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ON THE SEMICLASSICAL CHARACTER OF ORTHOGONAL POLYNOMIALS SATISFYING STRUCTURE RELATIONS

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ABSTRACT: We prove the semiclassical character of some sequences of orthogonal polynomials, say $\{P_n\}$, $\{R_n\}$, related through relations of the following type: $\sum_{k=0}^{N} \zeta_{n,k} R^{(\alpha)}_{n+i-k} = \sum_{k=0}^{M} \xi_{n,k} P_{n+j-k}$, where i, j, M, N, α are non-negative integers, $\zeta_{n,k}, \xi_{n,k}$ are complex numbers, and $R^{(\alpha)}$ denotes the α-derivative of R. The case $M = j = 0, \alpha = 2, i = 2$ is studied for a pair of orthogonal polynomials whose corresponding orthogonality measures are coherent. The relation $\sum_{k=0}^{s} \xi_{n,k} P_{n+s-k} = \sum_{k=0}^{s+2} \zeta_{n,k} P'_{n+s+1-k}$ is shown to give a characterization for the semiclassical character of $\{P_n\}$.

KEYWORDS: Orthogonal polynomials; Recurrence relations; Semiclassical linear functionals; Structure relations.
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1. Introduction

This paper is devoted to the study of sequences of orthogonal polynomials on the real line, say $\{R_n\}$, $\{P_n\}$, satisfying differential-difference equations of the following type:

$$\sum_{k=0}^{N} \zeta_{n,k} R^{(\alpha)}_{n+i-k} = \sum_{k=0}^{M} \xi_{n,k} P_{n+j-k} \,, \tag{1}$$

where i, j, M, N, α are non-negative integers, $\zeta_{n,k}, \xi_{n,k}$ are complex numbers, and $R^{(\alpha)}$ denotes the α-derivative of R. These type of relations are members of the well-known *structure relations* for orthogonal polynomials.

The structure relations appear in a wide range of topics in the literature of orthogonal polynomials. For example, they appear in the study of coherent pairs of measures, arising within the theory of Sobolev inner products [11, 12] (see also [3], where a variational isoperimetric problem is studied in the context of the Sobolev orthogonality). Further, the structure relations have

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been used in the study of orthogonal polynomials associated with exponential weights, for example, in the study of the asymptotic, of estimates for the orthogonal polynomials, and second-order differential equations satisfied by these polynomials (see [4, 5, 9] and the references therein).

A theme of research in the theory of orthogonal polynomials is the characterization of orthogonal polynomial sequences satisfying a given structure relation. In this issue we refer the reader to [1, 2], [4]-[8], [16]-[19] (see also [13], one of the first papers formulating inverse problems for orthogonal polynomials).

In the present paper we are interested in the analysis of the semiclassical character of orthogonal polynomial sequences satisfying structure relations of the above referred type, which will be specified later in the text. Before proceeding to the results of the present paper let us give a brief account on some of the known results in this topic.

We begin by mentioning [4, 5], where it was given a characterization for the orthogonal polynomials related through structure relations

$$\phi\, R_n^{(\alpha)} = \sum_{k=n-\alpha-t}^{n-\alpha+s} \xi_{n,k} P_k\,, \quad n=0,1,\dots\,, \tag{2}$$

where ϕ is a fixed polynomial, $\xi_{n,k}$ are real numbers, and s, t are non-negative integers. In [4] (see also [5]), the measures of orthogonality for pairs of orthogonal polynomial sequences related through (2) are given explicitly, where it turns out its semiclassical character, that is, the log-derivative of the absolutely continuous parts is a rational function. The analogue result in the setting of the theory of distributions was established in [7].

Note that the structure relation (2) with $\alpha = 1$ and $P_n = R_n$, known as *first structure relation* for semiclassical orthogonal polynomial, generalizes the well-known first structure relation with $\deg(\phi) \leq 2, t = 0, s \leq 2$, that characterizes the classical orthogonal polynomials (that is, semiclassical of class zero) [1, 2, 8, 17, 18].

Another well-known characterization in the theory of orthogonal polynomials is the one of the classical orthogonal polynomials, $\{P_n\}$, in terms of the *second structure relation*

$$P_n(x) = \sum_{k=0}^{2} \xi_{n,k} P_{n-k}^{[1]}(x)\,, \quad n \geq 2\,, \tag{3}$$

where $P_n^{[1]} = P'_{n+1}/(n+1),\ n \geq 0$ (see [8, 18]).

Recently, in [16], it was formulated the structure relation

$$\sum_{k=0}^{2\sigma} \xi_{n,k} P_{n+\sigma-k}(x) = \sum_{k=0}^{\sigma+t} \varsigma_{n,k} P_{n+\sigma-k}^{[1]}(x)\,, \quad n \geq \max\{t+1, \sigma\}\,,$$

that, under certain conditions, establishes the semiclassical character of orthogonal polynomial sequences $\{P_n\}$, and which generalizes the above mentioned second structure relation for classical orthogonal polynomials.

In the present paper we establish the semiclassical character of sequences of monic orthogonal polynomials satisfying structure relations of the same type as (1). We begin by establishing the semiclassical character of pairs of orthogonal polynomial sequences, say $\{P_n\}, \{R_n\}$, associated with a coherent pair of measures arising from a Sobolev inner product specified in §3, related trough

$$R_n(x) = \sum_{k=0}^{s} \zeta_{n,k}\, P''_{n+2-k}(x)\,, \quad n \geq s$$

(cf. Lemma 5). We will use an analytic method based on the recurrence relation satisfied by the orthogonal polynomials written in the matrix form (cf. § 2). Then, we prove that the structure relation

$$\sum_{k=0}^{s} \xi_{n,k} P_{n+s-k} = \sum_{k=0}^{s+2} \zeta_{n,k} P_{n+s-k}^{[1]}\,, \quad n \geq 1\,, \tag{4}$$

where s is some non-negative integer, and $\xi_{n,k}, \zeta_{n,k}$ are complex numbers, is sufficient to establish the semiclassical character of $\{P_n\}$. The converse result is also established, that is, semiclassical sequences of monic orthogonal polynomials of class $s,\ s \geq 0$, are shown to satisfy structure relations such as (4) (cf. Theorem 1). Note that when $s = 0$ in (4) we recover the above-mentioned characterization of classical orthogonal polynomials in terms of structure relations (3).

This paper is organized as follows. In §2 we give the definitions and state the basic results which will be used in the forthcoming sections. In §3 we study the semiclassical character of the coherent pairs arising from the above mentioned Sobolev inner product. In § 4 we establish the characterization of semiclassical orthogonal polynomials in terms of structure relations such as (4).

2. Preliminary results and notations

Let $\mathbb{P} = \text{span}\{z^k : k \in \mathbb{N}_0\}$ be the space of polynomials with complex coefficients, and let $\mathbb{P}'$ be its algebraic dual space, that is, the linear space of linear functionals defined on $\mathbb{P}$. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$. For a polynomial g, we define the linear functional gu as

$$\langle g\, u, f\rangle = \langle u, gf\rangle\,,\ f \in \mathbb{P}\,,$$

and we define $\mathcal{D}u$ as

$$\langle \mathcal{D}u, f\rangle = -\langle u, f'\rangle\,,\ f \in \mathbb{P}\,,$$

where f' denotes the derivative of f.

Given the sequence of moments (c_n) of u, $c_n = \langle u, x^n\rangle,\ n \geq 0$, the minors of the corresponding *Hankel matrix* are defined by $H_n = \det(((c)_{i+j})_{i,j=0}^n), n \geq 0$. The linear functional u is said to be quasi-definite (respectively, positive-definite) if $H_n \neq 0$ (respectively, $H_n > 0$), for all integer $n \geq 0$.

Definition 1. Let $u \in \mathbb{P}'$. A sequence $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to u if the following two conditions hold:
(i) $\deg(P_n) = n,\ n \geq 0\,,$
(ii) $\langle u, P_nP_m\rangle = k_n\delta_{n,m}, k_n = \langle u, P_n^2\rangle \neq 0,\ n \geq 0\,.$
If the leading coefficient of each P_n is 1, then $\{P_n\}$ is said to be a sequence of monic orthogonal polynomials with respect to u, and it will be referred to as SMOP.

If u is quasi-definite, then there exists a unique SMOP with respect to u (see, for example, [10, 20]).

The SMOP $\{P_n\}$ satisfies a three-term recurrence relation

$$P_{n+1} = (x - \beta_n)P_n - \gamma_n P_{n-1},\ \ n \geq 0, \tag{5}$$

with $P_{-1}(x) = 0, P_0(x) = 1$, and $\gamma_n \neq 0,\ n \geq 1$. The converse result, known as the Favard Theorem (see [10]), is also true, that is, given a SMOP $\{P_n\}$ satisfying a three-term recurrence relation of the preceding type, there exists a unique quasi-definite linear functional u such that $\{P_n\}$ is the SMOP with respect to u. The γ_n's and β_n's in (5) are given by

$$\gamma_0 = c_0\,,\ \ \gamma_n = H_{n-2}H_n/H_{n-1}^2\,,\ \ \beta_n = \langle u, xP_n^2\rangle/\langle u, P_n^2\rangle\,,\ \ n \geq 1\,.$$

Furthermore, the linear functional u is positive-definite if, and only if, $\gamma_n > 0,\ n \geq 1\,$. In this case, u has an integral representation in terms of a positive

Borel measure, μ, supported on the real line with an infinite set of points of increase, S, such that $\langle u, x^n \rangle = \int_S x^n \, d\mu \,, \;\; n \geq 0\,.$

Definition 2. (see [2, 17]) A quasi-definite linear functional $u \in \mathbb{P}'$ is said to be *semiclassical* if there exist $\phi, \varphi \in \mathbb{P}$, $\deg(\varphi) \geq 1\,$, such that u satisfies $\mathcal{D}(\phi\, u) = \varphi\, u$. The corresponding sequence of orthogonal polynomials is called *semiclassical*.

The pair of polynomials (ϕ, φ) satisfying $\mathcal{D}(\phi\, u) = \varphi\, u$ is not unique (see [17, 18]). To a semiclassical functional u one associates the *class of* u, defined as the minimum value of $\max\{\deg(\phi)-2, \deg(\varphi)-1\}$, for all pairs of polynomials (ϕ, φ) satisfying $\mathcal{D}(\phi\, u) = \varphi\, u\,, \deg(\varphi) \geq 1\,$, and $n\phi^{(s+2)}(0)/(s+2)! + \varphi^{(s+1)}(0)/(s+1)!$, $\; n \geq 1$ (i.e. the *admissibility condition* of the pair (ϕ, φ)).

When the class of u is zero, that is, $\deg(\varphi) = 1$ and $\deg(\phi) \leq 2$, the well-known classical orthogonal polynomials appear, and these are, up to a linear change of the variables, members of one of the Hermite, Laguerre, Jacobi or Bessel families.

In the sequel we will use the vectors defined by

$$\psi_n(x) = [P_{n+1}(x) \;\; P_n(x)]^T \,, \; n \geq 0\,, \tag{6}$$

where T denotes the transpose. With this notation, the recurrence relation (5) can be written as

$$\psi_n(x) = \mathcal{A}_n \psi_{n-1}(x)\,, \;\; \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}\,, \; n \geq 0\,, \tag{7}$$

with initial conditions $\psi_{-1} = [1 \;\; 0]^T$. Notice that the matrices $\mathcal{A}_n$ are nonsingular, as $\det(\mathcal{A}_n) = \gamma_n \neq 0\,, \; \forall n \geq 0$. As a consequence of (7) the following holds.

Lemma 1. *Let $\{P_n\}$ be a SMOP and $\{\psi_n\}$ the corresponding sequence given by (6). For any fixed integers s, m, λ, M, the following holds, for all $n \geq 1$:*

$$\psi_{n-m} = (\prod_{l=0}^{\lambda-m-1} \mathcal{A}_{n-m-l})\psi_{n-\lambda}\,, \;\; m < \lambda\,, \tag{8}$$

$$\psi_{n-\lambda-M} = (\prod_{l=0}^{M-1} \mathcal{A}_{n-\lambda-l})^{-1}\psi_{n-\lambda}\,, \;\; M \geq 1\,. \tag{9}$$

The lemmas that follow will be used in the forthcoming sections. Henceforth we will write $X^{(i,j)}$ to denote the element of a matrix X in the position (i,j).

Lemma 2. *Let $\{P_n\}$ be a SMOP and $\{\psi_n\}$ the corresponding sequence given by (6). If $\{\psi_n\}$ satisfies*

$$\phi_{n,1}\psi_n' = \mathcal{L}_{n,1}\psi_n \tag{10}$$

$$\phi_{n,2}\psi_n' = \mathcal{L}_{n,2}\psi_n \tag{11}$$

where $\phi_{n,1}, \phi_{n,2} \in \mathbb{P}$ and and $\mathcal{L}_{n,1}, \mathcal{L}_{n,2}$ are matrices of order two whose entries are polynomials with degree uniformly bounded by a number independent of n, then

$$\frac{\phi_{n,1}}{\phi_{n,2}} = \frac{\mathcal{L}_{n,1}^{(i,j)}}{\mathcal{L}_{n,2}^{(i,j)}}\,, \quad i,j = 1,2. \tag{12}$$

Proof: If we multiply (10) by $\phi_{n,2}$, (11) by $\phi_{n,1}$, and subtract the resulting equations, we get

$$(\phi_{n,2}\mathcal{L}_{n,1} - \phi_{n,1}\mathcal{L}_{n,2})\psi_n = 0\,,$$

thus

$$(\phi_{n,2}\mathcal{L}_{n,1}^{(1,1)} - \phi_{n,1}\mathcal{L}_{n,2}^{(1,1)})P_{n+1} = (\phi_{n,1}\mathcal{L}_{n,2}^{(1,2)} - \phi_{n,2}\mathcal{L}_{n,1}^{(1,2)})P_n\,,$$
$$(\phi_{n,2}\mathcal{L}_{n,1}^{(2,1)} - \phi_{n,1}\mathcal{L}_{n,2}^{(2,1)})P_{n+1} = (\phi_{n,1}\mathcal{L}_{n,2}^{(2,2)} - \phi_{n,2}\mathcal{L}_{n,1}^{(2,2)})P_n\,.$$

Since P_n and P_{n+1} do not share zeroes (see, for example, [10]), then we conclude that there exist polynomials π, η such that

$$\phi_{n,2}\mathcal{L}_{n,1}^{(1,1)} - \phi_{n,1}\mathcal{L}_{n,2}^{(1,1)} = \pi P_n\,, \quad \phi_{n,1}\mathcal{L}_{n,2}^{(1,2)} - \phi_{n,2}\mathcal{L}_{n,1}^{(1,2)} = \pi P_{n+1}\,,$$
$$\phi_{n,2}\mathcal{L}_{n,1}^{(2,1)} - \phi_{n,1}\mathcal{L}_{n,2}^{(2,1)} = \eta P_n\,, \quad \phi_{n,1}\mathcal{L}_{n,2}^{(2,2)} - \phi_{n,2}\mathcal{L}_{n,1}^{(2,2)} = \eta P_{n+1}\,.$$

But this is a contradiction to the fact that the degrees of $\phi_{n,1}\mathcal{L}_{n,2}^{(i,2)} - \phi_{n,2}\mathcal{L}_{n,1}^{(i,2)}$ and $\phi_{n,2}\mathcal{L}_{n,1}^{(i,1)} - \phi_{n,1}\mathcal{L}_{n,2}^{(i,1)}$, $i = 1,2$, are bounded. Therefore, π and η must be identically zero, hence

$$\phi_{n,2}\mathcal{L}_{n,1}^{(1,1)} - \phi_{n,1}\mathcal{L}_{n,2}^{(1,1)} \equiv 0\,, \quad \phi_{n,1}\mathcal{L}_{n,2}^{(1,2)} - \phi_{n,2}\mathcal{L}_{n,1}^{(1,2)} \equiv 0\,,$$
$$A_{n,2}\mathcal{L}_{n,1}^{(2,1)} - A_{n,1}\mathcal{L}_{n,2}^{(2,1)} \equiv 0\,, \quad A_{n,1}\mathcal{L}_{n,2}^{(2,2)} - A_{n,2}\mathcal{L}_{n,1}^{(2,2)} \equiv 0\,,$$

and (12) follows. ∎

Now we follow [14].

Lemma 3. *Let $\{P_n\}$ be a SMOP and let $\{\psi_n\}$ be the corresponding sequence given by (6). Let $\{\psi_n\}$ satisfy*

$$\tilde{\phi}_n \psi_n' = \tilde{\mathcal{L}}_n \psi_n \,, \quad n \geq 1 \,, \tag{13}$$

where $\tilde{\phi}_n \in \mathbb{P}$ and the degrees of the entries of $\tilde{\mathcal{L}}_n$ are uniformly bounded by a number independent of n. Then, (13) is equivalent to

$$\phi \, \psi_n' = \mathcal{L}_n \psi_n \tag{14}$$

where ϕ is a polynomial that does not depend on n. Thus, $\{P_n\}$ is semi-classical.

Furthermore, the matrix $\mathcal{L}_n$ satisfies:

$$\mathcal{L}_{n+1}\mathcal{A}_{n+1} - \mathcal{A}_{n+1}\mathcal{L}_n = \phi \, \mathcal{A}_{n+1}' \,, \tag{15}$$

$$\det(\mathcal{L}_{n+1}) = \det(\mathcal{L}_1) + \phi \sum_{k=1}^{n} \mathcal{L}_k^{(1,2)} / \gamma_{k+1} \,, \tag{16}$$

$$\operatorname{tr}(\mathcal{L}_n) = \operatorname{tr}(\mathcal{L}_1) \,. \tag{17}$$

where $\mathcal{A}_n$ are the matrices of the recurrence relation (7), and the γ_n's are the parameters appearing in (7), det *and* tr *denote the determinant and the trace, respectively.*

Proof: If we write (13) to $n+1$ and use the recurrence relations for ψ_n we get

$$\tilde{\phi}_{n+1} \psi_n' = \mathcal{A}_{n+1}^{-1} (\tilde{\mathcal{L}}_{n+1}\mathcal{A}_{n+1} - \tilde{\phi}_{n+1}\mathcal{A}_{n+1}') \psi_n \,. \tag{18}$$

Now, from (13) and (18) we conclude that there exists a polynomial l_n such that

$$\tilde{\phi}_{n+1} = l_n \tilde{\phi}_n \,, \quad \mathcal{A}_{n+1}^{-1} (\tilde{\mathcal{L}}_{n+1}\mathcal{A}_{n+1} - \tilde{\phi}_{n+1}\mathcal{A}_{n+1}') = l_n \tilde{\mathcal{L}}_n \,, \forall n \geq 1,$$

because the first order differential equation for ψ_n is unique, up to a multiplicative factor. But from $\tilde{\phi}_{n+1} = l_n \tilde{\phi}_n$ we obtain

$$\tilde{\phi}_{n+1} = (l_n \cdots l_2) \, \tilde{\phi}_1 \,, \ \forall n \geq 1 \,.$$

Since, for all $n \geq 1$, the degree of $\tilde{\phi}_n$ is bounded by a number independent of n, then the degree of the l_n's must be zero, that is, l_n is constant, for all $n \geq 1$. Hence we obtain (14) with

$$\phi = \tilde{\phi}_1 \,, \quad \mathcal{L}_n = \mathcal{A}_{n+1}^{-1} (\tilde{\mathcal{L}}_{n+1}\mathcal{A}_{n+1} - \tilde{\phi}_{n+1}\mathcal{A}_{n+1}') / (l_n \cdots l_2) \,.$$

The semi-classical character of SMOP satisfying an equation such as (14) is established in [17, 18].

To obtain (15) we take derivatives on $\psi_{n+1} = \mathcal{A}_{n+1}\psi_n$ and multiply the result by ϕ, to get

$$\phi\,\psi'_{n+1} = \phi\,\mathcal{A}'_{n+1}\psi_n + \mathcal{A}_{n+1}\phi\,\psi'_n\,.$$

Using (36) in the previous equation we get

$$\mathcal{L}_{n+1}\psi_{n+1} = \phi\mathcal{A}'_{n+1}\psi_n + \mathcal{A}_{n+1}\mathcal{L}_n\psi_n\,.$$

From the recurrence relation (7) there follows

$$\mathcal{L}_{n+1}\mathcal{A}_{n+1}\psi_n = \phi\mathcal{A}'_{n+1}\psi_n + \mathcal{A}_{n+1}\mathcal{L}_n\psi_n\,,$$

and, since ψ_n is nonsingular, for all $n \in \mathbb{N}$, as $\gamma_n \neq 0$, then (15) follows.

Now we prove (16). From (15) we have

$$\det(\mathcal{L}_{n+1}\mathcal{A}_{n+1}) = \det(\mathcal{A}_{n+1}\mathcal{L}_n + \phi\,\mathcal{A}'_{n+1})\,.$$

Taking into account that $\det(\mathcal{A}_n) = \gamma_n$ and $\mathcal{A}'_n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\forall n \in \mathbb{N}$, we obtain

$$\det(\mathcal{L}_{n+1})\det(\mathcal{A}_{n+1}) = \det(\mathcal{A}_{n+1})\det(\mathcal{L}_n) + \phi\,\mathcal{L}_n^{(1,2)},$$

hence

$$\det(\mathcal{L}_{n+1}) = \det(\mathcal{L}_n) + \frac{\phi}{\gamma_{n+1}}\,\mathcal{L}_n^{(1,2)}\,,$$

and (16) follows.

To prove (17) we begin by writing (15) as

$$\begin{cases} \phi = (x-\beta_{n+1})(\mathcal{L}_{n+1}^{(1,1)} - \mathcal{L}_n^{(1,1)}) + \mathcal{L}_{n+1}^{(1,2)} + \gamma_{n+1}\mathcal{L}_n^{(2,1)}\,, \\ -\gamma_{n+1}\mathcal{L}_{n+1}^{(1,1)} - (x-\beta_{n+1})\mathcal{L}_n^{(1,2)} + \gamma_{n+1}\mathcal{L}_n^{(2,2)} = 0\,, \\ (x-\beta_{n+1})\mathcal{L}_{n+1}^{(2,1)} + \mathcal{L}_{n+1}^{(2,2)} - \mathcal{L}_n^{(1,1)} = 0\,, \\ -\gamma_{n+1}\mathcal{L}_{n+1}^{(2,1)} - \mathcal{L}_n^{(1,2)} = 0\,. \end{cases}$$

If we multiply the third equation of the above system by γ_{n+1} and add to the second one we get

$$(x-\beta_{n+1})(\gamma_{n+1}\mathcal{L}_{n+1}^{(2,1)} + \mathcal{L}_n^{(1,2)}) = \gamma_{n+1}(\mathcal{L}_n^{(1,1)} + \mathcal{L}_n^{(2,2)} - \mathcal{L}_{n+1}^{(1,1)} - \mathcal{L}_{n+1}^{(2,2)}).$$

Taking into account the fourth equation of the above system we obtain

$$\gamma_{n+1}(\mathcal{L}_n^{(1,1)} + \mathcal{L}_n^{(2,2)} - \mathcal{L}_{n+1}^{(1,1)} - \mathcal{L}_{n+1}^{(2,2)}) = 0,$$

that is, $\gamma_{n+1}(\mathrm{tr}(\mathcal{L}_n) - \mathrm{tr}(\mathcal{L}_{n+1})) = 0$. Since $\gamma_n \neq 0\,, \forall n \geq 1\,,$ there follows $\mathrm{tr}(\mathcal{L}_{n+1}) = \mathrm{tr}(\mathcal{L}_n)\,,\ \ \forall n \geq 1\,,$ thus we get (17). ∎

3. The semiclassical character of coherent pairs of measures

In this section we study the semiclassical character of the coherent pairs that emerge from a Sobolev inner product given below. A background for Sobolev inner products and related problems can be found in [11, 12] and also in [15, Lesson V].

Let μ_0 and μ_1 be positive Borel measures supported on the real line. Consider the following inner product in $\mathbb{P}$,

$$(f, g) = \int_{\mathbb{R}} f g \, d\mu_0 + \lambda \int_{\mathbb{R}} f'' g'' \, d\mu_1 \,, \quad f, g \in \mathbb{P}, \ \lambda > 0 \,. \tag{19}$$

We shall denote by $\{P_n\}, \{R_n\}, \{Q_n^\lambda\}$ the sequences of monic polynomials orthogonal with respect to μ_0, μ_1 and $(\cdot, \cdot)$, respectively.

Let $c_{m,n} = (x^m, x^n)$ be the moment of order (m, n) associated with the inner product (19) and let us denote by $(c_j^{(i)}), i = 0, 1$, the moments of the measures μ_0, μ_1, respectively. Then, for every $m, n = 1, 2, \ldots,$ we have

$$c_{m,n} = c_{m+n}^{(0)} + \lambda \ell_{m,n} c_{m+n-4}^{(1)}, \quad \ell_{m,n} = m(m-1)n(n-1).$$

Let us determine the Fourier coefficients of

$$Q_n^\lambda = \sum_{k=0}^{n} \xi_{n,k} x^k \,, \quad \xi_{n,n} = 1 \,.$$

By Cramer's rule we obtain the following representation for Q_n^λ, for all $n \geq 1$:

$$Q_n^\lambda = \frac{\begin{vmatrix} c_0^{(0)} & c_1^{(0)} & c_2^{(0)} & \cdots & c_m^{(0)} \\ c_1^{(0)} & c_2^{(0)} & c_3^{(0)} & \cdots & c_{m+1}^{(0)} \\ c_2^{(0)} & c_3^{(0)} & c_4^{(0)} + \lambda \ell_{2,2} c_0^{(1)} & \cdots & c_{m+2}^{(0)} + \lambda \ell_{m,2} c_{m-2}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}}{\begin{vmatrix} c_0^{(0)} & c_1^{(0)} & c_2^{(0)} & \cdots & c_m^{(0)} \\ c_1^{(0)} & c_2^{(0)} & c_3^{(0)} & \cdots & c_{m+1}^{(0)} \\ c_2^{(0)} & c_3^{(0)} & c_4^{(0)} + \lambda \ell_{2,2} c_0^{(1)} & \cdots & c_{m+2}^{(0)} + \lambda \ell_{m,2} c_{m-2}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1}^{(0)} & c_n^{(0)} & c_{n+1}^{(0)} + \lambda \ell_{2,n-1} c_{n-3}^{(1)} & \cdots & c_{m+n-1}^{(0)} + \lambda \ell_{m,n-1} c_{m+n-5}^{(1)} \end{vmatrix}} \,. \tag{20}$$

Lemma 4. *Let* $\{Q_n\}$ *be the sequence of polynomials defined by*

$$Q_n(x) = \lim_{\lambda\to\infty} Q_n^\lambda(x)\,, \quad n \geq 1\,.$$

Then, for each $n \geq 1$, Q_n *is monic of degree* n, *and it satisifes:*

$$\int_{\mathbb{R}} Q_n\, d\mu_0 = 0, \quad n = 1, 2, \ldots \tag{21}$$

$$\int_{\mathbb{R}} xQ_n\, d\mu_0 = 0, \quad n = 2, 3, \ldots \tag{22}$$

$$\int_{\mathbb{R}} (Q_n)''(x^k)''\, d\mu_1 = 0, \quad k = 0, 1, \ldots, n-1, \quad n \geq 2\,. \tag{23}$$

Proof: If we divide both determinants of (20) by λ, from the second to the nth-row, and take limits when $\lambda \to \infty$, we get the result. ∎

Now, (23) implies

$$Q_n'' = n(n-1)R_{n-2}\,.$$

Let

$$R_{n-2} = \sum_{k=2}^{n} l_{n,k} \frac{P_k''}{k(k-1)}\,.$$

Then,

$$\frac{Q_n''}{n(n-1)} = \sum_{k=2}^{n} l_{n,k} \frac{P_k''}{k(k-1)}\,.$$

If we integrate twice we get

$$\frac{Q_n}{n(n-1)} = \sum_{k=2}^{n} l_{n,k} \frac{P_k}{k(k-1)} + l_{n,1}x + l_{n,0}\,, \tag{24}$$

with $l_{n,1} = l_{n,0} = 0$ (this follows from (21) and (22)).

On the other hand, we have that

$$Q_n = Q_n^\lambda + \sum_{j=0}^{n-1} \beta_j\, Q_j^\lambda\,, \quad \beta_j = \frac{(Q_n, Q_j^\lambda)}{(Q_j^\lambda, Q_j^\lambda)}\,. \tag{25}$$

Therefore, if $l_{n,k} = 0,\ k = 2, \ldots, n-s-1$ and $l_{n,n-s} \neq 0$, for some $s \geq 1$, then, taking into account (24) and (25), there follows $\beta_j = 0,\ k = 2, \ldots, n-s-1$, and $\beta_{n,n-s} \neq 0$, for some $s \geq 1$, that is,

$$\sum_{k=n-s}^{n} l_{n,k} \frac{P_k}{k(k-1)} = Q_n^{\lambda} + \sum_{j=n-s}^{n-1} l_{n,k} \beta_j \, Q_j^{\lambda} \,, \quad n \geq s \,.$$

Thus we have the following definition.

Definition 3. A pair of positive Borel measures (μ_0, μ_1) is said to be quadratic s-coherent if

$$R_n = \sum_{k=n+2-s}^{n+2} l_{n,k} \frac{P_k''}{k(k-1)} \,. \tag{26}$$

Without loss of generality we consider the case $s = 1$ in (26), that is,

$$R_n = \frac{P_{n+2}''}{(n+2)(n+1)} + a_n \frac{P_{n+1}''}{(n+1)n} \,. \tag{27}$$

We define the vectors $\psi_n = \begin{bmatrix} P_{n+1} & P_n \end{bmatrix}^T$, $\eta_n = \begin{bmatrix} R_{n+1} & R_n \end{bmatrix}^T$, $n \geq 0$, which satisfy the recurrence relations

$$\psi_n = \mathcal{A}_n \psi_{n-1} \,, \quad \eta_n = \mathcal{B}_n \eta_{n-1} \quad n \geq 1 \,,$$

with

$$\mathcal{A}_n = \begin{bmatrix} x - \beta_{0,n} & -\gamma_{0,n} \\ 1 & 0 \end{bmatrix} , \quad \mathcal{B}_n = \begin{bmatrix} x - \beta_{1,n} & -\gamma_{1,n} \\ 1 & 0 \end{bmatrix} , \quad n \in \mathbb{N} \,.$$

Lemma 5. *Let (μ_0, μ_1) be a pair of quadratic 1-coherent measures. Then, μ_0 is semiclassical and μ_1 is a rational modification of μ_0, thus it is also semi-classical.*

Proof: Let $\{P_n\}$, $\{R_n\}$ be the sequences of monic orthogonal polynomials with respect to μ_0 an dμ_1, respectively, related through (27). Let us write (27) in the equivalent matrix form

$$\eta_n = M_n \psi_{n+2}'' + N_n \psi_n'' \,,$$

with $M_n = \begin{bmatrix} \frac{1}{(n+3)(n+2)} & \frac{a_{n+1}}{(n+2)(n+1)} \\ 0 & \frac{1}{(n+2)(n+1)} \end{bmatrix}$, $N_n = \begin{bmatrix} 0 & 0 \\ \frac{a_n}{(n+1)n} & 0 \end{bmatrix}$. Using the recurrence relations for ψ_n we get

$$\eta_n = G_{n,1} \psi_{n+1}'' + G_{n,2} \psi_{n+1}' \,, \tag{28}$$

where $G_{n,1} = M_n \mathcal{A}_{n+2} + N_n (\mathcal{A}_{n+1})^{-1}$, $G_{n,2} = 2(M_n \mathcal{A}_{n+2}' + N_n ((\mathcal{A}_{n+1})^{-1})')$.

The multiplication of (28) by the adjoint matrix of $G_{n,1}$, $\mathrm{adj}(G_{n,1})$, yields

$$\mathrm{adj}(G_{n,1})\eta_n = \det(G_{n,1})\psi''_{n+1} + \mathrm{adj}(G_{n,1})G_{n,2}\psi'_{n+1}\,. \tag{29}$$

If we take $n+1$ in the above equation and use the recurrence relations for ψ_n and η_n we obtain

$$\begin{aligned}\det(G_{n+1,1})\mathcal{A}_{n+2}\psi''_{n+1} &= \mathrm{adj}(G_{n+1,1})\mathcal{B}_{n+1}\eta_n \\ &\quad - \{2\det(G_{n+1,1})\mathcal{A}'_{n+2} + \mathrm{adj}(G_{n+1,1})G_{n+1,2}\mathcal{A}_{n+2}\}\psi'_{n+1} \\ &\quad - \mathrm{adj}(G_{n+1,1})G_{n+1,2}\mathcal{A}'_{n+2}\psi_{n+1}\,. \end{aligned} \tag{30}$$

Now we eliminate ψ''_{n+1} between (29) and (30), which gives us

$$L_n\eta_n = L_{n,1}\psi'_{n+1} + L_{n,2}\psi_{n+1} \tag{31}$$

with

$$\begin{aligned} L_n &= \det(G_{n,1})\,\mathrm{adj}(G_{n+1,1})\mathcal{B}_{n+1} - \det(G_{n+1,1})\mathcal{A}_{n+2}\,\mathrm{adj}(G_{n,1})\,, \\ L_{n,1} &= \det(G_{n,1})(2\det(G_{n+1,1})\mathcal{A}'_{n+2} + \mathrm{adj}(G_{n+1,1})G_{n+1,2}\mathcal{A}_{n+2}) \\ &\qquad - \det(G_{n+1,1})\mathcal{A}_{n+2}\,\mathrm{adj}(G_{n,1})G_{n,2}\,, \\ L_{n,2} &= \det(G_{n,1})\,\mathrm{adj}(G_{n+1,1})G_{n+1,2}\mathcal{A}'_{n+2}\,. \end{aligned}$$

But $\det(L_n) = \det(L_{n,1}) = \det(L_{n,2}) = 0$. If we consider the second line of (31), for n and $n+1$, we get

$$S_n\eta_n = S_{n,1}\psi'_{n+1} + S_{n,2}\psi_{n+1} \tag{32}$$

where

$$\begin{aligned} S_n &= \begin{bmatrix} L_{n+1}^{(2,1)} & L_{n+1}^{(2,2)} \\ 0 & L_n^{(2,1)} \end{bmatrix}\mathcal{B}_{n+1} + \begin{bmatrix} 0 & 0 \\ L_n^{(2,2)} & 0 \end{bmatrix}\mathcal{B}_n^{-1}\,, \\ S_{n,1} &= \begin{bmatrix} L_{n+1,1}^{(2,1)} & L_{n+1,1}^{(2,2)} \\ 0 & L_{n,1}^{(2,1)} \end{bmatrix}\mathcal{A}_{n+2} + \begin{bmatrix} 0 & 0 \\ L_{n,1}^{(2,2)} & 0 \end{bmatrix}\mathcal{A}_{n+1}^{-1}\,, \\ S_{n,2} &= \begin{bmatrix} L_{n+1,1}^{(2,1)} & L_{n+1,1}^{(2,2)} \\ 0 & L_{n,1}^{(2,1)} \end{bmatrix}\mathcal{A}'_{n+2} + \begin{bmatrix} 0 & 0 \\ L_{n,1}^{(2,2)} & 0 \end{bmatrix}(\mathcal{A}_{n+1}^{-1})' \\ &\qquad + \begin{bmatrix} L_{n+1,2}^{(2,1)} & L_{n+1,2}^{(2,2)} \\ 0 & L_{n,2}^{(2,1)} \end{bmatrix}\mathcal{A}_{n+2} + \begin{bmatrix} 0 & 0 \\ L_{n,2}^{(2,2)} & 0 \end{bmatrix}\mathcal{A}_{n+1}^{-1}\,. \end{aligned}$$

Now we multiply (32) by $\mathrm{adj}(S_n)$ to get

$$T_n\eta_n = T_{n,1}\psi'_{n+1} + T_{n,2}\psi_{n+1}\,, \tag{33}$$

with $T_n = \det(S_n)$, $T_{n,1} = \operatorname{adj}(S_n)S_{n,1}$, $T_{n,2} = \operatorname{adj}(S_n)S_{n,2}$.

Now we take (33) to n and to $n+1$ and use the recurrence relations for ψ_n and η_n to eliminate η_n, thus obtaining

$$U_n\psi'_{n+1} = V_n\psi_{n+1}$$

with

$$\begin{aligned} U_n &= T_{n+1}\mathcal{B}_{n+1}T_{n,1} - T_nT_{n+1,1}\mathcal{A}_{n+2}\,, \\ V_n &= T_n(T_{n+1,1}\mathcal{A}'_{n+2} + T_{n+1,2}\mathcal{A}_{n+2}) - T_{n+1}\mathcal{B}_{n+1}T_{n,2}\,, \end{aligned}$$

thus

$$\det(U_n)\psi'_{n+1} = \operatorname{adj}(U_n)V_n\psi_{n+1}\,.$$

Taking into account the Lemma 3 we conclude that there exist $\phi \in \mathbb{P}$ and a matrix $\mathcal{L}_n$ such that

$$\phi\,\psi'_n = \mathcal{L}_n\psi_n\,, \quad n \geq 1\,, \tag{34}$$

and, consequently, the semiclassical character of μ_0 follows. Furthermore, taking into account (34), the measure μ_1 is a rational modification of μ_0, hence semiclassical (see [4], where a representation for the measures is given). ■

4. A characterization for semiclassical orthogonal polynomials

Our goal is to establish the following theorem.

Theorem 1. *Let $\{P_n\}$ be a SMOP with respect to a linear functional u. Let $\{\psi_n\}$ be the corresponding sequence given by (6), and let $P_n^{[1]} = \frac{P'_{n+1}}{n+1}$, $n \geq 0$. The following statements are equivalent:*

a) There exist a non-negative integer s and sequences $(a_{n,k})$ and $(b_{n,k})$ such that $\{P_n\}$ satisfies the structure relation

$$\sum_{k=0}^{s} b_{n+s,k}P_{n+s-k} = \sum_{k=0}^{s+2} a_{n+s,k}P^{[1]}_{n+s-k}\,, \quad b_{n+s,0} = a_{n+s,0} = 1\,, \quad \forall n \geq 1\,. \tag{35}$$

b) $\{P_n\}$ satisfies

$$\phi\,\psi'_n = \mathcal{L}_n\psi_n\,, \quad \forall n \geq 1\,.$$

where ϕ is a polynomial with degree less or equal than $s+2$, and $\mathcal{L}_n$ is a 2×2 matrix whose entries are polynomials with degrees less or equal than $s+1$.

c) $\exists \phi, \varphi \in \mathbb{P}$, $\deg(\phi) \leq s+2, 1 \leq \deg(\varphi) \leq s+1$, such that $\mathcal{D}(\phi\,u) = \varphi\,u$. Furthermore, if $b_{n+s,s}\,a_{n+s-1,s+2} \neq 0$, then $\deg(\phi) = s+2\,, \deg(\varphi) = s+1$.

The proof of the previous theorem will use the lemmas that follow.

Lemma 6. *Let $\{P_n\}$ be a SMOP with respect to a linear functional u. Let $\{P_n\}$ satisfy (35) for some $s \geq 0$,*

$$\sum_{k=0}^{s} b_{n+s,k} P_{n+s-k} = \sum_{k=0}^{s+2} a_{n+s,k} P_{n+s-k}^{[1]}, \ \forall n \geq 1\,.$$

Then, for each λ in $\{0, \ldots, s+2\}$ (when s is even) or λ in $\{0, \ldots, s+3\}$ (when s is odd), there exist a polynomial, $\tilde{\phi}_n(\cdot, \lambda)$, and a matrix of order two with polynomial entries, $\tilde{\mathcal{L}}_{n+s-\lambda}$, such that

$$\tilde{\phi}_n(x,\lambda)\psi'_{n+s-\lambda}(x) = \tilde{\mathcal{L}}_{n+s-\lambda}(x)\psi_{n+s-\lambda}(x)\,, \ \ n \geq 1\,. \tag{36}$$

where $\deg(\tilde{\phi}_n) \leq 2s+2, \deg(\tilde{\mathcal{L}}^{(i,j)}_{n+s-\lambda}) \leq 2s+1, \ i,j = 1,2,$ *when s is even, and* $\deg(\tilde{\phi}_n) \leq 2s+3, \deg(\tilde{\mathcal{L}}^{(i,j)}_{n+s-\lambda}) \leq 2s+2, \ i,j = 1,2,$ *when s is odd. Moreover, if $a_{n+s-1,s+2} \neq 0, \ n \in \mathbb{N}$, then the following holds: when $\lambda = \frac{s+2}{2}$, s even, or $\lambda = \frac{s+3}{2}$, s odd,* $\deg(\tilde{\phi}_n)$ *is exactly $s+2$.*

Proof: **Step 1.** We deduce nonsingular matrices of order two, $\mathcal{M}_{n+s-\lambda}$ and $\mathcal{N}_{n+s-\lambda}$, with polynomial entries whose degree is uniformly bounded by a number independent of n, such that

$$\mathcal{M}_{n+s-\lambda}\psi'_{n+s-\lambda} = \mathcal{N}_{n+s-\lambda}\psi_{n+s-\lambda}\,. \tag{37}$$

<u>Case 1: s is even</u>. If we write (35) to n and to $n-1$ in the matrix form we get, using the notation introduced in (6),

$$\sum_{k=0}^{s/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} = \sum_{k=-1}^{s/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2}\,, \tag{38}$$

where

$$\mathcal{B}_{n,s} = \begin{bmatrix} 1 & b_{n+s,1} \\ 0 & 1 \end{bmatrix}, \ \ \mathcal{B}_{n,s-2k} = \begin{bmatrix} b_{n+s,2k} & b_{n+s,2k+1} \\ b_{n+s-1,2k-1} & b_{n+s-1,2k} \end{bmatrix}, \ k = 1, \ldots, s/2-1\,,$$

$$\mathcal{B}_{n,0} = \begin{bmatrix} b_{n+s,s} & 0 \\ b_{n+s-1,s-1} & b_{n+s-1,s} \end{bmatrix}, \ \ \mathcal{C}_{n,s+1} = \begin{bmatrix} \frac{1}{n+s+1} & \frac{a_{n+s,1}}{n+s} \\ 0 & \frac{1}{n+s} \end{bmatrix},$$

$$\mathcal{C}_{n,s-2k-1} = \begin{bmatrix} \frac{a_{n+s,2k+2}}{n+s-2k-1} & \frac{a_{n+s,2k+3}}{n+s-2k-2} \\ \frac{a_{n+s-1,2k+1}}{n+s-2k-1} & \frac{a_{n+s-1,2k+2}}{n+s-2k-2} \end{bmatrix}, \ k = 0, \ldots, s/2-1\,,$$

$$\mathcal{C}_{n,-1} = \begin{bmatrix} \frac{a_{n+s,s+2}}{n-1} & 0 \\ \frac{a_{n+s-1,s+1}}{n-1} & \frac{a_{n+s-1,s+2}}{n-2} \end{bmatrix}.$$

Our next step is to deduce $\mathcal{M}_{n+s-\lambda}\psi'_{n+s-\lambda} = \mathcal{N}_{n+s-\lambda}\psi_{n+s-\lambda}$, where $\mathcal{M}_{n+s-\lambda}$ and $\mathcal{N}_{n+s-\lambda}$ are matrices, and λ is some fixed integer, $0 \leq \lambda \leq s+2$. To this end we shall write the vectors ψ_{n+s-l} , $l = 0, \ldots, s+2$, appearing in (38), as multiples of $\psi_{n+s-\lambda}$. We consider the two sub-cases that follow.
Sub-case 1.1: s and λ are even. The Eq. (38) can be split as follows

$$\sum_{\substack{k=0 \\ n+s-2k-1>n+s-\lambda}}^{s/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} + \sum_{\substack{k=0 \\ n+s-2k-1<n+s-\lambda}}^{s/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1}$$
$$= \sum_{\substack{k=-1 \\ n+s-2k-2>n+s-\lambda}}^{s/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2} + \mathcal{C}_{n,s-\lambda+1}\psi'_{n+s-\lambda}$$
$$+ \sum_{\substack{k=-1 \\ n+s-2k-2<n+s-\lambda}}^{s/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2}\,,$$

thus we have

$$\sum_{k=0}^{\lambda/2-1} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} + \sum_{k=\lambda/2}^{s/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1}$$
$$= \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2} + \mathcal{C}_{n,s-\lambda+1}\psi'_{n+s-\lambda} + \sum_{k=\lambda/2}^{s/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2}\,,$$

that is,

$$\sum_{k=0}^{\lambda/2-1} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} + \sum_{j=0}^{(s-\lambda)/2} \mathcal{B}_{n,s-\lambda-2j}\,\psi_{n+s-\lambda-2j-1} = \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2}$$
$$+ \mathcal{C}_{n,s-\lambda+1}\psi'_{n+s-\lambda} + \sum_{j=0}^{(s-\lambda)/2} \mathcal{C}_{n,s-\lambda-2j-1}\psi'_{n+s-\lambda-2j-2}\,. \quad (39)$$

The use of (8) with $m = 2k+1, m = 2k+2$ and (9) with $M = 2j+1, M = 2j+2$ in (39) yields (37), $\mathcal{M}_{n+s-\lambda}\,\psi'_{n+s-\lambda} = \mathcal{N}_{n+s-\lambda}\,\psi_{n+s-\lambda}$, with

$$\mathcal{M}_{n+s-\lambda} = \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1} \prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l} \\ + \mathcal{C}_{n,s-\lambda+1} + \sum_{j=0}^{(s-\lambda)/2} \mathcal{C}_{n,s-\lambda-2j-1}\Big(\prod_{l=0}^{2j+1} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1}, \quad (40)$$

$$\mathcal{N}_{n+s-\lambda} = \sum_{k=0}^{\lambda/2-1} \mathcal{B}_{n,s-2k} \prod_{l=0}^{\lambda-2-2k} \mathcal{A}_{n+s-2k-1-l} + \sum_{j=0}^{(s-\lambda)/2} \mathcal{B}_{n,s-\lambda-2j}\Big(\prod_{l=0}^{2j} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1} \\ - \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1}\Big(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l}\Big)' - \sum_{j=0}^{(s-\lambda)/2} \mathcal{C}_{n,s-\lambda-2j-1}\Big\{\Big(\prod_{l=0}^{2j+1} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1}\Big\}'.$$

<u>Sub-case 1.2: s is even and λ is odd</u>. The Eq. (38) can be split as follows

$$\sum_{k=0}^{(\lambda-3)/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} + \mathcal{B}_{n,s-\lambda+1}\psi_{n+s-\lambda} + \sum_{j=0}^{(s-\lambda-1)/2} \mathcal{B}_{n,s-\lambda-2j-1}\,\psi_{n+s-\lambda-2j-2} \\ = \sum_{k=-1}^{(\lambda-3)/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2} + \sum_{j=0}^{(s-\lambda+1)/2} \mathcal{C}_{n,s-\lambda-2j}\psi'_{n+s-\lambda-2j-1}\,.$$

From the recurrence relations for ψ_n (cf. Lemma 1) we get (37), with

$$\mathcal{M}_{n+s-\lambda} = \sum_{k=-1}^{(\lambda-3)/2} \mathcal{C}_{n,s-2k-1}\Big(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l}\Big) \\ + \sum_{j=0}^{(s-\lambda+1)/2} \mathcal{C}_{n,s-\lambda-2j}\Big(\prod_{l=0}^{2j} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1}, \quad (41)$$

$$\mathcal{N}_{n+s-\lambda} = \sum_{k=0}^{(\lambda-3)/2} \mathcal{B}_{n,s-2k}(\prod_{l=0}^{\lambda-2-2k} \mathcal{A}_{n+s-2k-1-l}) + \mathcal{B}_{n,s-\lambda+1}$$
$$+ \sum_{j=0}^{(s-\lambda-1)/2} \mathcal{B}_{n,s-\lambda-2j-1}(\prod_{l=0}^{2j+1} \mathcal{A}_{n+s-\lambda-l})^{-1} - \sum_{k=-1}^{(\lambda-3)/2} \mathcal{C}_{n,s-2k-1}(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l})'$$
$$- \sum_{j=0}^{(s-\lambda+1)/2} \mathcal{C}_{n,s-\lambda-2j}\{(\prod_{l=0}^{2j} \mathcal{A}_{n+s-\lambda-l})^{-1}\}'.$$

<u>Case 2: s is odd</u>. If we write (35) to n and to $n-1$ in the matrix form we get, using the notation introduced in (6),

$$\sum_{k=0}^{(s+1)/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} = \sum_{k=-1}^{(s+1)/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2}\,, \tag{42}$$

where

$$\mathcal{B}_{n,s} = \begin{bmatrix} 1 & b_{n+s,1} \\ 0 & 1 \end{bmatrix}, \quad \mathcal{B}_{n,s-2k} = \begin{bmatrix} b_{n+s,2k} & b_{n+s,2k+1} \\ b_{n+s-1,2k-1} & b_{n+s-1,2k} \end{bmatrix}, \; k = 1, \ldots, \frac{s-1}{2},$$
$$\mathcal{B}_{n,-1} = \begin{bmatrix} 0 & 0 \\ b_{n+s-1,s} & 0 \end{bmatrix}, \quad \mathcal{C}_{n,s+1} = \begin{bmatrix} \frac{1}{n+s+1} & \frac{a_{n+s,1}}{n+s} \\ 0 & \frac{1}{n+s} \end{bmatrix},$$
$$\mathcal{C}_{n,s-2k-1} = \begin{bmatrix} \frac{a_{n+s,2k+2}}{n+s-2k-1} & \frac{a_{n+s,2k+3}}{n+s-2k-2} \\ \frac{a_{n+s-1,2k+1}}{n+s-2k-1} & \frac{a_{n+s-1,2k+2}}{n+s-2k-2} \end{bmatrix}, \; k = 0, \ldots, \frac{s-1}{2},$$
$$\mathcal{C}_{n,-2} = \begin{bmatrix} 0 & 0 \\ \frac{a_{n+s-1,s+2}}{n-2} & 0 \end{bmatrix}.$$

<u>Sub-case 2.1: s is odd and λ is even</u>. The Eq. (42) can be split as follows

$$\sum_{k=0}^{\lambda/2-1} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} + \sum_{j=0}^{(s+1-\lambda)/2} \mathcal{B}_{n,s-\lambda-2j}\,\psi_{n+s-\lambda-2j-1} = \mathcal{C}_{n,s-\lambda+1}\psi'_{n+s-\lambda}$$
$$+ \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2} + \sum_{j=0}^{(s+1-\lambda)/2} \mathcal{C}_{n,s-\lambda-2j-1}\psi'_{n+s-\lambda-2j-2}\,.$$

From the recurrence relation for ψ_n (cf. Lemma 1) we get (37), with

$$\mathcal{M}_{n+s-\lambda} = \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1}\Big(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l}\Big) + \mathcal{C}_{n,s-\lambda+1} + \sum_{j=0}^{(s+1-\lambda)/2} \mathcal{C}_{n,s-\lambda-2j-1}\Big(\prod_{l=0}^{2j+1} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1}, \quad (43)$$

$$\begin{aligned}\mathcal{N}_{n+s-\lambda} &= \sum_{k=0}^{\lambda/2-1} \mathcal{B}_{n,s-2k} \prod_{l=0}^{\lambda-2-2k} \mathcal{A}_{n+s-2k-1-l} \\ &+ \sum_{j=0}^{(s+1-\lambda)/2} \mathcal{B}_{n,s-\lambda-2j}\Big(\prod_{l=0}^{2j} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1} - \sum_{k=-1}^{\lambda/2-2} \mathcal{C}_{n,s-2k-1}\Big(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l}\Big)' \\ &- \sum_{j=0}^{(s+1-\lambda)/2} \mathcal{C}_{n,s-\lambda-2j-1}\Big\{\Big(\prod_{l=0}^{2j+1} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1}\Big\}'.\end{aligned}$$

Sub-case 2.2: s and λ are odd. The Eq. (42) can be split as follows

$$\begin{aligned}\sum_{k=0}^{(\lambda-3)/2} \mathcal{B}_{n,s-2k}\psi_{n+s-2k-1} + \mathcal{B}_{n,s-\lambda+1}\psi_{n+s-\lambda} + \sum_{j=0}^{(s-\lambda)/2} \mathcal{B}_{n,s-\lambda-2j-1}\,\psi_{n+s-\lambda-2j-2} \\ = \sum_{k=-1}^{(\lambda-3)/2} \mathcal{C}_{n,s-2k-1}\psi'_{n+s-2k-2} + \sum_{j=0}^{(s-\lambda+2)/2} \mathcal{C}_{n,s-\lambda-2j}\psi'_{n+s-\lambda-2j-1}\,.\end{aligned}$$

From the recurrence relation for ψ_n (cf. Lemma 1) we get (37), with

$$\mathcal{M}_{n+s-\lambda} = \sum_{k=-1}^{(\lambda-3)/2} \mathcal{C}_{n,s-2k-1}\Big(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l}\Big) + \sum_{j=0}^{(s-\lambda+2)/2} \mathcal{C}_{n,s-\lambda-2j}\Big(\prod_{l=0}^{2j} \mathcal{A}_{n+s-\lambda-l}\Big)^{-1}, \quad (44)$$

$$\mathcal{N}_{n+s-\lambda} = \sum_{k=0}^{(\lambda-3)/2} \mathcal{B}_{n,s-2k}(\prod_{l=0}^{\lambda-2-2k} \mathcal{A}_{n+s-2k-1-l}) + \mathcal{B}_{n,s-\lambda+1}$$
$$+ \sum_{j=0}^{(s-\lambda)/2} \mathcal{B}_{n,s-\lambda-2j-1}(\prod_{l=0}^{2j+1} \mathcal{A}_{n+s-\lambda-l})^{-1} - \sum_{k=-1}^{(\lambda-3)/2} \mathcal{C}_{n,s-2k-1}(\prod_{l=0}^{\lambda-3-2k} \mathcal{A}_{n+s-2k-2-l})'$$
$$- \sum_{j=0}^{(s-\lambda+2)/2} \mathcal{C}_{n,s-\lambda-2j}\{(\prod_{l=0}^{2j} \mathcal{A}_{n+s-\lambda-l})^{-1}\}'.$$

Step 2. We obtain (36).

From Step 1 we have (37), $\mathcal{M}_{n+s-\lambda}\psi'_{n+s-\lambda} = \mathcal{N}_{n+s-\lambda}\psi_{n+s-\lambda}$, where the matrices $\mathcal{M}_{n+s-\lambda}$ and $\mathcal{N}_{n+s-\lambda}$ are nonsingular. If we multiply the previous equation by the adjoint of $\mathcal{M}_{n+s-\lambda}$ then we get

$$\det(\mathcal{M}_{n+s-\lambda})\,\psi'_{n+s-\lambda} = \operatorname{adj}(\mathcal{M}_{n+s-\lambda})\,\mathcal{N}_{n+s-\lambda}\psi_{n+s-\lambda}\,,$$

and we obtain (36) with $\tilde{\phi}_n = \det(\mathcal{M}_{n+s-\lambda})$, $\tilde{\mathcal{L}}_{n+s-\lambda} = \operatorname{adj}(\mathcal{M}_{n+s-\lambda})\,\mathcal{N}_{n+s-\lambda}$.

Furthermore, an inspection of Eqs. (40), (41), (43), and (44) shows the assertion concerning the degrees of $\det(\mathcal{M}_{n+s-\lambda})$. When $\lambda = \frac{s+2}{2}$, s even, or $\lambda = \frac{s+3}{2}$, s odd, then $\det(\mathcal{M}_{n+s-\lambda})$ is exactly $s+2$, since $a_{n+s-1,s+2} \neq 0$. ∎

Now we discuss the structure relations (36) obtained in the preceding lemma.

Lemma 7. *Let $\{P_n\}$ be a SMOP with respect to a linear functional u satisfying (35). Let a structure relation of type (36) hold for two indexes λ_1, λ_2, with $\lambda_1, \lambda_2 \in \{0, \ldots, s+3\}$, that is,*

$$\tilde{\phi}_n(x,\lambda_1)\psi'_{n+s-\lambda_1}(x) = \tilde{\mathcal{L}}_{n+s-\lambda_1}(x)\psi_{n+s-\lambda_1}(x)\,, \tag{45}$$
$$\tilde{\phi}_n(x,\lambda_2)\psi'_{n+s-\lambda_2}(x) = \tilde{\mathcal{L}}_{n+s-\lambda_2}(x)\psi_{n+s-\lambda_2}(x)\,, \tag{46}$$

where, without loss of generality, we assume $\lambda_1 > \lambda_2$. Then, the following holds:

$$\frac{\tilde{\phi}_n(\cdot,\lambda_2)}{\tilde{\phi}_n(\cdot,\lambda_1)} = \frac{\mathcal{H}_n^{(i,j)}}{\mathcal{L}_{n+s-\lambda_1}^{(i,j)}}\,,\ i,j = 1,2\,, \tag{47}$$

with

$$\mathcal{H}_n = \Big(\prod_{l=0}^{\lambda-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big)^{-1}$$
$$\times \Big\{\tilde{\mathcal{L}}_{n+s-\lambda_2}\Big(\prod_{l=0}^{\lambda_1-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big) - \tilde{\phi}_n(\cdot,\lambda_2)\Big(\prod_{l=0}^{\lambda_1-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big)'\Big\}\,, \quad (48)$$

where $\mathcal{A}_n$ are the matrices of the recurrence relation (7).

Proof: Let us assume, without loss of generality, that $\lambda_1 > \lambda_2$. Then, from the recurrence relation for ψ_n (cf. Lemma 1) we have

$$\psi_{n+s-\lambda_2} = \Big(\prod_{l=0}^{\lambda_1-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big)\,\psi_{n+s-\lambda_1}\,.$$

The use of the previous equality in (46) yields

$$\tilde{\phi}_n(\cdot,\lambda_2)\Big(\prod_{l=0}^{\lambda_1-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big)\psi'_{n+s-\lambda_1} = \Big\{\tilde{\mathcal{L}}_{n+s-\lambda_2}\Big(\prod_{l=0}^{\lambda_1-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big)$$
$$- \tilde{\phi}_n(\cdot,\lambda_2)\Big(\prod_{l=0}^{\lambda_1-\lambda_2-1} \mathcal{A}_{n+s-\lambda_2-l}\Big)'\Big\}\psi_{n+s-\lambda_1},$$

thus,

$$\tilde{\phi}_n(\cdot,\lambda_2)\,\psi'_{n+s-\lambda_1} = \mathcal{H}_n\,\psi_{n+s-\lambda_1}\,, \quad (49)$$

where $\mathcal{H}_n$ is given by (48). From (45) and (49), and taking into account the Lemma 2, the relation (47) follows. ∎

In the next lemma we follow the same approach as [6].

Lemma 8. *Let $\{P_n\}$ be a SMOP with respect to a linear functional u. Let u be semiclassical of class s, satisfying*

$$\mathcal{D}(\phi u) = \varphi u\,. \quad (50)$$

Then there exist sequences $(a_{n,s})$, $(b_{n,s})$ such that

$$\sum_{k=0}^{s} b_{n+s,k}P_{n+s-k} = \sum_{k=0}^{s+2} a_{n+s,k}P^{[1]}_{n+s-k}\,,\ \forall n \geq 1\,. \quad (51)$$

Proof: Let us write

$$P_{n+l}=\sum_{j=0}^{n+l}c_{n+l,j}P^{[1]}_{n+l-j}\,,\ l=0,\ldots,s,$$

where the $c_{n+l,j}$ satisfy, for each $l=0,\ldots,s$,

$$0=\sum_{j=n+l+1-(k+s+1)}^{n+l}c_{n+l,j}\,\langle\phi\,u,P^{[1]}_{n+l-j}\,P_k\rangle\,,\ k=0,1,\ldots,n+l-(s+2)-1$$

(one has $n+l-(s+2)$ equations in $n+l-3$ unknowns).

Thus,

$$P_{n+s}+b_{n+s,1}P_{n+s-1}+\ldots+b_{n+s,s}P_n=P^{[1]}_{n+s}+\sum_{j=0}^{n+s-1}\mu_{n+s-1,j}P^{[1]}_{n+s-1-j}\qquad(52)$$

where

$$\mu_{n+s-1,j}=\begin{cases}c_{n+s,j+1}+\sum_{k=1}^{j+1}b_{n+s,k}c_{n+s-k,j+1-k}\,, & j=0,\ldots,s-1\,,\\ c_{n+s,j+1}+\sum_{k=1}^{s}b_{n+s,k}\,c_{n+s-k,j-(k-1)}\,, & j=s,\ldots,n+s-1\,.\end{cases}\qquad(53)$$

Let us multiply (52) by P_k and apply ϕu. Then, the left-hand side gives us

$$\langle\phi\,u,(P_{n+s}+b_{n+s,1}P_{n+s-1}+\cdots+b_{n+s,s}P_n)\,P_k\rangle=0,\ k+s+2<n\,.$$

Furthermore, since

$$\langle\phi\,u,P^{[1]}_{n+s-1-j}\,P_k\rangle=\frac{1}{n+s-j}(\langle\phi\,u,(P_{n+s-j}\,P_k)'\rangle-\langle\phi\,u,P_{n+s-j}\,P'_k\rangle)$$

and $\langle\phi\,u,(P_{n+s-j}\,P_k)'\rangle=-\langle\mathcal{D}(\phi\,u),P_{n+s-j}\,P_k\rangle$, taking into account (50), the right-hand side gives us

$$\langle\phi\,u,P^{[1]}_{n+s-1-j}\,P_k\rangle=\frac{1}{n+s-j}(-\langle\psi\,u,P_{n+s-j}\,P_k\rangle-\langle\phi\,u,P_{n+s-j}\,P'_k\rangle)\,,$$

hence $\langle\phi\,u,P^{[1]}_{n+s-1-j}\,P_k\rangle=0,\ k+s+1<n+s-j$. Therefore, the coefficients $\mu_{n+s-1,j}$ in (52) satisfy

$$0=\sum_{j=n-k-1}^{n+s-1}\mu_{n+s-1,j}\,\xi_{k,j},\ k=0,1,\ldots,n-s-3\,,\qquad(54)$$

where $\xi_{k,j}=(\langle\psi u,P_{n+s-j}\,P_k\rangle+\langle\phi\,u,P_{n+s-j}\,P'_k\rangle)/(n+s-j)$.

Our goal is to prove that there exist $b_{n+s,k}$, $k=1,\ldots,s$, such that only $s+3$ non-zero summands appear in the right-hand side of (52), with $\mu_{n+s-1,s+2} = \mu_{n+s-1,s+3} = \cdots = \mu_{n+s-1,n+s-1} = 0$.

Let us expand (54). We get $\mathcal{F}_{n,s-1}\mathcal{U}_{n,s-1} = 0_{n-2\times 1}$, with

$$\mathcal{F}_{n,s-1} = \begin{bmatrix} \xi_{n-s-3,s+2} & \cdots & \xi_{n-s-3,n-1} & \cdots & \xi_{n-s-3,n+s-1} \\ & \ddots & \vdots & & \vdots \\ & & \xi_{0,n-1} & \cdots & \xi_{0,n+s-1} \end{bmatrix}$$

and $\mathcal{U}_{n,s-1} = \begin{bmatrix} \mu_{n+s-1,s+2} & \cdots & \mu_{n+s-1,n+s-1} \end{bmatrix}^T$ (notice that in the above, and hereafter, the convention that the elements under the diagonal ($\xi_{n-s-3,s+2}$; $\ldots$; $\xi_{0,n-1}$) are zero is adopted). Furthermore, this system can be written as

$$\begin{bmatrix} \xi_{n-s-3,s+2} & \cdots & \xi_{n-s-3,n-1} \\ & \ddots & \vdots \\ & & \xi_{0,n-1} \end{bmatrix} \begin{bmatrix} \mu_{n+s-1,s+2} \\ \vdots \\ \mu_{n+s-1,n-1} \end{bmatrix} + \begin{bmatrix} \xi_{n-s-3,n} & \cdots & \xi_{n-s-3,n+s-1} \\ \vdots & & \vdots \\ \xi_{0,n} & \cdots & \xi_{0,n+s-1} \end{bmatrix} \begin{bmatrix} \mu_{n+s-1,n} \\ \vdots \\ \mu_{n+s-1,n+s-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{55}$$

We remark that

$$\mu_{n+s-1,n} = \mu_{n+s-1,n+1} = \cdots = \mu_{n+s-1,n+s-1} = 0 \tag{56}$$

implies

$$\mu_{n+s-1,s+2} = \mu_{n+s-1,s+3} = \cdots = \mu_{n+s-1,n-1} = 0\,,$$

because if (56) holds then (55) becomes

$$\begin{bmatrix} \xi_{n-s-3,s+2} & \cdots & \xi_{n-s-3,n-1} \\ & \ddots & \vdots \\ & & \xi_{0,n-1} \end{bmatrix} \begin{bmatrix} \mu_{n+s-1,s+2} \\ \vdots \\ \mu_{n+s-1,n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{57}$$

where the matrix of the system (57) is nonsingular (upper triangular), as

$$\xi_{j,n-1-j} = \langle u, (\psi\, P_j + \phi\, P_j')P_{j+s+1}\rangle \neq 0\,,\ j = 0,\ldots,n-s-3\,,$$

because $\deg(\psi P_j + \phi P_j') = j+s+1\,,\ j = 0,\ldots,n-s-3$ (note that this is the admissibility condition).

Let us return to (56). Taking into account (53) one can write (56) as

$$\mathcal{G}_{n,s-1}\,\mathcal{B}_{n,s} = -\begin{bmatrix} c_{n+s,n+1} & \cdots & c_{n+s,n+s} \end{bmatrix}^T, \tag{58}$$

where

$$\mathcal{G}_{n,s-1} = \begin{bmatrix} c_{n+s-1,n} & \cdots & c_{n,n-(s-1)} \\ \vdots & \ddots & \vdots \\ c_{n+s-1,n+s-1} & \cdots & c_{n,n} \end{bmatrix}, \quad \mathcal{B}_{n,s} = \begin{bmatrix} b_{n+s,1} \\ \vdots \\ b_{n+s,s} \end{bmatrix}.$$

Now we discuss the system (58). Let us denote by $\tilde{\mathcal{G}}_{n,s-1}$ the $s \times (s+1)$ matrix given by

$$\left[\begin{array}{c|c} & -c_{n+s,n+1} \\ \mathcal{G}_{n,s-1} & \vdots \\ & -c_{n+s,n+s} \end{array}\right].$$

If $\det(\mathcal{G}_{n,s-1}) \neq 0$, then (58) has a solution, and this means that there exist $b_{n+s,1}, \ldots, b_{n+s,s}$ such that $\mu_{n+s-1,n} = \mu_{n+s-1,n+1} = \cdots = \mu_{n+s-1,n+s-1} = 0$, thus, from our previous discussion, there follows $\mu_{n+s,s+2} = \mu_{n+s,s+3} = \cdots = \mu_{n+s,n-1} = 0$, and (51) holds.

If $\det(\mathcal{G}_{n,s-1}) = 0$, then (58) is possible iff the matrices $\mathcal{G}_{n,s-1}, \tilde{\mathcal{G}}_{n,s-1}$ have precisely the same number of independent rows.

Let us assume, without loss of generality, that the i-th and the j-th rows of $\mathcal{G}_{n,s-1}$ are linearly dependent, that is,

$$\frac{c_{n+s-1,n+i}}{c_{n+s-1,n+j}} = \frac{c_{n+s-2,n+i-1}}{c_{n+s-2,n+j-1}} = \cdots = \frac{c_{n,n+i-(s-1)}}{c_{n,n+j-(s-1)}}. \tag{59}$$

Note that n is arbitrary and the algorithm described above can be carried out to $n+1$, thus the same proportion as above appears to the matrix $\mathcal{G}_{n+1,s-1}$, thus we can take $n+1$ in (59),

$$\frac{c_{n+s,n+i+1}}{c_{n+s,n+j+1}} = \frac{c_{n+s-1,n+i}}{c_{n+s-1,n+j}} = \cdots = \frac{c_{n+1,n+1+i-(s-1)}}{c_{n+1,n+1+j-(s-1)}},$$

and we get

$$\frac{c_{n+s,n+i+1}}{c_{n+s,n+j+1}} = \frac{c_{n+s-1,n+i}}{c_{n+s-1,n+j}} = \cdots = \frac{c_{n,n+i-(s-1)}}{c_{n,n+j-(s-1)}},$$

that is, the i-th and the j-th rows of $\tilde{\mathcal{G}}_{n,s-1}$, are linearly dependent. With a similar reasoning one concludes that $\mathcal{G}_{n,s-1}, \tilde{\mathcal{G}}_{n,s-1}$ have precisely the same number of independent rows. Consequently, (58) is possible and, similarly to the previous discussion in the case $\det(\mathcal{G}_{n,s-1}) \neq 0$, we conclude that (51) holds. ■

Proof of the Theorem 1. Lemma 6 combined with the first part of Lemma 3 proves $a) \Rightarrow b)$; $b) \Rightarrow c)$ is proved in [17, 18]; Lemma 8 proves $c) \Rightarrow a)$.

Corollary 1. *Let $\{P_n\}$ be a SMOP with respect to a linear functional u, let $\{\psi_n\}$ be the corresponding sequence given by (6), and let $P_n^{[1]} = P'_{n+1}/(n+1)$, $n \geq 0$. The following statements are equivalent:*
a) There exist $(a_n), (b_n)$ such that $\{P_n\}$ satisfies

$$P_{n+1} + b_{n+1,1}P_n = P_{n+1}^{[1]} + a_{n+1,1}P_n^{[1]} + a_{n+1,2}P_{n-1}^{[1]} + a_{n+1,3}P_{n-2}^{[1]}\,, \forall n \geq 1\,, \quad (60)$$

b) there exist $\phi \in \mathbb{P}$, $\deg(\phi) \leq 3$, and a matrix $\mathcal{L}_n$ with polynomial entries whose degree is less or equal than two, such that

$$\phi\psi'_{n-1} = \mathcal{L}_{n-1}\psi_{n-1}\,, \quad n \geq 1\,. \tag{61}$$

c) u satisfies

$$\mathcal{D}(\phi\, u) = \varphi\, u\,, \tag{62}$$

where $\deg(\phi) \leq 3, 1 \leq \deg(\varphi) \leq 2$. *Furthermore,*

$$\deg(\phi) = \begin{cases} 3 \textit{ iff } a_{n+1,3} \neq 0 \\ 2 \textit{ iff } a_{n+1,3} = 0,\ a_{n+1,2} \neq 0 \\ 1 \textit{ iff } a_{n+1,3} = a_{n+1,2} = 0,\ a_{n+1,1} \neq 0 \\ 0 \textit{ iff } a_{n+1,3} = a_{n+1,2} = a_{n+1,1} = 0 \end{cases} . \tag{63}$$

Proof: Taking into account the preceding results, we just have to prove $a) \Rightarrow b)$ and $c) \Rightarrow a)$.
$a) \Rightarrow b)$. From Lemma 6 we obtain, for $s = 1$ and $\lambda = 2$ (cf. (43)),

$$\mathcal{M}_{n-1}\psi'_{n-1} = \mathcal{N}_{n-1}\psi_{n-1}\,, \tag{64}$$

where

$$\mathcal{M}_{n-1} = \mathcal{C}_{n,2}\mathcal{A}_{n+1}\mathcal{A}_n + \mathcal{C}_{n,0} + \mathcal{C}_{n,-2}(\mathcal{A}_{n-1}\mathcal{A}_{n-2})^{-1}\,,$$
$$\mathcal{N}_{n-1} = \mathcal{B}_{n,1}\mathcal{A}_n + \mathcal{B}_{n,-1}\mathcal{A}_{n-1}^{-1} - \mathcal{C}_{n,2}(\mathcal{A}_{n+1}\mathcal{A}_n)' - \mathcal{C}_{n,-2}((\mathcal{A}_{n-1}\mathcal{A}_{n-2})^{-1})'\,,$$
$$\mathcal{C}_{n,2} = \begin{bmatrix} \frac{1}{n+2} & \frac{a_{n+1,1}}{n+1} \\ 0 & \frac{1}{n+1} \end{bmatrix},\ \mathcal{C}_{n,0} = \begin{bmatrix} \frac{a_{n+1,2}}{n} & \frac{a_{n+1,3}}{n-1} \\ \frac{a_{n,1}}{n} & \frac{a_{n,2}}{n-1} \end{bmatrix},\ \mathcal{C}_{n,-2} = \begin{bmatrix} 0 & 0 \\ \frac{a_{n,3}}{n-2} & 0 \end{bmatrix},$$
$$\mathcal{B}_{n,1} = \begin{bmatrix} 1 & b_{n+1,1} \\ 0 & 1 \end{bmatrix},\ \mathcal{B}_{n,-1} = \begin{bmatrix} 0 & 0 \\ b_{n,1} & 0 \end{bmatrix}.$$

Therefore, we obtain

$$\phi_{n-1}\psi'_{n-1} = \mathrm{adj}(\mathcal{M}_{n-1})\mathcal{N}_{n-1}\psi_{n-1}\,, \tag{65}$$

where $\phi_{n-1} = \det(\mathcal{M}_{n-1})$, with

$$\begin{aligned}\det(\mathcal{M}_{n-1}) = & \frac{a_{n,3}}{\gamma_{n-1}(n+2)(n-2)}(x-\beta_{n+1})(x-\beta_n)(x-\beta_{n-1}) \\ & + \frac{a_{n,2}}{(n+2)(n-1)}(x-\beta_{n+1})(x-\beta_n) \\ & + e_{n,0}(x-\beta_{n+1}) + \frac{a_{n+1,1}a_{n,2}-a_{n+1,3}}{(n+1)(n-1)}(x-\beta_n) + e_{n,2}(x-\beta_{n-1}) \\ & +(\frac{a_{n+1,2}}{n}-\frac{\gamma_{n+1}}{n+2})(\frac{a_{n,2}}{n-1}-\frac{\gamma_n}{n+1})-(\frac{a_{n+1,3}}{n-1}-\frac{\gamma_n a_{n+1,1}}{n+1})(\frac{a_{n,1}}{n}-\frac{a_{n,3}}{\gamma_{n-1}(n-2)})\,,\end{aligned}$$

where

$$e_{n,0} = \frac{\gamma_n}{n+2}(\frac{a_{n,1}}{n}-\frac{a_{n,3}}{(n-2)\gamma_{n-1}})\,,\ e_{n,2} = \frac{a_{n,3}}{\gamma_{n-1}(n-2)}(\frac{a_{n+1,2}}{n}-\frac{\gamma_{n+1}}{n+2})\,.$$

Therefore we have $\deg(\phi_{n-1}) = \begin{cases} 3 \text{ iff } a_{n,3} \neq 0 \\ 2 \text{ iff } a_{n,3} = 0,\ a_{n,2} \neq 0 \\ 1 \text{ iff } a_{n,3} = a_{n,2} = 0,\ a_{n,1} \neq 0 \\ 0 \text{ iff } a_{n,3} = a_{n,2} = a_{n,1} = 0\,, \end{cases}$ and (63) follows.

Finally, from Lemma 3 we obtain that (65) is equivalent to (61) where the polynomial ϕ is independent of n.

$c) \Rightarrow a)$. Let us write

$$P_{n+1} = \sum_{j=0}^{n+1} c_{n+1,j} P^{[1]}_{n+1-j}\,,\quad P_n = \sum_{j=0}^{n} c_{n,j} P^{[1]}_{n-j}\,.$$

Thus,

$$P_{n+1} + b_{n+1,1}P_n = P^{[1]}_{n+1} + \sum_{j=0}^{n} \mu_{n,j} P^{[1]}_{n-j} \tag{66}$$

where

$$\mu_{n,l} = c_{n+1,l+1} + b_{n+1,1}\,c_{n,l}\,,\ l = 0,\dots,n\,. \tag{67}$$

Let us multiply (66) by P_k and apply $\phi\,u$. Then, the left-hand side gives us $\langle \phi\,u, (P_{n+1}+b_{n+1,1}P_n)\,P_k\rangle = 0,\ k+3<n$. Furthermore, since

$$\langle \phi\,u, P^{[1]}_{n-j}\,P_k\rangle = \frac{1}{n+1-j}(\langle \phi\,u, (P_{n+1-j}\,P_k)'\rangle - \langle \phi\,u, P_{n+1-j}\,P_k'\rangle)$$

and $\langle \phi\, u, (P_{n+s-j}\, P_k)' \rangle = -\langle \mathcal{D}(\phi\, u), P_{n+s-j}\, P_k \rangle$, taking into account (62), the right-hand side gives us

$$\langle \phi\, u, P^{[1]}_{n-j}\, P_k \rangle = \frac{1}{n+1-j}(-\langle \psi\, u, P_{n+1-j}\, P_k \rangle - \langle \phi\, u, P_{n+1-j}\, P'_k \rangle)\,,$$

hence $\langle \phi\, u, P^{[1]}_{n-j}\, P_k \rangle = 0,\ k+2 < n+1-j$. Therefore, the coefficients $\mu_{n,j}$ satisfy

$$0 = \sum_{j=n-k-1}^{n} \mu_{n,j}\, \xi_{k,j},\ k = 0, 1, \ldots, n-4\,, \tag{68}$$

where $\xi_{k,j} = (\langle \psi\, u, P_{n+1-j}\, P_k \rangle + \langle \phi\, u, P_{n+1-j}\, P'_k \rangle)/(n+1-j)$.

Our goal is to prove that there exists $b_{n+1,1}$ such that only 4 non-zero summands appear in (66), with $\mu_{n,3} = \mu_{n,4} = \cdots = \mu_{n,n} = 0$.

Let us expand (68). We get $\mathcal{F}_{n,0}\,\mathcal{U}_{n,0} = 0_{n-2\times 1}$, where

$$\mathcal{F}_{n,0} = \begin{bmatrix} \xi_{n-4,3} & \cdots & \xi_{n-4,n-1} & \xi_{n-4,n} \\ & \ddots & \vdots & \vdots \\ & & \xi_{0,n-1} & \xi_{0,n} \end{bmatrix},\ \mathcal{U}_{n,0} = \begin{bmatrix} \mu_{n,3} \\ \vdots \\ \mu_{n,n} \end{bmatrix}.$$

Furthermore, this system can be written as

$$\begin{bmatrix} \xi_{n-4,3} & \cdots & \xi_{n-4,n-1} \\ & \ddots & \vdots \\ & & \xi_{0,n-1} \end{bmatrix} \begin{bmatrix} \mu_{n,3} \\ \vdots \\ \mu_{n,n-1} \end{bmatrix} + \begin{bmatrix} \xi_{n-4,n} \\ \vdots \\ \xi_{0,n} \end{bmatrix} \begin{bmatrix} \mu_{n,n} \end{bmatrix} = 0\,. \tag{69}$$

Note that $\mu_{n,n} = 0$ implies $\mu_{n,3} = \mu_{n,4} = \cdots = \mu_{n,n-1} = 0$.

Taking into account Eqs. (67), we get $\mu_{n,n} = 0$ if, and only if, $b_{n+1,1} = -c_{n+1,n+1}/c_{n,n}$. Hence, if we choose $b_{n+1,1} = -c_{n+1,n+1}/c_{n,n}$, from our previous discussion there follows that the coefficients μ_k's in (66) satisfy $\mu_{n,3} = \mu_{n,4} = \cdots = \mu_{n,n-1} = 0$, thus (60) holds. ∎

Example 1. *Let $\{P_n\}$ be the SMOP with respect to the linear functional u that satisfies $\mathcal{D}(u) = \varphi\, u,\ \varphi(x) = -ix^2 + 2x - i(\alpha - 1)$, where $\alpha \notin \bigcup_{n\geq 0} E_n$, where $E_0 = \{\alpha \in \mathbb{C} : F(\alpha) = 0\}$, $F(\alpha) = \int_{\infty}^{+\infty} e^{ix^3/3 - x^2 + i(\alpha-1)} dx$, and, for $n \geq 1$, $E_n = \{\alpha \in \mathbb{C} : H_n(\alpha) = 0\}$, with H_n the Hankel determinant associated with u (these are the quasi-definiteness conditions) [19]. The functional u is semi-classical of class $s = 1$. From corollary 1 we obtain that the corresponding SMOP $\{P_n\}$ is characterized by:*
a) The structure relation

$$P_{n+1} + b_{n+1,1} P_n = P^{[1]}_{n+1}\,,\quad n \geq 1\,; \tag{70}$$

b) The equations $\phi\,\psi'_{n-1} = \mathcal{L}_{n-1}\psi_{n-1}$*, where* $\phi \equiv 1$ *and*

$$\begin{aligned}
\mathcal{L}_{n-1}^{(1,1)} &= -n\lambda_{n-1}/\gamma_{n-1}\,,\\
\mathcal{L}_{n-1}^{(1,2)} &= n(\gamma_{n-1} + \lambda_{n-1}(x-\beta_{n-1}))/\gamma_{n-1}\,,\\
\mathcal{L}_{n-1}^{(2,1)} &= -(n-1)/\gamma_{n-1} - (n-1)\lambda_{n-2}(x-\beta_{n-2})/(\gamma_{n-1}\gamma_{n-2})\,,\\
\mathcal{L}_{n-1}^{(2,2)} &= \frac{(n-1)(x-\beta_{n-1})}{\gamma_{n-1}} + \frac{(n-1)\lambda_{n-2}((x-\beta_{n-1})(x-\beta_{n-2})-\gamma_{n-1})}{\gamma_{n-1}\gamma_{n-2}}\,,
\end{aligned}$$

where $\lambda_n = -i\gamma_n\gamma_{n+1}/(n+1)$*,* $n \geq 0$*.*

Notice that the structure relation (70) leads to the structure relation for $\{P_n\}$ *studied in* [19]*,*

$$(x+\vartheta_{n,0})P_n(x) = P_{n+1}^{[1]}(x) + \varrho_n P_n^{[1]}(x)\,, \quad n \geq 1\,, \tag{71}$$

where

$$\vartheta_{n,0} = -i\gamma_{n+1}\gamma_{n+2}/(n+2) + i(n+1)/\gamma_{n+1} - \beta_n\,, \quad \varrho_n = i(n+1)/\gamma_{n+1}\,. \tag{72}$$

To that end we write (70) to n *and to* $n-1$*,*

$$P_{n+1} + b_{n+1,1}P_n = P_{n+1}^{[1]}, \quad P_n + b_{n,1}P_{n-1} = P_n^{[1]}\,.$$

If we multiply the second equation by s_n*, with* $s_n = \gamma_n/b_{n,1}$*, and add to the first one we get* $P_{n+1}+(b_{n+1,1}+s_n)P_n+s_n b_{n,1}P_{n-1} = P_{n+1}^{[1]}+s_nP_n^{[1]}$*, and the use of the recurrence relation for* P_{n+1} *yields (71) with* $\vartheta_{n,0} = b_{n+1,1}+s_n-\beta_n\,,\ \ \varrho_n = s_n\,.$ *Furthermore, if* $b_{n+1,1} = -i\gamma_{n+1}\gamma_{n+2}/(n+2)$*, then we obtain the* $\vartheta_{n,0}$ *and* ϱ_n *given by (72).*

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