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ON PRELINDELÖF METRIC SPACES AND THE AXIOM OF CHOICE

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ABSTRACT: A metric space is *Totally Bounded* (also called preCompact sometimes) if it has a finite ε-net for every $\varepsilon > 0$ and it is *preLindelöf* if it has a countable ε-net for every $\varepsilon > 0$. Using the *Axiom of Countable Choice* (**CC**), one can prove that a metric space is topologically equivalent to a Totally Bounded metric space if and only if it is a preLindelöf space if and only if it is a Lindelöf space.

In the absence of **CC**, it is not clear anymore what should be the definition of preLindelöfness. There are two distinguish options. One says that a metric space X is:

(a) preLindelöf if, for every $\varepsilon > 0$, there is countable cover of X by open balls of radius ε [11];
(b) Quasi Totally Bounded if, for every $\varepsilon > 0$, there is countable subset A of X such that the open balls with centers in A and radius ε cover X.

As we will see these two notions are distinct and both can be seen as a good generalization of Totally Boundness.

In this paper it is investigated the choice-free relations between the classes of preLindelöf spaces, Quasi Totally Bounded spaces and other related classes, both for metric and for pseudometric spaces. We will also study in which conditions the subspaces of $\mathbb{R}$ have these properties. To help us to do, it will be proven that the *Axiom of Countable Choice for subspaces* $\mathbb{R}$ ($\mathbf{CC}(\mathbb{R})$) is equivalent to the *partial Axiom of Countable Choice for dense subspaces of* $\mathbb{R}$ ($\mathbf{PCC}(d\mathbb{R})$).

KEYWORDS: preLindelöf space, Quasi Totally Bounded space, Axiom of Countable Choice.
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Introduction

A metric space is called preLindelöf if, for every $\varepsilon > 0$, can be covered by countably many open balls of radius ε [11]. Clearly, both the terminology and the definition are inspired by the definition of preCompact metric spaces which are more often called Totally Bounded metric spaces. Although both names refer to the same notion, one could say that the *term* preCompact is more "topological" and that Totally Bounded refers more to the metric. In

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other words, a metric space is preCompact if, for every $\varepsilon > 0$, can be covered by finitely many open balls of radius ε; and X is Totally Bounded if, for every $\varepsilon > 0$, there is a finite set F such that for every $x \in X$, exists $a \in F$ such that $d(x, a) < \varepsilon$. There is no doubt that both definitions coincide even in ZF, *Zermelo-Fraenkel set theory without the Axiom of Choice*. The same is not true when replacing finitely many by countably many. This leads to the definition of two different classes of metric spaces which are called *preLindelöf* and *Quasi Totally Bounded* spaces, respectively.

This paper was motivated by some results about preLindelöfness from K. Keremedis in [11]. There, he studied this condition and compared it with some other Lindelöf related conditions. Although the results are correct, the notions of preLindelöfness and Quasi Totally Boundness are not clearly separated what lead to some mistakes in the proofs when stating one and using the other. That proofs will be fixed.

It is known that, using the Axiom of Countable Choice, a metric space is preLindelöf iff it is Quasi Totally Bounded iff it is Topologically Totally Bounded iff it is Lindelöf iff it is Second Countable iff it is Separable (Proposition 1.4). We will investigate the relations between these notions in the choice-free environment, for metric and pseudometric spaces. Here, a pseudometric space is just a metric space where the distance between two distinct points can be 0. Priori to these study, one will look to these condition in the class of subspaces of $\mathbb{R}$.

We introduce next some definitions of set-theoretic axioms which will be used throughout the paper. All results take place in the setting of ZF.

Definitions 0.1.

(a) The *Axiom of Countable Choice* (**CC**) states that every countable family of non-empty sets has a choice function.
(b) The *Countable Union Condition* (**CUC**) states that the countable union of countable sets is countable.

We denote by $\mathbf{CC}(\aleph_0)$ the Axiom of Countable Choice restricted to families of countable sets and by $\mathbf{CC}(\mathbb{R})$ the Axiom of Countable Choice for families of subsets of $\mathbb{R}$. Note that $\mathbf{CC}\Rightarrow\mathbf{CUC}\Rightarrow\mathbf{CC}(\aleph_0)$.

Proposition 0.2. ([3, p.76], [8]) *The following conditions are equivalent to* **CC** *(respectively* $\mathbf{CC}(\mathbb{R})$*):*

(i) *every countable family of non-empty sets (resp. subsets of* $\mathbb{R}$*) has an infinite subfamily with a choice function;*

(ii) *for every countable family* $(X_n)_{n\in\mathbb{N}}$ *of non-empty sets (resp. subsets of* $\mathbb{R}$*), there is a sequence which take values in an infinite number of the sets* X_n*.*

Condition (i) of the previous proposition is known as the *partial Axiom of Countable Choice* (**PCC**).

1. Definitions and General Results

Given a (pseudo)metric space, an open ball of radius $\varepsilon > 0$ in X is a set B such that exists $x \in X$ with $B = \{y \in X \mid d(x, y) < \varepsilon\}$. We denote an open ball of radius $\varepsilon > 0$ and center x by $B_\varepsilon(x)$. Clearly, an open ball can have more than one center. This fact motivates us to distinguish between the following two notions ((a) and (b)).

Definitions 1.1. Let (X, d) be a (pseudo)metric space. We say that X is:

(a) *preLindelöf* (PL) if, for every $\varepsilon > 0$, exists a countable family $\mathcal{A}$ of open balls of radius ε such that $X = \bigcup \mathcal{A}$.

(b) *Quasi Totally Bounded* (QTB) if, for every $\varepsilon > 0$, exists a countable set $A \subseteq X$ such that $X = \displaystyle\bigcup_{a\in A} B_\varepsilon(a)$.

(c) *topologically preLindelöf* (TPL) if it is topologically equivalent to a preLindelöf space.

(d) *topologically Quasi Totally Bounded* (TQTB) if it is topologically equivalent to a Quasi Totally Bounded space.

(e) *Totally Bounded* (TB) if for every $\varepsilon > 0$, exists a finite set $A \subseteq X$ such that $X = \displaystyle\bigcup_{a\in A} B_\varepsilon(a)$.

(f) *topologically Totally Bounded* (TTB) if it is topologically equivalent to a Totally Bounded space.

Remark 1.2. Totally Bounded spaces are often called preCompact spaces. Even in ZF, there is no need to distinguished, in the definition of TB, between the cases where the centers of the open balls are given or not. Finite choice is enough in this case.

Definitions 1.3. Let (X, d) be a (pseudo)metric space and $\mathcal{T}(d)$ the induced topology. We say that X is:

(a) Lindelöf (L) if every cover of X by elements of $\mathcal{T}(d)$ has a countable subcover.
(b) Second Countable (SC) if $(X, \mathcal{T}(d))$ has a countable base.
(c) Separable (S) if it has a countable dense subset.

The fact that, for a (pseudo)metric space, these three definitions are equivalent and equivalent to topologically Totally Boundness in ZFC, *Zermelo-Fraenkel set theory with the Axiom of Choice*, is well-known. Maybe less known is that they are also equivalent to the definitions (a)-(d) of 1.1. It is surprising that if a metric space can covered by countably many open balls of radius ε, for every ε, so it can for any other equivalent metric.

Proposition 1.4 (ZF+**CC**)**.** *For a (pseudo)metric space X, the following properties are equivalent:*

(i) *X is Lindelöf;*
(ii) *X is Second Countable;*
(iii) *X is Separable;*
(iv) *X is topologically Totally Bounded;*
(v) *X is Quasi Totally Bounded;*
(vi) *X is topologically Quasi Totally Bounded;*
(vii) *X is preLindelöf;*
(viii) *X is topologically preLindelöf.*

Proof: The equivalences between (i), (ii), (iii) and (iv) can be seen in many introductory books of Topology (e.g. [2]). It is not difficult to see that **CC** suffices to prove these equivalences.

It is clear that, under **CC**, (iv)$\Rightarrow$(vi)$\Leftrightarrow$(viii) and (iii)$\Rightarrow$(v)$\Leftrightarrow$(vii)$\Rightarrow$ (viii).

It is now enough to prove that (vi) implies (iii), which is the same as to prove that (v) implies (iii). Let X be a QTB metric space. By definition, for all $n \in \mathbb{N}$, exists a countable set $A_n \subseteq X$ such that $X = \bigcup_{a \in A_n} B_{1/n}(a)$. The set $A = \bigcup_{n \in \mathbb{N}} A_n$ is dense in X and, by the Countable Union Condition, it is countable. ■

Some of these implications remain valid in ZF. From now on the notation A $\to$ B will be used to say that every space with the property A also have the property B.

Proposition 1.5 (ZF). *For pseudometric spaces, the following implications are valid.*

(a) S $\to$ SC.
(b) SC $\to$ TTB.
(c) TTB $\to$ TQTB $\to$ TPL.
(d) TB $\to$ QTB $\to$ PL.
(e) L $\to$ PL.
(f) S $\to$ QTB.

Proof: The usual proof of (a) remains valid in ZF and (c), (d) and (e) are trivial. The fact that every Second Countable metric space is topologically Totally Bounded relies on the fact that a Second Countable metric space can be embedded in $[0,1]^{\aleph_0}$. It was point out in [4] that this fact holds true in ZF. In [1], it is given a detailed proof.

(f) If (X,d) is Separable, then exists $A \subseteq X$ such that A is countable and dense in X. Clearly, for every $\varepsilon > 0$, $X = \bigcup_{a\in A} B_\varepsilon(a)$. ■

In [11], it is stated that S $\to$ PL, which it is a consequence of (d) and (f).

The implication P $\to$ PL is quite obvious, but it is not immediate to conclude that L $\to$ QTB. Although this last implication is not valid for pseudometric spaces, it is still valid for metric spaces. We will show these two facts in Theorem 1.10 and Proposition 1.6, respectively.

Proposition 1.6. *Every Lindelöf metric space is Quasi Totally Bounded.*

Proof: Let X be a Lindelöf metric space. Fix $\varepsilon > 0$. Since in a metric space the singleton sets are closed, the sets $B_\varepsilon(x)\setminus\{x\}$ are open. Define now $A := \{x \in X \,|\, (\forall y \neq x)\ B_\varepsilon(x) \neq B_\varepsilon(y)\}$.

Consider the open cover of X,

$$\mathcal{U} := \{B_\varepsilon(x)\,|\,x\in A\} \cup \{B_\varepsilon(x)\setminus\{x\}\,|\,x \notin A\}\ .$$

Note that for all x, y in X, $B_\varepsilon(x) \neq B_\varepsilon(y)\setminus\{y\}$ and that for all $x \neq y$ not in A, $B_\varepsilon(x)\setminus\{x\} \neq B_\varepsilon(y)\setminus\{y\}$. This means that every element in $\mathcal{U}$ can only be written in one form.

Since X is Lindelöf, there are $(x_n)_n$ in A and $(y_n)_n$ in $X \setminus A$ such that

$$\{B_\varepsilon(x_n) \,|\, n \in \mathbb{N}\} \cup \{B_\varepsilon(y_n) \setminus \{y_n\} \,|\, n \in \mathbb{N}\}$$

is a countable subcover of $\mathcal{U}$. This implies that $X = \bigcup_{n\in\mathbb{N}} B_\varepsilon(x_n) \cup B_\varepsilon(y_n)$ and then X is a Quasi Totally Bounded metric space. ■

Theorem 1.7. [9, 10] *Every Second Countable space is Lindelöf if and if the Axiom of Countable Choice for subsets of* $\mathbb{R}$ *(*$\mathbf{CC}(\mathbb{R})$*) holds.*

Theorem 1.8. [7] *Every Lindelöf* T_1*-space is compact if and only if* $\mathbf{CC}(\mathbb{R})$ *does not hold.*

Corollary 1.9. *Every Lindelöf space, for which the* T_0*-reflection is* T_1*, is compact if and only if* $\mathbf{CC}(\mathbb{R})$ *does not hold.*

The corollary is straightforward since the T_0-reflection does not modify the topology. In particular, we have that the failure of $\mathbf{CC}(\mathbb{R})$ implies that every Lindelöf pseudometric space is compact.

Theorem 1.10. *Every Lindelöf pseudometric space is Quasi Totally Bounded if and only if* $\mathbf{CC}$ *holds or* $\mathbf{CC}(\mathbb{R})$ *fails.*

Proof: ($\Leftarrow$) If $\mathbf{CC}$ holds, then the usual proof works. If $\mathbf{CC}(\mathbb{R})$ fails every Lindelöf pseudometric space is compact, and then Totally Bounded, which implies QTB.

($\Rightarrow$) Let $(X_n)_n$ be a countable family of non-empty sets. Define $X = \bigcup_n (X_n \times \{n\})$ and

$$d((x,n),(y,m)) := \begin{cases} 0 & \text{if } n = m \\ 1 & \text{if } n \neq m. \end{cases}$$

If $\mathbf{CC}(\mathbb{R})$ holds, by Theorem 1.7, (X, d) is Lindelöf and then by hypothesis is QTB. This means that exists $(x_n)_n$ in X such that $X = \bigcup_n B_{1/2}(x_n)$. Finally, the set $\{x_n \,|\, n \in \mathbb{N}\}$ must intersect all the sets X_n which means that $\mathbf{CC}$ also holds. That is $\mathbf{CC}(\mathbb{R})$ implies $\mathbf{CC}$. ■

The next diagram shows the implications which are known to be true in ZF.

$$\begin{tikzcd}
\mathrm{SC} \arrow[r] & \mathrm{TTB} \arrow[r] & \mathrm{TQTB} \arrow[r] & \mathrm{TPL} \\
\mathrm{S} \arrow[u] \arrow[rr] & & \mathrm{QTB} \arrow[u] \arrow[r] & \mathrm{PL} \arrow[u] \\
& & \mathrm{L} \arrow[u, rightsquigarrow] \arrow[ur] &
\end{tikzcd}$$

$\longrightarrow$ metrics $\rightsquigarrow$ pseudometrics

2. Subspaces of the Reals

Before studying the set-theoretic status of the implications between pre-Lindelöfness, Quasi Totally Boundness and the other properties for metric spaces, we will see what happen for subspaces of $\mathbb{R}$. Clearly, if an implication is not true for subspaces of $\mathbb{R}$ it is also not true for metric spaces in general.

The metric space $\mathbb{R}$, with the Euclidian metric, is Second Countable ($\Rightarrow$ S, TTB, PL, QTB) and it is Lindelöf if and only if $\mathbf{CC}(\mathbb{R})$ holds [9]. So, using Theorem 1.7, it is clear that the condition A $\rightarrow$ L for subspaces of $\mathbb{R}$ is equivalent to $\mathbf{CC}(\mathbb{R})$ with A being any other of conditions defined before.

Before proceeding, we will prove a result that we will help us to show the next theorem.

Proposition 2.1. *The following conditions are equivalent:*

(i) $\mathbf{CC}(\mathbb{R})$*, the Axiom of Countable Choice for subsets of* $\mathbb{R}$*;*
(ii) $\mathbf{PCC}(\mathbb{R})$*, the partial Axiom of Countable Choice for subsets of* $\mathbb{R}$*;*
(iii) $\mathbf{CC}(d\mathbb{R})$*, the Axiom of Countable Choice for dense subspaces of* $\mathbb{R}$*;*
(iv) $\mathbf{PCC}(d\mathbb{R})$*, the partial Axiom of Countable Choice for dense subspaces of* $\mathbb{R}$*.*

Proof: The equivalence (i)$\Leftrightarrow$(ii) is in the Proposition 0.2. It is clear that (i)$\Rightarrow$(iii)$\Rightarrow$(iv). Now, it is only necessary to prove that (iv)$\Rightarrow$(ii).

Let $(A_n)_n$ be a countable family of non-empty subsets of $\mathbb{R}$. Consider the collection $(q_k, s_k)_{k\in\mathbb{N}}$ of all open intervals with rational endpoints and the bijective functions $f_k : \mathbb{R} \to (q_k, s_k)$.

Now we define, for every n, the set $B_n = \bigcup_{k\in\mathbb{N}} f_k(A_n)$. By the way they were builded, each of the sets B_n is dense in $\mathbb{R}$. By (iv) there is a sequence $(b_{\varphi(n)})_n$ with φ an increasing sequence of naturals and $b_{\varphi(n)} \in B_{\varphi(n)}$, for every $n \in \mathbb{N}$. We define also $k(n) := \min\{k \in \mathbb{N} \,|\, b_{\varphi(n)} \in f_k(A_{\varphi(n)})\}$. The sequence $(a_{\varphi(n)})_n$, with $a_{\varphi(n)} := f^{-1}_{k(n)}(b_{\varphi(n)}) \in A_{\varphi(n)}$, is the desired partial choice function. ■

Theorem 2.2. *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every dense subspace of* $\mathbb{R}$ *is Lindelöf;*
(ii) *every dense subspace of* $\mathbb{R}$ *is Separable* [6]*;*
(iii) *every dense subspace of* $\mathbb{R}$ *is Quasi Totally Bounded;*
(iv) *every dense subspace of* $\mathbb{R}$ *is preLindelöf.*

Proof: Every subspace of $\mathbb{R}$ is Lindelöf if and only if every subspace of $\mathbb{R}$ is Separable if and only if $\mathbf{CC}(\mathbb{R})$ holds [9]. Then, it is clear that $\mathbf{CC}(\mathbb{R})$ implies (i) and (ii). From Propositions 1.5 and 1.6, (i)⇒(iii) and (ii)⇒(iii)⇒(iv).

Only remains to prove that (iv)⇒$\mathbf{CC}(\mathbb{R})$. By Proposition 2.1, it suffices to prove that (iv) implies the Axiom of Countable Choice for dense subspaces of $\mathbb{R}$. Let $(A_n)_n$ be a family of dense subspaces of $\mathbb{R}$. Using the fact that every open interval is homeomorphic to $\mathbb{R}$, one can consider A_{2n} dense in $(n, n+1)$ and A_{2n+1} dense in $(-n-1, -n)$. The set $A = \bigcup_n A_n$ is dense in $\mathbb{R}$. By (iv), there is a countable family $(B_n)_n$ such that each B_n is an open ball of radius $1/2$ and $A = \bigcup_n B_n$. Since A is dense in $\mathbb{R}$, each open ball has a unique center and then, for every $n \in \mathbb{N}$, there is only one x_n such that $B_n = B_{1/2}(x_n)$. The sequence $(x_n)_n$ induces a choice function in $(A_n)_n$. ■

Corollary 2.3. *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every subspace of* $\mathbb{R}$ *is Quasi Totally Bounded;*
(ii) *every subspace of* $\mathbb{R}$ *is preLindelöf.*

Proposition 2.4. *Every preLindelöf subspace of* $\mathbb{R}$ *is Quasi Totally Bounded if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

Proof: (⇐) Immediate from the previous result.

(⇒) Let $(A_n)_n$ be a countable family of dense subspaces of $\mathbb{R}$. Without loss of generality, one may consider, for every $n \geq 1$, the set A_n to be a dense subspace of $\left(\sum_{k=1}^{n} k, \sum_{k=1}^{n} k + \frac{1}{n}\right)$.

The set $A = \bigcup_n A_n$ is preLindelöf. For $\varepsilon \in [1/i, i-1]$ and $n > i$, the sets A_n are open ball of radius ε and $\bigcup_{n=1}^{i} A_n$ is Totally Bounded because, in $\mathbb{R}$, Totally Boundness coincide with Boundness. Now, by hypothesis A is also Quasi Totally Bounded which means that there is a sequence $(x_n)_n$ in A such that $A = \bigcup_n B_1(x_n)$. This sequence must take values in every of the sets A_n, which finishes the proof. ■

Corollary 2.5. *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every preLindelöf subspace of* $\mathbb{R}$ *is Separable;*
(ii) *every preLindelöf subspace of* $\mathbb{R}$ *is Lindelöf.*

Proposition 2.6. *Every Quasi Totally Bounded subspace of* $\mathbb{R}$ *is Separable if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

Proof: ($\Leftarrow$) Immediate from Theorem 2.2.

($\Rightarrow$) Every Bounded subset of $\mathbb{R}$ is Totally Bounded and then Quasi Totally Bounded. So, we have that every bounded subset of $\mathbb{R}$ is Separable. But this last condition it is equivalent to say that every subspace of $\mathbb{R}$ is Separable which is equivalent to $\mathbf{CC}(\mathbb{R})$ [9]. ■

To finish this section, we will address a problem raised in [12]. There it is stated that "every sequentially compact subspace of $\mathbb{R}$ is preLindelöf" if and only if the Axiom of Countable Choice holds for sequentially compact subspaces of $\mathbb{R}$ ($\mathbf{CC}(\mathrm{sc}\mathbb{R})$), enlarging a result from [5]. Although the result is true, the proof is not correct because what it is really shown is that "every sequentially compact subspace of $\mathbb{R}$ is QTB" if and only if $\mathbf{CC}(\mathrm{sc}\mathbb{R})$ holds.

Next, we will fix that proof. For doing so, we need some definitions and preliminary results.

Definitions 2.7.

(a) A topological space is *Sequentially Compact* if every sequence has a convergent subsequence.
(b) A metric space is *Complete* if every Cauchy sequence converges.

Lemma 2.8. [5]

(a) *Every Sequentially Compact metric space is Complete.*
(b) *A Bounded subspace of* $\mathbb{R}$ *is Complete if and only if it is Sequentially Compact.*

Proposition 2.9. *The following conditions are equivalent:*

(i) *the Axiom of (Countable) Choice for Complete(Sequentially Compact) subspaces of* $\mathbb{R}$*;*
(ii) $\mathbb{R}$ *is the only Complete and dense subspace of* $\mathbb{R}$*;*
(iii) *the partial Axiom of Countable Choice for Complete(Sequentially Compact) subspaces of* $\mathbb{R}$*;*
(iv) *the partial Axiom of Countable Choice for Sequentially Compact and dense subspaces of* $[0, 1]$.

Proof: The equivalence between (i) and (ii) is in [5] and between (i) and (iii) is in [12]. It is clear that (iii) implies (iv) and then it is only necessary to be shown that (iv) implies (ii).

Let A be a complete and dense subspace of $\mathbb{R}$ and $x \in \mathbb{R}$. Define $A_n := A \cap [x - \frac{1}{n}, x + \frac{1}{n}]$, $n \geq 1$ and $f_n : [x - \frac{1}{n}, x + \frac{1}{n}] \to [0, 1]$ a bijective continuous function. From the completeness of A, one can conclude that each of the spaces $f_n(A_n)$ is sequentially compact, and they are also dense in $[0, 1]$. Since $x \in \overline{A}$, the sets A_n are non-empty and then also the sets $f_n(A_n)$ are non-empty. By (iv) there is a partial choice in $(f_n(A_n))_n$ which induces a partial choice in $(A_n)_n$, $(a_{\varphi(k)})_{k\in\mathbb{N}}$ with φ an increasing sequence on the naturals. The sequence $(a_{\varphi(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence and converges to x, then $x \in A$. So, we have proved that $A = \mathbb{R}$ which means that $\mathbb{R}$ has no proper subspaces which are complete and dense. ■

Theorem 2.10. *The following conditions are equivalent:*
(i) *the Axiom of (Countable) Choice for complete subspaces of* $\mathbb{R}$*;*
(ii) *every Complete subspace of* $\mathbb{R}$ *is Separable;*
(iii) *every Complete subspace of* $\mathbb{R}$ *is Quasi Totally Bounded;*
(iv) *every Complete subspace of* $\mathbb{R}$ *is preLindelöf;*
(v) *every Sequentially Compact subspace of* $\mathbb{R}$ *is Quasi Totally Bounded;*
(vi) *every Sequentially Compact subspace of* $\mathbb{R}$ *is preLindelöf.*

Proof: Every Sequentially Compact metric space is Complete, every Quasi Totally Bounded metric space is preLindelöf and every Separable metric space is Quasi Totally Bounded. Then, (ii)⇒(iii)⇒(iv)⇒(vi) and (iii)⇒(v)⇒(vi). The equivalence between (i) and (ii) is in [5].

We will now show that (vi)⇒(i). From the previous proposition, it is only necessary to prove the partial Countable Choice for Sequentially Compact and dense subspaces of $[0, 1]$. Let $(X_n)_n$ be a countable family of Sequentially Compact and dense subspaces of $[0, 1]$. Without loss of generality, one can consider $X_{2n} \subseteq [n, n + 1]$ and $X_{2n+1} \subseteq [-n - 1, -n]$ dense in the respective intervals. Since each of the spaces X_n is Sequentially Compact, the space $X = \bigcup_n X_n$ is Sequentially Compact unless it has an unbounded sequence. If X has an unbounded sequence, then it induces a partial choice function in the family $(X_n)_n$. If such a sequence does not exist, X is Sequentially Compact and, by (vi), it is preLindelöf. The preLindelöf property implies that there is $(B_n)_n$ such that $X = \bigcup_n B_n$ and each B_n is an open ball of radius 1/2. Since X is dense in $\mathbb{R}$, the centers of the open balls are unique, which induces the desired choice function in $(X_n)_n$. ■

Remark 2.11. In [5] and [12] there are several other conditions equivalent to the ones of Proposition 2.9 and Theorem 2.10.

3. Metric Spaces

In the next two sections we will look at the relations between the notions we have been studied, in the realms of the metric spaces and of the pseudometric spaces. We will mainly be interested in the reverse implications of the ones in the diagram on page 7.

Some of the notions do not depend on the points, then in these cases does not matter if one works with metric or with pseudometric spaces. To make it precise, a pseudometric space is preLindelöf (Lindelöf, Second Countable) if and only if its metric reflection is preLindelöf (Lindelöf, Second Countable). Clearly the same is not true for Quasi Totally Boundness or Separability. So, results involving only preLindelöf, Lindelöf and Second Countable spaces are identical for metric or for pseudometric spaces.

Proposition 3.1. *If every preLindelöf metric space is Quasi Totally Bounded, then* $\mathbf{CC}(\mathbb{R})$ *and* $\mathbf{CC}(\aleph_0)$.

Proof: The first part come from Proposition 2.4. For the second part, let $(X_n)_n$ be a countable family of non-empty countable sets. Define the metric space (X, d) with $X = \bigcup_n (X_n \times \{n\})$ and

$$d((x,n),(y,m)) := \begin{cases} 2 & \text{if } n \neq m \\ \frac{1}{n} & \text{if } n = m \end{cases}.$$

The space (X, d) is preLindelöf. For $\varepsilon > 2$, X is an open ball of radius ε. Let $\varepsilon \leq 2$ and choose k the smallest natural number such that $\frac{1}{k} < \varepsilon$. For $n \geq k$, $X_n \times \{n\}$ is an open ball of radius ε and for $x \in X_n$, $n < k$, $\{x\} = B_\varepsilon(x)$. Since the finite union of countable sets is countable, (X, d) is preLindelöf.

The space (X, d) is QTB, which means that there is a sequence $(x_n)_n$ such that $X = \bigcup_n B_1(x_n)$. This sequence must intersect all of the sets X_n which implies that $(X_n)_n$ has a choice function. ■

Proposition 3.2. *For metric spaces, each of the following statements is true.*

(a) TPL → PL ⇒ **CC**(ℝ)+**CUC**.
(b) TQTB → QTB ⇒ **CC**(ℝ)+**CUC**.

Proof: (⇒**CC**(ℝ)) Every subspace of ℝ is topologically equivalent to a bounded subspace of ℝ and then topologically Totally Bounded. Corollary 2.3 says that every subspace of ℝ is PL (or QTB) if and only if **CC**(ℝ).

(⇒**CUC**) Let $(X_n)_n$ be a countable family of countable sets. We want to prove that its union is still countable. Let (X, d) be a metric space with $X = \bigcup_n (X_n \times \{n\})$ and d the discrete metric. Define also the metric d' on X with

$$d'((x, n), (y, m)) := \begin{cases} 0 & \text{if } (x, n) = (y, m) \\ \max\left\{\frac{1}{n}, \frac{1}{m}\right\} & \text{if } (x, n) \neq (y, m) \end{cases}.$$

It is not hard to see that both metrics induce the discrete topology and that (X, d') is QTB and then also PL. This means that (X, d) is both TQTB and TPL. From our assumptions, (X, d) is PL. The open balls of radius 1 for the metric d are the singleton sets. Since countably many open balls of radius 1 cover X, X must countable itself. ■

Proposition 3.3. *If every Totally Bounded metric space is Second Countable, then the Axiom of Countable Choice holds for families of finite sets.*

Proof: Let $(X_n)_n$ be a countable family of non-empty finite sets. Define $X = \bigcup_n (X_n \times \{n\})$, $Y = X \cup \{\star\}$ and a metric d on Y with $d((x, n), (y, m) = \max\{\frac{1}{n}, \frac{1}{m}\}$ for $(x, n) \neq (y, m)$ and $d((x, n), \star) = \frac{1}{n}$. If $(X_n)_n$ has no choice function, the space (Y, d) is compact(⇒ Totally Bounded), but not Second Countable [4, 1]. ■

Proposition 3.4. *If every Quasi Totally Bounded metric space is Second Countable, then the Countable Union Condition holds (***CUC***).*

Proof: Let $(X_n)_n$ be a countable family of countable sets. Define a metric space (Y, d) as in the proof of the previous proposition. The space (Y, d) is QTB because if one define $Z_n := \{\star\} \cup \bigcup_{k<n} X_k \times \{k\}$, for $\varepsilon > \frac{1}{n}$,

$X = \bigcup_{z \in Z_n} B_\varepsilon(z)$ and Z_n is countable. Since every point in $X = Y \setminus \{\star\}$ is open, X must countable for (Y, d) to be Second Countable. ■

Taking into consideration that S → SC and that QTB → PL, the next corollary is straightforward from the Corollary 2.5, Proposition 2.6 and Proposition 3.4.

Corollary 3.5. *For metric spaces, each of the following statements is true.*
(a) PL → SC ⇒ **CUC**.
(b) QTB → S ⇒ $\mathbf{CC}(\mathbb{R})$+**CUC**.
(c) PL → S ⇒ $\mathbf{CC}(\mathbb{R})$+**CUC**.

Proposition 3.6. *For metric spaces, each of the following statements is true.*
(a) QTB → L ⇒ $\mathbf{CC}(\mathbb{R})$+**CUC**.
(b) PL → L ⇒ $\mathbf{CC}(\mathbb{R})$+**CUC**.

Proof: Since QTB → PL, it is enough to prove (a).

Let $(X_n)_n$ be a countable family of countable sets and define the metric space (X, d) as in the proof of Proposition 3.4. It is not difficult to check that (X, d) is QTB. Using that QTB → L, we have that X is also Lindelöf. But X is a discrete space, and a discrete space only can be Lindelöf if it is countable. We have seen that X must be countable, that is **CUC** is true. Finally, a countable discrete space is Lindelöf if and only if $\mathbf{CC}(\mathbb{R})$ holds [9]. ■

Theorem 3.7. *For metric spaces, the following implications are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*
(i) SC → S;
(ii) SC → QTB;
(iii) SC → PL.

Proof: From Theorem 2.10, one has that $\mathbf{CC}(\mathbb{R})$ is equivalent to (i) [9]. Since S → QTB → PL (Proposition 1.5), (i)⇒(ii)⇒(iii). Every subspace of $\mathbb{R}$ is Second Countable and every subspace of $\mathbb{R}$ is preLindelöf if and only if $\mathbf{CC}(\mathbb{R})$ holds – Corollary 2.3. These two facts make that (iii) implies $\mathbf{CC}(\mathbb{R})$. ■

We finish this section by showing two equivalences between some of the implications we have seen are not provable in ZF, following what it was done in [11].

Proposition 3.8.
(a) *For metric spaces, if PL → L then PL → S.*
(b) *For (pseudo)metric spaces, if QTB → L then QTB → S.*

The result of (a) is stated in [11], but the proof presented there is uncorrect. We will follow that proof until the point some form of choice was used. To be easier for the reader, a complete proof is presented.

Proof: We will only prove (a), because the prove of (b) is similar and easier.

Let (X, d) be a preLindelöf metric space. One can consider the metric bounded by 1. Define $Y = X \times \mathbb{N}$ and a metric in Y

$$d'((x,n),(y,m)) := \begin{cases} \frac{1}{n}d(x,y) & \text{if } n = m \\ \\ \max\{\frac{1}{n}, \frac{1}{m}\} & \text{if } n \neq m \;. \end{cases}$$

With this metric Y is preLindelöf. For $\frac{1}{n} < \varepsilon \leq \frac{1}{n+1}$, $B_\varepsilon(x,n) = \bigcup\limits_{k \geq n} X \times \{k\}$ for any $x \in X$. Since X is preLindelöf, for all $k < n$, exists a countable cover $\mathcal{A}_k$ of X by open balls of radius $k\varepsilon$.

The set $\{A \times \{k\} \,|\, A \in \mathcal{A}_k,\ k < n\} \cup \left\{ \bigcup\limits_{k \geq n} X \times \{k\} \right\}$ is a countable cover by open balls of radius ε.

By hypothesis Y is Lindelöf. Define the sets

$$X_n := \{x \in X \,|\, (\forall y \neq x)\ B_{1/2^n}(x,n) \neq B_{1/2^n}(y,n)\}$$

and consider the open cover $\mathcal{U} = \bigcup_n \mathcal{U}_n$ of Y with

$$\mathcal{U}_n = \left\{B_{1/2^n}(x,n) \,|\, x \in X_n\right\} \cup \left\{B_{1/2^n}(x,n) \setminus \{(x,n)\} \,|\, x \notin X_n\right\} \;.$$

Since Y is Lindelöf, $\mathcal{U}$ has a countable subcover $\mathcal{V}$. As it was done in the proof of the Proposition 1.6, one can show that each element of $\mathcal{V}$ determines uniquely either its center or the element missing for the set to be an open ball of radius $1/2^n$. So, the set

$$C = \left\{x \in X \,|\, (\exists n \in \mathbb{N})\ B_{1/2^n}(x,n) \in \mathcal{V} \cap \mathcal{U}_n \text{ or } B_{1/2^n}(x,n) \setminus \{(x,n)\} \in \mathcal{V} \cap \mathcal{U}_n\right\}$$

is countable. To complete the proof, it is only necessary to see that C is dense in X.

Let $x \in X$. For every n, exists $y \in X$ such that $d'((x,n),(y,n)) < 1/2^n \Leftrightarrow d(x,y) < n/2^n$, which implies that C is dense in X because the sequence $n/2^n$ converges to 0. ∎

4. Pseudometric Spaces

Unlike with the metric spaces, the situation of the equivalence between preLindelöfness and QTBoundness is more clear for pseudometric spaces. We start with the result concerning that equivalence.

Theorem 4.1. *Every preLindelöf metric space is Quasi Totally Bounded if and only if the Axiom of Countable Choice holds.*

Proof: That Countable Choice suffices to prove the equivalence between PL and QTB, one has already seen in 1.4.

Let $(X_n)_n$ be a countable family of non-empty sets. Define a pseudometric d on $X = \bigcup_n (X_n \times \{n\})$ with

$$d((x,n),(y,m)) := \begin{cases} 0 & \text{if } n = m \\ 1 & \text{if } n \neq m. \end{cases}$$

Clearly (X,d) is preLindelöf. If (X,d) is also Quasi Totally Bounded, then there is a sequence $(x_k)_k$ in X such that $X = \bigcup_{k \in \mathbb{N}} B_1(x_k)$. This sequence gives a choice function in $(X_n)_n$. ∎

Proposition 4.2. *For pseudometric spaces, the following conditions are equivalent to* **CC***:*

(i) *every Topologically (Quasi) Totally Bounded space is Quasi Totally Bounded;*
(ii) *every Second Countable space is Quasi Totally Bounded.*

Proof: From Proposition 1.4, **CC**$\Rightarrow$(i) and (i)$\Rightarrow$(ii) because SC $\rightarrow$ TTB $\rightarrow$ TQTB (1.5).

It is only necessary to prove that (ii) implies **CC**. The space (X,d) defined in the proof of Theorem 4.1 is Second Countable and it is Quasi Totally Bounded if and only if **CC** is valid. ∎

Proposition 4.3. *Every (Quasi) Totally Bounded metric space is Separable if and only if the Axiom of Countable Choice holds.*

Proof: That Countable Choice suffices to prove the equivalence between QTB and S, one has already seen (1.4).

Let $(X_n)_n$ be a countable family of non-empty sets. Define a pseudometric d on $X = \bigcup_n (X_n \times \{n\})$ with

$$d((x,n),(y,m)) := \begin{cases} 0 & \text{if } n = m \\ \max\left\{\frac{1}{n}, \frac{1}{m}\right\} & \text{if } n \neq m. \end{cases}$$

We will see that (X, d) is (Quasi) Totally Bounded. Let $\varepsilon > \frac{1}{n}$. By finite choice, one has a finite sequence $(x_k)_{k \leq n}$ with $x_k \in X_k$. Then $X = \bigcup\limits_{k \leq n} B_\varepsilon(x_k, k)$. By our hypothesis, X is also separable which means that there is a sequence $(z_n)_n$ dense in X, that is that takes values in every set X_n what induces the desired choice function. ■

We finish this section enlarging the implication diagram of page 7 to show the set-theoretic status, for pseudometric spaces, of the reverse implications of that diagram.

$$\begin{tikzcd}
\mathrm{SC} \arrow[r, shift left] \arrow[d, squiggly, shift left] & \mathrm{TTB} \arrow[l, dashed, shift left] \arrow[r] & \mathrm{TQTB} \arrow[r] \arrow[d, squiggly, shift left] & \mathrm{TPL} \arrow[d, dashed, shift left] \\
\mathrm{S} \arrow[u, shift left] \arrow[rr, shift left] & & \mathrm{QTB} \arrow[ll, squiggly, shift left] \arrow[u, shift left] \arrow[r, shift left] \arrow[d, dashed, shift left] & \mathrm{PL} \arrow[l, squiggly, shift left] \arrow[u, shift left] \arrow[dl, dashed, shift left] \\
& & \mathrm{L} \arrow[u, dotted, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

$\longrightarrow$ ZF $\rightsquigarrow$ **CC** $\dashrightarrow$ not true in ZF $\cdots\!\!\rightarrow$ **CC** or not **CC**($\mathbb{R}$)

5. Last Remarks

This section is dedicated to discuss some problems related with covers by open balls in the same style it was done with preLindelöf and Quasi Totally Bounded spaces in the previous sections.

It is easy to see that, in a metric space X, if $X = \bigcup_n B_\varepsilon(x_n)$ then for any $\delta > \varepsilon$ the same is also true, i.e. $X = \bigcup_n B_\delta(x_n)$. In ZF, this result does not remain true if one replace the countable family of the centers by a countable family of the open balls.

Example 5.1. As we have seen in Proposition 2.1, if $\mathbf{CC}(\mathbb{R})$ fails there is a countably family of dense subspaces of $\mathbb{R}$ without a choice function. This is the same as to say that there is a family $(X_n)_n$ such that each X_n is dense in the open interval $(2n, 2n+1)$. For $X = \bigcup_n X_n$, each X_n is an open ball of radius 1. But there is no family $(A_n)_n$ of open balls of radius $3/2$ such that $X = \bigcup_n A_n$. The existence of such a family would imply the existence of a choice function in $(X_n)_n$ because, in X, the open balls of radius $3/2$ have unique centers.

It is known that Totally Boundness is preserved by uniform continuity. This is true in ZF and it is not a big surprise that Quasi Totally Boundness is also preserved by uniform continuity. At this point, it is clear that the same is not true for the property of being preLindelöf.

Example 5.2. Consider the subspace A of $\mathbb{R}$ defined as in the proof of Proposition 2.4, i.e. $A = \bigcup_n A_n$ with A_n dense in $\left(\sum_{k=1}^{n} k, \sum_{k=1}^{n} k + \frac{1}{n}\right)$. We have seen that A is preLindelöf. The function $f : A \to \mathbb{R}$, with $f\left(\sum_{k=1}^{n} k + x\right) = \sum_{k=1}^{n} \frac{1}{k} + x$ for $x \in \left(\sum_{k=1}^{n} k, \sum_{k=1}^{n} k + \frac{1}{n}\right)$, is uniformly continuous because it is non-expansive. If $(A_n)_n$ has no partial choice function, $f(A)$ is not preLindelöf because it is dense in $(0, \infty)$ and the the open balls have unique centers.

Our last proposition is dedicated to compare the existence of a countable cover by open balls of radius 1 (or any other fix $\varepsilon > 0$) with the existence of a countable set A such that the space is the union of the open balls of radius 1 with centers in A.

Proposition 5.3. *For metric spaces, the following conditions are equivalent:*

(i) *the Axiom of Countable Choice;*
(ii) *if $X = \bigcup \mathcal{A}$ such that $\mathcal{A}$ is a countable family of open balls of radius 1, then exists a countable set $A \subseteq X$ such that $X = \bigcup_{a \in A} B_1(a)$.*

Proof: (i)⇒(ii) Clear.
(ii)⇒(i) Let $(X_n)_n$ be a countable family of non-empty sets, $X = \bigcup_n X_n \times \{n\}$ and d a metric such that $d((x,n),(y,m)) = 1/2$ if

$(x, n) \neq (y, m)$. Each set $X_n \times \{n\}$ is an open ball of radius 1, and having a countable set of $A \subseteq X$ such that $X = \bigcup_{a \in A} B_1(a)$ is equivalent to have a choice on $(X_n)_n$. ■

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