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CONGRUENCES AND IDEALS ON PEIRCE ALGEBRAS: A HETEROGENEOUS/HOMOGENEOUS POINT OF VIEW

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ABSTRACT: For a Peirce algebra $\mathcal{P}$, lattices $Cong\mathcal{P}$ of all heterogenous Peirce congruences and $Ide\mathcal{P}$ of all heterogenous Peirce ideals are presented. The notions of kernel of a Peirce congruence and the congruence induced by a Peirce ideal are introduced to describe an isomorphism between $Cong\mathcal{P}$ and $Ide\mathcal{P}$. This isomorphism leads us to conclude that the class of the Peirce algebras is ideal determined. Opposed to Boolean modules case, each part of a Peirce ideal $I = (I_1, I_2)$ determines the other one. A similar result is valid to Peirce congruences. A characterization of the simple Peirce algebras is presented coinciding to that given by Brink, Britz and Schmidt in a homogeneous approach.

KEYWORDS: Relation algebras; boolean modules; Peirce algebras; Peirce heterogeneous congruence; Peirce heterogeneous ideal; simple Peirce algebras.
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1. Introduction

Boolean modules were defined and studied by Brink in [1]. A Boolean module is a two-sorted algebra $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ of a Boolean algebra $\mathcal{B}$ and a relation algebra $\mathcal{R}$ (Tarski [9] and Chin and Tarski [3]) that are combined by an operator : (the Peircean operator) a map $\mathcal{R} \times \mathcal{B} \to \mathcal{B}$ taking a relation algebra element and a Boolean algebra element and returning a Boolean algebra element.

A Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is a Boolean module $(\mathcal{B}, \mathcal{R}, :)$ with an additional operator ^{c} (the right cylindrification) a map $\mathcal{B} \to \mathcal{R}$ that creates a relation algebra element from a Boolean algebra element. There is a close relationship between the class of relation algebras and the class of Peirce algebras since every relation algebra gives rise to a Peirce algebra. In [2], Brink, Britz and Schmidt defined a simple Peirce algebra as a Peirce algebra whose underlying Boolean module is simple. There they claim that this definition is equivalent to requiring that the underlying relation algebra is

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simple. But a homogeneous approach was taken on the characterization of a simple Boolean module. A heterogeneous approach will be taken throughout our work: in the study of the lattices of heterogeneous Peirce congruences and of heterogeneous Peirce ideals and in the classification of the simple Peirce algebras.

2. Preliminaries

Peirce algebras are closely related to Boolean modules. We present here the required notions to establish the definition of a Peirce algebra.

Definition 2.1. A *relation algebra* is an algebra $\mathcal{R} = (R, \vee, \wedge, ', \mathrm{o}, 1, ;, \breve{}, e)$ satisfying for each $a, b, c \in R$ the following axioms

R1 $(R, \vee, \wedge, ', \mathrm{o}, 1)$ is a Boolean algebra
R2 $a; (b; c) = (a; b); c$
R3 $a; e = a = e; a$
R4 $a^{\smile\smile} = a$
R5 $(a \vee b); c = a; c \vee b; c$
R6 $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
R7 $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
R8 $a^{\smile}; (a; b)' \leq b'$

Notation. For $a, b \in R$ we also write ab instead of $a; b$.

As usual, for every elements p, q on a Boolean algebra B we define $p \oplus q = (p \wedge q') \vee (p' \wedge q)$. In particular, for every elements a, b on a relation algebra R we define $a \oplus b = (a \wedge b') \vee (a' \wedge b)$.

The standard class of models of relation algebras is the class of proper relation algebras.

Definition 2.2. A *proper relation algebra* over a non-empty set U is a set of binary relations on U that contains the identity relation and is closed with respect to union, intersection, complementation, relational composition and converse. If a proper relation algebra consists of all binary relations defined on U, then this algebra is called the *full relation algebra* and is denoted by $\mathcal{R}(U)$. More precisely, $\mathcal{R}(U)$ is the power set algebra over U^2 endowed with composition ("$;$"), converse ("$\smile$") and identity ("Id") operations defined, for $a, b \subseteq U^2$, by

$a; b = \{(s, t) : \text{ exists } u \in U \text{ such that } (s, u) \in a \text{ and } (u, t) \in b\}$
$a^{\smile} = \{(s, t) : (t, s) \in a\}$
$Id = \{(s, s) : s \in U\}$.

Definition 2.3. An element a of a relation algebra is a *right ideal element* if and only if $a;1 = a$.

The arithmetic of relation algebras can be described by the facts assembled on the following theorem.

Theorem 2.4. *On any relation algebra $\mathcal{R}$ the following hold for any $a, b, c, d \in R$*

$R9$ $\quad e^{\smile} = e, \quad o^{\smile} = o, \quad 1^{\smile} = 1$

$R10$ $\quad a \leq b$ *if and only if* $a^{\smile} \leq b^{\smile}$

$R11$ $\quad (a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}, \quad a'^{\smile} = a^{\smile\prime}$

$R12$ $\quad a; o = o = o; a, \quad 1;1 = 1$

$R13$ $\quad a(b \vee c) = ab \vee ac$

$R14$ $\quad$ *If* $a \leq b$ *then* $ca \leq cb$ *and* $ac \leq bc$.

$R15$ $\quad (ab) \wedge c = o$ *if and only if* $(a^{\smile}c) \wedge b = o$ *if and only if* $(cb^{\smile}) \wedge a = o$

$R16$ $\quad (ab) \wedge (cd) \leq a[(a^{\smile}c) \wedge (bd^{\smile})]d$

$R17$ $\quad$ *If b is a right ideal element then* $a \wedge b = (b \wedge e)a$.

$R18$ $\quad (a \oplus b)^{\smile} = a^{\smile} \oplus b^{\smile}$

Proof: R9-R16, R17 and R18 are proved in [3], [2] and [6], respectively. ∎

Associated to a relation algebra $\mathcal{R}$ Brink introduced the notion of a Boolean $\mathcal{R}$-module $\mathcal{B}$ as a homogeneous algebra, a Boolean algebra $\mathcal{B}$ where each element of $\mathcal{R}$ define an action on $\mathcal{B}$. The roles of $\mathcal{B}$ and $\mathcal{R}$ as universes of a single two-sorted algebra are taken evenly on the next notion.

Definition 2.5. A *Boolean module* is a two-sorted algebra $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ where $\mathcal{B}$ is a Boolean algebra, $\mathcal{R}$ is a relation algebra and $:$ is a mapping $\mathcal{R} \times \mathcal{B} \longrightarrow \mathcal{B}$ (written $a : p$) such that for any $a, b \in R$ and $p, q \in B$ the following assertions are satisfied.

M1 $\quad a : (p \vee q) = a : p \vee a : q$

M2 $\quad (a \vee b) : p = a : p \vee b : p$

M3 $\quad a : (b : p) = (a; b) : p$

M4 $\quad e : p = p$

M5 $\quad \mathrm{o} : p = 0$

M6 $\quad a^{\smile} : (a : p)' \leq p'$

Notation. For $a, b \in R$ and $p \in B$ we also use ap to represent $a : p$.

The standard models of Boolean modules are the proper Boolean modules.

Definition 2.6. A *proper Boolean module* is a two-sorted algebra of a proper Boolean algebra (a field of sets) and a proper relation algebra together with

Peirce product defined on sets and relations. For any relation a over some non-empty set U and any subset p of U, the *Peirce product* : of a and p is defined by

$$a : p = \{s \in U : \text{ there exists } t \in p \text{ such that } (s, t) \in a\}.$$

A *full Boolean module* $\mathcal{M}(U)$ over a non-empty set U is the Boolean module $(\mathcal{B}(U), \mathcal{R}(U), :)$, where $\mathcal{B}(U)$ is the power set algebra over U, $\mathcal{R}(U)$ is the full relation algebra over U, and : is the *Peirce product* defined set-theoretically.

On a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ a part of $\mathcal{B}$, the set of all ideal elements, will take a fundamental role later on.

Definition 2.7. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a Boolean module. An *ideal element* in $\mathcal{M}$ is a $p \in B$ satisfying $1p = p$.

Some facts satisfied on Boolean modules deserve mention.

Theorem 2.8. *On any Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ the following hold for any $a, b \in R$ and $p, q \in B$*

$M7$ *If* $p \leq q$ *then* $ap \leq aq$.
$M8$ *If* $a \leq b$ *then* $ap \leq bp$.
$M9$ $a(p \wedge q) \leq (ap \wedge aq)$
$M10$ $(a \wedge b)p \leq (ap \wedge bp)$
$M11$ $ap \wedge q = 0$ *if and only if* $\breve{a}q \wedge p = 0$
$M12$ *If* $\sum_{i \in I} p_i$ *exists, then so does* $\sum_{i \in I} ap_i$, *and* $a \sum_{i \in I} p_i = \sum_{i \in I} ap_i$.
$M13$ $a0 = 0$
$M14$ $1 : 1 = 1$
$M15$ $(a1)' \leq a'1$
$M16$ $ap \wedge q \leq a(p \wedge \breve{a}q)$
$M17$ $p \leq 1p$
$M18$ *If* p *is an ideal element so is* p'.
$M19$ *If* p *and* q *are ideal elements so is* $p \vee q$.
$M20$ $1p$ *is an ideal element.*
$M21$ *If* p *is an ideal element, then* $aq \wedge p = a(q \wedge p)$.
$M22$ *If* p *is an ideal element, then* $ap = a1 \wedge p$.

Proof: Proved in [1]. ∎

3. Peirce algebras

A Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is a Boolean module $(\mathcal{B}, \mathcal{R}, :)$ with an additional operator ^{c} (the right cylindrification) a map $\mathcal{B} \to \mathcal{R}$ that creates a relation algebra element from each Boolean algebra element.

Definition 3.1. Let $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ be a Boolean algebra and $\mathcal{R} = (R, \vee, \wedge, ', \mathrm{o}, 1, ;, \breve{}, e)$ be a relation algebra. A Peirce algebra is a two-sorted algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ where $(\mathcal{B}, \mathcal{R}, :)$ is a Boolean module and $^c : B \to R$ is a mapping such that for every $p \in B$ and $a \in R$

P1 $\quad p^c : 1 = p$

P2 $\quad (a : 1)^c = a; 1.$

The standard models of Peirce algebras are provided by the class of proper Peirce algebras.

Definition 3.2. A *proper Peirce algebra* is an algebra $(\mathcal{B}, \mathcal{R}, :, ^c)$ in which $(\mathcal{B}, \mathcal{R}, :)$ is a proper Boolean module and ^{c} is the cylindrification operation on sets, defined by $p^c = p \times V$, for V the universal set of $\mathcal{B}$ and p any subset of V.

The *full Peirce algebra* $\mathcal{P}(U) = (\mathcal{B}(U), \mathcal{R}(U), :, ^c)$ over some non-empty set U is the full Boolean module $(\mathcal{B}(U), \mathcal{R}(U), :)$ closed with respect to set-theoretical cylindrification ^{c}. Here $p^c = p \times U = \{(s,t) \in U \times U : s \in p\}$ for p any subset of U.

Example 3.3. The following example can be found on [2]. We can construct a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ through a relation algebra $\mathcal{R} = (R, \vee, \wedge, ', \mathrm{o}, 1, ;, \breve{}, e)$. In fact, if $\mathcal{B}$ is the Boolean algebra of right ideal elements of $\mathcal{R}$, : is ; on $\mathcal{R}$ and ^{c} is de map $\mathcal{B} \to \mathcal{R}$ defined by $p^c = p$, then $\mathcal{P}$ is a Peirce algebra.

Later on, a subclass of Peirce algebras will be quite useful.

Definition 3.4. A Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is *bijective* if and only if, for all $a, b \in R$ we have $a = b$ whenever $ap = bp$ for all $p \in B$.

Theorem 3.5. *On any Peirce algebra $(\mathcal{B}, \mathcal{R}, :, ^c)$ the following hold for each $p, q \in B$ and $a, b \in R$*

P3 $\quad p^c$ *is a right ideal element.*

P4 $\quad 0^c = o, \quad 1^c = 1$

P5 $\quad (p \vee q)^c = p^c \vee q^c$

P6 $p'^c = p^{c'}$
P7 $(p \wedge q)^c = p^c \wedge q^c$
P8 $p = q$ *if and only if* $p^c = q^c$
P9 $p \leq q$ *if and only if* $p^c \leq q^c$
P10 $(a; p^c) : 1 = a : p$
P11 $(a : p)^c = a; p^c$
P12 $a : p = q$ *if and only if* $a; p^c = q^c$
P13 $a : 1 = 0$ *if and only if* $a = o$
P14 $a \wedge p^c = (p^c \wedge e); a, \;\; p^c = (p^c \wedge e); 1$
P15 $a \wedge p^{c\smile} = a; (p^c \wedge e), \;\; p^{c\smile} = 1; (p^c \wedge e)$
P16 $(p^c \wedge e) : q = p \wedge q, \;\; (p^c \wedge e) : 1 = p$
P17 $(a \wedge p^{c\smile}) : 1 = a : p$
P18 $(a \wedge p^{c\smile}) : q = a : (p \wedge q)$
P19 $(p \oplus q)^c = p^c \oplus q^c$.

Proof: P3-P18 are proved in [2]. To prove P19 we use P5, P6 and P7. Thus $(p \oplus q)^c = [(p \wedge q') \vee (p' \wedge q)]^c = (p \wedge q')^c \vee (p' \wedge q)^c = (p^c \wedge q'^c) \vee (p'^c \wedge q^c) = (p^c \wedge q^{c'}) \vee (p^{c'} \wedge q^c) = p^c \oplus q^c$. ∎

In [6] we proved that, for $R = \{\text{o}\}$ any Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is the degenerate Boolean module. The same proof can be used to validate a similar result for Peirce algebras. Next we present a specific proof to this class of algebras.

Proposition 3.6. *In $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ a Peirce algebra, if $R = \{o\}$, then $\mathcal{P}$ is the degenerate Peirce algebra $\mathcal{P} = (\{0\}, \{o\}, :, ^c)$.*

Proof: For any $p \in B$ we have $p^c \in R$ and then $p^c = \text{o}$. Since $0^c = \text{o}$ then $p^c = 0^c$. By P8 we obtain $p = 0$. ∎

4. The lattice $Cong\mathcal{P}$

The lattice of congruences on a given general structure plays a central role both on lattice theory and in the theory of the structure under consideration.

Definition 4.1. Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be a Peirce algebra. The pair $\theta = (\theta_1, \theta_2)$ is a *(Peirce) congruence relation* on $\mathcal{P}$ if θ is a modular congruence on the Boolean module reduct of $\mathcal{P}$ and moreover $p^c \theta_2 q^c$ whenever $p \theta_1 q$, i.e., if θ_1 is a congruence relation on $\mathcal{B}$, θ_2 is a congruence relation on $\mathcal{R}$, $ap\ \theta_1\ bq$ whenever ($p\ \theta_1\ q$ and $a\ \theta_2\ b$) and $p^c \theta_2 q^c$ whenever $p \theta_1 q$.

Let us denote by $Cong\mathcal{P}$ the set of all Peirce congruences defined on a Peirce algebra $\mathcal{P}$.

The set $Cong\mathcal{P}$ is partially ordered by $(\theta_1, \theta_2) \leq (\gamma_1, \gamma_2)$ if and only if $\theta_1 \subseteq \gamma_1$ and $\theta_2 \subseteq \gamma_2$. Our next aim is to define the lattice structure $(Cong\mathcal{P}, \wedge_{\mathcal{P}}, \vee_{\mathcal{P}})$. Since the intersection $\theta \cap \gamma = (\theta_1 \cap \gamma_1, \theta_2 \cap \gamma_2)$ of any two Peirce congruences θ and γ defined on $\mathcal{P}$ is, itself, a Peirce congruence on $\mathcal{P}$, let $\theta \wedge_{\mathcal{P}} \gamma = \theta \cap \gamma$. Let us use $\langle\theta\rangle_{\mathcal{A}}$ to represent the congruence relation generated by the binary relation θ on any (homogeneous or heterogeneous) algebra $\mathcal{A}$, i.e., the intersection of all congruence relations $\theta^{'}$ on $\mathcal{A}$ containing θ,

$$\langle\theta\rangle_{\mathcal{A}} = \cap\{\theta^{'} : \theta^{'} \in Cong\mathcal{A} \text{ and } \theta \subseteq \theta^{'}\}.$$

Now we need to define $\theta \vee_{\mathcal{P}} \gamma = (\tau_1, \tau_2)$.

Attending to results valid on Boolean modules [6] (τ_1, τ_2) defined by

$$\tau_1 = \theta_1 \vee_{\mathcal{B}} \gamma_1 = \langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}$$

$$\tau_2 = \theta_2 \vee_{\mathcal{R}} \gamma_2 = \langle\theta_2 \cup \gamma_2\rangle_{\mathcal{R}}$$

is a modular congruence on the Boolean module reduct of $\mathcal{P}$ and then (τ_1, τ_2) will define a Peirce congruence if and only if for every $\theta, \gamma \in Cong\mathcal{P}$, if $(p, q) \in \langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}$, then $(p^c, q^c) \in \langle\theta_2 \cup \gamma_2\rangle_{\mathcal{R}}$. In fact we have

Proposition 4.2. *Let $\mathcal{P}$ be a Peirce algebra. For $\theta = (\theta_1, \theta_2)$, $\gamma = (\gamma_1, \gamma_2)$ Peirce congruences on a Peirce algebra $\mathcal{P}$, if $(p, q) \in \langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}$, then $(p^c, q^c) \in \langle\theta_2 \cup \gamma_2\rangle_{\mathcal{R}}$.*

Proof: Let $(p, q) \in \langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}$. By [4], we know that there exists a natural number n, a sequence $p_1, p_2, p_3, \cdots, p_n$ of elements in B such that

$$p\theta_1 p_2, p_2\gamma_1 p_3, \cdots, p_{n-1}\gamma_1 q.$$

Since θ and γ are Peirce congruences, then

$$p^c\theta_2 p_2^c, p_2^c\gamma_2 p_3^c, \cdots, p_{n-1}^c\gamma_2 q^c.$$

Since $p_i^c \in R$ for $i = 2, \cdots, n-1$ we have $(p^c, q^c) \in \langle\theta_2 \cup \gamma_2\rangle_{\mathcal{R}}$. ■

The structure $(Cong\mathcal{P}, \wedge_{\mathcal{P}}, \vee_{\mathcal{P}})$ where, for every $\theta, \gamma \in Cong\mathcal{P}$ the operations are defined by

$$\begin{aligned} \theta \wedge_{\mathcal{P}} \gamma &= \theta \cap \gamma = (\theta_1 \cap \gamma_1, \theta_2 \cap \gamma_2) \\ \theta \vee_{\mathcal{P}} \gamma &= \langle\theta \cup \gamma\rangle_{\mathcal{P}} = (\langle\theta_1 \cup \gamma_1\rangle_{\mathcal{B}}, \langle\theta_2 \cup \gamma_2\rangle_{\mathcal{R}}) \end{aligned}$$

is a lattice called the *congruence lattice* $Cong\mathcal{P}$ of $\mathcal{P}$.

5. The lattice $Ide\mathcal{P}$

Usually ideals and congruences are closely related, in the sense that the zero-class of any congruence is an ideal. Here we present the notion of Peirce ideals that, later on, will enable us to confirm that such a relationship exists for Peirce algebras.

Definition 5.1. A (*Peirce*) *ideal* on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is a pair $I = (I_1, I_2)$ satisfying the following conditions

(1) I_1 is a Boolean ideal on $\mathcal{B}$;
(2) If $p \in I_1$ and $a \in R$ then $p^c \in I_2$ and $ap \in I_1$;
(3) (a) I_2 is a Boolean ideal on $\mathcal{R}$;
  (b) If $a \in I_2, c \in R$ then $ac, ca, a^{\smile} \in I_2$;
(4) If $a \in I_2$ and $p \in B$ then $ap \in I_1$.

Such a subset I_2 of R satisfying condition (3) is called an *ideal* of R.

We note that the pair $I = (I_1, I_2)$ is a Peirce ideal on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ if it is a modular ideal [6] on the Boolean module reduct of P and if $p^c \in I_2$ whenever $p \in I_1$.

We denote by $Ide\mathcal{P}$ the set of all ideals on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$. We intend to insert a lattice structure into $Ide\mathcal{P}$. To do so we need to define, for arbitrary Peirce ideals I and J, $I \wedge_{\mathcal{P}} J$ and $I \vee_{\mathcal{P}} J$. It is immediate to put $I \wedge_{\mathcal{P}} J = (I_1 \cap J_1, I_2 \cap J_2)$.

We denote by $\langle X \rangle_{\mathcal{A}}$ the ideal generated by a subset X of any (homogeneous or heterogeneous) algebra $\mathcal{A}$, i.e., the intersection of all ideals I on $\mathcal{A}$ containing X,

$$\langle X \rangle_{\mathcal{A}} = \cap\{I : I \text{ ideal on } \mathcal{A} \text{ and } X \subseteq I\}.$$

In [6] we saw that for $I = (I_1, I_2)$ and $J = (J_1, J_2)$ elements of $Ide\mathcal{M}$, with $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ a Boolean module reduct of $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ we have
$\langle I_1 \cup J_1 \rangle_{\mathcal{B}} = \{p \in B : p \le p_1 \vee p_2, \text{ for some } p_i \in I_1 \cup J_1, i = 1, 2\}$
$\langle I_2 \cup J_2 \rangle_{\mathcal{R}} = \{a \in R : a \le a_1 \vee a_2, \text{ for some } a_i \in I_2 \cup J_2, i = 1, 2\}$

Since for $I, J \in Ide\mathcal{P}$ we have $I \vee_{\mathcal{P}} J = \langle I \cup J \rangle_{\mathcal{P}} = (\langle I_1 \cup J_1 \rangle_{\mathcal{B}}, \langle I_2 \cup J_2 \rangle_{\mathcal{R}})$ on the Boolean module reduct of $\mathcal{P}$ [6] we need to prove that if $p \in \langle I_1 \cup J_1 \rangle_{\mathcal{B}}$ then $p^c \in \langle I_2 \cup J_2 \rangle_{\mathcal{R}}$ to infer that this definition is valid on $\mathcal{P}$.

Proposition 5.2. *For $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ a Peirce algebra and $I = (I_1, I_2)$ and $J = (J_1, J_2)$ Peirce ideals on $\mathcal{P}$ we have $p^c \in \langle I_2 \cup J_2 \rangle_{\mathcal{R}}$ whenever $p \in \langle I_1 \cup J_1 \rangle_{\mathcal{B}}$.*

Proof: If $p \in \langle I_1 \cup J_1 \rangle_{\mathcal{B}}$, then there exists $p_1, p_2 \in I_1 \cup J_1$ such that $p \leq p_1 \vee p_2$. Then $p^c \leq p_1^c \vee p_2^c$ with $p_1^c, p_2^c \in I_2 \cup J_2$ (since I and J are Peirce ideals on $\mathcal{P}$). ■

Therefore, the structure $\mathcal{I}de\mathcal{P} = (Ide\mathcal{P}, \wedge_{\mathcal{P}}, \vee_{\mathcal{P}})$ with, for every $I = (I_1, I_2), J = (J_1, J_2) \in Ide\mathcal{P}$, the operations defined by

$$\begin{aligned} I \wedge_{\mathcal{P}} J &= I \cap J = (I_1 \cap J_1, I_2 \cap J_2) \\ I \vee_{\mathcal{P}} J &= \langle I \cup J \rangle_{\mathcal{P}} = (\langle I_1 \cup J_1 \rangle_B, \langle I_2 \cup J_2 \rangle_R) \end{aligned}$$

is a lattice called the *lattice of ideals* of $\mathcal{P}$.

6. Peirce congruence versus Peirce ideals

The purpose of this paragraph is to establish the notions of kernel of a congruence and of Peirce congruence induced by a Peirce ideal. That will enable us to prove the existence of an isomorphism between the lattices $Cong\mathcal{P}$ and $Ide\mathcal{P}$. This isomorphism will lead us to conclude that the class of Peirce algebra is ideal determined [5], i.e., each ideal is the zero-class of a unique Peirce congruence.

Definition 6.1. If $\theta = (\theta_1, \theta_2) \in Cong\mathcal{P}$ where $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is a Peirce algebra we say that $\mathcal{I}(\theta) = \mathcal{I}^\theta = (\mathcal{I}_1^\theta, \mathcal{I}_2^\theta)$ defined by
$\mathcal{I}_1^\theta = \{p \in B : p\,\theta_1 0\} = [0]_{\theta_1}$
$\mathcal{I}_2^\theta = \{a \in R : a\,\theta_2 \mathrm{o}\} = [\mathrm{o}]_{\theta_2}$
is the *kernel of the congruence* θ.

Proposition 6.2. *The kernel $\mathcal{I}(\theta)$ of a congruence θ on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is an ideal on $\mathcal{P}$.*

Proof: Since the kernel of a modular congruence θ is a modular ideal on the Boolean module reduct of $\mathcal{P}$ [6] we only have to prove that if $p \in \mathcal{I}_1^\theta$ then $p^c \in \mathcal{I}_2^\theta$. If $p \in \mathcal{I}_1^\theta$, then $p\theta_1 0$. Since (θ_1, θ_2) is a Peirce congruence $p^c\theta_2 0^c$, i.e., $p^c\theta_2 \mathrm{o}$. So $p^c \in \mathcal{I}_2^\theta$. ■

Definition 6.3. The *kernel* of a Peirce homomorphism $h = (h_1, h_2) : \mathcal{P} \longrightarrow \mathcal{P}'$ between Peirce algebras is the pair $(\{p \in B : h_1(p) = 0\}, \{a \in R : h_2(a) = \mathrm{o}\})$.

Proposition 6.4. *The kernel of a Peirce homomorphism $h : \mathcal{P} \longrightarrow \mathcal{P}'$ between Peirce algebras is a modular ideal on $\mathcal{P}$.*

Proof: Trivial. ■

Definition 6.5. If $I = (I_1, I_2)$ is a Peirce ideal on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ we define $\mathcal{C}(I) = \mathcal{C}^I = (\mathcal{C}_1^I, \mathcal{C}_2^I)$ by
$p\,\mathcal{C}_1^I q$ if and only if $p \vee i = q \vee i$ for some $i \in I_1$,
$a\,\mathcal{C}_2^I b$ if and only if $a \vee j = b \vee j$ for some $j \in I_2$,
for $p, q \in B$ and $a, b \in R$.

Proposition 6.6. *If $I = (I_1, I_2)$ is a Peirce ideal on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$, then $\mathcal{C}(I)$ is a Peirce congruence on $\mathcal{P}$.*

Proof: Since $\mathcal{C}(I)$ is a modular congruence on the Boolean module reduct of $\mathcal{P}$ we only have to prove that if $p\,\mathcal{C}_1^I q$ then $p^c\,\mathcal{C}_2^I q^c$. If $p\,\mathcal{C}_1^I q$ then there exists $i \in I_1$ such that $p \vee i = q \vee i$. So $(p \vee i)^c = (q \vee i)^c$, i.e., $p^c \vee i^c = q^c \vee i^c$. Since $i \in I_1$ then $i^c \in I_2$ and we have $p^c\,\mathcal{C}_2^I q^c$. ■

Proposition 6.7. *If $I = (I_1, I_2)$ is an ideal on a Peirce algebra, then $\mathcal{I}(\mathcal{C}(I)) = I$.*

Proof: Similar to Boolean algebras. ■

Proposition 6.8. *If $\theta = (\theta_1, \theta_2)$ is a congruence on a Peirce algebra, then $\mathcal{C}(\mathcal{I}(\theta)) = \theta$.*

Proof: Similar to Boolean algebras. ■

Theorem 6.9. *The pair of maps $\mathcal{C} : Ide\mathcal{P} \longrightarrow Cong\mathcal{P}$ (that for each $I \in Ide\mathcal{P}$ assigns the congruence $\mathcal{C}(I)$) and $\mathcal{I} : Cong\mathcal{P} \longrightarrow Ide\mathcal{P}$ (that for each $\theta \in Cong\mathcal{P}$ assigns the ideal $\mathcal{I}(\theta)$) defines an isomorphism between the lattices $Ide\mathcal{P}$ and $Cong\mathcal{P}$.*

Remark 6.10. Corresponding to any assertion valid for Peirce ideals there exists a valid assertion for Peirce congruence and vice versa.

Remark 6.11. The Peirce algebra class is ideal determined.

7. Peirce ideal/congruence determined by one of its two parts

As on Boolean modules [6], on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$, Boolean ideals on $\mathcal{B}$ can exist that are not the Boolean part of any Peirce ideal on $\mathcal{P}$. In fact, let $U = \{p, q\}$ and $\mathcal{P}$ the full Peirce algebra over U. The set $I_1 = \{\emptyset, \{p\}\}$ is a Boolean ideal on $\mathcal{B}(U)$ but, since, for $a \in \mathcal{R}(U)$ given by $a = \{(q, p)\}$, we have $a : \{p\} = \{q\} \notin I_1$, so the pair (I_1, I_2) is not a Peirce ideal on $\mathcal{P}$, for any subset I_2 of R (by 2 of Definition 5.1).

This gives rise to the following definitions.

Definition 7.1. Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be a Peirce algebra.

(1) A Boolean congruence θ_1 on $\mathcal{B}$ is called *pro-Peirce* congruence on $\mathcal{P}$ whenever there exists a congruence θ_2 on $\mathcal{R}$ such that (θ_1, θ_2) is a Peirce congruence on $\mathcal{P}$.

(2) A Boolean ideal I_1 on $\mathcal{B}$ is called *pro-Peirce* ideal on $\mathcal{P}$ if there exists an ideal I_2 of $\mathcal{R}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$.

Now we give a characterization of pro-Peirce ideals on a Peirce algebra.

Proposition 7.2. *On a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ a Boolean ideal I_1 is a pro-Peirce ideal on $\mathcal{P}$ if and only if the pair $(I_1, \{a \in R : ap, a\breve{\ } p \in I_1$ for every $p \in B\})$ is a Peirce ideal on $\mathcal{P}$.*

Proof: It is trivial that if the pair $(I_1, \{a \in R : ap, a\breve{\ } p \in I_1$ for every $p \in B\})$ is a Peirce ideal on $\mathcal{P}$, then the Boolean ideal I_1 is a pro-Peirce ideal on $\mathcal{P}$.

Now suppose that I_1 is a pro-Peirce ideal on $\mathcal{P}$ and let $F_2 = \{a : ap, a\breve{\ } p \in I_1$ for every $p \in B\}$. In [6] we proved that (I_1, F_2) is a modular ideal on the Boolean module reduct of $\mathcal{P}$ so we only have to prove that if for $p \in I_1$ then $p^c \in F_2$. Since I_1 is pro-Peirce ideal on $\mathcal{P}$ there exists a ideal I_2 of $\mathcal{R}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$ and then for $p \in I_1$ we have $p^c \in I_2$ and then $p^{c\breve{\ }} \in I_2$. Since (I_1, I_2) is a Peirce ideal then $p^c s, p^{c\breve{\ }} s \in I_1$ for every $s \in B$. Therefore $p^c \in F_2$. ∎

Next example illustrates Proposition 7.2.

Example 7.3. Let $\mathcal{R}$ be the relation algebra with $R = \{\Lambda, a, b, c\}$, Λ the empty relation, $a = \{(p,p)\}$, $b = \{(q,q)\}$ and $c = \{(p,p), (q,q)\}$ where the operations are defined as in a full relation algebra. Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be the Peirce algebra constructed through $\mathcal{R}$ as in Example 3.3. Since every element of R is a right ideal element then $B = \{\Lambda, a, b, c\}$. Let $I_1 = \{\Lambda, a\}$. Since

$\Lambda\breve{\ }\Lambda = \Lambda\Lambda = \Lambda$	$a\breve{\ }\Lambda = a\Lambda = \Lambda$	$b\breve{\ }\Lambda = b\Lambda = \Lambda$	$c\breve{\ }\Lambda = c\Lambda = \Lambda$
$\Lambda\breve{\ }a = \Lambda a = \Lambda$	$a\breve{\ }a = aa = a$	$b\breve{\ }a = ba = \Lambda$	$c\breve{\ }a = ca = a$
$\Lambda\breve{\ }b = \Lambda b = \Lambda$	$a\breve{\ }b = ab = \Lambda$	$b\breve{\ }b = bb = b$	$c\breve{\ }b = cb = b$
$\Lambda\breve{\ }c = \Lambda c = \Lambda$	$a\breve{\ }c = ac = a$	$b\breve{\ }c = bc = b$	$c\breve{\ }c = cc = c$

then $F_2 = \{d \in R : dp, d\breve{\ } p \in I_1$ for every $p \in B\} = \{\Lambda, a\}$ and so (I_1, F_2) is a Peirce ideal on $\mathcal{P}$ and I_1 is a pro-Peirce ideal.

Corollary 7.4. *On a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ a Boolean congruence θ_1 is a pro-Peirce congruence on $\mathcal{P}$ if and only if the pair $(\theta_1, \{(a, b) \in R \times R :$ there exists $j \in R$ such that $a \vee j = b \vee j$, $jp\theta_1 0$ and $j\breve{\ }p\theta_1 0$ for every $p \in B\}$ is a Peirce congruence on $\mathcal{P}$.*

Proof: Let $F = ([0]_{\theta_1}, \{a : ap, a\breve{\ }p \in [0]_{\theta_1}$ for every $p \in B\})$. By Proposition 7.2 we know that $[0]_{\theta_1}$ is a pro-Peirce ideal on $\mathcal{P}$ if and only if F is a Peirce ideal on $\mathcal{P}$. So it is sufficient to acknowledge that $\mathcal{C}(F) = (\theta_1, \{(a, b) \in R \times R :$ there exists $j \in R$ such that $a \vee j = b \vee j$, $jp\theta_1 0$ and $j\breve{\ }p\theta_1 0$ for every $p \in B\})$. ■

In the theory of Boolean modules we recognized [6] the existence, under assumed conditions, of several modular ideals (I_1, I_2) with the same Boolean part I_1. In particular, we were able to construct the smallest and the greatest modular ideals with the same Boolean part. This does not happen for Peirce algebras. In fact, we can establish the following result.

Proposition 7.5. *For each pro-Peirce ideal I_1 on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ there exists a unique ideal I_2 of $\mathcal{R}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$.*

Proof: Let I_1 be a pro-Peirce ideal on $\mathcal{P}$ and suppose that (I_1, I_2) and (I_1, F_2) are distinct Peirce ideals on $\mathcal{P}$. If $j \in I_2$, then $jp \in I_1$ for every $p \in B$. In particular, for $p = 1$ we have $j1 \in I_1$. Since (I_1, F_2) is a Peirce ideal, then $(j1)^c \in F_2$. But $(j1)^c = j; 1^c = j; 1$ and then $j; 1 \in F_2$. Since $e \leq 1$ we have $j; e \leq j; 1$, i.e., $j \leq j; 1$ and so $j \in F_2$. ■

Corollary 7.6. *For each pro-Peirce congruence θ_1 on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ there exists a unique congruence θ_2 on $\mathcal{R}$ such that (θ_1, θ_2) is a Peirce congruence on $\mathcal{P}$.*

We have already seen that on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$, Boolean ideals on $\mathcal{B}$ can exist that are not the Boolean part of any Peirce ideal on $\mathcal{P}$. We can ask if the same happens for ideals of $\mathcal{R}$, i.e., if there are ideals of $\mathcal{R}$ that are not the relation part of any Peirce ideal on $\mathcal{P}$. Proposition 7.7 states that every ideal of $\mathcal{R}$ is the relation part of a Peirce ideal on $\mathcal{P}$ and gives us the corresponding Peirce ideal construction.

Proposition 7.7. *For each ideal I_2 of $\mathcal{R}$ on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ the pair $(\{p \in B : p^c \in I_2\}, I_2)$ is a Peirce ideal on $\mathcal{P}$.*

Proof: Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be a Peirce algebra, I_2 be an ideal of $\mathcal{R}$ and $I_1 = \{p \in B : p^c \in I_2\}$. Since $0^c = \mathrm{o} \in I_2$ we have $0 \in I_1$ and then $I_1 \neq \emptyset$. Let $p, q \in I_1$. then $p^c, q^c \in I_2$ and $(p \vee q)^c = p^c \vee q^c \in I_2$. So $p \vee q \in I_1$. Let $p \in I_1$ and $q \leq p$. Then $q^c \leq p^c$ and since $p^c \in I_2$ and I_2 is an ideal of R we have $q^c \in I_2$ and then $q \in I_1$. Therefore I_1 is a Boolean ideal on $\mathcal{B}$.

Now we have to prove that if $a \in R$ and $p \in I_1$ then $ap \in I_1$ and $p^c \in I_2$. In fact, $(ap)^c = a; p^c$ and since $p^c \in I_2$ and I_2 is closed by composition by any element of $\mathcal{R}$ we have $a; p^c \in I_2$, i.e., $(ap)^c \in I_2$. So $ap \in I_1$. Trivially if $p \in I_1$ then $p^c \in I_2$.

It remains to be proved that if $a \in I_2$ and $p \in B$, then $ap \in I_1$. In fact, $(ap)^c = a; p^c$ and since $a \in I_2$ and I_2 is closed by composition by any element of $\mathcal{R}$ we have $a; p^c \in I_2$, i.e., $(ap)^c \in I_2$ and then $ap \in I_1$. ∎

Corollary 7.8. *For each congruence θ_2 on $\mathcal{R}$ on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ the pair $(\{(p, q) \in B \times B :$ there exists $i \in B$ such that $p \vee i = q \vee i$ and $i^c \theta_2 \mathrm{o}\}, \theta_2)$ is a Peirce congruence on $\mathcal{P}$.*

Proof: For θ_2 a congruence on $\mathcal{R}$ we know that $F = (\{p \in B : p^c \in [\mathrm{o}]_{\theta_2}\}, [\mathrm{o}]_{\theta_2})$ is a Peirce ideal on $\mathcal{P}$ (Proposition 7.7). So it is sufficient to acknowledge that $\mathcal{C}(F) = (\{(p, q) \in B \times B :$ there exists $i \in B$ such that $p \vee i = q \vee i$ and $i^c \theta_2 \mathrm{o}\}, \theta_2)$. ∎

Next example illustrates Proposition 7.7.

Example 7.9. Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be the Peirce algebra defined in Example 7.3, i.e., $\mathcal{R}$ is the relation algebra with $R = \{\Lambda, a, b, c\}$, Λ the empty relation, $a = \{(p, p)\}$, $b = \{(q, q)\}$ and $c = \{(p, p), (q, q)\}$ where the operations are defined as in a full relation algebra and $\mathcal{P}$ is the Peirce algebra constructed through $\mathcal{R}$ as in Example 3.3. So $B = \{\Lambda, a, b, c\}$, the Peirce product : is ; on $\mathcal{R}$ and ^{c} is de map $\mathcal{B} \to \mathcal{R}$ defined by $s^c = s$ for every $s \in B$. The set $I_2 = \{\Lambda, a\}$ is a ideal of $\mathcal{R}$ and $\{s \in B : s^c \in I_2\} = \{s \in B : s \in I_2\} = \{\Lambda, a\}$. Therefore $(\{\Lambda, a\}, I_2)$ is a Peirce ideal on $\mathcal{P}$ with I_2 as its relation part.

Proposition 7.5 states that for a pro-Peirce ideal I_1 on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ there exists only an ideal I_2 of $\mathcal{R}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$. Similarly, for any ideal I_2 of $\mathcal{R}$ we will show that there exists only a Boolean ideal I_1 on $\mathcal{B}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$.

Proposition 7.10. *For each ideal I_2 of $\mathcal{R}$ on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ there exists a unique Boolean ideal I_1 on $\mathcal{B}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$.*

Proof: Proposition 7.7 assures us that, for every ideal I_2 of $\mathcal{R}$ there exists a Boolean ideal I_1 on $\mathcal{B}$ such that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$. Suppose there is another Boolean ideal F_1 on $\mathcal{B}$ such that (F_1, I_2) is a Peirce ideal on $\mathcal{P}$. If $i \in I_1$, then $i^c \in I_2$. Since (F_1, I_2) is a Peirce ideal on $\mathcal{P}$, then $i^c p \in F_1$ for every $p \in B$. In particular, for $p = 1$ we have $i^c 1 \in F_1$. But by P1 we have $i^c 1 = i$, so $i \in F_1$. ■

Corollary 7.11. *For each congruence θ_2 on $\mathcal{R}$ on a Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ there exists a unique Boolean congruence θ_1 on $\mathcal{B}$ such that (θ_1, θ_2) is a Peirce congruence on $\mathcal{P}$.*

8. Simple Peirce algebras

A simple Peirce algebra is defined in [2] by Brink, Britz and Schmidt as a Peirce algebra whose underlying Boolean module is simple. And in their characterization of a simple Boolean module a homogeneous approach is taken. Although a heterogeneous point of view was followed on our study, our classification (Proposition 8.3) of a simple Peirce algebra agrees with that reached by them.

Definition 8.1. A Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is simple if and only if $Cong\mathcal{P} = \{(\Delta_B, \Delta_R), (\nabla_B, \nabla_R)\}$ (or equivalently, $Ide\mathcal{P} = \{(\{0\}, \{\text{o}\}), (B, R)\}$).

Next result, that will be required in the proof of Proposition 8.3, is proved in [1].

Proposition 8.2. *On a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ a element $p \in B$ is an ideal element if and only if $I_p = \{s \in B : s \leq p\}$ is a Boolean ideal on $\mathcal{B}$ satisfying the condition ($as \in I_1$ whenever $a \in R$ and $s \in I_1$).*

Proposition 8.3. *A Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is simple if and only if $1 : p = 1$ for every $p \neq 0$ in B.*

Proof: Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be a Peirce algebra where $1 : p = 1$ for every $p \neq 0$ in B and let be $I_1 \neq \{0\}$ a pro-Peirce ideal on $\mathcal{P}$. So, there exists a Boolean element $p \neq 0$ such that $p \in I_1$. Since I_1 is pro-Peirce ideal then $ap \in I_1$ for every $a \in R$. In particular, for $a = 1$ we have $1 : p \in I_1$. But $1 : p = 1$, so $1 \in I_1$ and so $I_1 = B$. Trivially $(\{0\}, \{\text{o}\})$ and (B, R) are Peirce ideals and the only ones with, respectively, Boolean part $\{0\}$ and B (Proposition 7.5). Conversely, suppose that exists $p_0 \neq 0$ in B such that $1 : p_0 \neq 1$. Let $q_0 = 1 : p_0$. By M20 we know that q_0 is a ideal element and using Proposition 8.2 we conclude that the set $I_1 = \{s \in B : s \leq q_0\}$ is a Boolean ideal on $\mathcal{B}$ and $as \in I_1$ whenever $a \in R$ and $s \in I_1$. Let $I_2 = \{a \in R : ap \leq q_0 \text{ and } a\breve{\ } p \leq q_0 \text{ for every } p \in B\}$. We will prove that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$. (The use of Proposition 7.2 to prove that (I_1, I_2) is a Peirce ideal on $\mathcal{P}$ is not allowed since the assumption of I_1 being a pro-Peirce ideal on $\mathcal{P}$ is not taken.)

(a) We have to prove that if $s \in I_1$, then $s^c \in I_2$. So we have to prove that if $s \leq q_0$, then $(s^c : p) \leq q_0$ and $(s^{c\smile} : p) \leq q_0$ for every $p \in B$.

We know that $p^c \leq 1$ for every $p \in B$ and using R14 we have $q_0^c; p^c \leq q_0^c; 1$. But by P11 we have $q_0^c; p^c = (q_0^c : p)^c$ and since q_0^c is a right ideal element (P3) we have $q_0^c; 1 = q_0^c$. So $(q_0^c : p)^c \leq q_0^c$ and by P9 we have $(q_0^c : p) \leq q_0$ for every $p \in B$. As $s \leq q_0$ we have $(s^c : p) \leq (q_0^c : p)$ for every $p \in B$, and since $(q_0^c : p) \leq q_0$ then $(s^c : p) \leq q_0$.

We know that $p^c \leq 1$ for every $p \in B$ and using R14 we have $q_0^{c\smile}; p^c \leq q_0^{c\smile}; 1$. By P15 we have $q_0^{c\smile}; 1 = 1; (q_0^c \wedge e); 1$. Since $q_0^c \wedge e \leq q_0^c$ we obtain $q_0^{c\smile}; 1 \leq 1; q_0^c; 1$. Since q_0 is a ideal element then $q_0 = 1 : q_0$ and so $q_0^c = (1 : q_0)^c = 1; q_0^c$ (by P11). As q_0^c is a right ideal element (P3) then $1; q_0^c = 1; q_0^c; 1$ and so $q_0^c = 1; q_0^c; 1$. Therefore $q_0^{c\smile}; 1 \leq q_0^c$ and then $q_0^{c\smile}; p^c \leq q_0^c$. But $q_0^{c\smile}; p^c = (q_0^{c\smile} : p)^c$ so $(q_0^{c\smile} : p)^c \leq q_0^c$, and by P9 we have $(q_0^{c\smile} : p) \leq q_0$. Since $s \leq q_0$ we have $(s^{c\smile} : p) \leq (q_0^{c\smile} : p)$ for every $p \in B$ and then $(s^{c\smile} : p) \leq q_0$.

(b) We have to prove that the set I_2 is a Boolean ideal. In fact, $\mathrm{o} \in I_2$, so $I_2 \neq \emptyset$.

If $a, b \in I_2$, then $ap \leq q_0$, $a\breve{\ } p \leq q_0$, $bp \leq q_0$ and $b\breve{\ } p \leq q_0$ for every $p \in B$. So $(a \vee b)p = ap \vee bp \leq q_0$ and $(a \vee b)\breve{\ } p = a\breve{\ } p \vee b\breve{\ } p \leq q_0$ and then $(a \vee b) \in I_2$.

If $a \in I_2$ and $d \in R$ with $d \leq a$, then for every $p \in B$ we have $dp \leq ap \leq q_0$ and $d\breve{\ } p \leq a\breve{\ } p \leq q_0$. So $d \in I_2$.

(c) We have to prove that if $a \in I_2$ and $b \in R$, then $a\breve{\ }, ab, ba \in I_2$. In fact, if $a \in I_2$, then $ap \leq q_0$ and $a\breve{\ } p \leq q_0$ for every $p \in B$. So $a\breve{\ }\breve{\ } p = ap \leq q_0$. Therefore $a\breve{\ } \in I_2$.

Since $ap \leq q_0$ for every $p \in B$ then $(ab)p = a(bp) \leq q_0$. As $a\breve{\ } p \leq q_0$ then $a\breve{\ } p \in I_1$ and since $ds \in I_1$ whenever ($d \in R$ and $s \in I_1$) we have $(ab)\breve{\ } p = b\breve{\ }(a\breve{\ } p) \in I_1$, i.e., $(ab)\breve{\ } p \leq q_0$. Therefore $ab \in I_2$.

Since $ap \leq q_0$ for every $p \in B$ then $ap \in I_1$, and since $ds \in I_1$ whenever ($d \in R$ and $s \in I_1$) then $(ba)p = b(ap) \in I_1$, i.e., $(ba)p \leq q_0$. As $a\breve{\ } p \leq q_0$ for every $p \in B$, then $(ba)\breve{\ } p = a\breve{\ }(b\breve{\ } p) \leq q_0$. Therefore $ba \in I_2$.

(d) We have to prove that $ap \in I_1$ whenever $a \in I_2$ and $p \in B$. In fact, if $a \in I_2$ then $ap \leq q_0$ for every $p \in B$ so $ap \in I_1$.

Therefore (I_1, I_2) is a Peirce ideal on $\mathcal{P}$. Since $q_0 \neq 1$ and $q_0 \neq 0$ (since by M17 we have $q_0 = 1 : p_0 \geq p_0 \neq 0$) then $I_1 \neq B$ and $I_1 \neq \{0\}$ and so $\mathcal{P}$ is not simple. ■

Corollary 8.4. *Every full Peirce algebra $\mathcal{P}(U)$ over a set U is simple.*

Proof: The relation $\nabla_R \in \mathcal{R}(U)$ satisfied $\nabla_R : p = \nabla_B$ for every $p \neq 0$ in $\mathcal{B}(U)$. ■

Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be a Peirce algebra where its relation algebra $\mathcal{R}$ contains an element $\exists_s$ satisfying $\exists_s 0 = 0$ and $\exists_s p = 1$ for every boolean element $p \neq 0$. As in Boolean modules, we say that this element of R is the *simple quantifier* on $\mathcal{P}$.

Remark 8.5. Let $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ be a bijective Peirce algebra with the relation algebra $\mathcal{R}$ containing the simple quantifier on $\mathcal{P}$. Then $\exists_s = 1$ and $1p = 1$ for every $p \neq 0$. (Since for Boolean element $p \neq 0$, $\exists_s p = 1$ and $1p \geq \exists_s p = 1$ we have $1p = 1$.)

Corollary 8.6. *A bijective Peirce algebra $\mathcal{P} = (\mathcal{B}, \mathcal{R}, :, ^c)$ is simple if and only if $\exists_s \in R$.*

We remark that in [7] we were able to classify the class of simple separable dynamic algebras (following Pratt's definition [8]) as the algebras $\mathcal{D} = (\mathcal{B}, \mathcal{R} = \{\exists_s\}, \langle\rangle)$ with $\mathcal{B}$ arbitrary Boolean algebras. Here, the simple quantifier $\exists_s$ played again a central role.

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