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ANISOTROPIC ELLIPTIC PROBLEMS WITH NATURAL GROWTH TERMS

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Abstract: In this paper we prove existence and regularity of solutions for nonlinear anisotropic elliptic equations of the type

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] + g(x,u,\nabla u) = f$$

in a bounded, smooth, domain Ω, in $\mathbb{R}^N$, with homogeneous Dirichlet boundary conditions. The right hand side f is assumed to belong to some Lebesgue space and the function g is a nonlinear lower order term.

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1. Introduction

In this work we study the existence and the regularity of solutions for the following nonlinear anisotropic elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] + g(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^N$, f is a given function belonging to some Lebesgue space. Concerning the exponents p_i, we assume $1 < p_1 \leq ... \leq p_N$ and $\overline{p} < N$, where $\overline{p}$ is the harmonic mean of p_i, that is

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}. \tag{2}$$

Furthermore, g is a nonlinear term having natural growth with respect to the gradient, which satisfies the sign condition, i. e. $g(x,\sigma,\xi)\sigma \geq 0$. This assumption allows us to obtain a priori estimates from the equation; if it

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is not satisfied, the problem may not even have a solution. Indeed in the isotropic case, i.e. $p_i = 2$ for any i, if we consider u, a bounded solution of problem

$$\begin{cases} -\Delta u = |\nabla u|^2 + \alpha & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $\alpha > 0$, then $u \geq 0$ and it easily follows that $v = e^u - 1$ is bounded, positive and solves

$$\begin{cases} -\Delta v = \alpha(1+v) & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

But it is well known that the previous problem has no positive solution for α sufficiently large.

Problems as (1) are very interesting because they naturally appear if we write the Euler-Lagrange equations of suitable functionals of the Calculus of Variations. As a matter of fact, if we consider the following functional

$$J(v) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega a(x,v) \left| \frac{\partial v}{\partial x_i} \right|^{p_i} - \int_\Omega f v, \tag{4}$$

where a is a bounded, smooth function, the Euler-Lagrange equation is

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right] + \sum_{i=1}^{N} a'(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p_i} = f,$$

and in the left hand side a sum of lower order terms appears. We also note that, obviously, problem (1), with $g \equiv \sum_{i=1}^N g_i$, does not correspond to the Euler-Lagrange equation of functionals like (4). Indeed if $a(x,u) \equiv 1$ then $a'(x,u) \equiv 0$ and so we do not find (1). Therefore we will not use techniques of Calculus of Variations to solve this type of problem. We will build approximating problems, we will derive a priori estimates from the equations and then we will use compactness results for anisotropic Sobolev spaces to pass to the limit in the approximating problems.

In the isotropic case, i.e. $p_i = 2$ or $p_i = p$ for all i, this kind of problem has been studied by many authors. With no hope of being thorough, we mention some papers regarding the study of these problems [4], [5], [6], [7], [9], [11], [12] and [14] (see also the references therein). While, in the anisotropic case, problems with natural growth terms are not still deeply studied. We recall some recent papers dealing with similar problems [1], [2], [3], [19], [20], [28]. Problems as (6) in the following, without lower order terms, have already been investigated. We only remember some of these: [8], [10], [16], [18], [22], [23], [27], [29].

This paper is organized as follows. In Section 2 we give some definitions and we recall some useful results concerning the anisotropic Sobolev spaces where it is natural to look for the solutions to problem (1). In Section 3 we study problems as (1), with zero lower order terms, that is g depending only on x and u. If we only assume g satisfying the sign condition and $f \in L^1(\Omega)$ then we show the existence of at least one distributional solution for (1), belonging to the anisotropic Sobolev space $W_0^{1,(q_i)}(\Omega)$ for every

$$1 \leq q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i$$

and

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \forall \ i = 1, ..., N, \tag{5}$$

as in the monotone case, that is

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right] = f \quad \text{in } \Omega, \\ \\ \qquad\qquad\qquad\qquad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

see [8], [15], [16]. Moreover if we assume a growth condition with respect to $|u|$ on g, i.e. $g(x,\sigma)\mathrm{sgn}(\sigma) \geq |\sigma|^s$, for some $s > s(N, \overline{p})$, we improve the regularity of u. To fix the ideas we consider as model problem the following

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right] + |u|^{s-1}u = f \quad \text{in } \Omega, \\ \\ \qquad\qquad\qquad\qquad u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

In Section 4 we consider a general nonlinearity

$$g \equiv g(x, u, \nabla u) = \sum_{i=1}^{N} g_i(x, u, \nabla u).$$

with natural growth with respect to the gradient (i.e. $|g_i(x, \sigma, \xi)| \leq b(|\sigma|)|\xi_i|^{p_i}$ for any i). With respect to $|u|$ we do not assume any growth restrictions but we assume the sign condition (i.e. $g_i(x,\sigma,\xi)\mathrm{sgn}(\sigma) \geq 0$). The right hand side f of (1) is assumed to belong either to $[W_0^{1,(p_i)}(\Omega)]^*$ ($*$ denotes the dual space) or to $L^1(\Omega)$. In the latest case we have to assume a sort of coercivity condition on g, i.e. $|g_i(x,\sigma,\xi)| \geq \gamma|\xi_i|^{p_i}$ for any i and $|\sigma|$ sufficiently large, in order to prove the existence of weak solutions, that is solutions with finite energy in the space $W_0^{1,(p_i)}(\Omega)$. The role of the coercivity condition is to give an a priori estimate in the energy space $W_0^{1,(p_i)}(\Omega)$. So the term with natural growth brings an extra regularity to the solutions for the problem (6) with L^1-data. Moreover in this case we do not need to make further assumptions on p_i (see (5)), apart from $\overline{p} < N$.

We can take as a model problem the following

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] + u\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} = f \quad \text{on } \Omega, \\ \\ \hfill u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

2. Preliminary results

This Section is dedicated to give some definitions and to recall some preliminary results useful in the following pages.

We start remembering the functional analytic framework in which we look for solutions to problems as (1). Let Ω be a bounded, smooth domain of $\mathbb{R}^N$, $N \geq 3$, $1 < p_1 \leq p_2 \leq ... \leq p_N$. The anisotropic Sobolev spaces are defined as follow

$$W_0^{1,(p_i)}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) \,:\, \frac{\partial v}{\partial x_i} \in L^{p_i}(\Omega) \right\}. \tag{9}$$

$W_0^{1,(p_i)}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}(\Omega)}. \tag{10}$$

In [21], [24], [26], [30] the theory of these spaces is developed and in particular the corresponding Sobolev embedding theorems are studied. Let

$$\overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}, \quad \text{for} \quad \overline{p} < N \quad \text{and} \quad \frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}. \tag{11}$$

In [30] it is proved that if $\overline{p} < N$, then

$$W_0^{1,(p_i)}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall\ r \in [1, \overline{p}^*]. \tag{12}$$

This embedding is continuous and also compact if $r < \overline{p}^*$. The following Sobolev type inequality is also proved: there exists a positive constant C, depending only on Ω such that

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \quad \forall\ r \in [1, \overline{p}^*], \tag{13}$$

for any $v \in C_0^1(\Omega)$. By density, (13) also holds for any $v \in W_0^{1,(p_i)}(\Omega)$. Inequality (13) also implies that

$$\|v\|_{L^r(\Omega)} \leq C \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)}, \quad \forall\ r \in [1, \overline{p}^*]. \tag{14}$$

Subsequently in [21] it is proved that the critical exponent depends on the kind of anisotropy. If the p_i are not Òtoo far apartÓ (i.e. the anisotropy is concentrated) the critical exponent is $\overline{p}^*$, as in [30], that is the ÒusualÓ critical exponent related to the harmonic mean $\overline{p}$ of the p_i. While if the p_i are Òtoo spread outÓ it coincides with the maximum of the p_i, i.e. p_N.

In the following, we will consider the composition of functions in $W_0^{1,(p_i)}(\Omega)$ with some useful auxiliary functions of real variable: the truncation function at level $k > 0$, T_k, that is

$$T_k(s) = \begin{cases} k\,\mathrm{sgn}(s) & \text{if } |s| > k, \\ s & \text{if } |s| \leq k \end{cases} \tag{15}$$

and

$$G_k(s) = s - T_k(s), \quad \text{with } k \geq 0. \tag{16}$$

We remember also a Poincaré type inequality, valid for all $v \in W_0^{1,(p_i)}(\Omega)$:

$$\|v\|_{L^r(\Omega)} \leq C(|\Omega|)\, r \left\| \frac{\partial v}{\partial x_i} \right\|_{L^r(\Omega)}, \quad \forall\ r \geq 1, \quad \forall\ i = 1, ..., N, \tag{17}$$

see [21].

In the following, we will write C to denote positive constants, possibly different, depending on the data, that is they will be fixed in the assumptions we will make, as the dimension N, the set Ω, the exponents p_i, etc. but in any case the constants are always meant to not depend on n.

Moreover, in the following, we will denote

$$\partial_i := \frac{\partial}{\partial x_i}, \quad i = 1, ..., N.$$

3. Problems with zero lower order terms

In this section we consider problem (1) with $g(x, u, \nabla u) \equiv g(x, u)$ that satisfies

(G1): $g(x, \sigma) : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Omega$ for any fixed $\sigma \in \mathbb{R}$ and continuous in σ for a.e. $x \in \Omega$.

(G2): There exists $s > 0$ such that for all σ and a.e. $x \in \Omega$

$$g(x, \sigma)\mathrm{sgn}(\sigma) \geq |\sigma|^s.$$

(G3): For all $t > 0$ the function

$$F_t(x) = \sup_{|\sigma| \leq t} |g(x, \sigma)|$$

belongs to $L^1(\Omega)$.

Definition 3.1. We say that u is a *distributional solution* for problem (1) if $u \in W_0^{1,1}(\Omega)$, $g(x, u) \in L^1(\Omega)$, and u satisfies the following inequality

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i - 2} \partial_i u \partial_i \phi + \int_\Omega g(x, u)\phi = \int_\Omega f\phi \tag{18}$$

for all $\phi \in C_0^\infty(\Omega)$.

Theorem 3.2. *Let $f \in L^1(\Omega)$, g be a function satisfying the conditions **(G1)-(G3)**, p_i and s (that appears in **(G2)**) be such that*

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i < \frac{\overline{p}(N-1)}{N-\overline{p}}, \qquad s > \frac{N(\overline{p}-1)}{N-\overline{p}}. \tag{19}$$

Then there exists a distributional solution u to the problem (1), belonging to $W_0^{1,(q_i)}(\Omega)$ for every

$$1 \leq q_i < \frac{sp_i}{s+1}. \tag{20}$$

Remark 3.3. We note that the assumption from below on p_i, is usual when we search for distributional solutions to problems, as (1) or (6), with L^1-data. It corresponds to the well known isotropic condition

$$p > 2 - \frac{1}{N},$$

see [6], necessary to have all the terms that appear in Definition 3.1 in $L^1(\Omega)$. The assumption from above is needed to pass to the limit in the approximating problems and it is always satisfied if $p_i = p$ for any i. This assumption implies that

$$p_N < \frac{\overline{p}(N-1)}{N-\overline{p}} = \overline{p}^* - \frac{\overline{p}}{N-\overline{p}}$$

and so p_N can not be greater than $\overline{p}^*$, the critical exponent for the Sobolev embedding theorems, as in (11). Moreover it can be removed if we look for entropy solutions (see [15] and [16]), which will not be studied in this paper.

Remark 3.4. We underline that to prove existence of a distributional solution for problem (1), assumption **(G2)** can be substituted by a weaker one, that is

(G2'): $g(x,\sigma)\mathrm{sgn}(\sigma) \geq 0$ for all $\sigma \in \mathbb{R}$, a.e. $x \in \Omega$.

In fact it easy to prove that **(G2)** implies **(G2')** but the contrary is false. If we only assume **(G2')**, we obtain the existence of a distributional solution u, but u only belongs to $W_0^{1,(q_i)}(\Omega)$ for all

$$1 \leq q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i.$$

We note that

$$\frac{sp_i}{s+1} > \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i \qquad \Leftrightarrow \qquad s > \frac{N(\overline{p}-1)}{N-\overline{p}},$$

as we are assuming. So we obtain a better result than the case $g(x,\sigma) \equiv 0$, as in (6). Obviously, a function identically equal to zero does not satisfy assumption **(G2)**.

Proof of Theorem 3.2: First of all we prove the existence of $u \in W_0^{1,(q_i)}(\Omega)$,

$$1 \leq q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i,$$

a distributional solution for the problem (1), when g satisfies **(G1)**, **(G2')**, **(G3)**. For this part we strictly follow [7], in which the authors studied the isotropic case.

Approximating problems. We define for $n \in \mathbb{N}$, $g_n(x,\sigma) = T_n(g(x,\sigma))$ for any $(x,\sigma) \in \Omega \times \mathbb{R}$, T_n as in (15). It easy to see that the assumptions on g, **(G1)**, **(G2')** and **(G3)**, also hold for g_n for any $n \in \mathbb{N}$. Moreover let $f_n = T_n(f)$, for any $n \in \mathbb{N}$, with

$$\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{21}$$

By a simple modification of Leray-Lions theorem (see [25]) it is possible to obtain a weak solution u_n of (1) with $g = g_n$ and $f = f_n$, that is a function $u_n \in W_0^{1,(p_i)}(\Omega)$, that satisfies

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i v + \int_\Omega g_n(x,u_n) v = \int_\Omega f_n v, \quad \forall \ v \in W_0^{1,(p_i)}(\Omega). \tag{22}$$

A priori estimate. We prove

(a): $\int_{\{|u_n|>t\}} |g_n(x,u_n)| \leq \int_{\{|u_n|>t\}} |f_n|$, forall n and $t \in \mathbb{R}^+$.

(b): The sequence $\{u_n\}$ is relatively compact in $W_0^{1,(q_i)}(\Omega)$ with

$$1 \leq q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i, \quad \forall \ i = 1, ..., N.$$

The proof of **(a)** is classical. We consider a sequence of real smooth increasing functions $\{\psi_h(s)\}$ that converges to the function $\psi_t(s) = T_t(s)/t$, we choose $\psi_h(u_n)$ as test function in (22) and we arrive, dropping nonnegative terms, to

$$\int_\Omega g_n(x,u_n)\psi_h(u_n) \leq \int_\Omega f_n \psi_h(u_n).$$

Letting h goes to infinity we obtain **(a)**. To prove **(b)** we proceed as for problem (6), see [16], with $f = h_n = f_n - g_n$. We note that if we choose $t = 0$ in **(a)** we get

$$\|g_n(\cdot,u_n)\|_{L^1(\Omega)} \leq \|f_n\|_{L^1(\Omega)},$$

so that

$$\int_\Omega |h_n| \leq 2\|f_n\|_{L^1(\Omega)} \leq C,$$

by (21), and $h_n \in [W_0^{1,(p_i)}(\Omega)]^* \cap L^1(\Omega)$. Hence we obtain **(b)**.

Passing to the limit. By **(b)** we have

$$u_n \to u \quad \text{in} \quad W_0^{1,(q_i)}(\Omega), \quad \forall\ 1 \le q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i, \tag{23}$$

$$u_n \to u, \quad \text{a.e.} \tag{24}$$

$$|\partial_i u_n|^{p_i-2}\partial_i u_n \to |\partial_i u|^{p_i-2}\partial_i u, \quad \text{in } L^{r_i}(\Omega), \quad \forall\ 1 \le r_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i' \tag{25}$$

We note that

$$1 < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i', \quad \Leftrightarrow \quad p_i < \frac{(N-1)\overline{p}}{N-\overline{p}} \quad \forall\ i = 1, ..., N.$$

By (24) we have

$$g_n(\cdot, u_n) \to g(\cdot, u), \quad a.e.. \tag{26}$$

Now we need to prove that $g_n(\cdot, u_n)$ converges to $g(\cdot, u)$ in $L^1(\Omega)$. Since (26) is true, in view of Vitali theorem, it is sufficient to show that $g_n(\cdot, u_n)$ is equiintegrable on Ω. For any measurable $E \subset \Omega$ and for any $t \in \mathbb{R}^+$, we get

$$\int_E |g_n(x, u_n)| = \int_{E\cap\{|u_n|>t\}} |g_n(x, u_n)| + \int_{E\cap\{|u_n|\le t\}} |g_n(x, u_n)| \le C,$$

using **(a)**, **(G3)**, the equiintegrability on Ω of the sequence f_n and the fact that the measure of the set $\{x \in \Omega : |u_n(x)| > t\}$ goes to zero uniformly respect to n when t goes to infinity. So we can pass to the limit in the approximating problems and we obtain a distributional solution u, belonging to $W_0^{1,(q_i)}(\Omega)$, for all

$$1 \le q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i \quad \text{and} \quad \frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i < \frac{\overline{p}(N-1)}{N-\overline{p}},$$

as desired.

Improved regularity. By the previous part of the proof we have

$$\int_\Omega |g(x, u)| \le C \tag{27}$$

and

$$\int_{B_k} |\partial_i u|^{p_i} \le C, \quad \forall\ k \in \mathbb{N} \quad \text{and} \quad \forall\ i = 1, ..., N \tag{28}$$

where $B_k = \{x \in \Omega : k \leq |u(x)| < k+1\}$. Let $m > 0$, that we will choose later; we have

$$\int_\Omega \frac{|\partial_i u|^{p_i}}{(1+|u|)^{m+1}} = \sum_{k=0}^{\infty} \int_{B_k} \frac{|\partial_i u|^{p_i}}{(1+|u|)^{m+1}} \leq C \sum_{k=0}^{\infty} \frac{1}{(k+1)^{m+1}} < C(m), \quad (29)$$

by (28), the definition of B_k and the convergence of the series that appears above.

Moreover by (27) and **(G2)** we deduce

$$\int_\Omega |u|^s \leq C. \quad (30)$$

Now, let $1 \leq q_i < p_i$ for any i, by Young inequality, we get

$$|\partial_i u|^{q_i} \leq \frac{|\partial_i u|^{p_i}}{(1+|u|)^{m+1}} + (1+|u|)^{\frac{(m+1)q_i}{p_i - q_i}}.$$

Integrating on Ω the previous expression and using (29), (30) and the second assumption in (19), we obtain

$$\int_\Omega |\partial_i u|^{q_i} \leq C, \quad \forall\ i = 1, ..., N.$$

We underline that

$$\frac{(m+1)q_i}{p_i - q_i} < s, \quad \forall\ i = 1, ..., N$$

if we choose

$$0 < m < \frac{s(p_i - q_i)}{q_i} - 1$$

and this is possible because we are assuming that

$$q_i < \frac{sp_i}{s+1} \quad \forall\ i = 1, ..., N.$$

So we obtain the result. ■

4. Problems with lower order terms having natural growth

In this Section we prove the existence of a solution for (1) with

$$g(x, u, \nabla u) = \sum_{i=1}^{N} g_i(x, u, \nabla u),$$

where g_i satisfies for any $i = 1, ..., N$

(G1'): $g_i(x,\sigma,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is measurable in Ω, for any fixed $\sigma\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ and continuous in σ and ξ for a.e. $x\in\Omega$.

(G2'): For almost every $x\in\Omega$, for all $\sigma\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$

$$g_i(x,\sigma,\xi)\mathrm{sgn}(\sigma)\geq 0.$$

(G3'): For almost every $x\in\Omega$, for all $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$

$$|g_i(x,\sigma,\xi)|\leq b(|\sigma|)|\xi_i|^{p_i},$$

where $b:\mathbb{R}\to\mathbb{R}^+$ is a continuous, nondecreasing function such that $b(\sigma)>\gamma>0$ for $|\sigma|$ sufficiently large.

Definition 4.1. We say that u is a *weak solution* for problem (1) if $u\in W_0^{1,(p_i)}(\Omega)$, $g_i(x,u,\nabla u)\in L^1(\Omega)$, for any $i=1,...,N$ and u satisfies the following inequality

$$\sum_{i=1}^N\int_\Omega|\partial_i u|^{p_i-2}\partial_i u\partial_i\varphi+\sum_{i=1}^N\int_\Omega g_i(x,u,\nabla u)\varphi=\int_\Omega f\varphi \tag{31}$$

for all $\varphi\in W_0^{1,(p_i)}(\Omega)\cap L^\infty(\Omega)$.

Theorem 4.2. *Let $f\in[W_0^{1,(p_i)}(\Omega)]^*$ (or in $L^{(\overline{p}^*)'}(\Omega)$) and let g_i be N functions satisfying the conditions **(G1')-(G3')**. Then there exists a weak solution u for problem* (1).

Proof: As in the previous section, we divide the proof in three parts.

Approximating problems. We consider the sequence of approximating equations

$$\begin{cases}-\sum_{i=1}^N\partial_i[|\partial_i u_n|^{p_i-2}\partial_i u_n]+\sum_{i=1}^N g_i^n(x,u_n,\nabla u_n)=f_n & \text{in }\Omega,\\ u_n\in W_0^{1,(p_i)}(\Omega)\cap L^\infty(\Omega),\qquad g_i^n(x,u_n,\nabla u_n)\in L^1(\Omega), & \forall\ i.\end{cases} \tag{32}$$

where

$$g_i^n(x,\sigma,\xi)=\frac{g_i(x,\sigma,\xi)}{1+\frac{1}{n}|g_i(x,\sigma,\xi)|},\qquad\forall\ i=1,...,N\quad\text{and}\quad\forall\ n\in\mathbb{N} \tag{33}$$

and $\{f_n\}$ a sequence of L^∞-functions such that $f_n\to f$ in $[W_0^{1,(p_i)}(\Omega)]^*$ (or in $L^{(\overline{p}^*)'}(\Omega)$). We remark that for all $i=1,...,N$

$$g_i^n(x,\sigma,\xi)\mathrm{sgn}(\sigma)\geq 0,\quad |g_i^n(x,\sigma,\xi)|\leq|g_i(x,\sigma,\xi)|\ \text{and}\ |g_i^n(x,\sigma,\xi)|\leq n.$$

Since g_i^n is bounded for all i, for any fixed $n \in \mathbb{N}$, (32) has at least one weak solution u_n by a simple modification of the result of J. Leray and J.L. Lions. Moreover by assumption of f_n, $u_n \in L^\infty(\Omega)$ (cf. [10], [16] and [29]).

A priori estimates. We take u_n as test function in the weak formulation of (32), we obtain

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} + \sum_{i=1}^N \int_\Omega g_i^n(x, u_n, \nabla u_n) u_n \leq \int_\Omega |f_n u_n| \leq$$

$$\leq \left(\int_\Omega |f_n|^{(\overline{p}^*)'} \right)^{\frac{1}{(\overline{p}^*)'}} \left(\int_\Omega |u_n|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*}} \leq C \sum_{i=1}^N \|\partial_i u_n\|_{L^{p_i}(\Omega)}$$

by Hölder end Sobolev inequalities. This implies, using the sign condition on g_i^n, that

$$\|u_n\|_{W_0^{1,(p_i)}(\Omega)} \leq C, \tag{34}$$

and consequently

$$\int_\Omega g_i^n(x, u_n, \nabla u_n) u_n \leq C, \qquad \forall \ i = 1, ..., N. \tag{35}$$

Passing to the limit. By the previous estimates we have that there exists $u \in W_0^{1,(p_i)}(\Omega)$ and a subsequence (still denoted by $\{u_n\}$) such that

$$u_n \to u \text{ weakly in } W_0^{1,(p_i)}(\Omega) \tag{36}$$

and

$$u_n \to u \quad \text{ a.e.} \tag{37}$$

This result is not sufficient to pass to the limit in the approximating problems. Indeed, we also need to prove that $\partial_i u_n \to \partial_i u$ a.e. for all i and $g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u)$ strongly in $L^1(\Omega)$ for any i. We begin by proving that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,(p_i)}(\Omega)$. We choose in the weak formulation of (32) as a test function $\varphi_n = \psi(T_k(u_n) - T_k(u))$ where $\psi(s) = se^{\lambda s^2}$. It is simple to see that if $\lambda \geq (b(k)/2)^2$ the following inequality holds for all $s \in \mathbb{R}$

$$\psi'(s) - b(k)|\psi(s)| \geq \frac{1}{2}. \tag{38}$$

Thanks to the previous step we already know that

$$\varphi_n \to 0 \quad \text{weakly} \;\; \text{in} \; W_0^{1,(p_i)}(\Omega)$$

and $*$-weakly in $L^\infty(\Omega)$, so that we have

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \varphi_n + \sum_{i=1}^N \int_\Omega g_i^n(x,u_n,\nabla u_n)\varphi_n \to 0. \tag{39}$$

Since $g_i^n(x,u_n,\nabla u_n)\varphi_n \geq 0$ on the set $\{x\in\Omega : |u_n(x)| \geq k\}$, we obtain by (39) that

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \varphi_n + \sum_{i=1}^N \int_{\{|u_n|\leq k\}} g_i^n(x,u_n,\nabla u_n)\varphi_n \leq \omega_1(n), \tag{40}$$

where $\omega_1(n)$ is a sequence of real numbers which converges to zero as n goes to infinity. In the following we will denote with $\omega_i(n)$, $i=1,2,...$ such sequences. For the first term in the left hand side of (40), we have, since $\partial_i\varphi_n = \psi'(T_k(u_n)-T_k(u))\partial_i(T_k(u_n)-T_k(u))$ and by easy calculations,

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial\varphi_n \geq \tag{41}$$

$$\geq \sum_{i=1}^N \int_\Omega [|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\times$$
$$\times \partial_i(T_k(u_n)-T_k(u))\psi'(T_k(u_n)-T_k(u)) + \omega_2(n).$$

On the other hand, by assumption **(G3')**,

$$\left|\sum_{i=1}^N \int_{\{|u_n|\leq k\}} g_i^n(x,u_n,\nabla u_n)\varphi_n\right| \leq \tag{42}$$

$$\leq b(k)\sum_{i=1}^N \int_\Omega [|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\times$$
$$\times \partial_i(T_k(u_n)-T_k(u))|\varphi_n| + \omega_3(n).$$

Putting together (40), (41) and (42), we obtain

$$\sum_{i=1}^N \int_\Omega [|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\times$$
$$\times \partial_i(T_k(u_n)-T_k(u))[\psi'(u_n) - b(k)|\varphi(u_n)|] \leq \omega_3(n).$$

Suppose now that $p_i \geq 2$ for some i. Then

$$[|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n) - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\times$$

$$\times\partial_i(T_k(u_n)-T_k(u))\geq C|\partial_i(T_k(u_n)-T_k(u))|^{p_i},$$

using (38), we obtain

$$\sum_{i=1}^{N}\int_\Omega|\partial_i(T_k(u_n)-T_k(u))|^{p_i}\leq 2C\,\omega_3(n)$$

that implies

$$T_k(u_n)\to T_k(u)\qquad\text{strongly in } W_0^{1,(p_i)}(\Omega). \tag{43}$$

We arrive to the same result also if $1<p_i<2$, for some i, by a slight modification and using the following inequality

$$[|\partial_i T_k(u_n)|^{p_i-2}\partial_i T_k(u_n)-|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)]\times$$

$$\times\partial_i(T_k(u_n)-T_k(u))\geq C\frac{|\partial_i(T_k(u_n)-T_k(u))|^2}{(|\partial_i T_k(u_n)|+|\partial_i T_k(u)|)^{2-p_i}}.$$

The strong convergence of $T_k(u_n)$ implies that for some subsequence, that we still denote by u_n,

$$\partial_i u_n\to\partial_i u\quad\text{a.e.}\qquad\forall\ i=1,...,N \tag{44}$$

and so

$$\nabla u_n\to\nabla u\quad\text{a.e.}. \tag{45}$$

It yields, since g_i is a continuous function in σ and ξ for a.e. $x\in\Omega$,

$$g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)\quad\text{a.e.}. \tag{46}$$

Now we prove that $g_i^n(x,u_n,\nabla u_n)$ is uniformly equiintegrable for $i=1,...,N$. For any measurable E of Ω and for any $m\in\mathbb{R}^+$, we have

$$\int_E|g_i^n(x,u_n,\nabla u_n)|\leq$$

$$\leq\int_{E\cap\{|u_n|\leq m\}}b(m)|\partial_i T_m(u_n)|^{p_i}+\int_{E\cap\{|u_n|>m\}}|g_i^n(x,u_n,\nabla u_n)|, \tag{47}$$

for fixed m and for $i=1,...,N$. The first term that appears in the right hand side of (47) is small uniformly in n and in E, since $\partial_i T_m(u_n)$ strongly converges to $\partial_i T_m(u)$ in $L^{p_i}(\Omega)$ for all i. Also the second term is small uniformly in n and in E, since it can be estimated as follows

$$\int_{E\cap\{|u_n|>m\}}|g_i^n(x,u_n,\nabla u_n)|\leq\int_{\{|u_n|>m\}}|g_i^n(x,u_n,\nabla u_n)|\leq$$

$$\leq \int_{\{|u_n|>m\}} \frac{1}{|u_n|} u_n\, g_i^n(x,u_n,\nabla u_n) \leq \frac{1}{m}\int_{\{|u_n|>m\}} u_n\, g_i^n(x,u_n,\nabla u_n) \leq \frac{C_2}{m},$$

thanks to (35). This completes the uniform equintegrability of g_i^n for any i. So thanks to (46) we get

$$g_i^n(x,u_n,\nabla u_n) \to g_i(x,u,\nabla u), \quad \text{strongly in } L^1(\Omega), \qquad \forall\ i=1,...,N.$$

Using the strong L^1-convergence of g_i and the fact that

$$|\partial_i u_n|^{p_i-2}\partial_i u_n \to |\partial_i u|^{p_i-2}\partial_i u \quad \text{weakly in } L^{p_i'}(\Omega), \qquad \forall\ i=1,...,N,$$

it is simple to pass to the limit in (32). ■

Theorem 4.3. *Let $f\in L^1(\Omega)$, and let g_i be N functions satisfying the conditions **(G1')**-**(G3')** and*

(G4'): *there exists $\gamma>0$ such that*

$$|g_i(x,\sigma,\xi)| \geq \gamma|\xi_i|^{p_i}, \quad \textit{for } |\sigma| \textit{ sufficiently large.}$$

Then there exists a weak solution u for problem (1).

Remark 4.4. We observe that the solution to (1) is a solution of finite energy ($u\in W_0^{1,(p_i)}(\Omega)$) even if $f\in L^1(\Omega)$ and without assumptions (5) on p_i. This seems strange at first glance, because if $f\in L^1(\Omega)$ the solution u of (6) is known to only belong to $W_0^{1,(q_i)}(\Omega)$ for all $1\leq q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)}\, p_i$. This improved regularity of u is due to **(G4')**. In other words the sense of the result that we will prove is that the term with natural growth, satisfying **(G4')**, brings an extra regularity to the solutions for the problem (1) with L^1-data, yielding the existence of solutions in $W_0^{1,(p_i)}(\Omega)$, without further assumptions on p_i, as in (5). The role of **(G4')** is to give an a priori estimate in the energy space $W_0^{1,(p_i)}(\Omega)$, which allows us to deal with the lower order terms with natural growth.

Proof of Theorem 4.3: As before we divide the proof in three steps.

Approximating problems. We consider the sequence of the approximating problems

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u_n|^{p_i-2}\partial_i u_n] + \sum_{i=1}^N g_i(x,u_n,\nabla u_n) = f_n & \text{in } \Omega,\\ u_n\in W_0^{1,(p_i)}(\Omega) \qquad g_i(x,u_n,\nabla u_n)\in L^1(\Omega) & \forall\ i=1,...,N, \end{cases} \tag{48}$$

where $\{f_n\}$ is a sequence of L^∞-functions such that $f_n \to f$ in $L^1(\Omega)$. The solutions of these problems exist by Theorem 4.2.

A priori estimates. In this case, the use in the weak formulation of (48) of the test function $T_k(u_n)$ yields for any $k > 0$

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)| + \sum_{i=1}^{N} \int_\Omega g_i(x, u_n, \nabla u_n) T_k(u_n) \le k \|f\|_{L^1(\Omega)},$$

that implies, thanks to the sign condition on g_i,

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i} \le C\, k, \tag{49}$$

and

$$k \int_{\{|u_n|>k\}} |g_i(x, u_n, \nabla u_n)| \le$$

$$\le \int_{\{|u_n|>k\}} |g_i(x, u_n, \nabla u_n)| k + \int_{\{|u_n|\le k\}} g_i(x, u_n, \nabla u_n) u_n =$$

$$= \int_\Omega g_i(x, u_n, \nabla u_n) T_k(u_n) \le C\, k, \qquad \forall\ i = 1, ..., N. \tag{50}$$

Now choosing $k > \overline{k}$, where $\overline{k}$ is such that **(G4')** holds, and combining (49), (50) and **(G4')** we again obtain

$$\|u_n\|_{W_0^{1,(p_i)}(\Omega)} \le C. \tag{51}$$

Passing to the limit. By (51), there exists $u \in W_0^{1,(p_i)}(\Omega)$ and a subsequence (still denoted by $\{u_n\}$) such that u_n weakly converges to u in $W_0^{1,(p_i)}(\Omega)$ and a.e. The proof of the strong convergence of $T_k(u_n)$ in $W_0^{1,(p_i)}(\Omega)$ is the same as in Theorem 4.2. Now the strong convergence of $T_k(u_n)$ implies that for some subsequence

$$\partial_i u_n \to \partial_i u \quad \text{a.e.} \qquad \forall\ i = 1, ..., N \tag{52}$$

and so

$$\nabla u_n \to \nabla u \quad \text{a.e.,} \tag{53}$$

which implies

$$g_i(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u) \quad \text{a.e..} \tag{54}$$

Now we prove that $g_i(x, u_n, \nabla u_n)$ is uniformly equiintegrable for $i = 1, ..., N$. For any measurable E of Ω and for any $m \in \mathbb{R}^+$. As before, we have

$$\int_E |g_i(x, u_n, \nabla u_n)| \le$$

$$\le \int_{E\cap\{|u_n|\le m\}} b(m)|\partial_i T_m(u_n)|^{p_i} + \int_{E\cap\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)|, \qquad (55)$$

for fixed m and for $i = 1, ..., N$. The first term of the expression above is small uniformly in n and in E, recalling that $\partial_i T_m(u_n)$ strongly converges to $\partial_i T_m(u)$ in $L^{p_i}(\Omega)$ for all i. For the second term in this case we use $T_1(G_{m-1}(u_n))$ as test function in the weak formulation of the problem (48). We obtain

$$\sum_{i=1}^{N} \int_{\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)| \le \int_{\{|u_n|\ge m-1\}} |f_n|$$

and hence

$$\limsup_{n\to+\infty} \int_{\{|u_n|>m\}} |g_i(x, u_n, \nabla u_n)| \le \int_{\{|u|\ge m-1\}} |f|, \qquad \forall\ i = 1, ..., N.$$

So also the second term, which appears in the right hand side of (55), is small uniformly in n and in E when m is sufficiently large. Hence by (54) we obtain

$$g_i(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u), \quad \text{strongly in } L^1(\Omega), \qquad \forall\ i = 1, ..., N,$$

and so it is simple to pass to the limit in (48). This fact concludes the proof. ■

5. Some final remarks

First of all we want to observe what happens if we do not assume sign conditions on the lower order terms. This result also appears in [17]. Here we only report the statement of the Theorem. The proof is rather standard and it strictly follows [12] (see also [5]).

Theorem 5.1. *Let* $f \in L^m(\Omega)$, $m > \overline{p}^*/(\overline{p}^* - p_N)$, $\overline{p}^* > p_N$ *as in* (11), $\mu_0 > 0$ *and*

i) $b_i(x, \sigma, \xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ *is measurable in* Ω, *for any fixed* $\sigma \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^N$ *and continuous in* σ *and* ξ *for a.e.* $x \in \Omega$, *for all* $i = 1, ..., N$;

ii) *there exists* $\gamma > 0$*, such that the following inequality is true for all* $(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$ *and a.e.* $x \in \Omega$

$$|b_i(x, \sigma, \xi)| \leq \gamma |\xi_i|^{p_i}, \qquad \forall \ i = 1, ..., N. \tag{56}$$

Then there exists a function $u \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$*, weak solution for the following problem*

$$\begin{cases} -\sum_{i=1}^N \partial_i[|\partial_i u|^{p_i-2}\partial_i u] + \mu_0 u = \sum_{i=1}^N b_i(x, u, \nabla u) + f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{57}$$

namely

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \varphi + \mu_0 \int_\Omega u\varphi = \sum_{i=1}^N \int_\Omega b_i(x, u, \nabla u)\varphi + \int_\Omega f\varphi \tag{58}$$

for all $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$.

We underline that in this case it is impossible to use the Leray-Lions Theorem, as in the previous section, because the functions b_i are not bounded and we do not have any information on their sign; therefore, to prove this result we use approximation techniques and some regularity results presented in [16]. Moreover the absence of the sign condition requires the addition of the term on the left hand side of the equation in (57), $\mu_0 u$. It is crucial in order to prove an existence result. Indeed if $\mu_0 = 0$, already in the isotropic case (i.e. $p_i = 2$ for all i), the example of J. L. Kazdan and E. Kramer (quoted in the Introduction, see (3)) shows that, even with bounded smooth data, no bounded solution exists. In this case we have the existence of solutions to (57) only if we assume that the norm of the data f is "small".

We highlight that it is also possible to take a unique function b instead of a sum of b_i in (57), if the following condition

$$|b(x, \sigma, \xi)| \leq \gamma \sum_{i=1}^N |\xi_i|^{p_i}, \qquad \forall \ i = 1, ..., N, \tag{59}$$

is satisfied, $\forall \ (\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$, a.e. $x \in \Omega$, and $\gamma > 0$. But the sum is however more natural, when considering the relation between these problems and some functionals of the Calculus of Variations, as said in the Introduction.

We can consider, as a special example of (57), the Dirichlet problem:

$$\begin{cases} -\sum_{i=1}^{N}\partial_i[|\partial_i u|^{p_i-2}\partial_i u]+\mu_0 u=\gamma\sum_{i=1}^{N}|\partial_i u|^{p_i}+f & \text{on } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{60}$$

Moreover, we remark that since we assume $f\in L^m(\Omega)$, with $m>\overline{p}^*/(\overline{p}^*-p_N)$, we must also assume $\overline{p}^*>p_N$. We think that this assumption is technical and that it should be sufficient $f\in L^m(\Omega)$ with $m>N/\overline{p}$, the condition to have bounded solutions in the monotone case (6) (see [16] and [29]).

We note that in all of the previous problems, the anisotropic operator can be substituted by a more general one, that is

$$A(u)=-\mathrm{div}(a(x,u,\nabla u)),$$

where $a(x,\sigma,\xi)=(a_i(x,\sigma,\xi))$ is vector valued function on $\Omega\times\mathbb{R}\times\mathbb{R}^N$, measurable in Ω, for any fixed $\sigma\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ and continuous in σ and ξ for a.e. $x\in\Omega$, such that, for some constant $\beta\geq\alpha>0$

$$\sum_{i=1}^{N}a_i(x,\sigma,\xi)\xi_i\geq\alpha\sum_{i=1}^{N}|\xi_i|^{p_i},$$

$$|a_i(x,\sigma,\xi)|\leq\beta\left(\sum_{j=1}^{N}|\xi_j|^{p_j}\right)^{1-1/p_i},\qquad\forall\ i=1,..,N$$

and for a.e. $x\in\Omega$ and $\forall\ \sigma\in\mathbb{R}$, ξ, $\eta\in\mathbb{R}^N$, $\xi\neq\eta$

$$\sum_{i=1}^{N}(a_i(x,\sigma,\xi)-a_i(x,\sigma,\eta))(\xi_i-\eta_i)>0.$$

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