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ON STRONG INCLUSIONS AND ASYMMETRIC PROXIMITIES IN FRAMES

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ABSTRACT: The strong inclusion, a specific type of subrelation of the order of a lattice with pseudocomplements, has been used in the concrete case of the lattice of open sets in topology for an expedient definition of proximity, and allowed for a natural point-free extension of this concept. A modification of a strong inclusion for biframes then provided a point-free model also for the non-symmetric variant. In this paper we show that a strong inclusion can be non-symmetrically modified to work directly on frames, without prior assumption of a biframe structure. The category of quasi-proximal frames thus obtained is shown to be concretely isomorphic with the biframe based one, and shown to be related to that of quasi-uniform frames in a full analogy with the symmetric case.

KEYWORDS: Frame, biframe, pseudocomplement, strong inclusion, paircover, quasi-uniform frame, quasi-proximal frame, total boundedness.
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Introduction

In a general setting, a *strong inclusion* is a subrelation $\lhd$ of the order of a lattice with pseudocomplements that

- is a sublattice of $L \times L$,
- interpolates,
- satisfies the implication

$$a \leq b \lhd c \leq d \;\Rightarrow\; a \leq d,$$

  and
- $a \lhd b \;\Rightarrow\; b^* \lhd a^*$.

This concept (and term) was, first, introduced by Dowker ([4]) for purposes of enriched topology. There it naturally appears, e.g as the "completely below" relation $U \prec\!\prec V$ (where U is separated from the complement of V by

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a real-valued continuous function), or "U is uniformly below V" in a uniform space.

In particular, the *proximity*, a certain enrichment of a topological structure, originally described by specifying for sets when they are "near" ("proximal" [12]) can be alternatively, and to advantage, described in terms of a strong inclusion stronger than the inclusion order.

This has proved to be particularly useful in the point-free setting where the strong inclusion, strengthening the order in a frame (a lattice with $\bigvee\wedge$-distributivity, see 1.1) makes for a suitable extension of the classical concept ([7]); imitating the "proximity of elements" makes here good sense only in (complete) Boolean algebras where it is then equivalent with the strong inclusion approach ([13]). It should be noted that besides of the properties above, the strong inclusion one deals with is further required that

- $a \lhd b \;\Rightarrow\; a^* \vee b = 1$ (implying the "rather below" relation), and
- for each a, $a = \bigvee\{b \mid b \lhd a\}$ (admissibility of the additional structure).

It turns out that the resulting category of proximity frames is concretely isomorphic to that of totally bounded uniform frames (see [7]) and that the compactifiable frames are exactly those that admit strong inclusions (see [1]).

In classical topology it was found useful to generalize proximities by dropping symmetry. This can be modeled in the point-free (frame) setting ([8], [19]) by introducing modified strong inclusions on biframes (triples (L, L_1, L_2) where L_i are specific subframes of L — see 1.1 below) as couples of subrelations $\lhd_i$ of the orders of L_i with certain intertwined properties (see 1.2). Thus generalized strong inclusions are, again, closely connected with compactifications (of biframes — see Schauerte [19]).

In our recent paper [15] we have shown, for uniformities, another environment of the frame structure, that when dropping the symmetry the biframes can be, essentially, avoided. The question naturally arises whether this can be done with the (quasi-) proximity as well. In the present paper we answer this question in the affirmative introducing (in Section 2) a category of quasi-proximal frames that enriches the plain frames directly. Such proximal structures make the picture in the point-free setting more similar to the classical one (where the bitopologies appear only a posteriori and clarify the discussion in the introduction of Doitchinov [3]). In Section 3 we prove that this category is concretely isomorphic with the biframe based one. Finally, in Section 4 the new category of quasi-proximal frames is related with that of quasi-uniform frames. It turns out that the relation is analogous with the symmetric case: namely, there is a one-to-one correspondence between quasi-proximities and totally bounded quasi-uniformities, yielding a concrete isomorphism of the categories.

1. Preliminaries

1.1. Frames and biframes. Recall that a *frame* is a complete lattice satisfying the distributivity law

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

and that *frame homomorphisms* preserve all joins (including the bottom element 0) and finite meets (including the top element 1). Frames and frame homomorphisms are the objects and morphisms of the category Frm.

A *biframe* is a triple (L, L_1, L_2) in which L is a frame and L_1 and L_2 are subframes of L such that $L_1 \cup L_2$ generates L (in the sense that any element of L can be expressed as a join of finite meets of elements of $L_1 \cup L_2$); a *biframe homomorphism* $h : (L, L_1, L_2) \to (M, M_1, M_2)$ is a frame homomorphism from L to M such that the image of L_i $(i = 1, 2)$ under h is contained in M_i. Biframes and biframe homomorphisms are the objects and morphisms of the category BiFrm. If (L, L_1, L_2) is a biframe and $a \in L_i$ $(i = 1, 2)$, the element

$$a^\bullet = \bigvee \{b \in L_j \mid a \wedge b = 0\} \quad (j \in \{1, 2\}, j \neq i)$$

is the analogue in biframes of the *pseudocomplement*

$$a^* = \bigvee \{b \in L \mid a \wedge b = 0\}$$

of an element a of a frame L.

For more about frames the reader can consult [16] or [17], for biframes see [2] and [19].

1.2. Strong inclusions on biframes: quasi-proximities. A *strong inclusion* [19] on a biframe (L, L_1, L_2) is a pair $(\lhd_1, \lhd_2)$ of relations on L_1 and L_2 respectively satisfying the following conditions (for $i = 1, 2$):

(S1) $\lhd_i$ is a sublattice of $L_i \times L_i$.
(S2) $a \leq b \lhd_i c \leq d$ implies that $a \lhd_i d$.
(S3) $a \lhd_i b$ implies that $a^\bullet \vee b = 1$ (usually denoted by $a \prec_i b$).
(S4) $a \lhd_i b$ implies that there exists $c \in L_i$ with $a \lhd_i c \lhd_i b$.

(S5) If $a \lhd_i b$ then $b^\bullet \lhd_j a^\bullet$ for $j \in \{1,2\}$ and $j \neq i$.

(S6) For every $a \in L_i$, $a = \bigvee\{b \in L_i \mid b \lhd_i a\}$.

Note that the more standard strong inclusion in a frame is the $\lhd$ from $(\lhd, \lhd)$ on (L, L, L).

A triple $((L, L_1, L_2), \lhd_1, \lhd_2)$ where $(\lhd_1, \lhd_2)$ is a strong inclusion on the biframe (L, L_1, L_2) is called a *quasi-proximal frame* [7] (*proximal biframe* in the more recent [8]). Given proximal biframes

$$((L, L_1, L_2), \lhd_1^L, \lhd_2^L) \quad \text{and} \quad ((M, M_1, M_2), \lhd_1^M, \lhd_2^M),$$

a biframe homomorphism $h : (L, L_1, L_2) \to (M, M_1, M_2)$ is a *proximal biframe homomorphism* if

$$a \lhd_i^L b \text{ implies } h(a) \lhd_i^M h(b) \quad (\text{for } i = 1, 2 \text{ and every } a, b \in L_i).$$

The category of proximal biframes and proximal biframe homomorphisms will be denoted by

$$\mathsf{PBiFrm}.$$

1.3. Quasi-uniform frames. Let L be a frame. A subset $C \subseteq L \times L$ is a *paircover* [15] of L if

$$\bigvee\{c_1 \wedge c_2 \mid (c_1, c_2) \in C\} = 1.$$

A paircover C of L is *strong* if, for any $(c_1, c_2) \in C$, $(c_1, c_2) = (0, 0)$ whenever $c_1 \wedge c_2 = 0$. For any $C, D \subseteq L \times L$ we write $C \leq D$ (and say that C *refines* D) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further we write

$$C \wedge D = \{(c_1 \wedge d_1, c_2 \wedge d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\};$$

obviously it is a is a paircover again.

For $a \in L$ and $C, D \subseteq L \times L$, we set

$$\begin{aligned}
&\mathsf{st}_1(a, C) = \bigvee\{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \wedge a \neq 0\},\\
&\mathsf{st}_2(a, C) = \bigvee\{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \wedge a \neq 0\},\\
&C^{-1} = \{(c_2, c_1) \mid (c_1, c_2) \in C\}, \text{and}\\
&\mathsf{st}(D, C) = \{(\mathsf{st}_1(d_1, C), \mathsf{st}_2(d_2, C)) \mid (d_1, d_2) \in D\}
\end{aligned}$$

and write

$$C^* \text{ for } \mathsf{st}(C, C).$$

We shall need the following facts from [15, Proposition 2.2]:

Proposition 1.3.1. *Let $C, D \subseteq L \times L$ and $a, b \in L$. Then:*

(1) *If $a \le b$ then $\mathsf{st}_i(a, C) \le \mathsf{st}_i(b, C)$.*
(2) *If $C \le D$ then $\mathsf{st}_i(a, C) \le \mathsf{st}_i(a, D)$.*
(3) *$a \wedge \mathsf{st}_1(b, C) = 0$ iff $b \wedge \mathsf{st}_2(a, C) = 0$.*
(4) *If C is a paircover then $a \le \mathsf{st}_i(a, C)$ and $C \le C^*$.*
(5) *If C is a paircover then $\mathsf{st}_i(\mathsf{st}_i(a, C), C) \le \mathsf{st}_i(a, C^*)$.*
(6) *For any frame homomorphism $h : L \to M$,*

$$\mathsf{st}_i(h(a), h[C]) \le h(\mathsf{st}_i(a, C)).$$

(7) *For any frame homomorphism $h : L \to M$, $h[C]^* \le h[C^*]$.* □

Given a non-empty family $\mathcal{U}$ of paircovers of L, we write $a \triangleleft_i^{\mathcal{U}} b$ ($i = 1, 2$) whenever $\mathsf{st}_i(a, C) \le b$ for some $C \in \mathcal{U}$, and set

$$L_i(\mathcal{U}) = \{a \in L \mid a = \bigvee\{b \in L \mid b \triangleleft_i^{\mathcal{U}} a\}\} \quad (i = 1, 2).$$

From [15, Proposition 2.4] we know the following:

Proposition 1.3.2. *Let $\mathcal{U}$ be a basis for a filter of paircovers of L. Then, for $i = 1, 2$:*

(1) *The relations $\triangleleft_i^{\mathcal{U}}$ are sublattices of $L \times L$, both stronger than $\le$.*
(2) *For any $a, b, c, d \in L$, $a \le b \triangleleft_i^{\mathcal{U}} c \le d$ implies $a \triangleleft_i^{\mathcal{U}} d$.*
(3) *$L_i(\mathcal{U})$ are subframes of L.* □

A system $\mathcal{U}$ of paircovers of L is *admissible* [15] if $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ is a biframe or, equivalently, if for every $a \in L$, $a = \bigvee\{b \in L \mid b \triangleleft_i^{\overline{\mathcal{U}}} a\}$, where $\overline{\mathcal{U}}$ denotes the filter of paircovers of L generated by $\{C \wedge C^{-1} \mid C \in \mathcal{U}\}$.

Now, an admissible system $\mathcal{U}$ of paircovers of L is a *quasi-uniformity* on L if it satisfies the following conditions:

(QU1) For any $C \in \mathcal{U}$ and any paircover D with $C \le D$, then $D \in \mathcal{U}$.
(QU2) For any $C, D \in \mathcal{U}$ there exists a strong $E \in \mathcal{U}$ such that $E \le C \wedge D$.
(QU3) For any $C \in \mathcal{U}$ there is a $D \in \mathcal{U}$ such that $D^* \le C$.

The pair $(L, \mathcal{U})$ is called a *quasi-uniform frame* [15]. Let $(L, \mathcal{U})$ and $(M, \mathcal{V})$ be quasi-uniform frames. A frame homomorphism $h : L \to M$ is *uniform* if $h[C] \in \mathcal{V}$ for every $C \in \mathcal{U}$. The resulting category will be denoted by

$$\mathsf{QUFrm}.$$

We say that a quasi-uniform frame $(L, \mathcal{U})$ is *totally bounded* if for every $C \in \mathcal{U}$ there is a finite paircover $D \in \mathcal{U}$ such that $D \le C$.

Quasi-uniform frames $(L,\mathcal{U})$ have the following crucial properties (see [15, lemmata 2.6, 3.2, 3.4]):

Consider the interior operator on $\mathcal{P}(L\times L)$ defined by

$$\mathsf{int}(C)=\bigcup\{D\subseteq L\times L \mid D \overset{\mathcal{U}}{\Subset} C\},$$

where

$$C \overset{\mathcal{U}}{\Subset} D \equiv_{\text{def}} \mathsf{st}(C,U)\leq D \text{ for some } U\in\mathcal{U}.$$

Proposition 1.3.3. *For every $C\in\mathcal{U}$ we have:*

(1) $\mathsf{int}(C)\leq C\leq \mathsf{int}(C^*)$.
(2) $\mathsf{st}_i(a,\mathsf{int}(C))\in L_i(\mathcal{U})$ $(i=1,2)$ *for every $a\in L$.* □

Given $U\in\mathcal{U}$, $a\in L$ is said to be U-*small* if

$$a\leq\bigvee\{u_1\wedge u_2 \mid (u_1,u_2)\in U, u_1\wedge u_2\wedge b\neq 0\}$$

whenever $a\wedge b\neq 0$. Let

$$C_U=\{(\mathsf{st}_1(a,\mathsf{int}(U)),\mathsf{st}_2(a,\mathsf{int}(U)))\mid a \text{ is an } U\text{-small member of } L\}.$$

Proposition 1.3.4. (1) *Each C_U is a strong paircover of L contained in $L_1(\mathcal{U})\times L_2(\mathcal{U})$.*

(2) $\mathsf{st}_i(a,C_U)\leq \mathsf{st}_i(a,U^{**})$ $(i=1,2)$.
(3) $\mathsf{st}_i(a,U)\leq \mathsf{st}_i(a,C_{U^*})$ $(i=1,2)$.
(4) $C_U\leq U^{**}$; *if U is strong then $U\leq C_{U^*}$.* □

2. Quasi-proximities without biframes

Let L be a frame, $\lhd$ a binary relation in L and

$$L(\lhd)=\{a\in L\mid a=\bigvee\{b\in L\mid b\lhd a\}\}.$$

Lemma 2.1. *If $\lhd$ is a sublattice of $L\times L$, stronger than $\leq$ (that is, $\lhd\subseteq\leq$), satisfying*

$$a\leq b\lhd c\leq d\Rightarrow a\lhd d \tag{2.1.1}$$

then $L(\lhd)$ is a subframe of L.

Proof: Since $0 \lhd 0$ and $1 \lhd 1$, then $0, 1 \in L(\lhd)$. Since $\lhd \subseteq \leq$, we have $\bigvee\{b \in L \mid b \lhd a\} \leq a$. Let $a, b \in L(\lhd)$. Then, since $\lhd$ is closed under finite meets,

$$a \wedge b = \bigvee\{a' \in L \mid a' \lhd a\} \wedge \bigvee\{b' \in L \mid b' \lhd b\} =$$
$$= \bigvee\{a' \wedge b' \mid a', b' \in L, a' \lhd a, b' \lhd b\} \leq \bigvee\{c \in L \mid c \lhd a \wedge b\}$$

which shows that $a \wedge b \in L(\lhd)$.

Now, let $a_i \in L(\lhd)$ $(i \in I)$. Then $\bigvee_{i\in I} a_i = \bigvee_{i\in I} \bigvee\{b \in L \mid b \lhd a_i\}$. For each such b, $b \lhd a_i \leq \bigvee_{i\in I} a_i$. Consequently, by (2.1.1), $\bigvee_{i\in I} a_i \leq \bigvee\{b \in L \mid b \lhd \bigvee_{i\in I} a_i\}$ and $\bigvee_{i\in I} a_i \in L(\lhd)$. ■

In the sequel we will have to refer to pseudocomplements relatively to distinct subframes. Therefore we will adopt the following notation: for a subframe K of a frame L and $a \in L$, we denote by $\mathsf{c}_K(a)$ the element

$$\bigvee\{b \in K \mid b \wedge a = 0\}.$$

We have $\mathsf{c}_L(a) = a^*$, $\mathsf{c}_K(a) \leq a^*$, $\mathsf{c}_K(0) = 1$, $\mathsf{c}_K(1) = 0$ and $a \leq \mathsf{c}_K(\mathsf{c}_K(a))$ for every $a \in K$.

Definition 2.2. Let L be a frame. A pair $(\lhd_1, \lhd_2)$ of relations in L will be called a *strong bi-inclusion* on L if for $i, j \in \{1, 2\}$ we have:

(SB1) $\lhd_i$ is a sublattice of $L \times L$.
(SB2) $a \leq b \lhd_i c \leq d$ implies that $a \lhd_i d$.
(SB3) $a \lhd_i b$ implies that $\mathsf{c}_{L(\lhd_j)}(a) \vee b = 1$ $(j \neq i)$.
(SB4) $a \lhd_i b$ implies that there exists $c \in L(\lhd_i)$ with $a \lhd_i c \lhd_i b$.
(SB5) If $a \lhd_i b$ then $b^* \lhd_j a^*$ for $j \neq i$.
(SB6) $(L, L(\lhd_1), L(\lhd_2))$ is a biframe.

Remarks 2.3. (1) (SB3) implies that each $\lhd_i$ $(i = 1, 2)$ is stronger than $\leq$: if $\mathsf{c}_{L(\lhd_j)}(a) \vee b = 1$ then $a = a \wedge (\mathsf{c}_{L(\lhd_j)}(a) \vee b) = a \wedge b$. It also implies that $b^* \leq \mathsf{c}_{L(\lhd_j)}(a)$ whenever $a \lhd_i b$ (because $\mathsf{c}_{L(\lhd_j)}(a) \vee b = 1$ and $b^* \wedge b = 0$).

(2) For any $a, b \in L$,

$$a \lhd_i b^* \Rightarrow a \lhd_i \mathsf{c}_{L(\lhd_i)}(b) \quad (i = 1, 2).$$

Indeed, by (SB4) there exists $c \in L(\lhd_i)$ such that $a \lhd_i c \lhd_i b^*$. Since $c \leq b^*$, then $c \wedge b = 0$ and consequently $c \leq \mathsf{c}_{L(\lhd_i)}(b)$. Therefore $a \lhd_i \mathsf{c}_{L(\lhd_i)}(b)$ by (SB2).

(3) Also, it may be worth mentioning that any pair $(\lhd_1, \lhd_2)$ satisfying (SB2), (SB3) and (SB4) satisfies (SB5) if and only if it satisfies

$$\text{(SB5a) if } a \lhd_i b \text{ then } \mathsf{c}_{L(\lhd_j)}(b) \ \lhd_j \ \mathsf{c}_{L(\lhd_j)}(a) \text{ for } j \neq i.$$

Indeed:

$\Rightarrow$: If $a \lhd_i b$ then, by (SB5), $\mathsf{c}_{L(\lhd_j)}(b) \le b^* \lhd_j a^*$ and therefore by the preceding remark $\mathsf{c}_{L(\lhd_j)}(b) \ \lhd_j \ \mathsf{c}_{L(\lhd_j)}(a)$.

$\Leftarrow$: In order to prove (SB5), suppose $a \lhd_i b$ and apply (SB4) to get c satisfying $a \lhd_i c \lhd_i b$. By (SB5a) and Remark (1) we obtain $b^* \le \mathsf{c}_{L(\lhd_j)}(b) \lhd_j \mathsf{c}_{L(\lhd_j)}(a) \le a^*$.

(4) (SB3) is obviously stronger than

$$\text{(SB3a) } a \lhd_i b \text{ implies that } a^* \vee b = 1$$

(since $\mathsf{c}_{L(\lhd_j)}(a) \le a^*$). However, if $(\lhd_1, \lhd_2)$ satisfies (SB2) and (SB4) then it satisfies (SB3) and (SB5) if and only if it satisfies (SB3a) and (SB5a): the implication "$\Rightarrow$" was already proved in (3) and, conversely, if $a \lhd_i b$ then, by (SB5a), $\mathsf{c}_{L(\lhd_j)}(b) \ \lhd_j \ \mathsf{c}_{L(\lhd_j)}(a)$; applying (SB3a) we get $(\mathsf{c}_{L(\lhd_j)}(b))^* \vee \mathsf{c}_{L(\lhd_j)}(a) = 1$ and thus $b \vee \mathsf{c}_{L(\lhd_j)}(a) = 1$ (since $b \le (\mathsf{c}_{L(\lhd_j)}(b))^*$).

(5) In addition, (SB3) may be equivalently replaced by the conjunction

(SB0) for every $a, b \in L$, $a \lhd_i b \Rightarrow a \le b$, and

(SB3b) for every $a, b \in L(\lhd_i)$, $a \lhd_i b$ implies that $\mathsf{c}_{L(\lhd_j)}(a) \vee b = 1 \, (j \neq i)$.

Indeed, the implication

$$\text{(SB3)} \ \Rightarrow \ \text{(SB0)+(SB3b)}$$

is obvious by the preceding observation. On the other hand, if $a \lhd_i b$ then there exist by (SB4) $c, d \in L(\lhd_i)$ such that $a \lhd_i c \lhd_i d \lhd_i b$. Thus, by (SB3b), $\mathsf{c}_{L(\lhd_j)}(c) \vee d = 1$. But $d \le b$ and $\mathsf{c}_{L(\lhd_j)}(c) \le \mathsf{c}_{L(\lhd_j)}(a)$ (because $a \le c$) and hence $\mathsf{c}_{L(\lhd_j)}(a) \vee b = 1$.

Any strong bi-inclusion $(\lhd_1, \lhd_2)$ on a frame L induces two subframes $L(\lhd_1)$ and $L(\lhd_2)$ of L. The triple $(L, L(\lhd_1), L(\lhd_2))$ is a biframe by (SB6). The following proposition provides an alternative to condition (SB6) that avoids biframes.

Proposition 2.4. *Let* $(\lhd_1, \lhd_2)$ *be a pair of binary relations in* L*, both stronger than* $\leq$ *and satisfying* (SB1) *and* (SB2)*. Then* $(\lhd_1, \lhd_2)$ *satisfies* (SB6) *if and only if*

$$\text{(SB6}'\text{)} \textit{ for each } a \in L, a = \bigvee\{b \in L \mid b \lhd a\},$$

where $\lhd$ *is the binary relation in* L *defined by*

$$b \lhd a \ \equiv_{\text{def}} \ \exists a_1 \in L(\lhd_1), \exists a_2 \in L(\lhd_2),\ a_1 \wedge a_2 \leq a,\ b \lhd_1 a_1 \textit{ and } b \lhd_2 a_2.$$

Proof: $\Rightarrow$: For each $a \in L$ we can write $a = \bigvee_{i \in I}(a_i^1 \wedge a_i^2)$ for some

$$\{a_i^1 \mid i \in I\} \subseteq L(\lhd_1) \quad \text{and} \quad \{a_i^2 \mid i \in I\} \subseteq L(\lhd_2).$$

Taking into account that, for any $i \in I$,

$$a_i^1 = \{b \in L \mid b \lhd_1 a_i^1\} \quad \text{and} \quad a_i^2 = \{b \in L \mid b \lhd_2 a_i^2\},$$

it suffices to show that $b_1 \wedge b_2 \lhd a_1 \wedge a_2$ whenever $b_1 \lhd_1 a_1$ and $b_2 \lhd_2 a_2$. This, however, is an immediate consequence of (SB1) and the definition of $\lhd$.

$\Leftarrow$: By 2.1, each $L(\lhd_i)$ $(i = 1, 2)$ is a subframe of L. It remains to show that each $a \in L$ is a join of finite meets in $L(\lhd_1) \cup L(\lhd_2)$.

Let $a \in L$. Then $a = \bigvee S$ where $S = \{b \in L \mid b \lhd a\}$. For each $b \in S$ there exist $a_1^b \in L(\lhd_1)$ and $a_2^b \in L(\lhd_2)$ satisfying $b \lhd_1 a_1^b$, $b \lhd_2 a_2^b$ and $a_1^b \wedge a_2^b \leq a$. Hence $a = \bigvee_{b \in S} b \leq \bigvee_{b \in S}(a_1^b \wedge a_2^b) \leq a$. ∎

A frame L with a strong bi-inclusion $(\lhd_1, \lhd_2)$ will be called a *quasi-proximal frame.* Given quasi-proximal frames $(L, \lhd_1^L, \lhd_2^L)$ and $(M, \lhd_1^M, \lhd_2^M)$, a *quasi-proximal map*

$$h : (L, \lhd_1^L, \lhd_2^L) \to (M, \lhd_1^M, \lhd_2^M)$$

is a frame homomorphism $h : L \to M$ such that $a \lhd_i^L b \Rightarrow h(a) \lhd_i^M h(b)$ for every $a, b \in L$. The resulting category will be denoted by

$$\mathsf{QPFrm}.$$

Note that our definition of a quasi-proximal frame contains, of course, the symmetric case of proximal frames (as defined by strong inclusions) [7]: it is a frame equipped with a strong bi-inclusion $(\lhd_1, \lhd_2)$ such that $\lhd_1 = \lhd_2$.

Quasi-uniform frames provide canonical examples of quasi-proximal frames:

Proposition 2.5. *For each quasi-uniform frame* $(L, \mathcal{U})$*, the triple* $(L, \lhd_1^{\mathcal{U}}, \lhd_2^{\mathcal{U}})$ *is a quasi-proximal frame.*

Proof: The properties (SB1) and (SB2) follow from Proposition 1.3.2, and (SB6) is obvious.

(SB3) Suppose $a \lhd_i^{\mathcal{U}} b$. Then $\mathsf{st}_i(a,U) \le b$ for some $U \in \mathcal{U}$. It suffices to show that $\mathsf{c}_{L(\lhd_j)}(a) \vee \mathsf{st}_i(a,U) = 1$. Consider $V \in \mathcal{U}$ such that $V^{***} \le U$. By Proposition 1.3.4, $V \le C_{V^*}$ and $C_{V^*} \le V^{***} \le U$. Therefore $\mathsf{st}_i(a, C_{V^*}) \le \mathsf{st}_i(a,U)$. Let

$$(v_1, v_2) \in C_{V^*} \subseteq L_1(\mathcal{U}) \times L_2(\mathcal{U}).$$

If $u_j \wedge a = 0$ then $u_j \le \mathsf{c}_{L(\lhd_j)}(a)$; otherwise, $u_i \le \mathsf{st}_i(a, C_{V^*}) \le \mathsf{st}_i(a,U)$. This shows that

$$\bigvee\{u_1 \wedge u_2 \mid (u_1,u_2) \in C_{V^*}\} \le \mathsf{c}_{L(\lhd_j)}(a) \vee \mathsf{st}_i(a,U).$$

Hence $\mathsf{c}_{L(\lhd_j)}(a) \vee \mathsf{st}_i(a,U) = 1$, since C_{V^*} is a paircover.

(SB4) Let $\mathsf{st}_i(a,U) \le b$ for some $U \in \mathcal{U}$ and take $V \in \mathcal{U}$ such that $V^{**} \le U$. By Proposition 1.3.3,

$$\mathsf{st}_i(a, \mathsf{int}(V^*)) \in L_i(\mathcal{U}) = L_{\lhd_i^{\mathcal{U}}}.$$

Of course $a \lhd_i^{\mathcal{U}} \mathsf{st}_i(a, \mathsf{int}(V^*))$. On the other hand, by 1.3.1(5),

$$\mathsf{st}_i(\mathsf{st}_i(a, \mathsf{int}(V^*)), V^*)) \le \mathsf{st}_i(\mathsf{st}_i(a, V^*), V^*)) \le \mathsf{st}_i(a, V^{**})\mathsf{st}_i(a,U) \le b$$

and hence $\mathsf{st}_i(a, \mathsf{int}(V^*)) \lhd_i^{\mathcal{U}} b$.

(SB5) Let $\mathsf{st}_i(a,U) \le b$ for some $U \in \mathcal{U}$. Then $\mathsf{st}_i(a,U) \wedge b^* = 0$ $(j \ne i)$. By 1.3.1(3), $a \wedge \mathsf{st}_j(b^*, U) = 0$, that is, $\mathsf{st}_j(b^*,U) \le a^*$. ■

3. The concrete isomorphism $\mathsf{QPFrm} \cong \mathsf{QPBiFrm}$

Given a proximal biframe $((L, L_1, L_2), \lhd_1, \lhd_2)$, let

$$\Phi((L, L_1, L_2), \lhd_1, \lhd_2) = (L, \overline{\lhd}_1, \overline{\lhd}_2)$$

where, for any $a, b \in L$,

$$a \,\overline{\lhd}_i\, b \equiv_{\text{def}} \exists\, c, d \in L_i : a \le c \lhd_i d \le b \quad (i = 1, 2).$$

Proposition 3.1. *For any proximal biframe* $((L, L_1, L_2), \lhd_1, \lhd_2)$,

$$\Phi((L, L_1, L_2), \lhd_1, \lhd_2)$$

is a quasi-proximal frame.

Proof: (SB1) $0\overline{\lhd}_i 0$ and $1\overline{\lhd}_i 1$ are trivial. Let $a_1, a_2 \mathrel{\overline{\lhd}_i} b$. Then $a_1 \leq c_1 \lhd_i d_1 \leq b$ and $a_2 \leq c_2 \lhd_i d_2 \leq b$ for some $c_1, c_2, d_1, d_2 \in L_i$. Consequently, $a_1 \leq c_1 \lhd_i d_1 \vee d_2 \leq b$ and $a_2 \leq c_2 \lhd_i d_1 \vee d_2 \leq b$ with $c_1, c_2, d_1 \vee d_2 \in L_i$. By hypothesis, $c_1 \wedge c_2 \lhd_i d_1 \vee d_2$ and $c_1 \wedge c_2 \in L_i$. Thus $a_1 \wedge a_2 \mathrel{\overline{\lhd}_i} b$. Similarly, $a_1 \vee a_2 \leq c_1 \vee c_2 \lhd_i d_1 \vee d_2 \leq b$ and $a_1 \vee a_2 \mathrel{\overline{\lhd}_i} b$.

(SB2) is obvious.

(SB3) First note that $L_i \subseteq L(\overline{\lhd}_i)$. Indeed, for each $a \in L_i$, since $\lhd_i \subseteq \overline{\lhd}_i$, we have

$$a = \bigvee\{b \in L_i \mid b \lhd_i a\} \leq \bigvee\{b \in L \mid b \mathrel{\overline{\lhd}_i} a\} \leq a.$$

Now, let $a, b \in L(\overline{\lhd}_i)$ and $a \leq c \lhd_i d \leq b$ with $c, d \in L_i$. By hypothesis, $\mathsf{c}_{L(\lhd_j)}(c) \vee d = 1$ and from the inclusion $L_i \subseteq L(\overline{\lhd}_i)$ and the fact that $a \leq c$ it follows that

$$\mathsf{c}_{L(\overline{\lhd}_j)}(a) = \bigvee\{a' \in L(\overline{\lhd}_j) \mid a' \wedge a = 0\} \geq \bigvee\{c' \in L_j \mid c' \wedge c = 0\} = \mathsf{c}_{L(\lhd_j)}(c).$$

Hence $\mathsf{c}_{L(\overline{\lhd}_j)}(a) \vee b = 1$.

(SB4) follows immediately from (S4) and the fact that $L_i \subseteq L(\overline{\lhd}_i)$ proved above.

(SB5) By Remark 2.3(3) it suffices to prove (SB5a). Let $a, b \in L$ with $a \mathrel{\overline{\lhd}_i} b$, that is, $a \leq c \lhd_i d \leq b$ for some $c, d \in L_i$. Then, by hypothesis, $\mathsf{c}_{L(\lhd_j)}(d) \lhd_j \mathsf{c}_{L(\lhd_j)}(c)$ and, of course, $\mathsf{c}_{L(\lhd_j)}(d), \mathsf{c}_{L(\lhd_j)}(c) \in L_j$. Now it suffices to show that $\mathsf{c}_{L(\overline{\lhd}_j)}(b) \leq \mathsf{c}_{L(\lhd_j)}(d)$ and $\mathsf{c}_{L(\lhd_j)}(c) \leq \mathsf{c}_{L(\overline{\lhd}_j)}(a)$. The latter was already proved in (SB3) above and the former can be proved in a similar way.

(SB6) By Lemma 2.1, each $L(\overline{\lhd}_i)$ is a subframe of L. Since (L, L_1, L_2) is a biframe and $L_i \subseteq L(\overline{\lhd}_i)$ $(i = 1, 2)$, then immediately $(L, L(\overline{\lhd}_1), L(\overline{\lhd}_2))$ is also a biframe. ∎

Given a quasi-proximal frame $(L, \lhd_1, \lhd_2)$, let

$$\Psi(L, \lhd_1, \lhd_2) = ((L, L(\lhd_1), L(\lhd_2)), \lhd_1|_{L(\lhd_1)}, \lhd_2|_{L(\lhd_2)}).$$

Proposition 3.2. *For any quasi-proximal frame* $(L, \lhd_1, \lhd_2)$*,* $\Psi(L, \lhd_1, \lhd_2)$ *is a proximal biframe.*

Proof: By hypothesis, $(L, L(\lhd_1), L(\lhd_2))$ is a biframe and $\Psi(L, \lhd_1, \lhd_2)$ satisfies conditions (S1)-(S5) trivially. It remains to check (S6):

For every $a \in L(\lhd_i)$, $a = \bigvee\{b \in L \mid b \lhd_i a\}$. But by condition (SB4) there is some $c \in L(\lhd_i)$ satisfying $b \lhd_i c \lhd_i a$. Therefore $a = \bigvee\{b \in L(\lhd_i) \mid b \lhd_i a\}$, as desired. ■

Concerning morphisms, the next result allows us to define $\Phi(h) = h$ for every $h \in \mathsf{PBiFrm}$ and $\Psi(h) = h$ for every $h \in \mathsf{QPFrm}$.

Proposition 3.3. (1) *Let* $h : ((L, L_1, L_2), \lhd_1^L, \lhd_2^L) \to ((M, M_1, M_2), \lhd_1^M, \lhd_2^M)$ *be a proximal biframe homomorphism. Then*

$$h : \Phi((L, L_1, L_2), \lhd_1^L, \lhd_2^L) \to \Phi((M, M_1, M_2), \lhd_1^M, \lhd_2^M) \in \mathsf{QPFrm}.$$

(2) *Let* $h : (L, \lhd_1^L, \lhd_2^L) \to (M, \lhd_1^M, \lhd_2^M)$ *be a quasi-proximal map. Then*

$$h : \Psi(L, \lhd_1^L, \lhd_2^L) \to \Psi(M, \lhd_1^M, \lhd_2^M) \in \mathsf{PBiFrm}.$$

Proof: (1) We have to check that

$$\forall a, b \in L_i,\ \ a \mathrel{\overline{\lhd}_i^L} b \Rightarrow h(a) \mathrel{\overline{\lhd}_i^M} h(b) \quad (i = 1, 2).$$

Let $a \mathrel{\overline{\lhd}_i^L} b$, that is $a \leq c \lhd_i^L d \leq b$ for some $c, d \in L_i$. Then, by hypothesis, $h(c), h(d) \in M_i$ and $h(a) \leq h(c) \lhd_i^M h(d) \leq h(b)$, which shows that $h(a) \mathrel{\overline{\lhd}_i^M} h(b)$.

(2) It suffices to check that h is a biframe map

$$(L, L(\lhd_1^L), L(\lhd_2^L)) \to (M, M(\lhd_1^M), M(\lhd_2^M))$$

(the rest is obvious). Consider $a \in L(\lhd_i^L)$. Since $a = \bigvee\{b \in L \mid b \lhd_i^L a\}$ and $b \lhd_i^L a$ implies $h(b) \lhd_i^M h(a)$, then

$$h(a) = \bigvee\{h(b) \mid b \in L, b \lhd_i^L a\} \leq \bigvee\{c \in M \mid c \lhd_i^M h(a)\} \leq h(a).$$

Hence $h(a) \in M(\lhd_i^M)$. ■

Finally, we have:

Theorem 3.4. *The functors* Ψ *and* Φ *constitute a concrete isomorphism between* QPFrm *and* PBiFrm.

Proof: It suffices to show that

$$\text{(a) } \Phi\Psi = \mathrm{Id}_{\mathsf{QPFrm}} \ \text{ and (b) } \ \Psi\Phi = \mathrm{Id}_{\mathsf{PBiFrm}}$$

on objects.

(a) We will show that $\overline{\lhd}|_{L(\lhd_i)} = \lhd_i$. Consider $a, b \in L$ with $a \lhd_i b$. By (SB4), there is $c, d \in L(\lhd_i)$ such that $a \lhd_i c \lhd_i d \lhd_i b$. Since $c \lhd_i|_{L(\lhd_i)} d$

then, immediately, $a \mathrel{\overline{\lhd}}|_{L(\lhd_i)} b$. On the other hand, if $a, b \in L$ are such that $a \mathrel{\overline{\lhd}}|_{L(\lhd_i)} b$ then there exists a pair c, d of elements of $L(\lhd_i)$ satisfying $a \leq c \lhd_i d \leq b$. Thus $a \lhd_i b$.

(b) It suffices to check that $L(\overline{\lhd}_i) = L_i$ for $i = 1, 2$. Let $a \in L_i$. Then, since $\lhd_i \subseteq \overline{\lhd}_i$, we have

$$a = \bigvee\{b \in L_i \mid b \lhd_i a\} \leq \bigvee\{b \in L \mid b \mathrel{\overline{\lhd}_i} a\} \leq a.$$

Conversely, if $a \in L(\overline{\lhd}_i)$, meaning that $a \in L$ and $a = \bigvee\{b \in L \mid b \mathrel{\overline{\lhd}_i} a\}$, then for each such b there is $c_b, d_b \in L_i$ satisfying $b \leq c_b \lhd_i d_b \leq a$. Consequently,

$$a \leq \bigvee\{c \in L_i \mid c \lhd_i a\} \leq a$$

and, therefore, $a \in L_i$. ■

Remark 3.5. As a consequence of Proposition 1 of Schauerte [19], every quasi-proximity $(\lhd_1, \lhd_2)$ on a frame L induces a compactification of the associated biframe $(L, L(\lhd_1), L(\lhd_2))$. Then by Proposition 3 of [19] this is a zero-dimensional compactification if and only if $a \lhd_i b$ (for $a, b \in L(\lhd_i)$, $i = 1, 2$) implies the existence of $c \in L(\lhd_i)$ satisfying $a \leq c \lhd_i c \leq b$. Note that, by (SB3), $c \lhd_i c$ means that $c \in L(\lhd_i)$ is complemented in L with complement in $L(\lhd_j)$ $(j \neq i)$.

4. Quasi-proximities and quasi-uniformities: total boundedness

To finish we show, in analogy with the spatial case or the symmetric case (see [5, 7, 9, 12, 13]), that the category QPFrm is isomorphic to the full subcategory TBQUFm of QUFrm of all totally bounded quasi-uniform frames.

First, we need a few basic facts about paircovers. Let C be a paircover of the frame L. Set

$$C_s = \{(c_1, c_2) \in C \mid c_1 \wedge c_2 \neq 0\}.$$

Lemma 4.1. *Let C, D be paircovers of the frame L. Then:*

(1) *C_s is a strong paircover of L.*
(2) *$(C \wedge D)_s \leq C_s \wedge D_s$.*
(3) *$C^* \leq D \to (C_s)^* \leq D_s$.*
(4) *If $h : L \to M$ is a frame homomorphism, then $(h[C])_s \leq h[C_s]$.*

Proof: (1) and (2) are obvious.

(3) Let $(c_1, c_2) \in C$ with $c_1 \wedge c_2 \neq 0$. Since $\mathsf{st}_i(c_i, C_s) \leq \mathsf{st}_i(c_i, C) \leq d_i$ $(i = 1, 2)$ for some $(d_1, d_2) \in D$ and $c_1 \wedge c_2 \leq \mathsf{st}_1(c_1, C_s) \wedge \mathsf{st}_2(c_2, C_s) \leq d_1 \wedge d_2$, then $(d_1, d_2) \in D_s$.

(4) Suppose $h(c_1) \wedge h(c_2) \neq 0$ for $(c_1, c_2) \in C$. Then $h(c_1 \wedge c_2) \neq 0$, hence $c_1 \wedge c_2 \neq 0$ and consequently $(c_1, c_2) \in C_s$. Hence $(h(c_1), h(c_2)) \in h[C_s]$. ■

Lemma 4.2. *Let $\mathcal{U}$ be an admissible filter of paircovers of L with property* (QU3)*, and let $\mathcal{U}_s$ be the filter of paircovers of L that has $\{C_s \mid C \in \mathcal{U}\}$ as subbasis. Then $(L, \mathcal{U}_s)$ is a quasi-uniform frame.*

Proof: $\mathcal{U}_s$ is a filter of paircovers of L which by Lemma 4.1 satisfies the requirements (QU1)-(QU3). Since $\mathcal{U} \subseteq \mathcal{U}_s$, then $L_i(\mathcal{U}) \subseteq L_i(\mathcal{U}_s)$ $(i = 1, 2)$. Hence $(L, L_1(\mathcal{U}_s), L_2(\mathcal{U}_s))$ is also a biframe and $\mathcal{U}_s$ is admissible. ■

Let $(L, \triangleleft_1, \triangleleft_2)$ be a quasi-proximal frame. For any $a \triangleleft_1 b$ we define

$$C^1_{a,b} = \{(1, a^*), (b, 1)\}$$

and for $a \triangleleft_2 b$ we define

$$C^2_{a,b} = \{(a^*, 1), (1, b)\}.$$

Of course, $C^1_{a,b}$ and $C^2_{a,b}$ are paircovers of L.

Proposition 4.3. *The filter $\mathcal{U}$ of paircovers of L which has as subbasis the family of paircovers $\{C^i_{a,b} \mid a \triangleleft_i b, i = 1, 2\}$ satisfies the conditions of the preceding lemma.*

Proof: Let $a \triangleleft_1 b$. Use (SB4) to select $c_1, c_2 \in L(\triangleleft_1)$ such that $a \triangleleft_1 c_1 \triangleleft_1 c_2 \triangleleft_1 b$ and consider $C = C^1_{a,c_1} \wedge C^1_{c_1,c_2} \wedge C^1_{c_2,b}$. Then

$$C = \{(c_1, 1), (c_1, c_1^*), (c_1, c_2^*), (c_2, a^*), (c_2, c_2^*), (b, c_1^*), (1, c_2^*)\}.$$

Since

$$\mathsf{st}_1(c_1, C) \leq b, \quad \mathsf{st}_1(c_2, C) \leq b, \quad \mathsf{st}_2(c_1^*, C) \leq a^* \quad \text{and} \quad \mathsf{st}_2(c_2^*, C) \leq a^*,$$

we have $C^* \leq C^1_{a,b}$. Similarly, for $a \triangleleft_2 b$ and $a \triangleleft_2 c_1 \triangleleft_2 c_2 \triangleleft_2 b$,

$$(C^2_{a,c_1} \wedge C^2_{c_1,c_2} \wedge C^2_{c_2,b})^* \leq C^2_{a,b}.$$

This shows that $\mathcal{U}$ satisfies (QU3). In order to prove the admissibility of $\mathcal{U}$ it suffices to show that $a \triangleleft_i b$ implies $a \triangleleft_i^{\mathcal{U}} b$ which is obvious since $a \triangleleft_i b$ implies $\mathsf{st}_i(a, C^i_{a,b}) \leq b$. ■

Hence the corresponding $\mathcal{U}_s$ given by the lemma is a quasi-uniformity on L (which is of course totally bounded). We shall denote it by $\mathcal{U}_F(L, \lhd_1, \lhd_2)$.

Proposition 4.4. *The correspondence* $(L, \lhd_1, \lhd_2) \rightsquigarrow (L, \mathcal{U}_F(L, \lhd_1, \lhd_2))$ *determines a concrete functor* $\Phi : \mathsf{QPFrm} \to \mathsf{TBQUFrm}$.

Proof: It remains to check that for any quasi-proximal map

$$h : (L, \lhd_1^L, \lhd_2^L) \to (M, \lhd_1^M, \lhd_2^M),$$

$$h : (L, \mathcal{U}_F(L, \lhd_1^L, \lhd_2^L)) \to (M, \mathcal{U}_F(M, \lhd_1^M, \lhd_2^M))$$

is a quasi-uniform map. By Lemma 4.1(4) it suffices to show that $h[C^i_{a,b}] \in \mathcal{U}_F(M, \lhd_1^M, \lhd_2^M)$ whenever $a \lhd_i b$ $(i = 1, 2)$.

Suppose $a \lhd_i b$ and apply (SB4) to select $c \in L(\lhd_i)$ such that $a \lhd_i c \lhd_i b$. By (SB3), $a^* \vee c = 1$, which in turn implies that $h(c)^* \leq h(a^*)$:

$$a^* \vee c = 1 \Rightarrow h(a^*) \vee h(c) = 1 \Rightarrow h(c)^* = h(c)^* \wedge (h(a^*) \vee h(c)) = h(c)^* \wedge h(a^*).$$

Hence

$$C^1_{h(c),h(b)} = \{(1, h(c)^*), (h(b), 1)\}$$

refines $h[C^1_{a,b}] = \{(1, h(a^*), (h(b), 1)\}$ and

$$C^2_{h(c),h(b)} = \{(h(c)^*, 1), (1, h(b))\}$$

refines $h[C^2_{a,b}] = \{(h(a^*), 1), (1, h(b))\}$. ■

On the other hand, going back to Proposition 2.5, we have:

Proposition 4.5. *The correspondence* $(L, \mathcal{U}) \rightsquigarrow (L, \lhd_1^{\mathcal{U}}, \lhd_2^{\mathcal{U}})$ *determines a concrete functor* $\Psi : \mathsf{TBQUFrm} \to \mathsf{QPFrm}$.

Proof: Let $h : (L, \mathcal{U}) \to (M, \mathcal{V})$ be a quasi-uniform map. We need to show that

$$h : (L, \lhd_1^{\mathcal{U}}, \lhd_2^{\mathcal{U}}) \to (M, \lhd_1^{\mathcal{V}}, \lhd_2^{\mathcal{V}})$$

is a quasi-proximal map, that is, $a \lhd_i^{\mathcal{U}} b$ implies $h(a) \lhd_i^{\mathcal{V}} h(b)$. But $a \lhd_1^{\mathcal{U}} b$ means that $\mathsf{st}_i(a, U) \leq b$ for some $U \in \mathcal{U}$ and, by the statement (6) in 1.3.1, $\mathsf{st}_i(h(a), h[U]) \leq h(b)$. Since $h[U] \in \mathcal{V}$ this makes $h(a) \lhd_i^{\mathcal{V}} h(b)$, as required. ■

Theorem 4.6. *The functors* Φ *and* Ψ *constitute a concrete isomorphism between* QPFrm *and* $\mathsf{TBQUFrm}$.

Proof: We want to show that $\Psi\Phi = \mathrm{Id}_{\mathsf{QPFrm}}$ and $\Phi\Psi = \mathrm{Id}_{\mathsf{TBQUFrm}}$. After 4.4 and 4.5 there is nothing left to prove for morphisms.

Now for the objects. We have

$$\Phi\Psi((L,\mathcal{U})) = \Phi((L,\lhd_1^{\mathcal{U}},\lhd_2^{\mathcal{U}})) = (L,\mathcal{U}_F(L,\lhd_1^{\mathcal{U}},\lhd_2^{\mathcal{U}})) \quad \text{and}$$

$$\Psi\Phi((L,\lhd_1,\lhd_2)) = \Psi((L,\mathcal{U}_F(L,\lhd_1,\lhd_2))) = (L,\lhd_1^{\mathcal{U}_F(L,\lhd_1,\lhd_2)},\lhd_2^{\mathcal{U}_F(L,\lhd_1,\lhd_2)}),$$

so that we need to prove that

$$\text{(a) } \mathcal{U}_F(L,\lhd_1^{\mathcal{U}},\lhd_2^{\mathcal{U}}) = \mathcal{U} \quad \text{and} \quad \text{(b) } \lhd_i^{\mathcal{U}_F(L,\lhd_1,\lhd_2)} = \lhd_i \text{ for } i=1,2.$$

(a) Let $C^i_{a,b}$ be a subbasic paircover of $\mathcal{U}_F(L,\lhd_1^{\mathcal{U}},\lhd_2^{\mathcal{U}})$. Then $a \lhd_i^{\mathcal{U}} b$, that is, $\mathsf{st}_i(a,U) \leq b$ for some $U \in \mathcal{U}$.

Suppose $i=1$ and let $(u_1,u_2) \in U$. If $u_2 \wedge a \neq 0$ then $u_1 \leq \mathsf{st}_1(a,U) \leq b$ and $(u_1,u_2) \leq (b,1)$; otherwise, $u_2 \leq a^*$ so $(u_1,u_2) \leq (1,a^*)$. Hence $U \leq C^1_{a,b}$ which shows that $C^1_{a,b} \in \mathcal{U}$. The case $i=2$ proceeds similarly. Thus $\mathcal{U}_F(L,\lhd_1^{\mathcal{U}},\lhd_2^{\mathcal{U}}) \subseteq \mathcal{U}$.

Now let $U \in \mathcal{U}$ and select a strong paircover $V \in \mathcal{U}$ such that $V^* \leq U$. Since $\mathcal{U}$ is totally bounded, there is a finite $F \subseteq V$ such that F is still a paircover:

$$F = \{(c_1,d_1),(c_2,d_2)\dots,(c_n,d_n)\}, \quad c_i \wedge d_i \neq 0, \quad \bigvee_{i=1}^{n}(c_i \wedge d_i) = 1.$$

Since $(\mathsf{st}_1(c_i,V),\mathsf{st}_2(d_i,V)) \leq (u_i,v_i)$ for some $(u_i,v_i) \in U$ $(i=1,2,\dots,n)$, it follows that $c_i \lhd_1^{\mathcal{U}} u_i$ and $d_i \lhd_2^{\mathcal{U}} v_i$. It suffices now to show that for

$$C := C^1_{c_1,u_1} \wedge C^1_{c_2,u_2} \wedge \cdots \wedge C^1_{c_n,u_n} \wedge C^2_{d_1,v_1} \wedge C^2_{d_2,v_2} \wedge \cdots \wedge C^2_{d_n,v_n},$$

the corresponding strong paircover C_s (which belongs to $\mathcal{U}_F(L,\lhd_1^{\mathcal{U}},\lhd_2^{\mathcal{U}})$) is a refinement of U.

Any element (x,y) in C is of the form

$$(u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_k}\wedge d^*_{j'_1}\wedge d^*_{j'_2}\wedge\cdots\wedge d^*_{j'_{n-l}}, v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_l}\wedge c^*_{i'_1}\wedge c^*_{i'_2}\wedge\cdots\wedge c^*_{i'_{n-k}})$$

where

$$I = \{i_1,i_2,\dots,i_k\}, \quad I' = \{i'_1,i'_2,\dots,i'_{n-k}\}$$

and

$$J = \{j_1,j_2,\dots,j_l\}, \quad J' = \{j'_1,j'_2,\dots,j'_{n-l}\}$$

are partitions of $\overline{n} = \{1,2,\dots,n\}$.

If $I \cap J \neq \varnothing$ then $(x,y) \leq (u_\alpha, v_\alpha) \in U$ for $\alpha \in I \cap J$. Otherwise, if $I \cap J = \varnothing$ then $I' \cup J' = \overline{n}$ (and $k + l = n$). Since

$$(x,y) \leq (d^*_{j'_1} \wedge d^*_{j'_2} \wedge \cdots \wedge d^*_{j'_{n-l}}, c^*_{i'_1} \wedge c^*_{i'_2} \wedge \cdots \wedge c^*_{i'_{n-k}})$$

this in turn implies that

$$x \wedge y \leq \bigwedge_{i=1}^{n} (c_i^* \vee d_i^*).$$

But

$$\bigvee_{i=1}^{n} (c_i \wedge d_i) = 1 \;\Rightarrow\; \bigwedge_{i=1}^{n} (c_i \wedge d_i)^* = 0 \;\Rightarrow\; \bigwedge_{i=1}^{n} (c_i^* \vee d_i^*) = 0.$$

Hence $x \wedge y = 0$ and $(x,y) \notin C_s$.

(b) If $a \lhd_i b$ then $C^i_{a,b} \in \mathcal{U}_F(L, \lhd_1, \lhd_2)$. Since $\mathsf{st}_i(a, C^i_{a,b}) \leq b$ we have

$$a \;\lhd_i^{\mathcal{U}_F(L,\lhd_1,\lhd_2)}\; b.$$

Conversely, suppose for some $U \in \mathcal{U}_F(L, \lhd_1, \lhd_2)$ we have $\mathsf{st}_i(a, U) \leq b$ ($i = 1$ or $i = 2$). Our aim is to show that $a \lhd_i b$. We may assume that

$$U = C^1_{a^1_1, b^1_1} \wedge C^1_{a^1_2, b^1_2} \wedge \cdots \wedge C^1_{a^1_n, b^1_n} \wedge C^2_{a^2_1, b^2_1} \wedge C^2_{a^2_2, b^2_2} \wedge \cdots \wedge C^2_{a^2_m, b^2_m}$$

where $a^1_\alpha \lhd_1 b^1_\alpha$ for $\alpha = 1, 2, \ldots, n$ and $a^2_\beta \lhd_2 b^2_j$ for $\beta = 1, 2, \ldots m$. Any element (u_1, u_2) in U is of the form

$$\Big(b^1_{\alpha_1} \wedge b^1_{\alpha_2} \wedge \cdots \wedge b^1_{\alpha_k} \wedge (a^2_{\beta'_1})^* \wedge (a^2_{\beta'_2})^* \wedge \cdots \wedge (a^2_{\beta'_{m-l}})^*,$$
$$b^2_{\beta_1} \wedge b^2_{\beta_2} \wedge \cdots \wedge b^2_{\beta_l} \wedge (a^1_{\alpha'_1})^* \wedge (a^1_{\alpha'_2})^* \wedge \cdots \wedge (a^1_{\alpha'_{n-k}})^*\Big)$$

where

$$A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, \quad A' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-k}\}$$

is a partition of $\overline{n} = \{1, 2, \ldots, n\}$ and

$$B = \{\beta_1, \beta_2, \ldots, \beta_l\}, \quad B' = \{\beta'_1, \beta'_2, \ldots, \beta'_{m-l}\}$$

is a partition of $\overline{m} = \{1, 2, \ldots, m\}$. Select $c^1_\alpha, d^1_\alpha \in L$ such that

$$a^1_\alpha \lhd_1 c^1_\alpha \lhd_1 d^1_\alpha \lhd_1 b^1_\alpha \quad \text{for every } \alpha \in \overline{n}$$

and $c^2_\beta, d^2_\beta \in L$ such that

$$a^2_\beta \lhd_2 c^2_\beta \lhd_2 d^2_\beta \lhd_2 b^2_\beta \quad \text{for every } \beta \in \overline{m}.$$

By (SB5), $(c^2_\beta)^* \lhd_1 (a^2_\beta)^*$ for every $\beta \in \overline{m}$ and $(c^1_\alpha)^* \lhd_2 (a^1_\alpha)^*$ for every $\alpha \in \overline{n}$.

Suppose $(u_1, u_2) \in U$ is such that

$$u_1 = \bigwedge_{\alpha\in A} b^1_\alpha \wedge \bigwedge_{\beta\in B'} (a^2_\beta)^* \quad \text{and} \quad u_2 = \bigwedge_{\beta\in B} b^2_\beta \wedge \bigwedge_{\alpha\in A'} (a^1_\alpha)^*,$$

with $A \cup A' = \overline{n}$, $A \cap A' = \varnothing$, $B \cup B' = \overline{m}$, $B \cap B' = \varnothing$. Then set

$$\widetilde{u}_1 = \bigwedge_{\alpha\in A} d^1_\alpha \wedge \bigwedge_{\beta\in B'} (c^2_\beta)^* \quad \text{and} \quad \widetilde{u}_2 = \bigwedge_{\beta\in B} d^2_\beta \wedge \bigwedge_{\alpha\in A'} (c^1_\alpha)^*.$$

Clearly $\widetilde{u}_1 \lhd_1 u_1$ and $\widetilde{u}_2 \lhd_2 u_2$. Therefore $u_1^* \lhd_2 (\widetilde{u}_1)^*$ and $u_2^* \lhd_1 (\widetilde{u}_2)^*$.

Let $j \in \{1, 2\}$, $j \neq i$, and

$$U_1 = \{(u_1, u_2) \in U \mid u_j \wedge a = 0\}, \quad U_2 = \{(u_1, u_2) \in U \mid u_j \wedge a \neq 0\}.$$

This is a partition of U. Now, by the first De Morgan law (which holds in any frame),

$$a \leq \bigwedge\{u_j^* \mid (u_1, u_2) \in U_1\} = (\bigvee\{u_j \mid (u_1, u_2) \in U_1\})^*. \tag{4.6.1}$$

Since U is finite and $\widetilde{u}_j \lhd_j u_j$, we have

$$\bigvee\{\widetilde{u}_j \mid (u_1, u_2) \in U_1\} \ \lhd_j \ \bigvee\{u_j \mid (u_1, u_2) \in U_1\}$$

and then by (SB5)

$$(\bigvee\{u_j \mid (u_1, u_2) \in U_1\})^* \ \lhd_i \ (\bigvee\{\widetilde{u}_j \mid (u_1, u_2) \in U_1\})^*. \tag{4.6.2}$$

But $\{(\widetilde{u}_1, \widetilde{u}_2) \mid (u_1, u_2) \in U\}$ is a paircover of L (it coincides with the paircover

$$C^1_{c^1_1,d^1_1} \wedge C^1_{c^1_2,d^1_2} \wedge \cdots \wedge C^1_{c^1_n,d^1_n} \wedge C^2_{c^2_1,d^2_1} \wedge C^2_{c^2_2,d^2_2} \wedge \cdots \wedge C^2_{c^2_m,d^2_m}).$$

This means that

$$1 = \bigvee_{(u_1,u_2)\in U} (\widetilde{u}_1 \wedge \widetilde{u}_2) \leq \bigvee\{\widetilde{u}_j \mid (u_1, u_2) \in U_1\} \vee \bigvee\{\widetilde{u}_i \mid (u_1, u_2) \in U_2\}$$

which immediately implies that

$$\begin{aligned}(\bigvee\{\widetilde{u}_j \mid (u_1, u_2) \in U_1\})^* &\leq \bigvee\{\widetilde{u}_i \mid (u_1, u_2) \in U_2\} \\ &\leq \bigvee\{u_i \mid (u_1, u_2) \in U_2\} = \mathsf{st}_i(a, U) \leq b.\end{aligned} \tag{4.6.3}$$

By (4.6.1), (4.6.2) and (4.6.3) we have $a \lhd_i b$ as desired. ■

Remark 4.7. The dual adjoint situation between quasi-uniform spaces and quasi-uniform frames established in [15] by functors

$$\Omega : \mathsf{QUnif} \to \mathsf{QUFrm} \quad \text{and} \quad \Sigma : \mathsf{QUFrm} \to \mathsf{QUnif}$$

restricts immediately to a dual adjunction

$$\mathsf{QProx} \leftrightarrows \mathsf{QPFrm}$$

between the categories of quasi-proximal spaces and quasi-proximal frames (since for each totally bounded quasi-uniform space (X, μ), $\Omega(X, \mu)$ is a totally bounded quasi-uniform frame and for each totally bounded quasi-uniform frame $(L, \mathcal{U})$, the quasi-uniform space $\Sigma(L, \mathcal{U})$ is also totally bounded).

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