid is the $TSO(3)$ extension of $TSO(3)$, which is equivalent to $TTSO(3)$. The control bundle corresponds to $T^2SO(3)$. The diagram describing the system turns out to be:

$$\begin{tikzcd} & TTSO(3) \arrow[r, "id"] \arrow[d, "\tau"'] & TTSO(3) \arrow[dl, "\tau"] \\ T^2SO(3) \arrow[ur, "inc"] \arrow[r] & TSO(3) & \end{tikzcd}$$

The cost function we shall consider now is the total torque $k(x^i, y^a, u^a) = \frac{1}{2}\left[(M_1)^2 + (M_2)^2 + (M_3)^2\right]$, where M_1, M_2 and M_3 are

$$M_1 = I_1u^1+(I_3-I_2)y^2y^3,\ M_2 = I_2u^2+(I_1-I_3)y^1y^3\ M_3 = I_3u^3+(I_2-I_1)y^1y^2. \tag{41}$$

So,

$$\begin{aligned}k(x^i, y^a, u^a) &= \tfrac{1}{2}\left[(I_1)^2(u^1)^2 + (I_2)^2(u^2)^2 + (I_3)^2(u^3)^2+\right.\\ &+2I_1(I_3-I_2)u^1y^2y^3 + 2I_2(I_1-I_3)u^2y^1y^3 + 2I_3(I_2-I_1)u^3y^1y^2+\\ &\left.+(I_3-I_2)^2(y^2)^2(y^3)^2 + (I_1-I_3)^2(y^1)^2(y^3)^2 + (I_2-I_1)^2(y^1)^2(y^2)^2\right]\end{aligned}$$

The optimality condition becomes in this case

$$\begin{aligned}\pi_1 &= (I_1)^2u^1 + I_1(I_3-I_2)y^2y^3\\ \pi_2 &= (I_2)^2u^2 + I_2(I_1-I_3)y^1y^3\\ \pi_3 &= (I_3)^2u^3 + I_3(I_2-I_1)y^1y^2\end{aligned}$$

and this determines the submanifold W which we specify by writing $u_a = u_a(x, y, \pi)$. On W, the Pontryagin-Hamilton equations become the control system (40) and

$$\begin{aligned}\dot{\pi}_1 &= \frac{M_2(M_2 - I_2u^2) + M_3(M_3 - I_3u^3)}{y^1} - \mu_1\\ \dot{\pi}_2 &= \frac{M_1(M_1 - I_1u^1) + M_3(M_3 - I_3u^3)}{y^2} - \mu_2\\ \dot{\pi}_3 &= \frac{M_1(M_1 - I_1u^1) + M_2(M_2 - I_2u^2)}{y^3} - \mu_3\end{aligned}$$

$$\dot{\mu}_1 + \mu_3y^2 - \mu_2y^3 = 0 \quad \dot{\mu}_2 - \mu_3y^1 + \mu_1y^3 = 0 \quad \dot{\mu}_3 + \mu_2y^1 - \mu_1y^2 = 0$$

The equations above are equivalent to equation (27) in [48].

8.2.3. *Constrained kinematic problem.* Let us study now the constrained system, i.e., the system with admissible velocities belonging to the subbundle $\mathcal{D} \subset TSO(3)$ defined by the condition $y^1 = 0$. Thus the system is

$$\dot{x}^1 = \sec x^2 \cos x^3 u^2, \quad \dot{x}^2 = -\sin x^3 u^2, \quad \dot{x}^3 = \tan x^2 \cos x^3 u^2 + u^3, \tag{42}$$

or in terms of the diagram,

$$\begin{tikzcd} & TSO(3) \arrow[r, "id"] \arrow[d, "\tau"'] & TSO(3) \arrow[dl, "\tau"] \\ \mathcal{D} \arrow[ur, "inc"] \arrow[r] & SO(3) & \end{tikzcd}$$

The cost function corresponds to the energy provided by the controls $k(x^i, u^a) = \frac{1}{2}\left[I_2(u^2)^2 + I_3(u^3)^2\right]$. The Hamiltonian in this case is written as $H = \mu_2 u^2 + \mu_3 u^3 - \frac{1}{2}\left[I_2(u^2)^2 + I_3(u^3)^2\right]$. Optimality conditions defining the submanifold W (18) are $\mu_2 = I_2 u^2 \quad \mu_3 = I_3 u^3$. Using the representation $u^2 = u^2(x, \mu)$ and $u^3 = u^3(x, \mu)$ for W, the equations (21) become then the control system (42) together with $\dot{\mu}_2 + \mu_1 u^3 = 0 \quad \dot{\mu}_3 - \mu_1 u^2 = 0 \quad \dot{\mu}_1 + (I_3 - I_2)u^2 u^3 = 0$.

In the case of the completely symmetric rigid body we get $\dot{\mu}_2 + \mu_1 u^3 = 0 \quad \dot{\mu}_3 - \mu_1 u^2 = 0 \quad \dot{\mu}_1 = 0$, which are equivalent to the equations obtained by Sastry and Montgomery in [53].

8.2.4. *Constrained dynamic problem.* Finally let us study the case of dynamic control for the constrained system. From the geometrical point of view, the control bundle B corresponds to $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap \mathrm{Adm(TSO(3))}$ and the system can be described as

$$\begin{tikzcd} & \mathcal{T}^{TSO(3)}\mathcal{D} \arrow[r, "\rho^1"] \arrow[d, "\tau"'] & T\mathcal{D} \arrow[dl] \\ B \arrow[ur, "inc"] \arrow[r] & \mathcal{D} & \end{tikzcd}$$

where

$$\dot{x}^1 = \sec x^2 \cos x^3 y^2, \quad \dot{x}^2 = -\sin x^3 y^2, \tag{43}$$

$$\dot{x}^3 = \tan x^2 \cos x^3 y^2 + y^3, \quad \dot{y}^2 = u^2,\ \dot{y}^3 = u^3 \tag{44}$$

We shall consider now as cost function, the restriction to $\mathcal{D}$ of the cost function defined in Section 8.2.2

$$k(x^i, y^a, u^a) = \frac{1}{2}\left[(I_2)^2(u^2)^2 + (I_3)^2(u^3)^2 + (I_3 - I_2)^2(y^2)^2(y^3)^2\right],$$

where we assume that the control functions are the components of the admissible angular accelerations of our system $u^2 = v^2$ and $u^3 = v^3$.

Optimality condition leads to the submanifold W defined as $\pi_2 = (I_2)^2 u^2 \quad \pi_3 = (I_3)^2 u^3$. Then W is defined by specifying $u^2 = u^2(x, y, \pi)$ and $u^3 = u^3(x, y, \pi)$. The motion on W corresponds then to the control system (43) and

$$\begin{aligned}
&\dot{\pi}_2 = \frac{(M_1)^2}{y^2} - \mu_2, \quad \dot{\pi}_3 = \frac{(M_1)^2}{y^3} - \mu_3,\\
&\dot{\mu}_2 + \mu_1 y^3 = 0, \quad \dot{\mu}_3 - \mu_1 y^2 = 0,\\
&\dot{\mu}_1 + \mu_3 y^2 - \mu_2 y^3 = 0
\end{aligned}$$

where $M_1 = (I_3 - I_2) y^2 y^3$ is a torque on $\mathcal{D}$, defined in (41).

In the case of the completely symmetric rigid body we obtain

$$\dot{\pi}_2 = -\mu_2, \quad \dot{\pi}_3 = -\mu_3, \quad \dot{\mu}_2 + \mu_1 y^3 = 0, \quad \dot{\mu}_3 - \mu_1 y^2 = 0, \quad \dot{\mu}_1 + \mu_3 y^2 - \mu_2 y^3 = 0.$$

This system gives the following equations obtained by Crouch and Silva Leite in [18], Ex. 6.4, Case II

$$\dddot{y}^2 - \mu_1 y^3 = 0, \quad \dddot{y}^3 + \mu_1 y^2 = 0, \quad \dot{\mu}_1 - \ddot{y}^3 y^2 + \ddot{y}^2 y^3 = 0$$

8.3. Systems with symmetry and constraints: quasi-coordinates for the Atiyah algebroid.

8.3.1. *The geometrical setting.* Consider a ball rolling without sliding on a fixed table (see Example 8.12 in [16]). The configuration space is $Q = \mathbb{R}^2 \times SO(3)$, where $SO(3)$ is parameterized by the Eulerian angles θ, ϕ and ψ. In quasi-coordinates $(x, y, \theta, \phi, \psi, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ the energy may be expressed by $T = \frac{1}{2}[\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)]$, where ω_x, ω_y and ω_z are the components of the angular velocity of the ball.

The system is invariant under $SO(3)$ transformations, and thus it is natural to consider the corresponding formulation on the Atiyah algebroid $E = TQ/SO(3) \equiv T\mathbb{R}^2 \times \mathbb{R}^3$. On that system we must still implement the nonholonomic constraint arising from the rolling-without-sliding condition $\dot{x}^1 - r\omega_2 = 0 \qquad \dot{x}^2 + r\omega_1 = 0$.

For the configuration space we can choose coordinates $M = Q/G = \mathbb{R}^2 \leftarrow \mathrm{x} = (x^1, x^2)$, with $x^1 = x$ and $x^2 = y$. In what regards the fiber, we can choose thus a transformation mapping the set of fiber coordinates $\{\dot{x}^1, \dot{x}^2, \omega_3, \omega_1, \omega_2\}$ onto a new set $\{y^\alpha\}$. These quasi-velocities become then $y^i = \dot{x}^i, \quad y^3 = \omega_3, \quad y^4 = \dot{x}^1 - r\omega_2, \quad y^5 = \dot{x}^2 + r\omega_1$.

Analogously we can consider the inverse transformation. Thus the original velocities can be written in terms of the quasi-velocities as $\dot{x}^i = y^i$, $\omega_3 = y^3$, $\omega_1 = -\frac{1}{r}y^2 + \frac{1}{r}y^5$, $\omega_2 = \frac{1}{r}y^1 - \frac{1}{r}y^4$

The local basis of sections of E determined by the quasi-velocities turns out to be

$$f_1 = e'_1 + \frac{1}{r}e'_4, \quad f_2 = e'_2 - \frac{1}{r}e'_3, \quad f_3 = e'_5, \quad f_4 = -\frac{1}{r}e'_4, \quad f_5 = \frac{1}{r}e'_3$$

where $\{e'_1, e'_2, \cdots, e'_5\}$ is the local basis defined in [16], page 36.

With respect to this basis, the structure constants and the anchor mapping of the Lie algebroid structure become

$$[f_2, f_1] = [f_1, f_5] = [f_4, f_2] = [f_5, f_4] = \frac{1}{r^2}f_3$$

$$[f_3, f_1] = [f_4, f_3] = f_5 \quad [f_2, f_3] = [f_3, f_5] = f_4$$

$$\rho(f_1) = \partial_x, \quad \rho(f_2) = \partial_y,$$

the remaining elements being zero.

The set of admissible velocities becomes thus the fiber of the distribution $\mathcal{D}$, which corresponds to

$$\mathcal{D} = \{(x^i, y^\alpha) \in E \mid y^4 = y^5 = 0\}$$

The coordinates for these points are therefore a $= (x^1, x^2, y^1, y^2, y^3) = (x^i, y^\alpha)$, where in terms of the original set of coordinates these correspond to $y^1 = \dot{x}^1 = u^1$, $y^2 = \dot{x}^2 = u^2$, $y^3 = \omega_3 = u^3$ and $y^4 = 0 = y^5$.

The dynamical system on the algebroid is defined by a Lagrangian function on E, which can be written in terms of the velocities as

$$L(x, y, \dot{x}, \dot{y}, \omega_1, \omega_2, \omega_3) = \frac{1}{2}[\dot{x}^2 + \dot{y}^2 + k^2(\omega_1^2 + \omega_2^2 + \omega_3^2)], \tag{45}$$

and in terms of the quasi-velocities

$$L(x^i, y^\alpha) = \frac{1}{2}\Big[(y^1)^2 + (y^2)^2 + \frac{k^2}{r^2}((y^1)^2 + (y^2)^2 + (y^4)^2 + (y^5)^2 - y^2y^5 - y^1y^4) + k^2(y^3)^2\Big]. \tag{46}$$

8.3.2. *Two different optimal control problems.* Let us consider now two different optimal control problems on this Lie algebroid.

Kinematic Control Problem

Consider the following problem: determine the minimal value among the set of admissible solutions $a : \mathbb{R} \to E$, of the controlled Euler-Lagrange equations of the form $\delta L(a(t)) = 0, \quad a(t) \in \mathcal{D}$, where the cost fuction is $\kappa(x^i, u^a) = \frac{1}{2}\left\{(u^1)^2 + (u^2)^2 + \frac{k^2}{r^2}\left[(u^2)^2 + (u^1)^2\right] + k^2(u^3)^2\right\}$. The control bundle is $\mathcal{D}$, and the section we consider $\sigma : (x^i, u^a) \in \mathcal{D} \to (x^i, u^a, 0, u^a) \in E$ is the canonical inclusion. Please notice that we use the notation u^a to denote the elements of the fiber of $\mathcal{D}$ when considered as the control bundle.

$$\begin{tikzcd} & E \arrow[r, "\rho"] \arrow[d, "\tau"'] & T\mathbb{R}^2 \arrow[dl] \\ \mathcal{D} \arrow[ur, "\sigma"] \arrow[r] & \mathbb{R}^2 & \end{tikzcd}$$

The Pontryagin Hamiltonian is then written as a function $H \in C^\infty(E^* \times_{\mathbb{R}^2} \mathcal{D})$. The optimality condition of the Maximum principle on this function implies

$$\frac{\partial H}{\partial u^a} = 0 \Leftrightarrow \mu_a = c_a u^a \tag{47}$$

where $c_1 = c_2 = 1 + k^2/r^2$ and $c_3 = k^2$.

If we write the set of Pontryagin equations we see:

$$\dot{x}^i = y^i \qquad \dot{\mu}_1 = \mu_3\frac{\mu_2}{c_2 r^2} + \mu_5\frac{\mu_3}{c_3} \quad \dot{\mu}_2 = -\mu_4\frac{\mu_3}{c_3} - \mu_3\frac{\mu_1}{c_1 r^2}$$

$$\dot{\mu}_3 = -\mu_5\frac{\mu_1}{c_1} + \mu_4\frac{\mu_2}{c_2} \quad \dot{\mu}_4 = -\mu_3\frac{\mu_2}{c_2 r^2} - \mu_5\frac{\mu_3}{c_3} \quad \dot{\mu}_5 = \mu_4\frac{\mu_3}{c_3} + \mu_3\frac{\mu_1}{c_1 r^2}$$

Thus we can use (47) and the above equations to define the resulting system on $\mathcal{D}$:

$$\ddot{x}^1 = \frac{1}{c_1}(d_2\omega_3 - \dot{x}^2\omega_3) \qquad \ddot{x}^2 = \frac{1}{c_2}(\dot{x}^1\omega_3 - d_1\omega_3) \qquad \dot{\omega}_3 = \frac{1}{c_3}(d_1\dot{x}^2 - d_2\dot{x}^1)$$

with d_1, d_2 constants and $c_1 = c_2 = 1 + k^2/r^2$ and $c_3 = k^2$.

Dynamic Optimal Control Problem

Let us consider now a different control problem, where we are able to control the forces acting on the system, i.e. we consider a system corresponding to

$(\delta L)_a = u_a$, where L is defined as (46) and δ represents the variational derivative. In this case, for the Lagrangian given above, this implies that $\dot{y}^a = \frac{u_a}{c_a} \quad a = 1, 2, 3$, where again $c_1 = c_2 = 1 + k^2/r^2$ and $c_3 = k^2$ (see equations 1.9.13 in [2]).

The control system is thus defined as a section

$$\begin{tikzcd} & \mathcal{T}^E\mathcal{D} \arrow[r, "\rho^1"] \arrow[d, "\tau"] & T\mathcal{D} \arrow[dl] \\ \mathcal{D} \oplus \mathcal{D}^* \arrow[ur, "\sigma"] \arrow[r] & \mathcal{D} & \end{tikzcd},$$

where $\sigma : (x^i, y^a, u_a) \in \mathcal{D} \oplus \mathcal{D}^* \mapsto (x^i, y^a, y^a, 0, u_a/c_a) \in \mathcal{T}^E\mathcal{D}$.

The cost function now shall be the energy provided by the control functions: $\boldsymbol{\kappa}(x^i, u_a) = \frac{1}{2}\sum_a u_a^2$. As a result, the Pontryagin Hamiltonian $H \in C^\infty((\mathcal{T}^E\mathcal{D})^* \times_{\mathcal{D}} B)$ reads now $H(x^i, y^a, \mu_\alpha, \pi_a, u_a) = \mu_a y^a + \pi_a \frac{u_a}{c_a} - \frac{1}{2}\sum_a u_a^2$. The maximum principle applied to this function results

$$\frac{\partial H}{\partial u_a} = 0 \Leftrightarrow u_a = \frac{\pi_a}{c_a}, \text{ with } a = 1, 2, 3$$

Then the optimal manifold correponds to this submanifold of $(\mathcal{T}^E\mathcal{D})^* \times_{\mathcal{D}} B$.

The Pontryagin equations on $(\mathcal{T}^E\mathcal{D})^* \times_{\mathcal{D}} B$ are:

$$\dot{x}^i = y^i \quad \dot{y}^a = \frac{u_a}{c_a} \quad \dot{\pi}_a = -\mu_a \quad \dot{\mu}_1 = \mu_3 \frac{y^2}{r^2} + \mu_5 y^3$$
$$\dot{\mu}_2 = -\mu_4 y^3 - \mu_3 \frac{y^1}{r^2} \quad \dot{\mu}_3 = -\mu_5 y^1 + \mu_4 y^2 \quad \dot{\mu}_4 = -\dot{\mu}_1 \quad \dot{\mu}_5 = -\dot{\mu}_2$$

But if we restrict them to the optimized submanifold we obtain the reduced system of equations:

$$\ddddot{x}^1 = [\frac{c_3}{c_1 r}]^2 \dot{x}^2 \ddot{\omega}_3 - \omega_3 \dddot{x}^2 - \frac{e_2}{c_1^2}\omega_3 \qquad \ddddot{x}^2 = -[\frac{c_3}{c_2 r}]^2 \dot{x}^1 \ddot{\omega}_3 + \omega_3 \dddot{x}^1 + \frac{e_1}{c_2^2}\omega_3$$
$$\dddot{\omega}_3 = [\frac{c_2}{c_3}]^2 \dot{x}^1 \dddot{x}^2 - [\frac{c_1}{c_3}]^2 \dot{x}^2 \dddot{x}^1 + \frac{e_2}{c_3^2}\dot{x}^1 - \frac{e_1}{c_3^2}\dot{x}^2$$

where $c_1 = c_2 = 1 + k^2/r^2$, $c_3 = k^2$ and e_1, e_2 are arbitrary constants.

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