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LIMITS OF ANISOTROPIC AND DEGENERATE ELLIPTIC PROBLEMS

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Abstract: This paper analyzes the behavior of solutions for anisotropic problems of (p_i)-Laplacian type as the exponents go to infinity. We show that solutions converge uniformly to a function that solves, in the viscosity sense, a certain problem that we identify. The results are presented in a two-dimensional setting but can be extended to any dimension.

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1. Introduction

Let Ω be a bounded, smooth and convex domain in $\mathbb{R}^N$, $f \in C(\overline{\Omega})$ a given function and consider the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the exponents satisfy the condition $N < p_i$, for all $i = 1, \cdots, N$.

We are interested in the study of the behavior of solutions of (1) as the exponents go to infinity. The results and arguments we will present are valid in arbitrary dimensions (see the last section) but we restrict the analysis to the two-dimensional setting for the sake of simplicity.

We start with some motivation for our study. The limit of the solutions to

$$-\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = f \tag{2}$$

as p goes to infinity, when $f \equiv 0$ and $u = g$ on $\partial\Omega$, has been extensively studied in the literature (see [2], [3], [5], [7]) and leads naturally to the

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infinity-Laplacian

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Infinity harmonic functions, solutions in the viscosity sense of $-\Delta_\infty u = 0$, solve the optimal Lipschitz extension problem (cf. [1], [2], [16], [17]) and are related to several applications, for instance optimal transportation, image processing and tug-of-war games (see e.g. [10], [12], [13], [24]). When $f > 0$ and $u = 0$ on $\partial\Omega$, the limit of (2) as $p \to +\infty$ has been analyzed in [5]. The solutions u_p converge uniformly to $u_\infty(x) = \operatorname{dist}(x, \partial\Omega)$, which solves the eikonal equation, $|Du_\infty| = 1$, in the viscosity sense.

In more recent years, problems related to PDEs involving variable exponents (like (2) with $p = p(x)$) have been deeply investigated, the interest stemming from applications to elasticity and the modeling of electrorheological fluids. The limit as $p(x) \to \infty$ in Ω, or in some subdomain, is treated in [21], [22], [26] and [27].

If $p_i = p$, for every i, the operator that appears in (1) is the pseudo p-Laplacian

$$-\widetilde{\Delta}_p u = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right].$$

For $-\widetilde{\Delta}_p u = f$, the limit as $p \to +\infty$ was considered in [4], [15]. For $f = 0$ and $u = g \neq 0$ on $\partial\Omega$, the limit equation is

$$-\widetilde{\Delta}_\infty u = -\sum_{i \in I(Du)} \left|\frac{\partial u}{\partial x_i}\right|^2 \frac{\partial^2 u}{\partial x_i^2} = 0, \tag{3}$$

where $I(\xi) = \{1 \le i \le N : |\xi_i| = \max_j |\xi_j|\}$. The operator is known as the pseudo infinity-Laplacian. In [15], also the case $f > 0$ and $u = 0$ on $\partial\Omega$ is discussed.

It is then natural to look for the limit problem of (1), where the anisotropic (p_i)-Laplacian weighs the partial derivatives with different powers. We assume in this paper (in the two-dimensional setting) that there exist sequences $p_{1,n} \to +\infty$ and $p_{2,n} \to +\infty$, with $p_{2,n} \ge p_{1,n} > 2$, and show that the sequence (u_n) of solutions of (1), with $p_1 = p_{1,n}$ and $p_2 = p_{2,n}$, converges uniformly to some function u_∞. Moreover, we either determine u_∞ or identify the limit problem it solves. The case $f \equiv 0$ is contained in [25], where the anisotropic (p, q)-Laplacian is studied.

In the following we will denote

$$\partial_i := \frac{\partial}{\partial x_i} \quad \text{and} \quad \partial_{ii} := \frac{\partial^2}{\partial x_i^2}, \quad i = 1, 2.$$

We next present the main results of this paper, starting with the convergence result. Let u_n be the solution to (1), with $p_1 = p_{1,n}$ and $p_2 = p_{2,n}$.

Theorem 1.1. *Let $f \in C(\overline{\Omega})$. There exists a subsequence of solutions (u_n) that converges to some nontrivial function u_∞ in $C^\beta(\Omega)$, for some $0 < \beta < 1$. Moreover, the limit u_∞ belongs to $W_0^{1,\infty}(\Omega)$, verifies*

$$\max\{\|\partial_1 u_\infty\|_{L^\infty(\Omega)}, \|\partial_2 u_\infty\|_{L^\infty(\Omega)}\} \leq 1,$$

and is a maximizer of the following variational problem

$$\max_K \int_\Omega f v \, dx, \tag{4}$$

where $K = \left\{ v \in W_0^{1,\infty}(\Omega) : \max\{\|\partial_1 v\|_{L^\infty(\Omega)}, \|\partial_2 v\|_{L^\infty(\Omega)}\} \leq 1 \right\}$.

Remark 1.2. We remark that the convergence result, unlike the next Theorem, also holds if the datum f only belongs to $L^q(\Omega)$, with $q > 1$.

In the next theorem we determine the equation verified by the limit u_∞.

Theorem 1.3. *Let $f \in C(\overline{\Omega})$. A function u_∞ obtained as the uniform limit of a subsequence of (u_n) verifies $u_\infty = 0$ on $\partial\Omega$ and is a viscosity solution of the following system of PDEs*

$$\begin{aligned}
&G_\infty(Du_\infty, D^2u_\infty) = 0, && \text{in } \Omega \setminus \text{supp } f,\\
&\max\{|\partial_1 u_\infty|, |\partial_2 u_\infty|\} = 1, && \text{in } \{f > 0\},\\
&-\max\{|\partial_1 u_\infty|, |\partial_2 u_\infty|\} = -1, && \text{in } \{f < 0\},\\
&G_\infty(Du_\infty, D^2u_\infty) \geq 0, && \text{in } \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\},\\
&G_\infty(Du_\infty, D^2u_\infty) \leq 0, && \text{in } \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\},
\end{aligned}$$

with

$$G_\infty(Du_\infty, D^2u_\infty) = \begin{cases} -\theta\,\partial_{11}u_\infty|\partial_1 u_\infty|^2, & \text{if } |\partial_1 u_\infty|^\theta > |\partial_2 u_\infty|, \\ -\partial_{22}u_\infty|\partial_2 u_\infty|^2, & \text{if } |\partial_1 u_\infty|^\theta < |\partial_2 u_\infty|, \\ -\theta\,\partial_{11}u_\infty|\partial_1 u_\infty|^2 & \\ -\partial_{22}u_\infty|\partial_2 u_\infty|^2, & \text{if } |\partial_1 u_\infty|^\theta = |\partial_2 u_\infty|, \end{cases} \tag{5}$$

where

$$\theta = \lim_{n\to+\infty} \frac{p_{1,n}}{p_{2,n}} \in (0,1].$$

This paper is organized as follows: in Section 2 we introduce some definitions and preliminary results. Section 3 is devoted to analyzing the convergence result, while Section 4 deals with the identification of the limit problem. Finally, in the last section, we consider the extension to higher dimensions.

2. Definitions and preliminary results

It is well-known (see [20], and also [6] and [9]) that, for any pair of real numbers $p_1 \le p_2$ and for any $f \in C(\overline{\Omega})$, there exists a unique weak solution of problem (1), that is a function $u \in W_0^{1,p_1,p_2}(\Omega)$ such that

$$\int_\Omega |\partial_1 u|^{p_1-2}\partial_1 u\partial_1 v + \int_\Omega |\partial_2 u|^{p_2-2}\partial_2 u\partial_2 v = \int_\Omega fv, \quad \forall\ v \in W_0^{1,p_1,p_2}(\Omega),$$

where $W_0^{1,p_1,p_2}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p_1,p_2} := \|\partial_1 u\|_{L^{p_1}(\Omega)} + \|\partial_2 u\|_{L^{p_2}(\Omega)}$$

or, equivalently,

$$W_0^{1,p_1,p_2}(\Omega) = \left\{ u \in W_0^{1,p_1}(\Omega) : \partial_i u \in L^{p_i}(\Omega),\ \ i = 1,2 \right\}.$$

The same result holds under less stringent assumptions on the regularity of the given function f. We also recall that, since $p_1 > 2$,

$$W_0^{1,p_1,p_2}(\Omega) \hookrightarrow C_0(\overline{\Omega})$$

and such embedding is compact (see [23], [28] and [29]). We note that the weak solution of (1) can be obtained as the minimizer of the functional

$$J(v) = \frac{1}{p_1}\int_\Omega |\partial_1 v|^{p_1} + \frac{1}{p_2}\int_\Omega |\partial_2 v|^{p_2} - \int_\Omega fv$$

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i, $\theta_{j,i} = 1$ for any j, i, and the operator

$$-\sum_{i\in I} \theta_{i,j}\partial_{ii}u_\infty|\partial_i u_\infty|^2$$

becomes the pseudo infinity-Laplacian given in (3).

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in $W_0^{1,p_1,p_2}(\Omega)$.

Let us now recall the definition of viscosity solution to a nonlinear problem of the form

$$F(x, Du, D^2u) = 0 \quad \text{in } \Omega \tag{6}$$

with a boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{7}$$

being F a continuous function

$$F : \Omega \times \mathbb{R}^2 \times S(2) \to \mathbb{R},$$

with Ω an open set of $\mathbb{R}^2$ and $S(2)$ denoting the set of symmetric matrices $S = \{s_{i,j}\}_{1\le i,j\le 2}$ in $\mathbb{R}^{2\times 2}$.

Definition 2.1. A lower semicontinuous function u defined in Ω is a viscosity supersolution of (6) and (7) (or equivalently a viscosity solution of $F \ge 0$ in Ω and $u \ge 0$ on $\partial\Omega$) if $u \ge 0$ on $\partial\Omega$ and, for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \Omega$, with $u(x_0) = \phi(x_0)$, we have

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \ge 0.$$

An upper semicontinuous function u is a viscosity subsolution of (6) and (7) (or equivalently a viscosity solution of $F \le 0$ in Ω and $u \le 0$ on $\partial\Omega$) if $u \le 0$ on $\partial\Omega$ and, for every $\psi \in C^2(\Omega)$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \Omega$, with $u(x_0) = \psi(x_0)$, we have

$$F(x_0, D\psi(x_0), D^2\psi(x_0)) \le 0.$$

Finally u is a viscosity solution of (6) and (7) if it is both a viscosity supersolution and a viscosity subsolution.

We refer to [8] for more details about the general theory of viscosity solutions, and to [17], [18] for viscosity solutions related to the ∞-Laplacian and the p-Laplacian operators.

We recall the following proposition, stating that weak solutions of problem (1) are also viscosity solutions. In this case, F is defined by

$$F(x, \xi, S) = -(p_1 - 1)|\xi_1|^{p_1-2}s_{11} - (p_2 - 1)|\xi_2|^{p_2-2}s_{22} - f(x).$$

The proof is obtained in a standard way (see for example [5], and also [21]).

Proposition 2.2. *Let u be a continuous weak solution of* (1)*. Then u is a viscosity solution in the sense of Definition 2.1.*

We close this section by introducing the concept of viscosity solution when the function given in (6) is not continuous and independent of x. More precisely, we have a discontinuous function $G : \mathbb{R}^2 \times S(2) \to \mathbb{R}$ and we wish to define the notion of viscosity solution of

$$G(Du, D^2u) = 0, \text{ in } \Omega. \tag{8}$$

We define G^* and G_*, the upper and the lower semicontinuous envelopes of G, respectively, by

$$G^*(\xi, S) = \limsup_{\varepsilon \to 0} \{G(\xi', S') : |\xi - \xi'| + |S - S'| < \varepsilon\}, \tag{9}$$

for $\xi \in \mathbb{R}^2$ and $S \in S(2)$, and

$$G_*(\xi, S) = -(-G)^*(\xi, S). \tag{10}$$

Obviously, if G is continuous, $G = G^* = G_*$.

Definition 2.3. A lower semicontinuous function u defined in Ω is a viscosity supersolution of (8) (or equivalently a viscosity solution of $G \geq 0$) if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \Omega$, with $u(x_0) = \phi(x_0)$, we have

$$G^*(D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

An upper semicontinuous function u is a viscosity subsolution of (8) (or equivalently a viscosity solution of $G \leq 0$) if for every $\psi \in C^2(\Omega)$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \Omega$, with $u(x_0) = \psi(x_0)$, we have

$$G_*(D\psi(x_0), D^2\psi(x_0)) \leq 0.$$

Finally u is a viscosity solution of (8) if it is both a viscosity supersolution and a viscosity subsolution.

We underline that this Definition is needed in Section 4, when computing the limit equation in the points where f vanishes. Indeed, in this case, the function G_∞ that appears in the limit problem, that u_∞ solves in the viscosity

Note that $\theta_{j,j} = 1$ and $\theta_{j,i} = 1/\theta_{i,j}$. All the results in Sections 2 and 3 can be extended in a natural way. Indeed, the following theorem holds.

Theorem 5.1. *There exists a unique weak solution u_n to (1), with $p_i = p_{i,n}$, which is also a viscosity solution of the same problem. Moreover, there exists a subsequence of (u_n) that converges uniformly to a function u_∞ that satisfies*

$$\max_{1 \leq i \leq N} \{\|\partial_i u_\infty\|_{L^\infty(\Omega)}\} \leq 1,$$

and maximizes the problem

$$\max_K \int_\Omega f v,$$

with

$$K = \left\{ v \in W_0^{1,\infty}(\Omega), \max_{1 \leq i \leq N} \{\|\partial_i v\|_{L^\infty(\Omega)}\} \leq 1 \right\}.$$

It is also possible to identify the limit problem.

Theorem 5.2. *Let $f \in C(\overline{\Omega})$. A function u_∞, obtained as the uniform limit of a subsequence of (u_n), verifies $u_\infty = 0$ on $\partial\Omega$ and the following system of PDEs, in the viscosity sense:*

$$\begin{aligned}
&-\sum_{i \in I} \theta_{i,j} \partial_{ii} u_\infty |\partial_i u_\infty|^2 = 0, && \text{in } \Omega \setminus \text{supp } f,\\
&\max_{1 \leq i \leq N} \{|\partial_i u_\infty|\} = 1, && \text{in } \{f > 0\},\\
&-\max_{1 \leq i \leq N} \{|\partial_i u_\infty|\} = -1, && \text{in } \{f < 0\},\\
&-\sum_{i \in I} \theta_{i,j} \partial_{ii} u_\infty |\partial_i u_\infty|^2 \geq 0, && \text{in } \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\},\\
&-\sum_{i \in I} \theta_{i,j} \partial_{ii} u_\infty |\partial_i u_\infty|^2 \leq 0, && \text{in } \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\},
\end{aligned}$$

for all $j = 1, \ldots, N$, where

$$I = \left\{ 1 \leq i \leq N : |\partial_i u_\infty| = \max_{1 \leq k \leq N} |\partial_k u_\infty|^{\theta_{k,i}} \right\},$$

and the $\theta_{j,i}$ are given by (19).

We remark that we recover the results already known for the pseudo p-Laplacian (see [4], [15] and references therein). In fact, when $p_i = p$ for any

Remark 4.5. Note that the arguments used in part two of the previous proof give an alternative proof of Proposition 5.1, Part II, of [5].

Remark 4.6. The proof of the first part of Theorem 1.3 is essentially contained in [25], where the authors study the case in which $f = 0$ in Ω, that is $\Omega \setminus \operatorname{supp} f = \Omega$; they assume a non homogeneous Dirichlet boundary condition, i.e. $u = g \not\equiv 0$ on $\partial\Omega$ and $g \in \mathrm{Lip}(\partial\Omega)$. Our proof extends equally to the non homogeneous Dirichlet setting but, for simplicity, we only considered the case $u_n = 0$ on $\partial\Omega$.

Remark 4.7.

1. If $f > 0$ then $u_\infty = \mathrm{dist}_\infty(\cdot, \partial\Omega)$ in Ω, as an immediate consequence of Lemma 4.2 with $D = \Omega$. Moreover, this is the unique possible limit, since u_∞ is a maximizer in (4), and $\mathrm{dist}_\infty(\cdot, \partial\Omega) \in K$. Then
$$\int_\Omega f \mathrm{dist}_\infty(\cdot, \partial\Omega)\, dx \leq \int_\Omega f u_\infty,\ \forall v \in K$$
and this inequality is strict unless $u_\infty = \mathrm{dist}_\infty(\cdot, \partial\Omega)$. In addition, $\mathrm{dist}_\infty(\cdot, \partial\Omega)$ is the unique solution to equation
$$\max\{|\partial_1 u_\infty|, |\partial_2 u_\infty|\} = 1$$
in the viscosity sense, see [14].
2. If $f \geq 0$, $u_\infty = \mathrm{dist}_\infty(\cdot, \partial\Omega)$ in $\operatorname{supp} f$, while it solves
$$G_\infty(Du_\infty, D^2 u_\infty) = 0 \tag{18}$$
in the interior of the set where $f = 0$. We have also uniqueness of the limit u_∞ in this case. If, in addition, the points in which $\mathrm{dist}_\infty(\cdot, \partial\Omega)$ is not differentiable lay in $\operatorname{supp} f$, then $u_\infty = \mathrm{dist}_\infty(\cdot, \partial\Omega)$ in Ω, since this function verifies (18), which has a unique solution.
3. If $f < 0$ or $f \leq 0$ we obtain analogous results with the function $-\mathrm{dist}_\infty(\cdot, \partial\Omega)$.

5. Higher dimensions

For completeness, we extend here the results of the previous sections to dimension $N \geq 3$. We recall that the exponents p_i are such that there exist sequences $p_{i,n} \to +\infty$, for all $i = 1, ..., N$, with
$$\lim_{n\to+\infty} \frac{p_{j,n}}{p_{i,n}} = \theta_{j,i} \in (0, \infty), \quad j, i = 1, ..., N. \tag{19}$$

sense (see Theorem 1.3), is
$$G_\infty(\xi, S) = \begin{cases} -\theta s_{11}|\xi_1|^2 & \text{for } |\xi_1|^\theta > |\xi_2| \\ -s_{22}|\xi_2|^2 & \text{for } |\xi_1|^\theta < |\xi_2| \\ -\theta s_{11}|\xi_1|^2 - s_{22}|\xi_2|^2 & \text{for } |\xi_1|^\theta = |\xi_2|, \end{cases}$$
which is discontinuous. So we have to characterize its upper and lower semicontinuous envelopes, $(G_\infty)^*$ and $(G_\infty)_*$. For the proof of the next lemma, see [25] and also [15].

Lemma 2.4. *The upper semicontinuous envelope of G_∞ is given by*
$$(G_\infty)^*(\xi, S) = \begin{cases} -\theta s_{11}|\xi_1|^2 & \textit{for } |\xi_1|^\theta > |\xi_2| \\ -s_{22}|\xi_2|^2 & \textit{for } |\xi_1|^\theta < |\xi_2| \\ \max\Big\{ -\theta s_{11}|\xi_1|^2, -s_{22}|\xi_2|^2, & \\ \qquad -\theta s_{11}|\xi_1|^2 - s_{22}|\xi_2|^2 \Big\} & \textit{for } |\xi_1|^\theta = |\xi_2|. \end{cases}$$
The lower semicontinuous envelope has the same expression except for the max *which is replaced by the* min.

3. A priori estimates and convergence

In this section, we prove Theorem 1.1, i.e., that there exists a subsequence of (u_n), the sequence of solutions to (1) with $p_1 = p_{1,n}$ and $p_2 = p_{2,n}$, that converges uniformly to some function u_∞.

Proof of Theorem 1.1: The weak solution u_n of problem (1), with $p_i = p_{i,n}$ fixed, is the minimum of the functional
$$J(v) = \frac{1}{p_{1,n}} \int_\Omega |\partial_1 v|^{p_{1,n}} + \frac{1}{p_{2,n}} \int_\Omega |\partial_2 v|^{p_{2,n}} - \int_\Omega f v$$
in $W_0^{1,p_{1,n},p_{2,n}}(\Omega)$. From $J(u_n) \leq J(0)$, we obtain
$$\frac{1}{p_{1,n}} \int_\Omega |\partial_1 u_n|^{p_{1,n}} + \frac{1}{p_{2,n}} \int_\Omega |\partial_2 u_n|^{p_{2,n}} \leq \int_\Omega f u_n$$

and so

$$\frac{1}{p_{i,n}}\int_\Omega |\partial_i u_n|^{p_{i,n}} \le \int_\Omega f u_n, \quad i=1,2.$$

Applying Hölder's inequality to the right hand side of the previous inequality and then a Poincaré type inequality (see [11]), we obtain

$$\frac{1}{p_{i,n}}\int_\Omega |\partial_i u_n|^{p_{i,n}} \le \|f\|_{L^{p'_{i,n}}(\Omega)}\|u_n\|_{L^{p_{i,n}}(\Omega)} \le C(|\Omega|,f)p_{i,n}\|\partial_i u_n\|_{L^{p_{i,n}}(\Omega)},$$

for $i=1,2$. Simplifying, we arrive at

$$\|\partial_i u_n\|_{L^{p_{i,n}}(\Omega)} \le (C(|\Omega|,f)p_{i,n}^2)^{\frac{1}{p_{i,n}-1}}, \quad i=1,2.$$

Then, for any $1<r<p_{1,n}\le p_{2,n}$ fixed,

$$\|\partial_i u_n\|_{L^r(\Omega)} \le \|\partial_i u_n\|_{L^{p_{i,n}}(\Omega)}|\Omega|^{\frac{p_{i,n}-r}{p_{i,n}r}} \le (C(|\Omega|,f)p_{i,n}^2)^{\frac{1}{p_{i,n}-1}}|\Omega|^{\frac{p_{i,n}-r}{p_{i,n}r}}, \tag{11}$$

which means that (u_n) is uniformly bounded in $W_0^{1,r}(\Omega)$, for any $1<r<p_{1,n}$. We may then select a subsequence, still indexed by n, such that

$$u_n \rightharpoonup u_\infty \quad \text{in } W_0^{1,r}(\Omega),$$

for some $u_\infty \in W_0^{1,r}(\Omega)$. By the lower semicontinuity of the $L^r(\Omega)$-norm, we obtain

$$\begin{aligned}\|\partial_i u_\infty\|_{L^r(\Omega)} &\le \liminf_{n\to+\infty}\|\partial_i u_n\|_{L^r(\Omega)} \le \liminf_{n\to+\infty}\|\partial_i u_n\|_{L^{p_{i,n}}(\Omega)}|\Omega|^{\frac{p_{i,n}-r}{p_{i,n}r}}\\ &\le \liminf_{n\to+\infty}(C(|\Omega|,f)p_{i,n}^2)^{\frac{1}{p_{i,n}-1}}|\Omega|^{\frac{p_{i,n}-r}{p_{i,n}r}} = |\Omega|^{\frac{1}{r}}. \end{aligned}\tag{12}$$

This inequality holds for any sequence $\{r_h\}_{h\in\mathbb{N}} \nearrow +\infty$. Indeed, such a sequence being fixed, we may select, by diagonalization, a subsequence of (u_n) such that

$$u_n \rightharpoonup u_\infty \quad \text{in } W_0^{1,r_h}(\Omega), \quad \forall\ h\in\mathbb{N}.$$

Writing (12) for all r_h and letting $r_h\to+\infty$ gives

$$\|\partial_i u_\infty\|_{L^\infty(\Omega)} \le 1,$$

for $i=1,2$, and so

$$\max\left\{\|\partial_1 u_\infty\|_{L^\infty(\Omega)}, \|\partial_2 u_\infty\|_{L^\infty(\Omega)}\right\} \le 1.$$

Moreover, by the compact Sobolev embedding,

$$u_n \to u_\infty \quad \text{in } C^\beta(\overline{\Omega}),\ 0<\beta<1. \tag{13}$$

Taking the limit as $n\to\infty$, we conclude that

$$\max\{|\partial_1\phi(x_0)|, |\partial_2\phi(x_0)|\} \ge 1,$$

otherwise the left-hand side in (17) goes to zero, while $f(x_0)>0$. Next, to prove u_∞ is a subsolution, we consider $x_0=(x_{0,1},x_{0,2})\in\{f>0\}$ and we take D the square with vertices $x^1_{0,\varepsilon}=(x_{0,1}+\varepsilon,x_{0,2})$, $x^1_{0,-\varepsilon}=(x_{0,1}-\varepsilon,x_{0,2})$, $x^2_{0,\varepsilon}=(x_{0,1},x_{0,2}+\varepsilon)$ and $x^2_{0,-\varepsilon}=(x_{0,1},x_{0,2}-\varepsilon)$, which is contained in $\{f>0\}$ for ε sufficiently small. By Lemma 4.2 and the definition of ψ, we know that

$$u_\infty(x_0) = \inf_{y\in\partial D}\{u_\infty(y)+|x_0-y|_\infty\} \le u_\infty(x^1_{0,-\varepsilon})+\varepsilon \le \psi(x^1_{0,-\varepsilon})+\varepsilon.$$

Taking into account that $u_\infty(x_0)=\psi(x_0)$ and rearranging the previous expression, we get

$$\frac{\psi(x_0)-\psi(x^1_{0,-\varepsilon})}{\varepsilon} \le 1.$$

Passing to the limit as $\varepsilon\to0$ we get $\partial_1\psi(x_0)\le 1$. Arguing analogously with the point $x^1_{0,\varepsilon}$, we get $\partial_1\psi(x_0)\ge -1$. So we have proved that $|\partial_1\psi(x_0)|\le 1$. The proof that $|\partial_2\psi(x_0)|\le 1$ runs in the same way.

3. Let x_0 be in $\{f<0\}$. This case is analogous to the previous one, so we omit the proof.

4. Let x_0 be in $\Omega\cap\partial\{f>0\}\setminus\partial\{f<0\}$. In other words we have that $f(x_0)=0$. Then there exists a sequence $x_n\to x_0$ such that $u_n-\phi$ attains a minimum in x_n and $f(x_n)\ge 0$ for any $n\in\mathbb{N}$. So we can argue as in the first step of the proof to obtain

$$(G_\infty)^*(Du_\infty, D^2u_\infty) \ge 0.$$

The subsolution case is analogous.

5. Let x_0 be in $\Omega\cap\partial\{f<0\}\setminus\partial\{f>0\}$. This case is analogous to the previous one, so we omit the proof.

■

Remark 4.4. We stress that for points in $\Omega\cap\partial\{f>0\}\cap\partial\{f<0\}$ the only thing we can say is that

$$\max\{\|\partial_1 u_\infty\|_{L^\infty(\Omega)}, \|\partial_2 u_\infty\|_{L^\infty(\Omega)}\} \le 1$$

since if $x_0\in\Omega\cap\partial\{f>0\}\cap\partial\{f<0\}$ then $f(x_0)=0$ but $f(x_n)$ can either be greater than or less than zero.

and we deduce from (16) that

$$-\partial_{22}\phi(x_0)|\partial_2\phi(x_0)|^2 \geq 0.$$

If $|\partial_1\phi(x_0)|^\theta > |\partial_2\phi(x_0)|$ (note that in this case $|\partial_1\phi(x_0)| \neq 0$),

$$\left(\frac{|\partial_1\phi(x_n)|^{\frac{p_{1,n}-4}{p_{2,n}-4}}}{|\partial_2\phi(x_n)|}\right)^{p_{2,n}-4} \to +\infty,$$

and then, again from (16)

$$-\theta\,\partial_{11}\phi(x_0)|\partial_1\phi(x_0)|^2 \geq 0.$$

In the case $|\partial_1\phi(x_0)|^\theta = |\partial_2\phi(x_0)|$, we argue by contradiction supposing

$$-\theta\partial_{11}\phi(x_0)|\partial_1\phi(x_0)|^2 < 0 \quad \text{and} \quad -\partial_{22}\phi(x_0)|\partial_2\phi(x_0)|^2 < 0.$$

Note that these inequalities imply

$$-\theta\partial_{11}\phi(x_0)|\partial_1\phi(x_0)|^2 - \partial_{22}\phi(x_0)|\partial_2\phi(x_0)|^2 < 0,$$

and also that

$$\partial_1\phi(x_0) \neq 0 \qquad \text{and} \qquad \partial_2\phi(x_0) \neq 0.$$

Suppose first that, for infinitely many n, we have

$$\frac{|\partial_1\phi(x_n)|^{p_{1,n}-4}}{|\partial_2\phi(x_n)|^{p_{2,n}-4}} \geq 1;$$

going back to (16), along a subsequence $n_i \to +\infty$, we get a contradiction. Also if

$$\frac{|\partial_1\phi(x_n)|^{p_{1,n}-4}}{|\partial_2\phi(x_n)|^{p_{2,n}-4}} < 1,$$

we reach a contradiction (as before), using the fact that

$$\frac{|\partial_2\phi(x_n)|^{p_{2,n}-4}}{|\partial_1\phi(x_n)|^{p_{1,n}-4}} > 1.$$

The fact that u_∞ is a viscosity subsolution of $G_\infty(Du_\infty, D^2u_\infty)$ can be proved analogously.

2. Let $x_0 \in \{f > 0\}$. There is a sequence $x_n \to x_0$, $x_n \in \{f > 0\}$ such that $u_n - \phi$ reaches a minimum at x_n, for any $n \in \mathbb{N}$. As u_n are viscosity solutions of (1), it holds that

$$-(p_{1,n}-1)|\partial_1\phi|^{p_{1,n}-2}\partial_{11}\phi(x_n) - (p_{2,n}-1)|\partial_2\phi|^{p_{2,n}-2}\partial_{22}\phi(x_n) \geq f(x_n). \quad (17)$$

Next, we show that u_∞ maximizes (4) and so u_∞ is nontrivial. We have, for n fixed,

$$\frac{1}{p_{1,n}}\int_\Omega |\partial_1 u_n|^{p_{1,n}} + \frac{1}{p_{2,n}}\int_\Omega |\partial_2 u_n|^{p_{2,n}} - \int_\Omega f u_n \leq \frac{|\Omega|}{p_{1,n}} + \frac{|\Omega|}{p_{2,n}} - \int_\Omega fv,$$

for any $v \in K$. Passing to the limit in the previous expression, we obtain, using (13), that

$$\int_\Omega fv \leq \int_\Omega fu_\infty,$$

for any function $v \in K$.

■

4. Identifying the limit u_∞

In this section, we first derive some properties of the function u_∞, which will be useful to determine the limit problem it satisfies. To this end, we first show that the limit u_∞ is not only the maximizer of (4) in Ω, but also in any subset $D \subset \Omega$.

Lemma 4.1. *If u_∞ is a maximizer of (4), then it is also a maximizer of*

$$\max_{v\in\widetilde{K}} \int_D fv,$$

where $D \subset \Omega$ is open and smooth, and

$$\widetilde{K} = \left\{ v \in W^{1,\infty}(D) \ : \ \max\{\|\partial_1 v\|_{L^\infty(D)}, \|\partial_2 v\|_{L^\infty(D)}\} \leq 1, \ v|_{\partial D} = u_\infty|_{\partial D} \right\}.$$

Proof: Since $u_\infty|_D \in \widetilde{K}$, it follows that

$$\max_{v\in\widetilde{K}} \int_D fv \geq \int_D fu_\infty.$$

By contradiction, suppose that

$$\max_{v\in\widetilde{K}} \int_D fv > \int_D fu_\infty.$$

This implies that there exists $v^* \in \widetilde{K}$ such that $\int_D fv^* > \int_D fu_\infty$. But then, if we define

$$u^* = \begin{cases} v^* & \text{in D} \\ u_\infty & \text{in } \Omega \setminus \text{D}, \end{cases}$$

it holds that $u^* \in K$ and $\int_\Omega fu^* > \int_\Omega fu_\infty$, which contradicts (4). ■

Now we consider the distance function to the boundary in the ∞-norm

$$\mathrm{dist}_\infty(x,\partial\Omega)=\inf_{y\in\partial\Omega}|x-y|_\infty,\quad x\in\Omega,$$

where

$$|x-y|_\infty=\max_i|x_i-y_i|.$$

With the help of the previous lemma, we are ready to prove the following property for the limit u_∞ in $\{f>0\}$.

Lemma 4.2. *Let D be a convex set such that $D\subset\{f>0\}$. For every $x\in D$, we have that*

$$u_\infty(x)=\inf_{y\in\partial D}\{u_\infty(y)+|x-y|_\infty\}.$$

Proof: Since $u_\infty\in K$ and D is assumed to be convex,

$$u_\infty(x)\le u_\infty(y)+|x-y|_\infty,\quad \text{for every } y\in\partial D.$$

Thus,

$$u_\infty(x)\le\inf_{y\in\partial D}\{u_\infty(y)+|x-y|_\infty\}. \tag{14}$$

Let us define

$$v(x)=\inf_{y\in\partial D}\{u_\infty(y)+|x-y|_\infty\}\text{ in }D.$$

Note that $v\in\widetilde{K}$ since $\max\{\|\partial_1 v\|_{L^\infty(D)},\|\partial_2 v\|_{L^\infty(D)}\}\le 1$, and $v|_{\partial D}\equiv u_\infty$. Then, by the previous lemma, we have

$$\int_D fv\le\int_D fu_\infty. \tag{15}$$

We recall that by (14) we have $v\ge u_\infty$ in D. But now, since $f>0$ in D, from (15) we deduce that $v=u_\infty$ in D, as desired. ∎

In a similar way, it is possible to prove the following property of u_∞ in the set $\{f<0\}$.

Lemma 4.3. *Let D be a convex set such that $D\subset\{f<0\}$. For every $x\in D$, we have that*

$$u_\infty(x)=\sup_{y\in\partial D}\{u_\infty(y)-|x-y|_\infty\}.$$

Now we have all the ingredients to identify the limit, i.e, to prove Theorem 1.3.

Proof of Theorem 1.3: It is clear that $u_\infty=0$ on $\partial\Omega$, since $u_n=0$ on $\partial\Omega$ for any n.

Now, as usual, we consider a point $x_0\in\Omega$. To prove that u_∞ is a viscosity supersolution, let ϕ be a function in $C^2(\Omega)$ such that $u_\infty(x_0)=\phi(x_0)$ and $u_\infty-\phi$ has a local minimum at x_0. To show that u_∞ is a viscosity subsolution, let ψ be a function in $C^2(\Omega)$ such that $u_\infty(x_0)=\psi(x_0)$ and $u_\infty-\psi$ has a local maximum at x_0. Depending on the location of x_0, that is the sign of the function f at this point, we have different situations. So let us consider each case separately.

1. Let $x_0\in\Omega\backslash\mathrm{supp}\,f$. We have to show that u_∞ is a viscosity supersolution of $G_\infty(Du_\infty,D^2u_\infty)=0$, G_∞ defined in (5), in the sense of Definition 2.3. So we need to prove that

$$(G_\infty)^*(D\phi(x_0),D^2\phi(x_0))\ge 0,$$

with $(G_\infty)^*$ as in Lemma 2.4. Since $u_n\to u_\infty$ uniformly, there is a sequence $x_n\to x_0$, $x_n\in\Omega\setminus\mathrm{supp}\,f$, such that $u_n-\phi$ has a local minimum at x_n, for any $n\in\mathbb{N}$. As u_n is a viscosity solution of (1) and $f(x_n)=0$ for any n, we have

$$-(p_{1,n}-1)|\partial_1\phi(x_n)|^{p_{1,n}-2}\partial_{11}\phi(x_n)-(p_{2,n}-1)|\partial_2\phi(x_n)|^{p_{2,n}-2}\partial_{22}\phi(x_n)\ge 0.$$

Assuming $\partial_2\phi(x_0)\neq 0$, we divide by $(p_{2,n}-1)|\partial_2\phi(x_n)|^{p_{2,n}-4}$ to obtain

$$-\frac{p_{1,n}-1}{p_{2,n}-1}\left(\frac{|\partial_1\phi(x_n)|^{\frac{p_{1,n}-4}{p_{2,n}-4}}}{|\partial_2\phi(x_n)|}\right)^{p_{2,n}-4}\partial_{11}\phi(x_n)|\partial_1\phi(x_n)|^2-\partial_{22}\phi(x_n)|\partial_2\phi(x_n)|^2\ge 0. \tag{16}$$

We observe that, as $n\to+\infty$,

$$\partial_{22}\phi(x_n)|\partial_2\phi(x_n)|^2\to\partial_{22}\phi(x_0)|\partial_2\phi(x_0)|^2$$

and

$$\frac{p_{1,n}-1}{p_{2,n}-1}\,\partial_{11}\phi(x_n)|\partial_1\phi(x_n)|^2\to\theta\,\partial_{11}\phi(x_0)|\partial_1\phi(x_0)|^2.$$

If $|\partial_1\phi(x_0)|^\theta<|\partial_2\phi(x_0)|$, then

$$\left(\frac{|\partial_1\phi(x_n)|^{\frac{p_{1,n}-4}{p_{2,n}-4}}}{|\partial_2\phi(x_n)|}\right)^{p_{2,n}-4}\to 0,$$