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CATEGORIES VS. GROUPOIDS VIA GENERALISED MAL'TSEV PROPERTIES

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Abstract: We study the difference between internal categories and internal groupoids in terms of generalised Mal'tsev properties—the weak Mal'tsev property on the one hand, and n-permutability on the other. In the first part of the article we give conditions on internal categorical structures which detect whether the surrounding category is naturally Mal'tsev, Mal'tsev or weakly Mal'tsev. We show that these do not depend on the existence of binary products. In the second part we prove that, in a weakly Mal'tsev context, categories and groupoids coincide precisely when every relation which is reflexive and transitive is also symmetric. In varieties of algebras this latter condition is shown to be equivalent to n-permutability.

Keywords: Mal'tsev condition, n-permutable variety, internal category.
AMS Subject Classification (2010): 17D10, 18B99, 18D35.

Introduction

In this article we study the difference between internal categories and internal groupoids through the generalised Mal'tsev properties their surrounding category may have—the weak Mal'tsev property on the one hand, and n-permutability on the other. Conversely, or equivalently, we try to better understand these Mal'tsev conditions by providing new characterisations and new examples for them, singling out distinctive properties of a given type of category via properties of its internal categorical structures: internal categories, (pre)groupoids, relations.

The first part of the text is devoted to a conceptual unification of three levels of Mal'tsev properties: *naturally Mal'tsev* categories [7] using groupoids, categories, pregroupoids, etc. (Theorem 2.2), *Mal'tsev* categories [2, 3] using equivalence relations, preorders, difunctional relations (Theorem 2.6), and *weakly Mal'tsev* categories [11] using strong equivalence relations, strong preorders, difunctional strong relations (Theorem 2.9). Each of the resulting

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collections of equivalent conditions is completely parallel to the others, and such that a weaker collection of conditions is characterised by a smaller class of internal structures.

Some of these characterisations are well established, whereas some others are less familiar; what is new in all cases is the context in which we prove them: we never use binary products, but restrict ourselves to categories in which kernel pairs and split pullbacks exist.

The notion of weakly Mal'tsev category is probably not as well known as the others. It was introduced in [11] as a setting where any internal reflexive graph admits at most one structure of internal category. It turned out that this new notion is weaker than the concept of Mal'tsev category. But, unlike in Mal'tsev categories, in this setting not every internal category is automatically an internal groupoid. This gave rise to the following problem: to characterise those weakly Mal'tsev categories in which internal categories and internal groupoids coincide.

In Section 3 we observe that, in a weakly Mal'tsev category with kernel pairs and equalisers, the following hold: (1) the forgetful functor from internal categories to multiplicative graphs is an equivalence; (2) the forgetful functor from internal groupoids to internal categories is an equivalence if and only if every internal preorder is an equivalence relation (Theorem 3.1).

We study the varietal implications of this result in Section 4. In finitary quasivarieties of universal algebra, the latter condition—that reflexivity and transitivity together imply symmetry—is shown to be equivalent to the variety being *n-permutable*, for some n (Proposition 4.5). Thus we obtain another interpretation of a Mal'tsev condition (being n-permutable, where Mal'tsev = 2-permutable) in terms of internal categorical structures (here, internal relations). On the way we prove Proposition 4.4, a result claimed by Hagemann and Mitschke [4] for which we could find no proof in the literature. We furthermore explain how to construct a weakly Mal'tsev quasivariety starting from a Goursat (= 3-permutable) quasivariety (Proposition 4.9), and use this procedure to show that categories which are both weakly Mal'tsev and Goursat still need not be Mal'tsev (Example 4.10).

Of course, via part (2) of Theorem 3.1, our Proposition 4.5 implies that, in an n-permutable weakly Mal'tsev variety, every internal category is an internal groupoid—but surprisingly, here in fact the weak Mal'tsev property is not needed: n-permutability suffices, as was recently proved by Rodelo [14].

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This indicates that there may still be hidden connections between these two (a priori independent) weakenings of the Mal'tsev axiom.

1. Preliminaries

We recall the definitions and basic properties of some internal categorical structures which we shall use throughout this article.

1.1. Split pullbacks. Let $\mathcal{C}$ be any category. A diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} E & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & C \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle s} \\ A & \underset{r}{\overset{f}{\rightleftarrows}} & B \end{array} \qquad (\mathbf{A})$$

such that

$$gp_2 = fp_1, \quad p_1e_2 = rg, \quad e_1r = e_2s, \quad p_2e_1 = sf$$

and

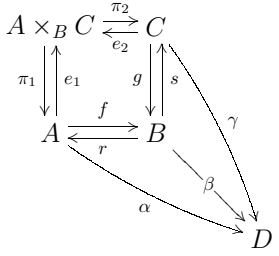
$$p_1e_1 = 1_A, \quad fr = 1_B, \quad gs = 1_B, \quad p_2e_2 = 1_C$$

is called a **double split epimorphism**. When we call a double split epimorphism a **pullback** we refer to the commutative square of split epimorphisms $fp_1 = gp_2$. Any pullback of a split epimorphism along a split epimorphism gives rise to a double split epimorphism; we say that $\mathcal{C}$ **has split pullbacks** when the pullback of a split epimorphism along a split epimorphism always exists.

In a category with split pullbacks $\mathcal{C}$, any diagram such as

$$\begin{array}{ccccc} A & \underset{r}{\overset{f}{\rightleftarrows}} & B & \underset{s}{\overset{g}{\rightleftarrows}} & C \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\gamma} & \\ & & D & & \end{array} \qquad (\mathbf{B})$$

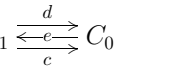
where $fr = 1_B = gs$ and $\alpha r = \beta = \gamma s$ induces a diagram



(**C**)

in which the square is a double split epimorphism. This kind of diagram will appear in the statements of Theorem 2.2, 2.6 and 2.9 as part of a universal property: under certain conditions one expects it to induce a (unique) morphism $\varphi\colon A \times_B C \to D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$. When such a unique morphism φ *always* exists, this of course just means that the commutative square of sections $e_1 r = e_2 s$ is a pushout. In this case we may say that the double split epimorphism induced by f and g **is a pushout**, but naturally we should be careful to avoid confusion about which square is meant.

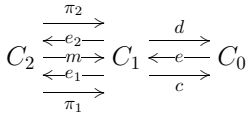
1.2. Internal groupoids. A **reflexive graph** in $\mathcal{C}$ is a diagram of the form



(**D**)

such that $de = 1_{C_0} = ce$.

A **multiplicative graph** in $\mathcal{C}$ is a diagram of the form



(**E**)

where

$$me_1 = 1_{C_1} = me_2, \quad dm = d\pi_2 \quad \text{and} \quad cm = c\pi_1$$

a	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
b	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3
c	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
x	1	2	3	1	1	3	1	2	1	2	-	-	1	2	3	2	-	2	3	-	-	3	3	-	1	2	3

TABLE 2. x is uniquely determined by a, b and c in B

also belong to the quasivariety $\mathcal{W}$: given any choice of a, b and c, the system of equations

$$\begin{cases} (ba)x = (ab)c \\ (bc)x = (cb)a \end{cases}$$

either has no solution or just one, as pictured in Table 1 for the algebra A and in Table 2 for B.

To see that the quasivariety $\mathcal{W}$ is not Mal'tsev, it now suffices to consider the homomorphisms f, $g\colon A \to B$ defined respectively by

$$f(1) = f(2) = 1, \quad f(3) = 2$$

and

$$g(1) = g(3) = 1, \quad g(2) = 2.$$

It is easy to check that the congruences $R[f]$ and $R[g]$ do not commute: $R[f]R[g]$ contains the element $(3, 2)$, but not $(2, 3)$, which is in $R[g]R[f]$.

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a	1	2	1	2	1	2	1	2
b	1	1	1	1	2	2	2	2
c	1	1	2	2	1	1	2	2
x	1	2	1	1	2	-	1	2

TABLE 1. x is uniquely determined by a, b and c in A

We could actually leave out the middle equalities (the ones not involving x and x') in the quasi-identity and still obtain a weakly Mal'tsev quasivariety, but the result of this procedure would be to small to include the following example, so we are not sure that it wouldn't force the quasivariety to become Mal'tsev.

Example 4.10. The example due to Mitschke [13] of a category which is Goursat but not Mal'tsev may be modified using Proposition 4.9 to yield an example of a category which is Goursat and weakly Mal'tsev but not Mal'tsev. In fact, Proposition 4.9 makes it possible to construct such examples ad libitum.

Let the variety $\mathcal{V}$ consist of **implication algebras**, i.e., $(I, \cdot)$ which satisfy

$$\begin{cases} (xy)x = x \\ (xy)y = (yx)x \\ x(yz) = y(xz) \end{cases}$$

where we write $x \cdot y = xy$. It is proved in [4, 13] that $\mathcal{V}$ is Goursat, and this is easily checked using Proposition 4.4 as witnessed by the terms $w_1(x, y, z) = (zy)x$ and $w_2(x, y, z) = (xy)z$. The further quasi-identity

$$\left.\begin{array}{l} (ba)x = (ab)c = (ba)x' \\ (bc)x = (cb)a = (bc)x' \end{array}\right\} \quad \Rightarrow \quad x = x'$$

determines a weakly Mal'tsev sub-quasivariety $\mathcal{W}$ of $\mathcal{V}$ by Proposition 4.9. This quasivariety certainly stays Goursat, and the counterexample given in [13] still works to prove that $\mathcal{W}$ is not Mal'tsev.

Indeed, the implication algebras $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ with respective multiplication tables

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

and the double split epimorphism

$$\begin{array}{ccc} C_2 & \underset{e_2}{\overset{\pi_2}{\rightleftarrows}} & C_1 \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle c}\downarrow\uparrow{\scriptstyle e} \\ C_1 & \underset{e}{\overset{d}{\rightleftarrows}} & C_0 \end{array}$$

is a pullback. Observe that a multiplicative graph is in particular a reflexive graph ($de = 1_{C_0} = ce$) and that the morphisms e_1 and e_2 are universally induced by the pullback:

$$e_1 = \langle 1_{C_1}, ed \rangle \quad \text{and} \quad e_2 = \langle ec, 1_{C_1} \rangle.$$

When the category $\mathcal{C}$ admits split pullbacks we shall refer to a multiplicative graph simply as

$$C_2 \xrightarrow{m} C_1 \underset{c}{\overset{d}{\underset{e}{\rightleftarrows}}} C_0.$$

An **internal category** is a multiplicative graph which satisfies the associativity condition $m(1 \times m) = m(m \times 1)$.

An **internal groupoid** is an internal category where both squares $dm = d\pi_2$ and $cm = c\pi_1$ are pullbacks (see for instance [1, Proposition A.3.7]). Equivalently, there should be a morphism $t \colon C_1 \to C_1$ with $ct = d$, $dt = c$ and $m\langle 1_{C_1}, t \rangle = ec$, $m\langle t, 1_{C_1} \rangle = ed$.

In the following sections we shall consider the obvious forgetful functors

$$\mathsf{Grpd}(\mathcal{C}) \xrightarrow{U_3} \mathsf{Cat}(\mathcal{C}) \xrightarrow{U_2} \mathsf{MG}(\mathcal{C}) \xrightarrow{U_1} \mathsf{RG}(\mathcal{C})$$

from groupoids in $\mathcal{C}$ to internal categories, to multiplicative graphs, to reflexive graphs.

1.3. Internal pregroupoids.

A **pregroupoid** [9, 8, 5] in $\mathcal{C}$ is a span

$$\begin{array}{ccc} & D & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} \\ D_0 & & D_0' \end{array}$$

together with a structure of the form

$$\begin{array}{ccccc}
R[d,c] & \underset{i_2}{\overset{p_2}{\rightleftarrows}} & R[c] & \xrightarrow{c_2} & D \\
{\scriptstyle p_1}\downarrow\uparrow{\scriptstyle i_1} & (1) & {\scriptstyle c_1}\downarrow & (2) & \downarrow{\scriptstyle c} \\
R[d] & \xrightarrow{d_2} & D & \xrightarrow[c]{} & D_0' \\
{\scriptstyle d_1}\downarrow & (3) & \downarrow{\scriptstyle d} & & \\
D & \xrightarrow[d]{} & D_0 & &
\end{array}$$

where (1), (2) and (3) are pullback squares, the morphisms i_1, i_2 are determined by

$$p_1 i_1 = 1_{R[d]}, \quad p_2 i_1 = \langle d_2, d_2 \rangle$$

and

$$p_2 i_2 = 1_{R[c]}, \quad p_1 i_2 = \langle c_1, c_1 \rangle$$

and there is a further morphism $p\colon R[d,c] \to D$ which satisfies the conditions

$$pi_1 = d_1 \quad \text{and} \quad pi_2 = c_2, \tag{\textbf{F}}$$

$$dp = dc_2 p_2 \quad \text{and} \quad cp = cd_1 p_1. \tag{\textbf{G}}$$

When $\mathcal{C}$ admits split pullbacks and kernel pairs, we shall refer to a pregroupoid structure simply as a structure

$$\begin{array}{ccccc}
 & & & \overset{c}{\nearrow} & D_0' \\
R[d,c] & \xrightarrow{p} & D & & \\
 & & & \underset{d}{\searrow} & D_0.
\end{array} \tag{\textbf{H}}$$

In order to have a visual picture, we may think of the object D as having elements of the form

$$c(x) \xleftarrow{x} d(x) \quad \text{or} \quad \cdot \xleftarrow{x} \cdot$$

and hence the "elements" of $R[d]$, $R[c]$ and $R[d,c]$ are, respectively, of the form

$$\cdot \xleftarrow{x} \cdot \xrightarrow{y} \cdot, \qquad \cdot \xrightarrow{x} \cdot \xleftarrow{y} \cdot$$

and

$$\cdot \xleftarrow{x} \cdot \xrightarrow{y} \cdot \xleftarrow{z} \cdot\ .$$

Proof: For any split pullback

$$\begin{array}{ccc}
A \times_B C & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & C \\
{\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle s} \\
A & \underset{r}{\overset{f}{\rightleftarrows}} & B
\end{array}$$

we have to show that e_1 and e_2 are jointly epic: any two morphisms φ, $\varphi'\colon A \times_B C \to D$ such that

$$\varphi e_1 = \alpha = \varphi' e_1 \quad \text{and} \quad \varphi e_2 = \gamma = \varphi' e_2$$

must coincide. We use the notations from Diagram (**C**) and consider $a \in A$ and $c \in C$ with $f(a) = b = g(c)$. Then

$$\begin{aligned}
w_1(\varphi(a,c), \alpha(a), \beta(b)) &= w_1(\varphi(a,c), \varphi(a, s(b)), \varphi(r(b), s(b))) \\
&= \varphi(w_1(a, a, r(b)), w_1(c, s(b), s(b))) \\
&= \varphi(w_2(a, r(b), r(b)), c) \\
&= \varphi(w_2(a, r(b), r(b)), w_2(s(b), s(b), c)) \\
&= w_2(\varphi(a, s(b)), \varphi(r(b), s(b)), \varphi(r(b), c)) \\
&= w_2(\alpha(a), \beta(b), \gamma(c))
\end{aligned}$$

and

$$\begin{aligned}
w_2(\beta(b), \gamma(c), \varphi(a,c)) &= w_2(\varphi(r(b), s(b)), \varphi(r(b), c), \varphi(a,c)) \\
&= \varphi(w_2(r(b), r(b), a), w_2(s(b), c, c)) \\
&= \varphi(a, w_1(s(b), s(b), c))) \\
&= \varphi(w_1(a, r(b), r(b)), w_1(s(b), s(b), c)) \\
&= w_1(\varphi(a, s(b)), \varphi(r(b), s(b)), \varphi(r(b), c)) \\
&= w_1(\alpha(a), \beta(b), \gamma(c)),
\end{aligned}$$

which proves that

$$w_1(\varphi(a,c), \alpha(a), \beta(b)) = w_2(\alpha(a), \beta(b), \gamma(c)) = w_1(\varphi'(a,c), \alpha(a), \beta(b))$$

and

$$w_2(\beta(b), \gamma(c), \varphi(a,c)) = w_1(\alpha(a), \beta(b), \gamma(c)) = w_2(\beta(b), \gamma(c), \varphi'(a,c)),$$

since both expressions only depend on $\alpha(a)$, $\beta(b)$ and $\gamma(c)$. Hence by definition of $\mathcal{W}$ we have that $\varphi(a,c) = \varphi'(a,c)$ for all $(a,c) \in A \times_B C$. ∎

algebra structures, which are not available in general. On the other hand, the implication (2) ⇒ (1) admits a proof which is *almost* categorical—but depends on a characterisation of n-permutability for regular categories as in Condition (3) of Proposition 4.4. We do not know whether this condition is still equivalent to n-permutability in this general situation. At least, such a result would involve a proof of the equivalence of Condition (1) with Condition (3) of Proposition 4.4 which does not pass via Condition (2).

Remark 4.7. Through Theorem 3.1, this result implies that in an n-permutable weakly Mal'tsev variety, every internal category is an internal groupoid. On the other hand, using different techniques, and without assuming the weak Mal'tsev condition, Rodelo recently proved that in any n-permutable variety, internal categories and internal groupoids coincide [14]. Whence the question: how different are n-permutable varieties from weakly Mal'tsev ones? The only thing we know about this so far is that the two conditions together are not strong enough to imply that the variety is Mal'tsev (see Example 4.10).

4.8. Constructing weakly Mal'tsev quasivarieties. Any 3-permutable (quasi)variety contains a canonical subvariety which is also weakly Mal'tsev. This allows us to construct examples of weakly Mal'tsev categories which are 3-permutable but not 2-permutable—thus we see, in particular, that in a weakly Mal'tsev category $\mathcal{C}$, categories and groupoids may coincide, even without $\mathcal{C}$ being Mal'tsev.

Proposition 4.9. *Let $\mathcal{V}$ be a Goursat finitary quasivariety with w_1, w_2 the terms obtained using Proposition 4.4. Then the sub-quasivariety $\mathcal{W}$ of $\mathcal{V}$ defined by the quasi-identity*

$$\left.\begin{aligned} w_1(x,a,b) = w_2(a,b,c) = w_1(x',a,b) \\ w_2(b,c,x) = w_1(a,b,c) = w_2(b,c,x') \end{aligned}\right\} \quad \Rightarrow \quad x = x'$$

is weakly Mal'tsev.

Observe that the morphism p is a kind of Mal'tsev operation in the sense that $p(x,y,y) = x$ and $p(x,x,y) = y$ (the conditions (**F**)). Furthermore, $dp(x,y,z) = dz$ and $cp(x,y,z) = cx$ by (**G**).

In the following sections we shall also consider the forgetful functor

$$V\colon \mathsf{PreGrpd}(\mathcal{C}) \to \mathsf{Span}(\mathcal{C})$$

from the category of pregroupoids to the category of spans in $\mathcal{C}$.

1.4. Relations. The notions of reflexive relation, preorder (or reflexive and transitive relation), equivalence relation, and difunctional relation, may all be obtained, respectively, from the notions of reflexive graph, internal category (or multiplicative graph), internal groupoid, and pregroupoid, simply by imposing the extra condition that the pair of morphisms (d,c) is jointly monomorphic.

2. Mal'tsev conditions

In this section we study some established and some less known characterisations of *Mal'tsev* and *naturally Mal'tsev* categories in terms of internal categorical structures. We extend these characterisations, which are usually considered in a context with finite limits, to a more general setting: categories with kernel pairs and split pullbacks. In particular we shall never assume that binary products exist. This allows for a treatment of *weakly Mal'tsev* categories in a manner completely parallel to the treatment of the two stronger notions.

2.1. Naturally Mal'tsev categories. We first consider the notion of naturally Mal'tsev category [7] in a context where binary products are not assumed to exist. This may seem strange, as the original definition takes place in a category with binary products (and no other limits). We can do this because the main characterisation of naturally Mal'tsev categories—as those categories for which the forgetful functor from internal groupoids to reflexive graphs is an equivalence—is generally stated in a finitely complete context. This context may be even further reduced: we shall show that the existence of kernel pairs and split pullbacks is sufficient.

Theorem 2.2. *Let $\mathcal{C}$ be a category with kernel pairs and split pullbacks. The following are equivalent:*

(i) *the functor $U_{123}\colon \mathsf{Grpd}(\mathcal{C}) \to \mathsf{RG}(\mathcal{C})$ is an equivalence;*

(ii) *the functor* $U_{12}\colon \mathsf{Cat}(\mathcal{C}) \to \mathsf{RG}(\mathcal{C})$ *is an equivalence;*
(iii) *the functor* $U_1\colon \mathsf{MG}(\mathcal{C}) \to \mathsf{RG}(\mathcal{C})$ *is an equivalence;*
(iv) *the functor* $V\colon \mathsf{PreGrpd}(\mathcal{C}) \to \mathsf{Span}(\mathcal{C})$ *is an equivalence;*
(v) *for every diagram such as* $(\mathbf{B})$ *in* $\mathcal{C}$*, given any span*

$$\begin{array}{ccc} & D & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} \\ D_0 & & D'_0 \end{array}$$

such that $d\alpha = d\beta f$ *and* $c\gamma = c\beta g$*, there is a unique morphism* $\varphi\colon A \times_B C \to D$ *such that*

$$\varphi e_1 = \alpha, \qquad \varphi e_2 = \gamma \qquad \text{and} \qquad d\varphi = d\gamma\pi_2, \qquad c\varphi = c\alpha\pi_1.$$

Proof: We shall prove the following implications.

$$\begin{array}{ccccccc} \text{(i)} & \searrow & & & & \nearrow & \text{(i)} \\ \text{(ii)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(v)} & \Longrightarrow & \text{(ii)} \\ \text{(iii)} & \nearrow & & & & \searrow & \text{(iii)} \end{array}$$

First suppose that the functor U_1 (or U_{12}, or U_{123}) is an equivalence. Then any reflexive graph admits a canonical morphism m

$$C_2 \xrightarrow{m} C_1 \underset{\xleftarrow[c]{}}{\overset{d}{\underset{\xleftarrow{e}}{\rightrightarrows}}} C_0$$

such that $me_1 = 1_{C_1} = me_2$ as in the definition of a multiplicative graph. Furthermore, this morphism is natural, in the sense that, for any morphism $f = (f_1, f_0)$ of reflexive graphs, the diagram

$$\begin{array}{ccccc} C_2 & \xrightarrow{m} & C_1 & \underset{\xleftarrow[c]{}}{\overset{d}{\underset{\xleftarrow{e}}{\rightrightarrows}}} & C_0 \\ {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ C'_2 & \xrightarrow[m']{} & C'_1 & \underset{\xleftarrow[c']{}}{\overset{d'}{\underset{\xleftarrow{e'}}{\rightrightarrows}}} & C'_0 \end{array}$$

with $f_2 = f_1 \times_{f_0} f_1$ commutes.

$(z, x) \in R^{-1}$ is an element of R^{n-1}, which means that there exist ternary terms $w_1, \ldots, w_{n-1}$ such that

$$z = w_{n-1}(x,x,z)Rw_{n-1}(x,z,z) = w_{n-2}(x,x,z)Rw_{n-2}(z,z,x) = \\ \cdots = w_1(x,x,z)Rw_1(x,z,z) = x.$$

In particular, there are $n-1$ terms $w_1, \ldots, w_{n-1}$ in $\mathcal{V}$ such that

$$\begin{cases} w_1(x,z,z) = x \\ w_i(x,x,z) = w_{i+1}(x,z,z) \\ w_{n-1}(x,x,z) = z, \end{cases}$$

and Condition (2) holds. ∎

A variation of the above proof now gives us the following.

Proposition 4.5. *For a finitary quasivariety* $\mathcal{V}$*, the following are equivalent:*
(1) *in* $\mathcal{V}$*, every internal preorder is an equivalence relation;*
(2) $\mathcal{V}$ *is* n*-permutable for some* n*.*

Proof: By Proposition 4.4, if Condition (2) holds then for every reflexive relation R in $\mathcal{V}$ we have that $R^{-1} \subset R^{n-1}$. Now if R is transitive then $R^{n-1} \subset R$, so that $R^{-1} \subset R$, which means that R is symmetric.

To prove the converse, suppose that every internal preorder in $\mathcal{V}$ is an equivalence relation. As above, let A be the free algebra on the set $\{x, z\}$ and let R be the reflexive relation on A consisting of all couples

$$(w(x,x,z), w(x,z,z))$$

for w a ternary term. Again, the couple (x, z) is in R. By assumption, the transitive closure $\overline{R}$ of R is also symmetric, hence contains the couple (z, x). This means that (z, x) may be expressed through a chain of finite length in R. More precisely, there exists a natural number n and ternary terms w_1, $\ldots$, w_{n-1} such that

$$z = w_{n-1}(x,x,z)Rw_{n-1}(x,z,z) = w_{n-2}(x,x,z)Rw_{n-2}(z,z,x) = \\ \cdots = w_1(x,x,z)Rw_1(x,z,z) = x.$$

By Proposition 4.4 this means that $\mathcal{V}$ is n-permutable. ∎

Remark 4.6. This of course raises the question whether a similar result would hold in a purely categorical context. It seems difficult to obtain the number n which occurs in Condition (2) of Proposition 4.5 without using free

4.3. n-Permutable varieties. The following equivalent conditions due to Hagemann and Mitschke [4] describe what it means for a variety to be **n-permutable**. (Recall that 2-permutability is just the Mal'tsev property and a regular category which is 3-permutable is called **Goursat** [2].)

Proposition 4.4. *For a finitary quasivariety $\mathcal{V}$ and a natural number $n \geq 2$, the following are equivalent:*

(1) *for any two equivalence relations R and S on an object A, we have $(R, S)_n = (S, R)_n$;*
(2) *there exist $n-1$ terms $w_1, \ldots, w_{n-1}$ in $\mathcal{V}$ such that*

$$\begin{cases} w_1(x,z,z) = x \\ w_i(x,x,z) = w_{i+1}(x,z,z) \\ w_{n-1}(x,x,z) = z; \end{cases}$$

(3) *for any reflexive relation R, we have $R^{-1} \subset R^{n-1}$.*

Proof: The first condition is the classical definition of an n-permutable variety. By Theorem 2 in [4], this is is equivalent to (2). Hagemann and Mitschke also claim the equivalence with Condition (3) in the case of finitary varieties, but we could not locate a proof, so we give it here for the sake of completeness.

Suppose that Condition (2) holds and let (z, x) be in R^{-1}. Then $(x, z) \in R$ and (x, x), $(z, z) \in R$ by reflexivity, so that $(w_i(x,x,z), w_i(x,z,z))$ is in R for all i. The equations in (2) now imply that $(z, x) \in R^{n-1}$.

Conversely, let A be the free algebra on the set $\{x, z\}$ and let R be the relation on A consisting of all couples

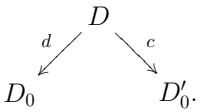
$$(w(x,x,z), w(x,z,z))$$

for w a ternary term. This set R indeed determines a subalgebra of $A \times A$: if v is a term of arity k then

$$\begin{aligned} &v((u_1(x,x,z), u_1(x,z,z)), \ldots, (u_k(x,x,z), u_k(x,z,z))) \\ =&(v(u_1(x,x,z), \ldots, u_k(x,x,z)), v(u_1(x,z,z)), \ldots, u_k(x,z,z))) \\ =&(w(x,x,z), w(x,z,z)), \end{aligned}$$

where $w(a,b,c) = v(u_1(a,b,c), \ldots, u_k(a,b,c))$, is still in R. Furthermore, R is a reflexive relation because any element of A is of the form $p(x, z)$ for some binary term p, and we can put $w(a,b,c) = p(a,c)$. Now note that the couple (x, z) is in R as we may choose $w(a,b,c) = b$. Hence by assumption

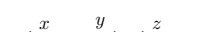
To prove that the functor V is an equivalence, we have to construct a pregroupoid structure for any given span



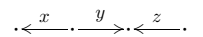
Let us consider the reflexive graph

$$R[d,c] \rightleftarrows D \qquad (\mathbf{I})$$

where an "element" of $R[d,c]$

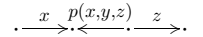


is viewed as an arrow y having domain x and codomain z. It is clearly reflexive, with (x,x,x) being the identity on x. By the assumed Condition (iii) (or (ii), or (i)) it is a multiplicative graph. The desired pregroupoid structure p for (D, d, c) is obtained by the following procedure (already used, for instance, in [12]): given



in $R[d,c]$, consider the pair of composable arrows

in the reflexive graph (**I**). Since this reflexive graph is multiplicative, multiply in order to obtain



and project to the middle component.

The equalities $p(x,y,y) = x$ and $p(x,x,y) = y$ simply follow from the multiplicative identities $me_1 = 1_{C_1} = me_2$ of the multiplicative graph. Likewise, $dp(x,y,z) = dz$ and $cp(x,y,z) = cx$. This construction is functorial because the multiplication is natural. Finally, p is uniquely determined since so is the multiplication on (**I**), and each one of them determines the other. This proves that if U_1 (or U_{12}, or U_{123}) is an equivalence then V is an equivalence.

Next we prove that, if V is an equivalence, then the category $\mathcal{C}$ satisfies Condition (v). Consider a diagram such as (**C**) above and a suitable span (d, c). We have to construct a morphism $\varphi\colon A \times_B C \to D$ which satisfies the needed conditions, and prove that this φ is unique. To do so, we use the

natural pregroupoid structure $p\colon R[d,c] \to D$. Since $d\alpha = d\beta f$, $c\gamma = c\beta g$ and $\alpha r = \beta = \gamma s$, there is an induced morphism

$$\langle \alpha\pi_1, \beta f \pi_1, \gamma\pi_2 \rangle \colon A \times_B C \to R[d,c].$$

It assigns to any (a,c) with $f(a) = b = g(c)$ in $A \times_B C$ a triple

$$\cdot \xleftarrow{\alpha(a)} \cdot \xrightarrow{\beta(b)} \cdot \xleftarrow{\gamma(c)} \cdot$$

in $R[d,c]$. The desired morphism $\varphi\colon A \times_B C \to D$ is then obtained by taking its composition in the pregroupoid, i.e., $\varphi(a,c) = p(\alpha(a), \beta(b), \gamma(c))$ or

$$\varphi = p\langle \alpha\pi_1, \beta f \pi_1, \gamma\pi_2 \rangle.$$

This proves existence; the equalities $\varphi(a,b,s(b)) = \alpha(a)$ and $\varphi(r(b),b,c) = \gamma(c)$ follow from the properties of p, as do $d\varphi = d\gamma\pi_2$ and $c\varphi = c\alpha\pi_1$.

Now we show that the equalities

$$\varphi e_1 = \alpha, \qquad \varphi e_2 = \gamma \qquad \text{and} \qquad d\varphi = d\gamma\pi_2, \qquad c\varphi = c\alpha\pi_1$$

determine φ uniquely. Let us consider the span

$$\begin{array}{ccc} & A \times_B C & \\ {\scriptstyle \pi_2}\swarrow & & \searrow{\scriptstyle \pi_1} \\ C & & A \end{array}$$

with its induced pregroupoid structure

$$q\colon R[\pi_2, \pi_1] \to A \times_B C;$$

if the morphisms in this pregroupoid are viewed as arrows

$$a \xleftarrow{(a,c)} c$$

then the operation q takes a composable triple

$$a \xleftarrow{(a,c)} c \xrightarrow{(a',c)} a' \xleftarrow{(a',c')} c'$$

and sends it to

$$a \xleftarrow{(a,c')} c'$$

4.1. Finitary quasivarieties. Just like a variety of algebras is determined by certain identities between terms, a quasivariety also admits quasi-identities in its definition, i.e., expressions of the form

$$\left.\begin{array}{c} v_1(x_1,\dots,x_k) = w_1(x_1,\dots,x_k) \\ \vdots \\ v_n(x_1,\dots,x_k) = w_n(x_1,\dots,x_k) \end{array}\right\} \quad \Rightarrow \quad v_{n+1}(x_1,\dots,x_k) = w_{n+1}(x_1,\dots,x_k)$$

—see, for instance, [10] for more details. It is well known that any quasivariety may be obtained as a regular epi-reflective subcategory of a variety, and more generally the sub-quasivarieties of a quasivariety correspond to its regular epi-reflective subcategories. In particular, sub-quasivarieties are closed under subobjects. We shall only consider finitary (quasi)varieties: all operations have finite arity.

4.2. The transitive closure of a reflexive relation. Given an increasing (countable) sequence of relations

$$R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$$

on an algebra A in a finitary quasivariety $\mathcal{V}$, the union $\bigcup_n R_n$ exists and is given by the set-theoretical union with its induced operations and identities—of course, this union is still a relation on A. (Note that in the infinitary case this need no longer be true: given an operation of countable arity θ in A, if $r_i \in R_i \setminus R_{i-1}$ for all $i \geq 1$ then $\theta(r_0, r_1, \dots, r_i, \dots)$ may be an element of $\bigcup_n R_n$ which does not belong to any R_i.) One example of such a sequence is

$$(R,S)_0 \subset (R,S)_1 \subset (R,S)_2 \subset (R,S)_3 \subset \cdots \subset (R,S)_n \subset \cdots$$

for reflexive relations R and S on A, namely, the sequence

$$\Delta_A \subset R \subset RS \subset RSR \subset RSRS \subset \cdots$$

from [2]. In particular, when R is a reflexive relation on A, the sequence defined by $R_n = (R,R)_n$, which is just

$$\Delta_A \subset R \subset R^2 \subset \cdots \subset R^n \subset \cdots,$$

has a union $\overline{R}$. Clearly, this $\overline{R}$ is still a reflexive relation on A; it is, moreover, transitive: if (x,y) and (y,z) are in $\overline{R}$, there exists some n such that (x,y), $(y,z) \in R^n$, so that $(x,z) \in R^n R^n = R^{2n} \subset \overline{R}$. In fact, $\overline{R}$ is the transitive closure of R. Indeed, any other transitive relation S on A which contains R also contains $\overline{R}$, as any n-fold composite R^n is smaller than $S^n = S$.

By Lemma 3.2 we already know that the morphisms $\langle m, \pi_2 \rangle$ and $\langle \pi_1, m \rangle$ are monomorphisms. This means that the reflexive graph

$$C_2 \underset{\pi_1}{\overset{m}{\underset{\longleftarrow e_1 \longrightarrow}{\rightleftarrows}}} C_1$$

is a reflexive relation, and since it is transitive—by assumption it is a multiplicative graph—it is an equivalence relation. Hence there is a morphism

$$\tau = \langle m, q \rangle \colon C_2 \to C_2$$

such that $m\tau = \pi_1$. Now $t = qe_2$ is the needed morphism $C_1 \to C_1$. Indeed $dm = cq$, because $\langle m, q \rangle$ is a morphism into the pullback C_2, so that

$$ct = cqe_2 = dme_2 = d;$$

furthermore,

$$m\langle 1_{C_1}, t \rangle = m\langle me_2, qe_2 \rangle = m\langle m, q \rangle e_2 = \pi_1 e_2 = ec,$$

which completes the proof. ∎

Remark 3.3. In general, a category can be weakly Mal'tsev without Condition (2) of Theorem 3.1 holding. For instance, in the category of commutative monoids with cancellation, the relation $\leq$ on the monoid of natural numbers $\mathbb{N}$ is a preorder which is not an equivalence relation.

Remark 3.4. It is possible for a category to satisfy both Condition (1) and Condition (2) of Theorem 3.1 without being Mal'tsev. We prove this claim in the following section (Example 4.10), in which we further refine the above result in a varietal context.

4. The varietal case

We conclude this article by observing that, in a finitary quasivariety of universal algebras, the condition "every internal preorder is an equivalence relation" singled out in part (2) of Theorem 3.1 is equivalent to the variety being n-permutable for some n. We first recall some basic definitions and results.

in $A \times_B C$. The morphism φ then gives rise to a morphism of pregroupoids, determined by the morphism of spans

$$\begin{array}{ccccc} C & \xleftarrow{\pi_2} & A \times_B C & \xrightarrow{\pi_1} & A \\ {\scriptstyle d\gamma}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle c\alpha} \\ D_0 & \xleftarrow[d]{} & D & \xrightarrow[c]{} & D'_0. \end{array}$$

Write $\varphi' \colon R[\pi_2, \pi_1] \to R[d, c]$ for the induced morphism. We see that

$$\begin{aligned} \varphi(a, c) &= \varphi q(a \longleftarrow sf(a) \longrightarrow rg(c) \longleftarrow c) \\ &= p\varphi'(a \longleftarrow sf(a) \longrightarrow rg(c) \longleftarrow c) \\ &= p(\cdot \xleftarrow{\varphi e_1(a)} \cdot \xrightarrow{\varphi e_1 r(b)} \cdot \xleftarrow{\varphi e_2(c)} \cdot) \\ &= p(\cdot \xleftarrow{\alpha(a)} \cdot \xrightarrow{\beta(b)} \cdot \xleftarrow{\gamma(c)} \cdot) \\ &= p(\alpha(a), \beta(b), \gamma(c)) \end{aligned}$$

and φ is uniquely determined.

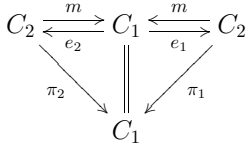
Next we prove that (v) implies Condition (iii) (and (ii), and (i)) in our theorem. Given a reflexive graph (**D**) the multiplication m is induced by the diagram

$$\begin{array}{ccccc} C_1 & \underset{e}{\overset{d}{\rightleftarrows}} & C_0 & \underset{e}{\overset{c}{\leftrightarrows}} & C_1 \\ & \searrow & \downarrow{\scriptstyle e} & \swarrow & \\ & & C_1 & & \end{array}$$

together with the reflexive graph itself considered as a span. All the required conditions are immediate. The associativity condition (needed for (ii)) follows from the uniqueness of the morphism induced by the following diagram.

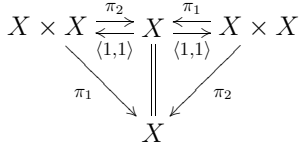
$$\begin{array}{ccccc} C_2 & \underset{e_2}{\overset{\pi_2}{\rightleftarrows}} & C_1 & \underset{e_1}{\overset{\pi_1}{\leftrightarrows}} & C_2 \\ & {\scriptstyle m}\searrow & \| & \swarrow{\scriptstyle m} & \\ & & C_1 & & \end{array}$$

The existence of inverses (needed for (i)) follows from the diagram



as explained in [11]. ∎

Observe that, in the case of finite limits, any one of the equivalent conditions of Theorem 2.2 is a characterisation for the notion of naturally Mal'tsev category introduced in [7]. Indeed, the Mal'tsev operation on an object X is determined by the following diagram.



In the presence of coequalisers we may simplify Condition (v), and obtain (cf. the Corollary in [7]):

Corollary 2.3. *If $\mathcal{C}$ is a category with kernel pairs, split pullbacks and coequalisers, the equivalent conditions of Theorem 2.2 hold if and only if*

(vi) *for every split pullback* (**A**) *in $\mathcal{C}$, the commutative square of sections $e_1 r = e_2 s$ is a pushout.*

Proof: Clearly (vi) implies (v); conversely, (v) gives (vi): for a span which satisfies the conditions in (v) one may simply choose $d = \mathrm{Coeq}(\alpha, \beta f)$ and $c = \mathrm{Coeq}(\gamma, \beta g)$. ∎

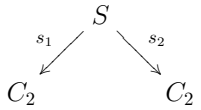
When every span in $\mathcal{C}$ is naturally equipped with a unique pregroupoid structure, there is an interchange law for composable strings valid in any pregroupoid in $\mathcal{C}$. In fact, the equality (**K**) is a partial version of the Mal'tsev operation p being *autonomous*, see [7].

Proposition 2.4 (Interchange law)**.** *Let $\mathcal{C}$ be a category with kernel pairs, split pullbacks and coequalisers satisfying the conditions (i)–(vi). Consider a*

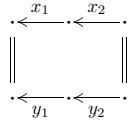
Indeed, $\pi_1 e_2 \pi_2 x = \pi_1 e_2 \pi_2 y$ as $\pi_1 e_2 \pi_2 = ec\pi_2 = ed\pi_1$. Considering the equaliser $(S, \langle s_1, s_2\rangle)$ of the pair of morphisms

$$R[\pi_1] \rightrightarrows C_2 \xrightarrow{m} C_1,$$

we obtain a strong relation



which may be pictured as



with $x_1 x_2 = y_1 y_2$ and $x_1 = y_1$ for $(x_1, x_2) S (y_1, y_2)$.

By Theorem 2.9, this relation, being a strong relation, is also difunctional and the argument used on page 103 of [3] also applies here to show that

$$\langle e_2\pi_2 x, e_2\pi_2 y\rangle = \langle s_1, s_2\rangle p i \overline{\langle x, y\rangle},$$

where $p\colon SS^{-1}S \to S$ is obtained by difunctionality, $\overline{\langle x, y\rangle}\colon Z \to S$ is the factorisation of $\langle x, y\rangle$ through the equaliser (we are assuming that $mx = my$), and the morphism $i\colon S \to SS^{-1}S$ may be pictured as

$$((x_1, x_2), (y_1, y_2)) \mapsto (((1,1),(1,x_1),(x_1,x_2)),((1,1),(1,y_1),(y_1,y_2))).$$

This proves that $\langle e_2\pi_2 x, e_2\pi_2 y\rangle$ factors through the equaliser S, so we may conclude that

$$m e_2 \pi_2 x = m e_2 \pi_2 y,$$

or $\pi_2 x = \pi_2 y$ as desired. ∎

Proof of Theorem 3.1: If the functor U_3 is an equivalence then in particular any preorder is an equivalence relation. For the converse, assume that every preorder is an equivalence relation (and every strong relation is difunctional). Given any category (**E**) we shall prove that it is a groupoid. For this to happen it suffices that there is a morphism $t\colon C_1 \to C_1$ with $ct = d$ and $m\langle 1_{C_1}, t\rangle = ec$ (see, for instance, [11]).

Proof: In the presence of equalisers, the weak Mal'tsev axiom is equivalent to Condition (iv")—see [6]. ∎

3. Internal categories vs. internal groupoids

We prove that, in a weakly Mal'tsev category with kernel pairs and equalisers, internal categories are internal groupoids if and only if every preorder is an equivalence relation.

Theorem 3.1. *Let $\mathcal{C}$ be a weakly Mal'tsev category with kernel pairs and equalisers. Then:*

(1) *the forgetful functor*

$$U_2 \colon \mathsf{Cat}(\mathcal{C}) \to \mathsf{MG}(\mathcal{C})$$

is an equivalence;

(2) *the forgetful functor*

$$U_3 \colon \mathsf{Grpd}(\mathcal{C}) \to \mathsf{Cat}(\mathcal{C})$$

is an equivalence if and only if every internal preorder in $\mathcal{C}$ is an equivalence relation.

Part (1) of this result was already obtained in [11]. The proof of Part (2) depends on the following lemma.

Lemma 3.2. *Let $\mathcal{C}$ be a weakly Mal'tsev category with equalisers. Given a category* (**E**) *in $\mathcal{C}$, the morphisms*

$$\langle \pi_1, m \rangle \colon C_2 \to R[c] \quad \text{and} \quad \langle m, \pi_2 \rangle \colon C_2 \to R[d]$$

are monomorphisms; this means that the multiplication is cancellable on both sides.

Proof: We shall prove $\langle \pi_1, m \rangle$ is a monomorphism. A similar argument shows the same for $\langle m, \pi_2 \rangle$.

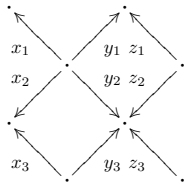
First observe that the kernel pairs $R[c]$, $R[d]$, $R[m]$, $R[\pi_1]$ and $R[\pi_2]$ exist because c, d, m, π_1 and π_2 are split epimorphisms. To prove that $\langle \pi_1, m \rangle$ is a monomorphism is the same as proving for every x, $y \colon Z \to C_2$ that

$$\left.\begin{aligned} \pi_1 x &= \pi_1 y \\ m x &= m y \end{aligned}\right\} \quad \Rightarrow \quad \pi_2 x = \pi_2 y.$$

Assuming that $\pi_1 x = \pi_1 y$ we have induced morphisms

$$\langle x, y \rangle \quad \text{and} \quad \langle e_2 \pi_2 x, e_2 \pi_2 y \rangle \colon Z \to R[\pi_1].$$

pregroupoid (**H**) *in $\mathcal{C}$. Then for any configuration of the shape*



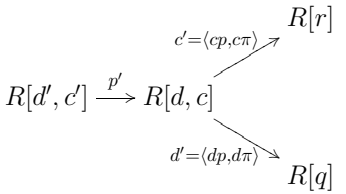
(**J**)

in this pregroupoid, the equality

$$\begin{aligned} p(p(x_1, x_2, x_3), p(y_1, y_2, y_3), p(z_1, z_2, z_3)) \\ = p(p(x_1, y_1, z_1), p(x_2, y_2, z_2), p(x_3, y_3, z_3)) \end{aligned} \quad (\mathbf{K})$$

holds.

Proof: It suffices to consider the pregroupoid in $\mathcal{C}$ in which the configurations (**J**) are the composable triples, and then the equality will follow by naturality of the pregroupoid structures. This pregroupoid



is determined by the span $(\langle dp, d\pi \rangle, \langle cp, c\pi \rangle)$ where $q = \mathrm{Coeq}(dp, d\pi)$ and $r = \mathrm{Coeq}(cp, c\pi)$, and the middle projection $\pi = d_2 p_1 = c_1 p_2 \colon R[d, c] \to D$ maps a composable triple (x_1, x_2, x_3) to x_2. It is easily checked that the morphism p' which sends (**J**) to its horizontal composite—the composable triple

$$(p(x_1, y_1, z_1), p(x_2, y_2, z_2), p(x_3, y_3, z_3))$$

in D—determines a pregroupoid structure (hence, the unique one) on this span. Furthermore, by naturality of pregroupoid structures, the morphism

of spans

$$\begin{array}{ccccc} R[q] & \xleftarrow{\langle dp, d\pi\rangle} & R[d,c] & \xrightarrow{\langle cp, c\pi\rangle} & R[r] \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r_1} \\ D_0 & \xleftarrow[d]{} & D & \xrightarrow[c]{} & D_0' \end{array}$$

induces a morphism $p''\colon R[d',c'] \to R[d,c]$ such that $pp'' = pp'$, which gives us the required equality (**K**). Indeed, the induced morphism p'' takes (**J**) and sends it to its vertical composite—the composable triple

$$(p(x_1,x_2,x_3), p(y_1,y_2,y_3), p(z_1,z_2,z_3))$$

in D. ∎

2.5. Mal'tsev categories. Restricting the previous results to the case where the morphisms d and c are jointly monomorphic we obtain the well known characterisation [3] for Mal'tsev categories.

Theorem 2.6. *Let $\mathcal{C}$ be a category with kernel pairs and split pullbacks. The following are equivalent:*

(i') *every reflexive relation is an equivalence relation;*
(ii') *every reflexive relation is a preorder;*
(iii') *every reflexive relation is transitive;*
(iv') *every relation is difunctional;*
(v') *for every diagram such as* (**B**) *in $\mathcal{C}$, given any relation*

$$\begin{array}{ccc} & D & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} \\ D_0 & & D_0' \end{array}$$

such that $d\alpha = d\beta f$ and $c\gamma = c\beta g$, there is a unique morphism $\varphi\colon A \times_B C \to D$ such that

$$\varphi e_1 = \alpha, \qquad \varphi e_2 = \gamma \qquad \text{and} \qquad d\varphi = d\gamma\pi_2, \qquad c\varphi = c\alpha\pi_1.$$

Proof: By restricting to relations one easily adapts the proof of Theorem 2.2 to the present situation. ∎

An important result on Mal'tsev categories is the following one, usually stated for finite limits [3]; it follows, for instance, from Theorem 3.1.

Theorem 2.7. *Let $\mathcal{C}$ be a category with kernel pairs, split pullbacks and equalisers, satisfying the equivalent conditions of Theorem 2.6. Then the forgetful functor*

$$U_3\colon \mathsf{Grpd}(\mathcal{C}) \to \mathsf{Cat}(\mathcal{C})$$

is an equivalence. ∎

2.8. Weakly Mal'tsev categories. A category is said to be **weakly Mal'tsev** when it has split pullbacks and every induced pair of morphisms into the pullback (e_1, e_2) as in Diagram (**A**) above is jointly epimorphic [11].

Further restricting the conditions of Theorem 2.2 to the case where the morphisms d and c are jointly strongly monomorphic—and calling such a span a **strong relation***—we obtain a characterisation of weakly Mal'tsev categories.

Theorem 2.9. *Let $\mathcal{C}$ be a category with kernel pairs and split pullbacks. The following are equivalent:*

(i") *every reflexive strong relation is an equivalence relation;*
(ii") *every reflexive strong relation is a preorder;*
(iii") *every reflexive strong relation is transitive;*
(iv") *every strong relation is difunctional;*
(v") *for every diagram such as* (**B**) *in $\mathcal{C}$, given any strong relation*

$$\begin{array}{ccc} & D & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} \\ D_0 & & D_0' \end{array}$$

such that $d\alpha = d\beta f$ and $c\gamma = c\beta g$, there is a unique morphism $\varphi\colon A \times_B C \to D$ such that

$$\varphi e_1 = \alpha, \qquad \varphi e_2 = \gamma \qquad \text{and} \qquad d\varphi = d\gamma\pi_2, \qquad c\varphi = c\alpha\pi_1.$$

Proof: By restricting to strong relations one easily adapts the proof of Theorem 2.2 to the present situation. ∎

Theorem 2.10. *Let $\mathcal{C}$ be a category with kernel pairs, split pullbacks and equalisers. The following are equivalent:*

(1) *$\mathcal{C}$ is a weakly Mal'tsev category;*
(2) *$\mathcal{C}$ satisfies the equivalent conditions of Theorem 2.9.*

*This concept is due to Z. Janelidze.