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OPTIMAL CONVERGENCE RATES FOR THE STRONG LAW OF LARGE NUMBERS UNDER ASSOCIATION

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ABSTRACT: We prove a convergence rate for the Strong Law of Large Numbers (SLLN) for associated and bounded variables of order $\frac{\log\log n}{n^{1/2}}$, thus without the logarithmic factors that have appeared in every previous characterizations. The approach combines the classical exponential inequalities with the subsequence method together with a maximal inequality.

KEYWORDS: association, Strong Law of Large Numbers, convergence rates, exponential inequality, maximal inequality.
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1. Introduction

Characterization of convergence rates in the Strong Law of large Numbers (SLLN) for associated variables has been studied thoroughly in recent years, trying to prove optimal or almost optimal rates, that is, rates that are arbitrarily close to the best known rates for independent variables. The first SLLN for associated variables, without any information about convergence rates, seems to have been proved by Newman and Wright [8], Newman [9] and Birkel [2] under suitable decrease rates on the covariance structure. Moment inequalities (see Birkel [1], Shao and Yu [13] and Shao [12]) and exponential inequalities (see Ioannides and Roussas [10], Oliveira [11]) for associated variables eventually were proved, thus enabling the extension of the classical approach to convergence rates for the associated framework. A first convergence rate was proved by Ioannides and Roussas [10], assuming the associated random variables to be bounded with geometrically decreasing covariances. Using the same approach, this result was extended dropping the boundedness assumption first by Oliveira [11] and later improving the convergence rate, by Sung [14] and Xing, Yang and Liu [15] who were able to optimize the proof to reach the rate $\frac{\log^2 n\,(\log\log n)^{1/2}}{n^{1/2}}$. A slightly better rate, still based on exponential inequalities, which implies assuming that the covariances decrease geometrically, was proved by Henriques and Oliveira [5]: $\frac{\log^2 n}{n^{1/2}}$. An

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approach to this same problem based on moment inequalities was first used by Masry [6] to characterize the convergence rate of the kernel density estimator for associated samples. This result applies to polynomially decreasing covariances. Henriques and Oliveira [4] still used exponential inequalities to study this same problem, obtaining similar results with a milder assumption on joint distributions of the sample. Going back to convergence rates for the SLLN, using moment inequalities together with a maximal approach Yang, Su and Yu [16] proved the almost optimal rate $\frac{(\log n)^{1/2}(\log\log n)^{\delta/2}}{n^{1/2}}$, where $\delta > 1$, under an assumption on the covariance that allows for polynomial decrease of the covariances. Being somewhat more precise on the decrease rate of the covariances, but still allowing for polynomial decrease, Henriques and Oliveira [5] proved the slightly better rate $\frac{(\log n)^{1/4+\delta}}{n^{1/2}}$, where $\delta > 0$ (in [5] there is an assumption about conditional moments, but it is not difficult to see that it is not needed for the result). Replacing, in this previous rate $1/4$ by $1/p$, where p is some convenient power of 2 is also possible, using an easy extension of the maximal inequality used by Henriques and Oliveira [5], thus indicating that it should be possible to prove a rate without logarithmic factors.

2. Preliminaries

Given a sequence of random variables X_n, $n \geq 1$, denote $S_n = X_1 + \cdots + X_n$. Recall that the random variables are associated if, for any $m \in \mathbb{N}$ and any two real-valued coordinatewise nondecreasing functions f and g, it holds

$$\operatorname{Cov}\Big(f\left(X_1, \ldots, X_m\right), g\left(X_1, \ldots, X_m\right)\Big) \geq 0,$$

whenever this covariance exists.

In the framework of association, assuming the stationarity of the sequence, it is usual to state conditions on the covariance structure in terms of decrease rate of

$$u(n) = \sum_{j=n+1}^{\infty} \operatorname{Cov}(X_1, X_j).$$

Throughout this paper we will assume that the random variables are uniformly bounded, that is, there exists $c > 0$ such that, with probability 1, $|X_n| \leq c$, for every $n \geq 1$.

We prove an easy but useful upper bound.

Lemma 2.1. *Assume the variables X_n, $n \geq 1$, are associated, bounded (by c), stationary and $u(0) < \infty$. Then $\mathrm{E}(S_n^2) \leq 2n(c^2 + u(0))$.*

Proof: Using the stationarity, it follows easily that

$$\mathrm{E}(S_n^2) = n\mathrm{Var}(X_1) + 2\sum_{j=1}^{n-1}(n-j)\mathrm{Cov}(X_1, X_{j+1}) \leq 2nc^2 + 2nu(0).$$

∎

3. An exponential inequality

We define now the decomposition of the partial sums to be used throughout. Consider a sequence of natural numbers p_n such that, for each $n \geq 1$, $p_n < \frac{n}{2}$ and define r_n as the greatest integer less or equal to $\frac{n}{2p_n}$. Define then, for $n \geq 1$ and $j = 1, \ldots, 2r_n$,

$$Y_{j,n} = \sum_{\ell=(j-1)p_n+1}^{jp_n} \Big(X_\ell - \mathrm{E}(X_\ell)\Big), \quad Z_{n,od} = \sum_{j=1}^{r_n} Y_{2j-1,n}, \quad Z_{n,ev} = \sum_{j=1}^{r_n} Y_{2j,n}. \quad (1)$$

Denote on the sequel $c^* = c^2 + u(0)$, where c is a constant bounding the random variables and assuming, of course that $u(0) < \infty$. We first use Lemma 2.1 to improve Lemma 3.1 in Oliveira [11].

Lemma 3.1. *Assume the variables X_n, $n \geq 1$, are associated, bounded (by c), stationary and $u(0) < \infty$. If there exists $0 < a_n \nearrow 1$ such that $0 < \lambda < \frac{a_n}{2cp_n}$, then*

$$\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right) \leq \exp\left(\frac{nc^*\lambda^2}{1-a_n}\right).$$

The same bound holds for $\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{2j,n}}\right)$.

Proof: Remembering that $|Y_{2j-1,n}| \leq 2cp_n$, it follows, using a Taylor expansion and Lemma 2.1, that, for each $j = 1, \ldots, r_n$,

$$\begin{aligned}
\mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right) &\leq 1 + \mathrm{E}(Y_{2j-1,n}^2)\lambda^2 \sum_{k=2}^{\infty}(2c\lambda p_n)^{k-2} \\
&\leq 1 + 2p_n c^* \lambda^2 \frac{1}{1 - 2c\lambda p_n} \\
&\leq \exp\left(\frac{2p_n c^* \lambda^2}{1-a_n}\right).
\end{aligned}$$

Now recall that we chose the sequences such that $2r_np_n \leq n$. ■

Theorem 3.2. *Assume the variables X_n, $n \geq 1$, are centered, associated, bounded (by c), stationary and $u(0) < \infty$. Let $d_n > 2$, for n large enough, and assume that*

$$\varepsilon_n^2 = \frac{8\alpha c^2\, d_n \operatorname{loglog} n}{n}, \quad \alpha > 1, \tag{2}$$

$$\frac{nd_n \operatorname{loglog} n}{p_n^2} \longrightarrow \infty, \tag{3}$$

$$\frac{p_n^2 \operatorname{loglog} n}{n} \leq \frac{d_n - 1}{4\alpha}, \tag{4}$$

$$\frac{\operatorname{loglog} n}{d_n} \exp\left(\left(\frac{\alpha}{2}\,\frac{n \operatorname{loglog} n}{d_n}\right)^{1/2}\right) u(p_n) \leq C_0\,. \tag{5}$$

Then, for sufficiently large n,

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,od}\right| \geq \varepsilon_n\right) \leq \left(1 + \frac{\alpha}{4c^2}C_0\right)\left(\log n\right)^{-\alpha}. \tag{6}$$

A similar inequality holds for $Z_{n,ev}$.

Proof: Write, using Markov's inequality and some $\lambda > 0$,

$$\begin{aligned}
\mathrm{P}\left(\frac{1}{n}\left|Z_{n,od}\right| \geq \varepsilon_n\right) &\leq e^{-\lambda n\varepsilon_n}\mathrm{E}\left(e^{\lambda Z_{n,od}}\right)\\
&\leq e^{-\lambda n\varepsilon_n}\left(\mathrm{E}\left(e^{\lambda Z_{n,od}}\right) - \prod_{j=1}^{r_n}\mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right)\right) + e^{-\lambda n\varepsilon_n}\prod_{j=1}^{r_n}\mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right)
\end{aligned}$$

Retracing the proofs in Section 3 of Oliveira [11] and using an inequality by Dewan and Prakasa Rao [3], it follows that, assuming that λ allows for using Lemma 3.1 with $d_n = (1 - a_n)^{-1}$,

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,od}\right| \geq \varepsilon_n\right) \leq \frac{\lambda^2 n}{2}e^{\lambda n(c-\varepsilon_n)}u(p_n) + e^{2c^2\lambda^2 nd_n - \lambda n\varepsilon_n}$$

Minimizing, with respect to λ, the second exponent leads to $\lambda = \frac{\varepsilon_n}{4c^2 d_n}$. We now verify that, for this choice of λ we fulfil the assumptions of Lemma 3.1. With respect to d_n, the assumption of this lemma rewrites as $\lambda < \frac{d_n-1}{d_n}\frac{1}{2c\,p_n}$.

This is easily verified to be equivalent to

$$\varepsilon_n < \frac{2c(d_n - 1)}{p_n}. \tag{7}$$

If we assume that

$$\frac{\lambda^2 n}{2} e^{\lambda n c} u(p_n) = \frac{n\varepsilon_n^2}{32c^4 d_n^2} \exp\left(\frac{n\varepsilon_n}{4cd_n}\right) \leq C_1, \tag{8}$$

then we have

$$\begin{aligned} \mathrm{P}\left(\frac{1}{n}\left|Z_{n,od}\right| \geq \varepsilon_n\right) &\leq C_1 \exp\left(-\frac{n\varepsilon_n^2}{4c^2 d_n}\right) + \exp\left(-\frac{n\varepsilon_n^2}{8c^2 d_n}\right) \\ &\leq (1 + C_1) \exp\left(-\frac{n\varepsilon_n^2}{8c^2 d_n}\right). \end{aligned} \tag{9}$$

Taking now into account (2), condition (7) follows from (4). As for (8), it becomes

$$\begin{aligned} &\frac{n\varepsilon_n^2}{32c^4 d_n^2} \exp\left(\frac{n\varepsilon_n}{4d_n}\right) \\ &\quad = \frac{\alpha \operatorname{loglog} n}{4c^2 d_n} \exp\left(\left(\frac{\alpha}{2}\frac{n \operatorname{loglog} n}{d_n}\right)^{1/2}\right) u(p_n) \leq \frac{\alpha}{4c^2} C_0. \end{aligned} \tag{10}$$

To complete the proof it remains to check that the terms of $Z_{n,od}$ that are left out of the sum of the blocks $Y_{2j-1,n}$ are negligible. According to the definition of the blocks $|Z_{n,od} - \sum_{\ell=1}^{r_n} Y_{2j-1,n}| \leq 2p_n c$, thus

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,od} - \sum_{\ell=1}^{r_n} Y_{2j-1,n}\right| > \varepsilon_n\right) \leq \mathrm{P}\left(2p_n c > n\varepsilon_n\right),$$

and this is, for n large enough, equal to 0 if $\frac{n\varepsilon_n}{p_n} \longrightarrow +\infty$. Replacing (2) in the previous expression,

$$\frac{n\varepsilon_n}{p_n} = (8\alpha)^{1/2} c \left(\frac{n \operatorname{loglog} n}{p_n^2}\right)^{1/2} \longrightarrow +\infty,$$

according to (3). Now replace (2) in (9) and proof is concluded. ∎

4. Almost sure convergence and rates

It follows immediately from (6) that, along a suitable subsequence, we have indeed almost convergence. In fact, choose the subsequence $n_k = \left[\theta^k\right]$, the largest integer less or equal than θ^k, for some $\theta > 1$. Then (6) rewrites as

$$\mathrm{P}\left(\frac{1}{n_k}\left|Z_{n_k,od}\right| > \varepsilon_{n_k}\right) \leq \left(1+\frac{\alpha}{4c^2}C_0\right) k^{-\alpha}(\log\theta)^{-\alpha},$$

thus, the Borel-Cantelli Lemma gives us the almost sure convergence. To get the convergence of the full sequence, define $S^*_{n_k} = \max_{n_{k-1}<n\leq n_k}\left|Z_{n,od}\right|$. Choosing the d_n to be constant or increasing slowly enough, the sequence ε_n is decreasing. Thus, for $n_{k-1} < n \leq n_k$,

$$\mathrm{P}\left(\frac{1}{n}\left|Z_{n,od}\right| > \varepsilon_n\right) \leq \mathrm{P}\left(S^*_{n_k} > n_{k-1}\varepsilon_{n_{k-1}}\right).$$

On the other hand,

$$\frac{n_k^2\varepsilon_{n_k}^2}{n_{k-1}^2\varepsilon_{n_{k-1}}^2}\frac{n_k d_{n_k}\operatorname{loglog} n_k}{n_{k-1} d_{n_{k-1}}\operatorname{loglog} n_{k-1}} \longrightarrow \theta.$$

Hence, for k large enough, $n_k^2\varepsilon_{n_k}^2 \leq 2n_{k-1}^2\varepsilon_{n_{k-1}}^2$, so, using Newman and Wright [7] maximal inequality (see the proof of Theorem 3 therein),

$$\begin{aligned}
&\mathrm{P}\left(S^*_{n_k} > n_{k-1}\varepsilon_{n_{k-1}}\right)\\
&\quad\leq \mathrm{P}\left(S^*_{n_k} > \frac{n_k\varepsilon_{n_k}}{\sqrt{2\theta}}\right)\\
&\quad\leq 2\mathrm{P}\left(Z_{n,od} > \left(\frac{1}{\sqrt{2\theta}}\frac{n_k\varepsilon_{n_k}}{t_{n_k}} - \sqrt{2}\right)\right),
\end{aligned}$$

where $t_n^2 = \mathrm{E}(Z_{n,od}^2)$. If we use now (9), it follows

$$\begin{aligned}
&\mathrm{P}\left(Z_{n,od} > \left(\frac{1}{\sqrt{2\theta}} - \frac{\sqrt{2}\,t_{n_k}}{n_k\varepsilon_{n_k}}\right)n_k\varepsilon_{n_k}\right)\\
&\quad\leq \left(1+\frac{\alpha}{4c^2}C_0\right)\exp\left(-\frac{n_k\varepsilon_{n_k}}{8d_{n_k}}\left(\frac{1}{\sqrt{2\theta}} - \frac{\sqrt{2}\,t_{n_k}}{n_k\varepsilon_{n_k}}\right)\right).
\end{aligned}$$

Assume that $\frac{t_n^2}{n}$ is bounded. Then, up to the multiplication by a constant,

$$\frac{t_{n_k}}{n_k\varepsilon_{n_k}} = \left(\frac{1}{d_{n_k}\operatorname{loglog} n_k}\right)^{1/2} \longrightarrow 0,$$

as long as d_n does not approach 0. Thus, if ε_n is chosen as in (2),

$$\mathrm{P}\left(S^*_{n_k} > n_{k-1}\varepsilon_{n_{k-1}}\right) \leq \left(1+\frac{\alpha}{4c^2}C_0\right)\left(\log\theta^k\right)^{-\alpha\left(\frac{1}{\sqrt{2\theta}}-\frac{\sqrt{2}\,t_{n_k}}{n_k\varepsilon_{n_k}}\right)} \sim k^{-\frac{\alpha}{\sqrt{2\theta}}}.$$

Hence, we have proved the following result.

Theorem 4.1. *Let the assumptions of Theorem 3.2 be satisfied, with* $\alpha > \sqrt{2}$. *Assume further that* $\frac{1}{n}\mathrm{E}(S_n^2)$ *is bounded. Then* $\mathrm{P}\left(\limsup|S_n| > n\varepsilon_n\right) = 0$, *thus we have almost sure convergence with convergence rate* ε_n.

Let us now assume that the covariances of the X_n decrease geometrically so that $u(n) \sim a^{-n}$, where $a \in (0,1)$. In such a case (5) follows if

$$\left(\frac{\alpha}{2}\right)^{1/2}\left(\frac{n\,\mathrm{loglog}\,n}{d_n}\right)^{1/2} - p_n\log a \leq C_0,$$

which leads to the choice

$$p_n \sim \left(\frac{n\,\mathrm{loglog}\,n}{d_n}\right)^{1/2}. \tag{11}$$

Recall that we must have $p_n \longrightarrow +\infty$, so we should be more precise about the choice of d_n. Condition (3) rewrites as $nd_n\,\mathrm{loglog}\,n\frac{d_n}{n\,\mathrm{loglog}\,n} = d_n^2 \longrightarrow +\infty$, while (4) becomes $\frac{(\mathrm{loglog}\,n)^2}{d_n} < \frac{d_n-1}{4\alpha}$, which follows from $(\mathrm{loglog}\,n)^2 \leq \frac{d_n^2}{4\alpha}$. The convergence rate that follows is then characterized by (2), so we are interested in choosing $d_n \longrightarrow +\infty$ as slowly as possible. The best possible choice, given the conditions above, is $d_n \sim \mathrm{loglog}\,n$, so the convergence rate that follows is of order $\frac{\mathrm{loglog}\,n}{n^{1/2}}$.

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