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A NEW CHARACTERISATION OF GOURSAT CATEGORIES

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Abstract: We present a new characterisation of Goursat categories in terms of special kind of pushouts, that we call *Goursat pushouts*. This allows one to prove that, for a regular category, the Goursat property is actually equivalent to the validity of the denormalised 3-by-3 Lemma. Goursat pushouts are also useful to clarify, from a categorical perspective, the existence of the quaternary operations characterising 3-permutable varieties.

Keywords: Goursat category, 3-by-3 Lemma, pushouts, Mal'tsev condition.
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Introduction

Over the past years, various authors have investigated the property of n-permutability of congruences in a variety of universal algebras from a categorical perspective (see [5, 9, 12, 13], and references therein). When $\mathbb{C}$ is a regular category, the 2-permutability property, well known as the Mal'tsev property [6], is a concept giving rise to a beautiful theory, whose main features are collected in [2]. Many important results still hold in the case a regular category $\mathbb{C}$ satisfies the strictly weaker 3-permutability property, also known as the Goursat property: in this case, $\mathbb{C}$ is called a *Goursat category*. Any Goursat category $\mathbb{C}$ is such that the semi-lattice of equivalence relations on any object in $\mathbb{C}$ is a modular lattice [5], a fact which is no longer true for the weaker 4-permutability property, and that has some important consequences in categorical Galois theory [11, 15]. Other aspects of the theory of Goursat categories which have been recently considered are: the validity of the denormalised 3-by-3 Lemma [13], the regularity of the category of internal groupoids [8], an explicit description of the closure operator associated with any Birkhoff subcategory of an exact Goursat category [3].

The aim of this work is to investigate some new aspects of the theory of Goursat categories, by providing a new and elementary characterisation of

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these categories, and then use it to show that a regular category satisfies the denormalised 3-by-3 Lemma exactly when it is a Goursat category.

In order to explain this characterisation, consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, two heads, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[r, two heads, "k"'] \arrow[u, shift right, "s"'] & D \arrow[u, shift right, "t"'] \end{tikzcd} \tag{1}$$

of vertical split epimorphisms and horizontal regular epimorphisms, which is necessarily a pushout. We call such diagram a *Goursat pushout* when the regular image of the kernel pair $R[f]$ of f along h is the kernel pair $R[g]$ of g, i.e. when the comparison morphism $\bar{h} : R[f] \twoheadrightarrow R[g]$ is a regular epimorphism. In Section 2 we prove that a regular category is Goursat if and only if any diagram (1) is a Goursat pushout (Theorem 2.3), therefore justifying this choice of designation.

In a non-pointed context, the classical 3-by-3 Lemma has a denormalised version where short exact sequences are replaced by exact forks of the form

$$R[f] \underset{p_2}{\overset{p_1}{\rightrightarrows}} A \overset{f}{\twoheadrightarrow} B,$$

where $(p_1, p_2) : R[f] \rightarrowtail A \times A$ is the kernel pair of its coequalizer f. It was shown by Lack in [13] that the denormalised 3-by-3 Lemma holds for any regular Goursat category. Thanks to the new characterisation in terms of pushouts (1), we prove in Section 3 that not only can this lemma be stated in terms of weaker equivalent conditions, but also that it characterises regular Goursat categories (Proposition 3.2).

Finally, in the last section, we turn our attention to Goursat algebraic varieties. We use a specific Goursat pushout in the category of free algebras to prove the existence of two quaternary operations p and q satisfying the identities $p(x, y, y, z) = x$, $q(x, y, y, z) = z$ and $p(x, x, y, y) = q(x, x, y, y)$, which characterise such varieties.

1. Preliminaries

In this article $\mathbb{C}$ will always be a finitely complete *regular* category: this means that regular epimorphisms are stable under pulling back, and that any effective equivalence relation has a quotient. This implies that any arrow $f\colon A \to B$ has a factorisation $f = i \cdot r$, where r is a regular epimorphism and i

is a monomorphism. It follows that the corresponding (regular epimorphism-monomorphism) factorisation system is stable under pulling back.

If R is a relation from A to B, namely a subobject $(r_1, r_2) : R \rightarrowtail A \times B$, its opposite relation, denoted R°, is a relation from B to A, the subobject $(r_2, r_1) : R \rightarrowtail B \times A$. We shall identify a morphism $f : A \to B$ with the relation $(1_A, f) : A \rightarrowtail A \times B$ and write f° for the opposite relation. Given another relation S from B to C, let SR be the relation from A to C which is the composite, as relations, of R and S. This notation allows one to write any relation R as $R = r_2 r_1^\circ$. The following properties are well known, and easy to prove. We collect them in the following lemma for future references:

Lemma 1.1. *Let $f : A \to B$ be an arrow in a regular category $\mathbb{C}$, and let $i \cdot r$ be its (regular epimorphism, monomorphism) factorisation. Then:*

1. *$f^\circ f$ is the kernel pair of f, thus $1_A \leq f^\circ f$, and $1_A = f^\circ f$ if and only if f is a monomorphism;*
2. *ff° is (i, i), thus $ff^\circ \leq 1_A$, and $ff^\circ = 1_A$ if and only if f is a regular epimorphism;*
3. *$ff^\circ f = f$ and $f^\circ f f^\circ = f^\circ$.*

Recall that a relation in $\mathbb{C}$ on an object A is *reflexive* if $1_A \leq R$, *symmetric* if $R^\circ = R$ and *transitive* when $RR = R$. As usual, a relation R on A is an *equivalence relation* when it is reflexive, symmetric and transitive. Any kernel pair of an arrow f is an equivalence relation, denoted by $(p_1, p_2) : R[f] \rightarrowtail A \times A$. When f is a regular epimorphism, then f is the coequalizer of (p_1, p_2) and the diagram

$$\begin{tikzcd} R[f] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, two heads, "f"] & B \end{tikzcd}$$

is called *exact*.

The standard reference for Goursat categories is the article by Carboni, Kelly and Pedicchio [5], where the notion of Goursat category was introduced:

Definition 1.2. A regular category $\mathbb{C}$ is called a *Goursat category* when the equivalence relations in $\mathbb{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.

We denote by $\mathrm{Equiv}(A)$ the poset of equivalence relations on an object A. By Theorem 6.8 in [5] a regular category $\mathbb{C}$ is a Goursat category if and only if for any regular epimorphism $r : A \twoheadrightarrow B$ and $S \in \mathrm{Equiv}(A)$, the image $r(S) = rSr^\circ$ of S along r is also an equivalence relation.

2. Goursat pushouts

In this section we introduce the definition of a *Goursat pushout*. It is a special kind of pushout that will be useful to obtain a new characterisation for Goursat categories.

Definition 2.1. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[u, shift right, "s"'] \arrow[r, two heads, "k"'] & D \arrow[u, shift right, "t"'] \end{tikzcd} \tag{2}$$

of vertical split epimorphisms f and g ($k \cdot f = g \cdot h$ and $t \cdot k = h \cdot s$) where h and k are regular epimorphisms. The square (2), which is always a pushout, will be called a *Goursat pushout* when the comparison arrow $\bar{h} : R[f] \twoheadrightarrow R[g]$ is a regular epimorphism.

Note that, the factorization $\bar{h}$ is a regular epimorphism if and only if the image of $R[f]$ by h is $R[g]$, i.e. $h(R[f]) = R[g]$.

Remark 2.2. It was shown by Bourn that, whenever $\mathbb{C}$ is a regular Mal'tsev category, any diagram (2) is a *regular pushout* [4]: this means precisely that the canonical factorisation $A \to B \times_D C$ induced by the universal property of the pullback $B \times_D C$ of k and g is a regular epimorphism. It is easy to prove that any regular pushout is a Goursat pushout. However, the converse does not hold, in general: this follows from the next theorem and the fact that a regular category $\mathbb{C}$ is Mal'tsev if and only if any pushout (2) in $\mathbb{C}$ is a regular pushout.

Theorem 2.3. *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(a) $\mathbb{C}$ *is a Goursat category;*
(b) *any pushout* (2) *in* $\mathbb{C}$ *is a Goursat pushout.*

Proof: The implication (a) $\Rightarrow$ (b) was already proved in Corollary 1.8 of [8]; we give a direct proof here to make this note self-contained. The induced factorisation $\bar{f} : R[h] \to R[k]$ is necessarily a split epimorphism and, consequently, the image of $R[h]$ by f is $R[k]$. Now, to see that $\bar{h}$ is a regular epimorphism, we use the following equalities:

$$\begin{aligned}
h(R[f]) &= hf^\circ fh^\circ \\
&= hh^\circ hf^\circ fh^\circ hh^\circ && \text{(by Lemma 1.1(3))} \\
&= hf^\circ fh^\circ hf^\circ fh^\circ && \text{(by the Goursat property)} \\
&= hf^\circ k^\circ kfh^\circ && \text{(since } f(R[h]) = R[k]) \\
&= hh^\circ g^\circ ghh^\circ && \text{(because } kf = gh) \\
&= g^\circ g && \text{(since } h \text{ is a regular epi)} \\
&= R[g]
\end{aligned}$$

Conversely, let $r : A \twoheadrightarrow B$ be a regular epimorphism, $S \in \mathrm{Equiv}(A)$ and consider the image $r(S) = T$ of S along r

$$\begin{tikzcd}
S \arrow[r, two heads, "\rho"] \arrow[d, tail, "{(s_1,s_2)}"'] & T \arrow[d, tail, "{(t_1,t_2)}"] \\
A \times A \arrow[r, two heads, "r\times r"'] & B \times B;
\end{tikzcd}$$

we are to prove that T is an equivalence relation on B.

The relation T is obviously reflexive. In order to prove that T is symmetric and transitive, it will suffice to show that there is an arrow t_T making the following diagram commute

$$\begin{tikzcd}
R[t_1] \arrow[r, dotted, "t_T"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
T \arrow[r, "t_2"'] & B.
\end{tikzcd} \tag{3}$$

Of course, since S is symmetric and transitive, we already know that there is an arrow t_S making the following diagram commute

$$\begin{tikzcd}
R[s_1] \arrow[r, "t_S"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
S \arrow[r, "s_2"'] & A.
\end{tikzcd}$$

Now, the following diagram is of the form (2)

$$\begin{tikzcd}
S \arrow[r, two heads, "\rho"] \arrow[d, shift right, "s_1"'] & T \arrow[d, shift right, "t_1"'] \\
A \arrow[r, two heads, "r"'] \arrow[u, shift right, "e_S"'] & B, \arrow[u, shift right, "e_T"']
\end{tikzcd}$$

where e_S and e_T are the arrows giving the reflexivity of S and T, respectively. Our assumption then implies that the factorization $\bar{\rho} : R[s_1] \twoheadrightarrow R[t_1]$ is a regular epimorphism. Since $\bar{\rho}$ is a strong epimorphism and (t_1, t_2) is a monomorphism, we see that the unique factorization t_T making the diagram

$$\begin{tikzcd}
R[s_1] \arrow[r, two heads, "\bar{\rho}"] \arrow[d, "\rho \cdot t_S"'] & R[t_1] \arrow[d, "t_2 \times t_2 \cdot (p_1, p_2)"] \arrow[dl, dotted, "t_T"'] \\
T \arrow[r, tail, "(t_1, t_2)"'] & B \times B
\end{tikzcd}$$

commute is precisely the arrow needed to complete diagram (3). ∎

3. On the 3-by-3 Lemma

In this section we observe that the validity of the (denormalised) 3-by-3 Lemma is actually equivalent to the Goursat property.

As shown by Lack, the denormalised version of the 3-by-3 Lemma holds for any Goursat category (see also [4] for the Mal'tsev case):

Theorem 3.1. (**3-by-3 Lemma**)[13] *Let $\mathbb{C}$ be a Goursat category. Consider a diagram*

$$\begin{tikzcd}
R[\varphi] \arrow[r, shift left, "\bar{h}_1"] \arrow[r, shift right, "\bar{h}_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & R[f] \arrow[r, "\bar{h}"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & R[g] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
R[h] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\varphi"'] & A \arrow[r, two heads, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & B \arrow[r, "k"'] & D
\end{tikzcd} \tag{4}$$

satisfying the natural commutativity conditions where the three columns and middle row are exact. Then the top row is exact if and only if the bottom row is exact.

By looking at diagram (4), we observe that the 3-by-3 Lemma can be divided into two (apparently) independent properties :

C1 : If the top row is exact, then the bottom row is exact.

C2 : If the bottom row is exact, then the top row is exact.

We shall now prove that these two conditions are, in fact, equivalent, and they are also equivalent to the Goursat property. To do so, we use the characterisation of Goursat categories given through Goursat pushouts (Theorem 2.3).

Proposition 3.2. *Let $\mathbb{C}$ be a regular category. Then the following properties are equivalent:*

(a) *$\mathbb{C}$ is a Goursat category;*
(b) *the 3-by-3 Lemma holds in $\mathbb{C}$;*
(c) *condition* **C1** *holds in $\mathbb{C}$;*
(d) *condition* **C2** *holds in $\mathbb{C}$.*

Proof: Implication (a) $\Rightarrow$ (b) holds by Theorem 3.1 and (b) obviously implies (c) and (d). Next, let us prove that (d) $\Rightarrow$ (a). For this, consider diagram (2), take the horizontal kernel pairs and the induced split epimorphism $\bar{f}$. We then complete this diagram by taking the vertical kernel pairs and the induced top row

$$\begin{tikzcd}
R[\bar{f}] \arrow[r, shift left, "\bar{p}_1"] \arrow[r, shift right, "\bar{p}_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & R[f] \arrow[r, "\bar{h}"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & R[g] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
R[h] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\bar{f}"'] & A \arrow[r, two heads, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\
R[k] \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[u, shift right, "\bar{s}"'] & B \arrow[r, "k"'] \arrow[u, shift right, "s"'] & D. \arrow[u, shift right, "t"']
\end{tikzcd}$$

Then condition **C2** implies that the top row is exact. Thus, the factorisation $\bar{h}$ is a regular epimorphism and $\mathbb{C}$ is a Goursat category by Theorem 2.3.

Finally, to prove that (c) $\Rightarrow$ (a) we take the lower part of the previous diagram, consider the image $h(R[f]) = T$ given through a regular epimorphism, say $\rho : R[f] \twoheadrightarrow T$, its kernel pair and the completed diagram

$$\begin{tikzcd}
R[\rho] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\bar{p}_1"'] \arrow[d, shift left, "\bar{p}_2"] & R[f] \arrow[r, two heads, "\rho"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & T \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] \\
R[h] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\bar{f}"'] & A \arrow[r, two heads, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\
R[k] \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[u, shift right, "\bar{s}"'] & B \arrow[r, "k"'] \arrow[u, shift right, "s"'] & D. \arrow[u, shift right, "t"']
\end{tikzcd}$$

The fact that $(t_1, t_2) : T \rightarrowtail C \times C$ is a monomorphism, implies that $(\bar{p}_1, \bar{p}_2) : R[\rho] \rightarrowtail R[h] \times R[h]$ is the kernel pair of the split epimorphism $\bar{f}$ and gives the exactness of the left column. By applying condition **C1**, where the role of columns and rows is exchanged, we conclude that the right column is exact.

Consequently, $T = R[g]$, the factorization $\bar{h} = \rho$ is a regular epimorphism and $\mathbb{C}$ is a Goursat category (again, Theorem 2.3). ■

4. On the characterisation of 3-permutable varieties

It is well known that a variety of universal algebras is n-permutable when its theory $\mathbb{T}$ contains $(n+1)$-ary operations satisfying appropriate identities (see [10], and references therein). The 2-permutability property is characterised by the existence of a ternary operation p such that $p(x,y,y) = x$ and $p(x,x,y) = y$ (see [7] for a conceptual proof of this characterisation). For the strictly weaker 3-permutability the theory $\mathbb{T}$ contains two quaternary operations p and q satisfying the identities $p(x,y,y,z) = x$, $q(x,y,y,z) = z$ and $p(x,x,y,y) = q(x,x,y,y)$. Among the examples of 3-permutable varieties which are not 2-permutable there is the variety of *implication algebras* [14].

Thanks to Theorem 2.3 this characterisation of 3-permutable varieties can now be obtained directly from a particular Goursat pushout lying in the category of free algebras:

Theorem 4.1. *Let $\mathbb{C}$ be a variety of universal algebras. Then the following conditions are equivalent:*

(a) $\mathbb{C}$ *is a 3-permutable variety;*
(b) $\mathbb{T}$ *contains two quaternary operations p and q satisfying the identities $p(x,y,y,z) = x$, $q(x,y,y,z) = z$ and $p(x,x,y,y) = q(x,x,y,y)$.*

Proof: (a) ⇒ (b) Let X be the free algebra on one element. We denote by ∇_i the codiagonal from the i-indexed copower of X to X, and by i_k the k-th injection into a copower. The following diagram

$$\begin{tikzcd}
X+X+X+X \arrow[r, two heads, "1+\nabla_2+1"] \arrow[d, shift right, "\nabla_2+\nabla_2"'] & X+X+X \arrow[d, shift right, "\nabla_3"'] \\
X+X \arrow[r, two heads, "\nabla_2"'] \arrow[u, shift right, "i_2+i_1"'] & X, \arrow[u, shift right, "i_2"']
\end{tikzcd}$$

is of type (2), since the horizontal morphisms $1+\nabla_2+1$ and ∇_2 are regular epimorphisms (they are split epimorphisms, see Remark 4.2). By Theorem 2.3, this is a Goursat pushout, and the induced morphism

$$\overline{1+\nabla_2+1}\colon R[\nabla_2+\nabla_2] \twoheadrightarrow R[\nabla_3]$$

is a surjective homomorphism. The terms $p_1(x,y,z) = x$ and $p_3(x,y,z) = z$ are such that $(p_1, p_3) \in R[\nabla_3]$. By surjectivity of the map $\overline{1+\nabla_2+1}$ there exists a pair $(p, q) \in R[\nabla_2 + \nabla_2]$ of quaternary terms such that

$$\overline{1 + \nabla_2 + 1}(p, q) = (p_1, p_3).$$

As a consequence, the two identities $p(x, y, y, z) = x$, $q(x, y, y, z) = z$ hold. The terms p and q also satisfy the identity $p(x, x, y, y) = q(x, x, y, y)$ since they belong to the kernel pair of $\nabla_2 + \nabla_2$.

(b) $\Rightarrow$ (a) Let R and S be equivalence relations on a same object. To show the 3-permutability condition it suffices to prove that $RSR \leq SRS$. For $(a, b) \in RSR$, there exists elements x and y such that $(a, x) \in R$, $(x, y) \in S$ and $(y, b) \in R$. On one hand, $(a, a), (x, a), (y, b), (b, b) \in R$ implies that $(p(a, x, y, b), p(a, a, b, b)) \in R$ and $(q(a, x, y, b), q(a, a, b, b)) \in R$. Since $p(a, a, b, b) = q(a, a, b, b)$, then $(p(a, x, y, b), q(a, x, y, b)) \in R$. On the other hand, $(a, a), (x, x), (x, y), (b, b) \in S$ implies that $(p(a, x, x, b), p(a, x, y, b)) \in S$ and $(q(a, x, x, b), q(a, x, y, b)) \in S$, i.e. $(a, p(a, x, y, b)), (b, q(a, x, y, b)) \in S$. From $(a, p(a, x, y, b) \in S$, $(p(a, x, y, b), q(a, x, y, b)) \in R$ and $(q(a, x, y, b), b) \in S$ we conclude that $(a, b) \in SRS$. ■

Remark 4.2. The morphism $1+\nabla_2+1$ is split by both $1+i_1+1$ and $1+i_2+1$, and similarly ∇_2 is split by both i_1 and i_2, but the downward diagram does not commute with any choice of horizontal splittings. This means that the factorization $\overline{1 + \nabla_2 + 1}$ is not a split epimorphism, in general.

Remark 4.3. The proof of Theorem 4.1 never uses the "exactness" of the variety $\mathbb{C}$, but only the fact that $\mathbb{C}$ is a (regular) Goursat category. The same arguments then provide a similar characterisation of the quasivarieties of universal algebras which are 3-permutable: $\mathbb{C}$ is a 3-permutable quasivariety if and only if in its theory there are two quaternary operations p and q satisfying the axioms in Theorem 4.1(b).

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