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PSEUDO-COMMUTATIVITY OF K-Z MONADS

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Abstract: In this paper we prove that K-Z monads (also known as lax idempotent 2-monads) are pseudo-commutative. The main examples of K-Z monads for us will be 2-monads whose algebras are $\mathscr{V}$-categories with chosen colimits of a given class; this provides a large family of examples of pseudo-commutative 2-monads. We also consider tensor products associated to pseudo-closed structures and show some results on preservation of colimits. To cover the general case of $\mathscr{V}$-enriched categories and not only ordinary categories we are led to consider monads enriched in a 2-category, and some of the associated two dimensional monad theory.

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1. Introduction

This paper studies categories equipped with extra structure satisfying a uniqueness condition that could be phrased as *if the extra structure exists then it arises in a specified way, unique up to isomorphism.* A typical example of this kind of category is a category with finite colimits: if these colimits exist they arise in a unique (up to isomorphism) way, given by the definition of colimit. If one thinks of extra structure imposed on a category as a family of operations satisfying some axioms, our main result roughly states that if the extra structure satisfies this uniqueness condition then the operations *commute* with each other up to isomorphism.

The fact that certain colimits commute with certain limits is fundamental in innumerable areas of mathematics. The most common manifestation of this phenomenon is the commutation of *filtered colimits* with *finite limits*; this is the fundamental in the theory of pro-finite objects and its variations (*e.g.*, pro-finite groups), the classical theory of sheaves on topological spaces (where the stack on a point is a filtered colimit) and hence in algebraic geometry, only to mention a few examples. The present paper could be considered to be the step zero in a program aimed to obtain an algebraic formulation and understanding of the commutation of some colimits with some limits. We

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say step zero because here we are concerned with commutation of colimits with colimits, or indeed of any other structure on a category that satisfies the uniqueness condition referred to at the beginning of this introduction. The mixed case of colimits and limits, although more complicated, should fit in the same abstract framework.

Let's first recall the lower dimensional case of sets with extra structure. One way of thinking of categories of sets with extra algebraic structure is by means of monads on the category of sets **Set**; this includes the usual intuition of a sets equipped with operations satisfying equations, and more besides. The idea of operations commuting with each other is encapsulated in the notion of a *commutative monad* introduced by Kock [24, 25, 26].

Similarly, but in one dimension up, categories with extra algebraic structure can be thought of in terms of 2-monads; a few examples are monoidal categories and their braided and symmetric variants, categories equipped with a monad, or two monads with a distributive law between them, categories with (finite or otherwise) chosen (co)products, or finite biproducts. When the algebraic structures are "commutative" the 2-monad is called pseudo-commutative, and this is the case studied in detail in [10], where the main example is provided by symmetric strict monoidal categories. Observe that the braided strong monoidal functors between two symmetric strict monoidal categories are the objects of a category that is also symmetric strict monoidal. This is also true in general: for a pseudo commutative 2-monad T, the pseudomorphisms of T-algebras $A \to B$ are the objects of a T-algebra $[\![A, B]\!]$, or more precisely, the 2-category of T-algebras and pseudomorphisms T-Alg is *pseudo-closed* [10].

Our main example in this paper, in fact a family of examples, are categories with chosen colimits of a given class, and the associated 2-monads. These are examples of *K-Z* or *lax idempotent* 2-monads. Our main result states that any such 2-monad is pseudo-commutative in a canonical way.

The 2-monads corresponding to a class of colimits are different from other examples in that, although we know their algebras, there is no easy description of the 2-monad itself. Indeed, if one tries to fashion a direct proof of the pseudo-commutativity of these 2-monads, one quickly finds numerous obstacles. We avoid them by considering the wider class of K-Z monads, obtaining at the same time cleaner statements and proofs.

A standard argument using the fact that J is filtered proves that neither $q_j(\mathrm{colim}(\phi, G_j))$ nor η depend on the choice of j. So we can now stipulate this object with the named unit as the chosen colimit in C of G weighted by ϕ, and furthermore, these choices make the $\mathscr{V}$-functors $q_j : C_j \to C$ strictly preserve colimits.

The definition colimits in the previous paragraph makes $q_j : C_j \to C$ a colimiting cocone in Φ-$\mathbf{Colim}_0$. Indeed, given another cocone $t_j : C_j \to B$ the respective induced $\mathscr{V}$-functor $t : C \to B$ strictly preserves Φ-colimits. For, any such colimit in C is of the form $q_j(\mathrm{colim}(\phi, G_j))$ as above, and then

$$\begin{aligned} t(\mathrm{colim}(\phi, G)) &= tq_j(\mathrm{colim}(\phi, G_j)) = t_j(\mathrm{colim}(\phi, G_j)) \\ &= \mathrm{colim}(\phi, t_jG_j) = \mathrm{colim}(\phi, tq_jG_j) = \mathrm{colim}(\phi, tG). \blacksquare \end{aligned}$$

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as $G_j : P \to C_j$ followed by $q_j : C_j \to C$, for some $j \in J$. Consider the chosen colimit $\mathrm{colim}(\phi, G_j)$ in C_j, with unit $\eta_j : \phi \Rightarrow C_j(G_j-, \mathrm{colim}(\phi, G_j))$. We shall show that $q_j(\mathrm{colim}(\phi, G_j))$ is a colimit of G weighted by ϕ, or in other words that there exists an isomorphism in in $\mathscr{V}\text{-}\mathbf{Cat}_1(C, \mathscr{V})$

$$[P^{\mathrm{op}}, \mathscr{V}](\phi-, C(G-, ?)) \cong C(q_j(\mathrm{colim}(\phi, G_j), ?). \tag{59}$$

Since C is a colimit of the functor D into $\mathscr{V}\text{-}\mathbf{Cat}_0$ and $P : (j \downarrow J) \to J$ is final, C is a colimit of DP. This is not only but also a 2-categorical colimit of the associated 2-functor into the 2-category $\mathscr{V}\text{-}\mathbf{Cat}_1$, it will be enough to exhibit $\mathscr{V}$-natural isomorphisms between functors $C_k \to \mathscr{V}$

$$[P^{\mathrm{op}}, \mathscr{V}](\phi-, C(G-, q_k?)) \cong C(q_j(\mathrm{colim}(\phi, G_j), q_k?)) \tag{60}$$

for each $\alpha : j \to k$ in J, and natural with respect to arrows in $j \downarrow J$. We define (60) by the following string of isomorphisms:

$$\begin{aligned}
[P^{\mathrm{op}}, \mathscr{V}](\phi-, &C(G-, q_k?)) = [P^{\mathrm{op}}, \mathscr{V}](\phi-, C(q_k(D\alpha)G_j-, q_k?)) \\
&\cong [P^{\mathrm{op}}, \mathscr{V}](\phi-, \operatorname*{colim}_{\beta:k\to\ell} C_\ell(D(\beta\alpha)G_j-, (D\beta)?)) && (61)\\
&\cong \operatorname*{colim}_{\beta:k\to\ell} [P^{\mathrm{op}}, \mathscr{V}](\phi-, C_\ell(D(\beta\alpha)G_j-, (D\beta)?)) && (62)\\
&\cong \operatorname*{colim}_{\beta:k\to\ell} C_\ell(\mathrm{colim}(\phi, D(\beta\alpha)G_j), (D\beta)?) && (63)\\
&\cong \operatorname*{colim}_{\beta:k\to\ell} C_\ell(D(\beta\alpha)(\mathrm{colim}(\phi, G_j)), (D\beta)?) && (64)\\
&\cong C(q_\ell D(\beta\alpha)(\mathrm{colim}(\phi, G_j)), q_\ell(D\beta)?) && (65)\\
&= C(q_j(\mathrm{colim}(\phi, G_j)), q_k?) && (66)
\end{aligned}$$

We briefly explain each isomorphism: (61) is an application of Lemma 9.8; ϕ is finitely presented in $[P^{\mathrm{op}}, \mathscr{V}]$ because it is a finite weight (see [23, section 3]), hence the isomorphism (62); (63) is just the definition of colimit and (64) is the isomorphism resulting from using the fact that $D(\beta\alpha)$ (strictly) preserves colimits; (65) is another application of Lemma 9.8 and finally the equality (66) holds by naturality of the cocone q_k. This shows that $q_j(\mathrm{colim}(\phi, G_j)$ is a colimit of G weighted by ϕ. To find the unit $\eta : \phi \Rightarrow C(G-, q_j(\mathrm{colim}(\phi, G_j))$ of this colimit it is enough to take the $\alpha = 1 : j \to j$ and from the identity morphism of $q_j(\mathrm{colim}(\phi, G_j)$ in (66) work our way up through the isomorphisms to obtain

$$\eta : \phi \xrightarrow{\eta_j} C_j(G_j-, \mathrm{colim}(\phi, G_j)) \xrightarrow{q_j} C(G-, q_j(\mathrm{colim}(\phi, G_j)).$$

The notion of colimit makes sense in the context of enriched categories and indeed in the examples our categories can be enriched in vector spaces, simplicial sets, in the category $\mathbb{2} = \{* \to \bullet\}$ (so $\mathbb{2}$-categories are partially ordered sets) and many other possibilities. This makes us consider the 2-monads T_Φ on the 2-category $\mathscr{V}\text{-}\mathbf{Cat}$ of $\mathscr{V}$-enriched categories, whose algebras are $\mathscr{V}$-categories with chosen colimits of the class Φ, as studied in [21]. In order to be able to extend the results on the pseudo-commutativity of T_Φ to this enriched case, we are forced to consider T_Φ as a monad enriched in $\mathscr{V}\text{-}\mathbf{Cat}$. Working with $\mathscr{V}\text{-}\mathbf{Cat}$ presents no advantages nor convenient features to working with a general 2-category $\mathscr{W}$ (required to be symmetric monoidal closed, complete and cocomplete); because of this we opt for using $\mathscr{W}$ if only because of the notational clarity it provides. We emphasise that enriching in $\mathscr{V}\text{-}\mathbf{Cat}$ would not have shorten or simplified any of the paper's material. To accommodate the existing theory of 2-monads to this enrichment in $\mathscr{W}$ we define the $\mathscr{W}$-category T-Alg, whose enriched homs are "objects of pseudomorphisms," and provide easy extensions of some of the results in [3] to the $\mathscr{W}$-enriched framework.

The paper is organised as follows.

After this introduction, Section 2 recalls some of the necessary background on two-dimensional monad theory. In Section 3 we describe the $\mathscr{W}$-category T-Alg of algebras (and pseudomorphisms) of a $\mathscr{W}$-monad T. The necessary adaptations of the pseudo-closed 2-categories and pseudo-commutative 2-monads of [10] to the $\mathscr{W}$-enriched context are described in Section 4. Section 5 proves one of the key results of this work, characterising pseudo-commutativities in terms of data in T-Alg. The pseudo-closed structure of the $\mathscr{W}$-category T-Alg for a pseudo-commutative T together with the induced tensor product can be found in Section 6; this is largely an easy adaptation of the 2-categorical case, but we add some results on preservation of colimits by the tensor product. In Section 7 we prove our main result stating that K-Z $\mathscr{W}$-monads are pseudo-commutative, while in Section 8 we look at the example of monads given by completion under a class of chosen colimits. Finally, there is an Appendix where we confine some standard extensions of the existence of flexible replacements to $\mathscr{W}$-enriched monads, and the proof that the 2-monad for chosen finite colimits is finitary.

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2. Background on 2-monads

In this section we summarise the concepts of two-dimensional monad theory necessary throughout the rest of the paper. The basic references on 2-categories are [13, 2] and for 2-monad theory [3]; [30] provides a good survey of both.

A 2-monad (T, η, μ) on a 2-category $\mathscr{K}$ is a 2-functor $T : \mathscr{K} \to \mathscr{K}$ together with 2-natural transformations $\mu : T^2 \Rightarrow T$, $\eta : 1 \Rightarrow T$ satisfying the usual monad axioms: $\mu_X \mu_{TX} = \mu_X(T\mu_X)$ and $\mu_X(T\eta_X) = 1_{TX} = \mu_X \eta_{TX}$.

Given a 2-monad T on $\mathscr{K}$, a T*-pseudoalgebra* is an object A of $\mathscr{K}$ with a 1-cell $a : TA \to A$ and invertible 2-cells

$$a(Ta) \cong a\mu_A \qquad 1_a \cong a\eta_X \tag{1}$$

satisfying two axioms (see [3]). When the these 2-cells are identities we say that (A, a) is a *strict algebra*. We will usually denote a T-algebra (A, a) simply by its object part A, omitting the explicit mention of the action, which will then be denoted by the lowercase of the letter we use for the object part.

For the benefit of the reader unfamiliar with 2-monads, we provide this basic example.

Example 2.1. A monoidal category can be identified with a pseudoalgebra for a 2-monad T on **Cat**. The category TC has objects and arrows, respectively, finite sequences of objects and arrows of C. Concatenation of lists endows TC with the structure of a strict monoidal category. The 1-cell $a : TC \to C$ is a functor, that can be thought as providing the tensor product of a list of objects

$$(x_1, x_2, \cdots, x_n) \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

while its value on the empty list can be thought as the unit object for the monoidal structure. The isomorphisms (1) provide the associativity and unit constraints. Observe that a pseudo-T-algebra is not exactly the same as a monoidal category but rather an *unbiased* monoidal category; see [32] for a full explanation. Monoidal categories are algebras for a 2-monad, the description of which is related to the original Mac Lane's proof of the coherence theorem for monoidal categories [33] and Kelly's notion of a *club* (see [15] and the references therein).

Example 2.2. We will refer to later to the following 2-monad S on **Cat**, that is the main example of a pseudo-commutative 2-monad in [10]. For a category

Now we turn out attention to the matter of the $\mathscr{V}$-naturality (in x, y) of the isomorphisms (51). We shall show the $\mathscr{V}$-naturality on one of the two variables, namely the commutativity of the square

$$\begin{tikzcd}
(\operatorname{colim} H^j)(x,y) \otimes C_j(y,y') \arrow[r] \arrow[d] & C(q_j x, q_j y) \otimes C_j(y,y') \arrow[d] \\
(\operatorname{colim} H^j)(x,y') \arrow[r] & C(q_j x, q_j y')
\end{tikzcd} \tag{58}$$

with horizontal arrows induced by the isomorphism (51) and vertical arrows given by the respective $\mathscr{V}$-functor structures. The case of the other variable is completely analogous. The diagram (58) commutes if for each $\alpha : j \to k$ in J the following diagram commutes:

$$\begin{tikzcd}
C_k((D\alpha)x, (D\alpha)y) \otimes C_j(y,y') \arrow[r, "q_k \otimes 1"] \arrow[d, "1 \otimes D\alpha"'] & C(q_j x, q_j y) \otimes C_j(y,y') \arrow[d, "1 \otimes D\alpha"] \\
C_k((D\alpha)x, (D\alpha)y) \otimes C_k((D\alpha)y, (D\alpha)y') \arrow[r, "q_k \otimes 1"] \arrow[d, "\mathrm{comp}"'] & C(q_j x, q_j y) \otimes C_k((D\alpha)y, (D\alpha)y') \arrow[d, "\mathrm{comp}"] \\
C_k((D\alpha)x, (D\alpha)y') \arrow[r, "q_k"] & C(q_j x, q_j y')
\end{tikzcd}$$

which does because q_k is a $\mathscr{V}$-functor. This finishes the proof of the lemma. ■

Theorem 9.9. *For any class of finite weights* Φ*, the forgetful* $\mathscr{V}$-**Cat***-functor* $U_s : \Phi\text{-}\mathbf{Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$ *creates filtered colimits. Equivalently, the* $\mathscr{V}$-**Cat**-*monad* T_Φ *on* $\mathscr{V}$-**Cat** *whose algebras are* $\mathscr{V}$*-categories with chosen* Φ*-colimits is finitary.*

Proof: The the theorem can be equivalently expressed as asserting that the ordinary functor (U_s creates filtered colimits; for Φ-**Colim** has cotensor products and hence any ordinary conical colimit in it is automatically an enriched colimit (see [17, Section 3.8]).

We follow the notation employed in Lemma 9.8: J will be a filtered category, $D : J \to \Phi\text{-}\mathbf{Colim}_0$ a functor whose composition with $(U_s)_0$ will be also denoted by D, $D(j)$ will be abbreviated by C_j and $\operatorname{colim} D \in \mathscr{V}\text{-}\mathbf{Cat}_0$ by C, with colimiting cocone $q_j : C_j \to C$.

First we must equip C with chosen Φ-colimits. Let $\phi : P^{\mathrm{op}} \to \mathscr{V}$ be a weight in Φ, and in particular a finite weight, and $G : P \to C$ a $\mathscr{V}$-functor. The $\mathscr{V}$-category P is finite, and then finitely presented in $\mathscr{V}\text{-}\mathbf{Cat}_0$, so G factors

and using the fact that P is final, $C(x,y) \cong \operatorname{colim} G_{x,y}P$. Now we show that

$$(\operatorname{colim} H^j)(x,y) \cong C(q_j(x), q_j(y)) \tag{51}$$

for all $x, y \in C_j$ by exhibiting a bijection between cocones $\rho : H^j(-)(x,y) \Rightarrow z$ and cocones $\tau : G_{q_j(x),q_j(y)}P \Rightarrow z$. To give τ is to give for each object $\alpha : j \to k$ in $(j \downarrow J)$ and $u, v \in C_k$ such that $q_k(u) = q_j(x)$, $q_k(v) = q_j(y)$, arrows in $\mathscr{V}_0$

$$\tau_\alpha^{u,v} : C_k(u,v) \to z \tag{52}$$

The naturality of τ with respect to α means that for any $\beta : k \to \ell$ in J we have

$$\begin{tikzcd} C_k(u,v) \arrow[rr, "D\beta"] \arrow[dr, "\tau_\alpha^{u,v}"'] & & C_\ell((D\beta)u, (D\beta)v) \arrow[dl, "\tau_{\beta\alpha}^{(D\beta)u,(D\beta)v}"] \\ & z & \end{tikzcd} \tag{53}$$

On the other hand to give ρ is equivalent to giving for each $\alpha : j \to k$ in $(j \downarrow J)$ an arrow

$$\rho_\alpha : C_k((D\alpha)x, (D\alpha)y) \to z \tag{54}$$

satisfying the following naturality condition for each arrow $\gamma : (\alpha : j \to k) \to (\beta : j \to \ell)$ in J.

$$\begin{tikzcd} C_k((D\alpha)x, (D\alpha)y) \arrow[rr, "D\gamma"] \arrow[dr, "\rho_\alpha"'] & & C_\ell((D\beta)x, (D\beta)y) \arrow[dl, "\rho_\beta"] \\ & z & \end{tikzcd} \tag{55}$$

Given ρ define (52) in the following way. Choose a arrow $\beta : k \to k'$ in J such that $(D\beta)u = D(\beta\alpha)x$ and $(D\beta)v = D(\beta\alpha)y$ and set

$$\tau_\alpha^{u,v} : C_k(u,v) \xrightarrow{D\beta} C_{k'}((D\beta)u, (D\beta)v) = C_{k'}(D(\beta\alpha)x, D(\beta\alpha)y) \xrightarrow{\rho_{\beta\alpha}} z. \tag{56}$$

Using the fact that J is filtered and the naturality of ρ (made explicit in (55)) its routine to verify that (56) does not depend on the choice of $\beta : k \to k'$. Conversely, given τ we can define ρ_α (54) as $\tau_\alpha^{(D\alpha)x,(D\alpha)y} : C_k((D\alpha)x, (D\alpha)y) \to z$. The naturality condition (55) is immediately implied from the naturality of τ (53). The correspondence between τ and ρ just described is a bijection, yielding an isomorphism (51) that is induced by the cocone

$$H^j(\alpha)(x,y) = C_k((D\alpha)x, (D\alpha)y) \xrightarrow{q_k} C(q_k(D\alpha)x, q_k(D\alpha)y) = C(q_jx, q_jy). \tag{57}$$

X, SX has objects the lists of objects of X and arrows $(x_1, \cdots, x_n) \to (y_1, \cdots, y_n)$ the $n+1$-tuples $(f_1, \cdots, f_n, s)$ where s is a permutation of n elements and $f_i : x_i \to y_{s(i)}$ is an arrow in X. Composition is induced by the multiplication of permutations and the composition in X. Note that there are no arrows between lists of different length. The category SX is not only strict (unbiased) monoidal via concatenation of lists (definition on arrows should be obvious) but is moreover *symmetric*. If $\underline{x} = (x_1, \cdots, x_n)$, $\underline{y} = (y_1, \cdots, y_m)$, the component of the symmetry $\underline{x} \otimes \underline{y} \to \underline{y} \otimes \underline{x}$ is the arrow $(1, \cdots, 1, s)$, where s is the permutation of $n+m$ elements: $s(i) = i+n$ if $1 \leq i \leq n$, $s(i) = i-n$ if $n+1 \leq i \leq n+m$. S-algebras can be identified with strict (unbiased) symmetric monoidal categories.

Example 2.3. The examples of 2-monad we are more interested in are 2-monads whose algebras are categories with chosen colimits of a certain class [21]. These examples are discussed in Section 8.

Example 2.4. Other structures that can be presented as algebras for a 2-monad on **Cat** are: braided monoidal categories; categories equipped with a endofunctor, a pointed endofunctor or a monad; a category equipped with two monads and a distributive law between them.

Given a pseudomonad $T : \mathscr{K} \to \mathscr{K}$ and two pseudo-T-algebras A and B, a lax morphism from A to B is a 1-cell $f : A \to B$ in $\mathscr{K}$ together with a 2-cell

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Swarrow \bar{f}"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

that satisfies two axioms of compatibility with the pseudoalgebra structures. When this 2-cell is invertible we say that $(f, \bar{f})$ is a *pseudomorphism*, and when $\bar{f} = 1$ we say that f is a *strict morphism*.

A 2-cell between two lax morphisms $(f, \bar{f}), (g, \bar{g}) : A \to B$ between (pseudo or strict) algebras is just a 2-cell $\alpha : f \Rightarrow g$ in $\mathscr{K}$ compatible with $\bar{f}, \bar{g}$.

Now we can combine algebras and morphisms to form 2-categories. For a given 2-monad T on $\mathscr{K}$ we denote by Ps-T-Alg the 2-category with objects pseudo-T-algebras, 1-cells pseudomorphisms and 2-cells the 2-cells defined above. We denote by T-Alg the 2-category of (strict) T-algebras and pseudomorphisms between them, and by T-Alg$_\mathrm{s}$ the 2-category of (strict)

T-algebras and strict morphisms between them. In all instances the 2-cells are the same, and all these 2-categories have a forgetful 2-functor into $\mathscr{K}$, and there is an inclusion 2-functor $J : T\text{-Alg}_{\mathrm{s}} \to T\text{-Alg}$. Full details can be found in [3].

Example 2.5. For the 2-monad T on **Cat** of Example 2.1, it is not hard to see that a pseudomorphism correspond to a strong monoidal functor (in the terminology of [12]), and in fact Ps-T-Alg is (equivalent) to the 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations. By methods developed in [3], with [14] as a predecessor, one can find another 2-monad T' and an equivalence between Ps-T-Alg and T'-Alg.

3. $\mathscr{W}$-enriched monads

Enriched categories provide a framework in which to study categories whose hom sets have more structure, for example, hom sets that are abelian groups; or even categories whose homs are not simply sets with extra structure but some other type of objects, as for example simplicial sets, chain complexes, non-negative real numbers.

The notion of enriched category we consider is the classical one due to Eilenberg-Kelly [9]. Thus will enrich in categories that are symmetric monoidal closed, complete and cocomplete. The power of the theory of enriched categories is exemplified by [17].

When we are dealing with $\mathscr{V}$-enriched categories with extra algebraic structure, usually we are not contemplating an ordinary 2-monad on $\mathscr{V}$-**Cat** but actually a $\mathscr{V}$-**Cat**-enriched monad. The case of most interest for us will be the 2-monads on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen colimits of a certain class, considered in [21].

Unless we impose some restrictive conditions on $\mathscr{V}$, there is no advantage in working with $\mathscr{V}$-**Cat** instead of a more general 2-category $\mathscr{W}$, so we will follow this second option, assuming that $\mathscr{W}$ is a complete and cocomplete symmetric monoidal closed **Cat**-enriched category. Before proceeding to the kernel of this paper we need to say some words on the two-dimensional monad theory associated to a $\mathscr{W}$-enriched monad.

As usual, the functor $\mathscr{W}(I,-) : \mathscr{W} \to \mathbf{Set}$ induces a 2-functor $(-)_0 : \mathscr{W}\text{-}\mathbf{Cat} \to \mathbf{Cat}$. But taking into account that $\mathscr{W}$ is a 2-category, and so $\mathscr{W}(I,-)$ is in fact a 2-functor into **Cat**, we get a 2-functor $(-)_1 : \mathscr{W}\text{-}\mathbf{Cat} \to$

hypotheses on $\mathscr{V}$, in order to have a good theory of enriched finite limits and colimits one has to require $\mathscr{V}$ to be locally finitely presentable as a monoidal category [18].

We want to show that the forgetful 2-functor $U_{\mathrm{s}} : \Phi\text{-Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$ creates filtered colimits. The forgetful 2-functor $\mathscr{V}\text{-}\mathbf{Cat}_0 \to \mathscr{V}\text{-}\mathbf{Gph}_0$ into the category of $\mathscr{V}$-graphs is finitarily monadic, as shown in [22]. Colimits in $\mathscr{V}\text{-}\mathbf{Gph}_0$ have the following simple description. If $D : J \to \mathscr{V}\text{-}\mathbf{Gph}_0$ is a functor with J small, write $G_j = D(j)$. Define $\mathrm{ob}\, G$ as $\mathrm{colim}_j\, \mathrm{ob}\, G_j$, with universal cocone $q_j : \mathrm{ob}\, G_j \to \mathrm{ob}\, G$. Define $G(x,y)$ as the colimit in $\mathscr{V}$ of the functor $G : J \to \mathscr{V}$ defined on objects by sending $j \in J$ to $\sum_{q_j(u)=x, q_j(v)=y} G_j(u,v)$ and on arrows in the obvious way. We obtain morphisms of $\mathscr{V}$-graphs $q_j : G_j \to G$ forming a colimiting cocone. Details, along with a more conceptual description using the bicategory $\mathscr{V}$-**Mat** of $\mathscr{V}$-matrices, can be found in [22].

Let $D : J \to \Phi\text{-Colim}_0$ be an ordinary functor with J filtered. We shall also denote by D the functor $J \to \mathscr{V}\text{-}\mathbf{Cat}_0$ resulting from composing with $(U_s)_0$. To abbreviate, we denote $D(j)$ by C_j. We know that D has a colimit since the 2-category $\mathscr{V}$-**Cat** is cocomplete; that is, there exists a $\mathscr{V}$-category $\mathscr{C}$ and a natural transformation $q_j : C_j \to C$ inducing an isomorphism $\mathscr{V}\text{-}\mathbf{Cat}(C,B) \cong \lim_j \mathscr{V}\text{-}\mathbf{Cat}(C_j,B)$ *2-natural* in $\mathscr{B}$. Then $\mathscr{C}$ is *a fortiori* a colimit in the ordinary category $\mathscr{V}\text{-}\mathbf{Cat}_0$ and hence in $\mathscr{V}\text{-}\mathbf{Gph}_0$. As J is filtered, the $\mathscr{V}$-enriched homs $C(x,y)$ have a simpler description than in the general case. Pick $j \in J$ such that there exist and define a functor

$$H^j : (j \downarrow J) \longrightarrow [C_j^{\mathrm{op}} \otimes C_j, \mathscr{V}]_0 \tag{49}$$

by $H^j(\alpha : j \to k) = C_k((D\alpha)-, (D\alpha)?)$; H^j is defined on an arrow $\gamma : (\alpha : j \to k) \to (\beta : j \to \ell)$ by the the effect on enriched homs of the $\mathscr{V}$-functor $D\gamma : C_k \to C_\ell$.

Lemma 9.8. $\mathrm{colim}\, H^j \cong C(q_j-, q_j?) : C_j^{\mathrm{op}} \otimes C_j \to \mathscr{V}$.

Proof: The category $(j \downarrow J)$ is filtered because J is so, and the projection functor $P : (j \downarrow J) \to J$ is final. Since the forgetful $\mathscr{V}\text{-}\mathbf{Cat} \to \mathscr{V}\text{-}\mathbf{Gph}$ is finitary, as previously mentioned $C(x,y)$ is the colimit of a functor $G_{x,y} : J \to \mathscr{V}_0$

$$G_{x,y}(k) = \sum_{q_k(u)=x, q_k(v)=y} C_k(u,v) \tag{50}$$

need to be verified. The $\mathscr{V}$-category $T\text{-Alg}_{\mathrm{s}}$ can be identified with a full sub-$\mathscr{V}$-category of $T/\mathscr{K}$ and this inclusion preserves cotensor products. Hence the inclusion has a left adjoint if its underlying functor does, and this is guaranteed by [16, Theorem 25.2]. ■

9.2. A parametrised biadjunction. A consequence of the existence of flexible replacements, and one of the main results of [3] is its Theorem 5.1. This is exactly the result used to show the existence of a tensor product associated to a pseudo-closed structure on T-Alg. As we explain below, we need the a parametrised version of Blackwell-Kelly-Power's result.

Recall that given 2-categories $\mathscr{P}, \mathscr{L}, \mathscr{M}$ and 2-functors $H : \mathscr{P} \times \mathscr{L} \to \mathscr{M}$ and $G : \mathscr{P}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{L}$, a *left parametrised adjunction* is a 2-natural isomorphism

$$\pi_{*,?,-} : \mathscr{M}(H(*,?),-) \cong \mathscr{L}(?,G(*,-)) \tag{47}$$

For each object P of $\mathscr{P}$ we obtain an adjunction

$$\pi_{P,?,-} : \mathscr{M}(H(P,?),-) \cong \mathscr{L}(?,G(P,-))$$

We shall work under the blanket assumptions of [3]: T is a 2-monad with a rank on a complete an cocomplete 2-category $\mathscr{K}$.

Theorem 9.7. *Let $G : \mathscr{P}^{\mathrm{op}} \times T\text{-Alg} \to \mathscr{L}$ be a 2-functor such that the composite $G(\mathscr{P}^{\mathrm{op}} \times J) : \mathscr{P}^{\mathrm{op}} \times T\text{-Alg}_{\mathrm{s}} \to \mathscr{L}$ has a left parametrised left adjoint $H : \mathscr{P} \times \mathscr{L} \to T\text{-Alg}_{\mathrm{s}}$, with unit $s_P : 1 \Rightarrow G(P, H(P,-))$. Then the 2-natural transformation*

$$T\text{-Alg}(JH(*,?),-) \xrightarrow{G(*,-)} \mathscr{L}(G(*,JH(*,?)),G(*,-)) \xrightarrow{\mathscr{L}(s_*,1)} \mathscr{L}(?,G(*,-)) \tag{48}$$

is a retract equivalence.

9.3. The monad for finite colimits is finitary. In this section we prove that the 2-monad on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen finite colimits is finitary. The observation that the category of (small) categories with certain chosen (co)limits is monadic over **Cat** is attributed in [1] to C. Liar [31]. The fact that the monad for certain finite (co)limits is finitary can be considered to be present in [4], modulo the subtleties mentioned in [1]. As the case of enriched categories seems to be missing from the literature we feel necessary to provide a complete proof that the 2-monad constructed in [21] associated to a class of finite colimits is finitary. In addition to the usual

2-**Cat**. Extending the usual notation of A_0 for the underlying (ordinary) category of a $\mathscr{W}$-category A, we call A_1 its *underlying 2-category,* and similarly for functors and transformations.

3.1. Strength and enrichment. Suppose $\mathscr{K}$ is a $\mathscr{W}$-category that admits cotensor products with objects of $\mathscr{W}$. Recall that a cotensor of an object B of $\mathscr{K}$ with an object X of $\mathscr{W}$, denoted by $\{X,B\}$, of $\mathscr{W}$ is defined by the existence of a $\mathscr{W}$-natural isomorphism $\mathscr{K}(A,\{X,B\}) \cong [X,\mathscr{K}(A,B)]$. Cotensor products are a particular instance of weighted limits (see [17]). As such, for any $\mathscr{W}$-functor $T : \mathscr{K} \to \mathscr{K}$ there is a canonical comparison $\mathscr{W}$-natural transformation

$$\bar{t}_{X,B} : T\{X,B\} \longrightarrow \{X,TB\}$$

whose component $\bar{t}_{X,B}$ can be characterised as the unique arrow making the following diagram commutative.

$$\begin{tikzcd}
{[X,\mathscr{K}(A,B)]} \arrow[rr, "{[X,\mathsf{T}]}"] \arrow[d, "\cong"'] & & {[X,\mathscr{K}(TA,TB)]} \arrow[d, "\cong"] \\
\mathscr{K}(A,\{X,B\}) \arrow[r, "\mathsf{T}"] & \mathscr{K}(TA,T\{X,B\}) \arrow[r, "{\mathscr{K}(TA,\bar{t})}"] & \mathscr{K}(TA,\{X,TB\})
\end{tikzcd} \tag{2}$$

Observe that the T-algebra structure on the contensor product $\{X,A\}$ of $X \in \mathscr{W}$ with a T-algebra A can be written in terms of $\bar{t}$ and $a : TA \to A$ as

$$T\{X,A\} \xrightarrow{\bar{t}_{X,A}} \{X,TA\} \xrightarrow{\{X,a\}} \{X,A\}$$

When $\mathscr{K} = \mathscr{W}$, cotensor products are just internal homs, and a further transformation is associated to the enrichment of T in $\mathscr{W}$, namely the *strength*

$$t_{X,Y} : X \otimes TY \longrightarrow T(X \otimes Y).$$

This transformation satisfies $t_{X,Y\otimes Z}.(X \otimes t_{Y,Z}) = t_{X\otimes Y,Z}$, and the composition of $t_{I,X}$ with the canonical unit isomorphisms is the identity arrow of TX. We denote by $t'_{X,Y} : TX \otimes Y \to T(X \otimes Y)$ the transformation obtained from t and the symmetry of $\mathscr{W}$ in the obvious way. An ordinary endo-functor equipped with a strength is called a *strong functor*. There is a bijection between strengths on $T : \mathscr{W} \to \mathscr{W}$ and enrichments of T in $\mathscr{W}$.

When (T,η,μ) is a $\mathscr{W}$-monad, the associated strength of T satisfies additional properties, equivalent to the $\mathscr{W}$-naturality of η and μ; namely, $\mu_{X\otimes Y}.T(t_{X,Y}).t_{X,TY} = t_{X,Y}.(X \otimes \mu_X)$ and $t_{X,Y}.(X \otimes \eta_Y) = \eta_{X\otimes Y}$. These equalities translate in terms of $\bar{t}_{X,Y} : T[X,Y] \to [X,TY]$ as $[X,\mu_Y].\bar{t}_{X,TY}.(T\bar{t}_{X,Y}) =$

$\bar{t}_{X,Y}.\mu_{[X,Y]}$ and $\bar{t}_{X,Y}.\eta_{[X,Y]} = [X, \eta_Y]$. Ordinary monads equipped with an strength satisfying the aforementioned equalities are called *strong monads*, and appear in the series [24, 25, 26]. Then, to give a strong monad on $\mathscr{W}$ is equivalent to giving a $\mathscr{W}$-monad.

3.2. The $\mathscr{W}$-category T-Alg. As in the rest of the section, $\mathscr{W}$ will be a complete and cocomplete monoidal closed 2-category. The theory of monads on categories and their algebras can be generalised to the enriched context [8]. For a $\mathscr{W}$-monad (T, η, μ) on a $\mathscr{W}$-category $\mathscr{K}$, the the Eilenberg-Moore $\mathscr{W}$-category of algebras of T, which we will denote by $T\text{-Alg}_{\mathrm{s}}$, has objects the T_0-algebras, where T_0 is the ordinary monad underlying T on $\mathscr{K}_0$. If A, B are T-algebras, the enriched hom $T\text{-Alg}_{\mathrm{s}}(A, B)$ is defined by the equalizer of the following pair:

$$\mathscr{K}(A,B) \xrightarrow{\mathsf{T}} \mathscr{K}(TA,TB) \xrightarrow{\mathscr{K}(1,b)} \mathscr{K}(TA,B) \qquad \mathscr{K}(A,B) \xrightarrow{\mathscr{K}(a,1)} \mathscr{K}(TA,B) \tag{3}$$

The corresponding forgetful $\mathscr{W}$-functor will be denoted by U_{s}. Observe that the 2-category $T\text{-Alg}_{\mathrm{s},1}$ is the 2-category of algebras and strict morphisms of algebras for the 2-monad T_1 on $\mathscr{K}_1$, and $U_{\mathrm{s},1}$ the corresponding forgetful 2-functor.

For each pair of T-algebras A, B, we have 1-cells in $\mathscr{W}$

$$\sigma_{A,B} : \mathscr{K}(A,B) \xrightarrow{\mathsf{T}} \mathscr{K}(TA,TB) \xrightarrow{\mathscr{K}(TA,b)} \mathscr{K}(TA,B) \tag{4}$$

that form a $\mathscr{W}$-natural transformation $\sigma : \mathscr{K}(U_s-, U_s-) \Rightarrow \mathscr{K}(TU_s-, U_s-) : T\text{-Alg}_s^{\mathrm{op}} \otimes T\text{-Alg}_s \to \mathscr{W}$. Observe that σ satisfy the following equations:

$$\sigma_{TA,B}\sigma_{A,B} = \mathscr{K}(\mu_A, B)\sigma_{A,B} \qquad \mathscr{K}(\eta_A, B)\sigma_{A,B} = 1 \tag{5}$$

This transformation will play a central role later sections.

Remark 3.1. When $\mathscr{K}$ admits cotensor products, $\sigma_{A,B}$ and $\sigma_{A,\{X,B\}}$ are related by the commutativity of the following square (a consequence of the commutativity of (2)).

$$\begin{tikzcd} {[X, \mathscr{K}(A,B)]} \arrow[r, "{[X,\sigma_{A,B}]}"] \arrow[d, "\cong"'] & {[X, \mathscr{K}(TA,B)]} \arrow[d, "\cong"] \\ \mathscr{K}(A, \{X,B\}) \arrow[r, "\sigma_{A,\{X,B\}}"] & \mathscr{K}(TA, \{X,B\}) \end{tikzcd} \tag{6}$$

Proof: Apply the standard fact that a cotensor product preserving enriched functor from a category that admits cotensor products has a left adjoint if and only if its underlying ordinary functor does. ■

The proof of the following fact is the same as what we indicated in Remark 9.1 but only for details.

Corollary 9.4. *Assume that T preserves ϕ-colimits for a weight $\phi : \mathscr{D} \to \mathscr{W}$. Then J preserves ϕ-colimits.*

Corollary 9.5. *If J has a left adjoint then there are canonical retract equivalences in $\mathscr{W}_1$*

$$T\text{-Alg}(FX, B) \simeq \mathscr{K}(X, UB) \tag{45}$$

Proof: Consider the following 1-cell in $\mathscr{W}_1$:

$$e_{X,B} : T\text{-Alg}(FZ, B) \xrightarrow{U} \mathscr{K}(TZ, UB) \xrightarrow{\mathscr{K}(\eta_Z, UB)} \mathscr{K}(Z, UB) \tag{46}$$

The functor $\mathscr{W}_1(I, e_{Z,B}) : T_1\text{-Alg}(F_1 Z, B) \to \mathscr{K}_1(Z, U_1 B)$ is a retract equivalence by Theorem 5.1 and Corollary 5.6 of [3]. Since U preserves cotensor products with objects of $\mathscr{W}$, we deduce that $\mathscr{W}_1(X, e_{Z,B})$ is an equivalence for all X in $\mathscr{W}$, as this functor is, up to composing with canonical isomorphisms, $\mathscr{W}_1(I, e_{Z,\{X,B\}})$. It follows that $e_{Z,B}$ is an equivalence in $\mathscr{W}_1$. To prove that it is a retract equivalence, it is enough to show that it has a right inverse (see [11, A.1.1.1]), which will be provided by the composite

$$\mathscr{K}(Z, UB) \xrightarrow{F_{\mathrm{s}}} T\text{-Alg}_{\mathrm{s}}(FZ, FUB) \xrightarrow{T\text{-Alg}_{\mathrm{s}}(FZ,b)} T\text{-Alg}_{\mathrm{s}}(FZ, B) \xrightarrow{J} T\text{-Alg}(FZ, B).$$

The fact that this is a right inverse is just a consequence of the adjunction $F_{\mathrm{s}} \dashv U_{\mathrm{s}}$. ■

We finish the section with an enriched version of [3, Theorem 3.8]. The enrichment can be in any complete and cocomplete symmetric monoidal closed category $\mathscr{V}$, not necessarily a 2-category.

Lemma 9.6. *Let T be a $\mathscr{V}$-enriched monad with a rank on a $\mathscr{V}$-category cocomplete $\mathscr{K}$. If $\mathscr{K}$ has cotensor products, then $T\text{-Alg}_{\mathrm{s}}$ is cocomplete.*

Proof: We only give a sketch of a proof, that is completely analogous to the 2-categorical case in [3]. The comma $\mathscr{V}$-category $T/\mathscr{K}$ can be easily seen to have cotensor products constructed from those of $\mathscr{K}$. Then one can copy the proof of [3, Proposition 3.4] to show that $T/\mathscr{K}$ has colimits; in the presence of cotensor products only the one-dimensional part of the colimit definition

equivalent to say that $J : T\text{-Alg}_s \to T\text{-Alg}$ preserves ϕ-colimits, because T-Alg can be seen as the Kleisli construction for the comonad $(-)'J$,

Lemma 9.2. *If T is a $\mathscr{W}$-monad on a cotensored $\mathscr{W}$-category $\mathscr{K}$, then T-Alg has and $J : T\text{-Alg}_s \to T\text{-Alg}$ preserves cotensor products with objects of $\mathscr{W}$.*

Proof: It is standard that the $\mathscr{W}$-functor $U_s : T\text{-Alg}_s \to \mathscr{W}$ creates cotensor products. So the cotensor product of a T-algebra A with an object X of $\mathscr{W}$ is the object $\{X, A\}$ with algebra structure $\{X, a\}.\bar{t}_{X,A} : T\{X, A\} \to \{X, TA\} \to \{X, A\}$. The $\mathscr{W}$-natural isomorphism $T\text{-Alg}_s(A, \{X, B\}) \cong [X, T\text{-Alg}_s(A, B)]$ is induced by a projection $p_B : X \to T\text{-Alg}_s(\{X, B\}, B)$ (that corresponds under the isomorphisms above to the identity of $\{X, B\}$).

We will show that the 1-cell

$$X \xrightarrow{p_B} T\text{-Alg}_s(\{X, B\}, B) \xrightarrow{J} T\text{-Alg}(\{X, B\}, B) \tag{42}$$

induces isomorphisms $T\text{-Alg}(A, \{X, B\}) \cong [X, T\text{-Alg}(A, B)]$. The existence of these isomorphisms follows easily from the definition of $T\text{-Alg}(A, B)$ as a limit in $\mathscr{W}_1$ (Section 3.2), the fact that $\{X, -\}$ preserves limits and Remark 3.1. It is not hard to see that these isomorphisms are $\mathscr{W}$-natural in A. As such, by Yoneda, they are induced by an arrow $q_B : X \to T\text{-Alg}(\{X, B\}, B)$. To finish the proof we must show that q_B is the arrow (42). The square (43) commutes by the definition of J (which is just a comparison 1-cell resulting from the universal property of the objects of pseudomorphisms) and the fact that $\{X, -\}$ preserves limits. Considering the case $A = \{X, B\}$, the arrow q_B is the result of applying the right vertical arrow of (43) to the identity $\mathrm{id} : I \to T\text{-Alg}(\{X, B\}, \{X, B\})$. Because identities are strict morphisms of algebras, id factors through J, yielding $Jp_B = q_B$.

$$\begin{tikzcd}
T\text{-Alg}_s(A, \{X, B\}) \arrow[r, "J"] \arrow[d, "\cong"'] & T\text{-Alg}(A, \{X, B\}) \arrow[d, "\cong"'] \\
{[X, T\text{-Alg}(A, B)_s]} \arrow[r, "{[X,J]}"] & {[X, T\text{-Alg}(A, B)]}
\end{tikzcd} \tag{43}$$

■

Corollary 9.3. *Suppose that T_1 algebras admit a flexible replacement, that is, there exists a left 2-adjoint $(-)' \dashv J_1$. Then this 2-adjunction lifts to a $\mathscr{W}$-enriched adjunction*

$$(-)' \dashv J : T\text{-Alg}_s \to T\text{-Alg}. \tag{44}$$

The 2-categories $T_1\text{-Alg}_\ell$ and $T_1\text{-Alg}$ of algebras and, respectively, lax morphisms and pseudomorphisms are the underlying 2-categories of two $\mathscr{W}$-categories, $T\text{-Alg}_\ell$ and T-Alg. For a pair of T-algebras A, B, the object $T\text{-Alg}_\ell(A, B)$ comes equipped with a 2-cell

$$\begin{tikzcd}[row sep=small]
 & \mathscr{K}(A, B) \arrow[dr, "\sigma_{A,B}"] & \\
T\text{-Alg}_\ell(A, B) \arrow[ur, "U_{\ell,A,B}"] \arrow[dr, "U_{\ell,A,B}"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & \mathscr{K}(TA, B) \\
 & \mathscr{K}(A, B) \arrow[ur, "\mathscr{K}(a,1)"'] &
\end{tikzcd} \tag{7}$$

universal with respect to the equalities in Figure 1. The one-dimensional part of the universal property says that given any other 2-cell $\delta : \sigma_{A,B}.p \Rightarrow \mathscr{K}(a, B).p : L \to \mathscr{K}(TA, B)$ satisfying the same equations, there exists a unique 1-cell $\hat{p} : L \to T\text{-Alg}_\ell(A, B)$ such that $\delta = \gamma.\hat{p}$. The two-dimensional part of the universal property says that given a 2-cell δ as above and another $\epsilon : \sigma_{A,B}.q \Rightarrow \mathscr{K}(a, B).q : L \to \mathscr{K}(TA, B)$, and a 2-cell $\alpha : p \Rightarrow q$ compatible with δ, ϵ in the sense that $\delta(\sigma_{A,B}.\alpha) = (\mathscr{K}(a, B).\alpha)\epsilon$, then $\alpha = U_{\ell,A,B}.\hat{\alpha}$ for a unique $\hat{\alpha} : \hat{p} \Rightarrow \hat{q}$.

If we further require the 2-cell (7) to be invertible, we obtain another object that we denote by $T\text{-Alg}(A, B)$; the object of pseudomorphisms.

Remark 3.2. The 2-cell (7) can be constructed by considering an inserter of the pair of 1-cells $\sigma_{A,B}, \mathscr{K}(a, B) : \mathscr{K}(A, B) \to \mathscr{K}(TA, B)$ and then two equifes to impose the equations of Figure 1. Hence it can also be constructed as a limit on one step: there exists a small 2-category $\mathscr{H}_\ell$, a weight $\chi_\ell : \mathscr{H}_\ell \to \mathbf{Cat}$ and a 2-functor $H_{\ell,A,B} : \mathscr{H}_{\ell,A,B} \to \mathscr{W}$ such that $\lim(\phi, H_{\ell,A,B})$ is $T\text{-Alg}_\ell(A, B)$. The same applies to $T\text{-Alg}(A, B)$, by using an iso-inserter instead of an inserter.

Now it is routine to see that the objects $T\text{-Alg}_\ell(A, B)$ and $T\text{-Alg}(A, B)$ are the enriched homs of two $\mathscr{W}$-categories, that we write $T\text{-Alg}_\ell$ and T-Alg respectively, both with objects the T-algebras in $\mathscr{K}$. For example, the composition $T\text{-Alg}_\ell(A, B) \otimes T\text{-Alg}_\ell(B, C) \to T\text{-Alg}_\ell(A, C)$ and identity $I \to T\text{-Alg}_\ell(A, A)$ correspond to the 2-cells in Figure 2.

The 1-cells $U_{\ell,A,B} : T\text{-Alg}_\ell(A, B) \to \mathscr{K}(A, B)$ and $U_{A,B} : T\text{-Alg}(A, B) \to \mathscr{K}(A, B)$ provide the effect on enriched homs of forgetful $\mathscr{W}$-functors $U_\ell : T\text{-Alg}_\ell \to \mathscr{K}$ and $U : T\text{-Alg} \to \mathscr{K}$. There are obvious identity on objects inclusions $J : T\text{-Alg}_s \to T\text{-Alg}$ and $T\text{-Alg} \to T\text{-Alg}_\ell$. The first exists simply

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[r, "\sigma_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma"] & \mathscr{K}(TA,B) \arrow[dr, "\sigma_{TA,B}"] & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_\ell"] \arrow[r, "U_\ell"] \arrow[dr, "U_\ell"'] & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] \arrow[dr, "\sigma A{,}B"] \arrow[d, phantom, "\Downarrow\gamma"] & & \mathscr{K}(T^2A,B) \\
 & \mathscr{K}(A,B) \arrow[r, "\mathscr{K}(a,1)"'] & \mathscr{K}(TA,B) \arrow[ur, "\mathscr{K}(Ta,1)"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] \arrow[dd, phantom, "\Downarrow\gamma"] & & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_\ell"] \arrow[dr, "U_\ell"'] & & \mathscr{K}(TA,B) \arrow[r, "\mathscr{K}(\mu_A,1)"] & \mathscr{K}(T^2A,B) \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] \arrow[dd, phantom, "\Downarrow\gamma"] & & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_\ell"] \arrow[dr, "U_\ell"'] & & \mathscr{K}(TA,B) \arrow[r, "\mathscr{K}(\eta_A,1)"] & \mathscr{K}(A,B) \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & &
\end{tikzcd}
= 1
$$

FIGURE 1. Equalities for T-Alg$(A,B)_\ell$.

because in the definition of the homs of T-Alg we used an iso-inserter, and identities are trivially invertible, or in other words, strict morphisms are pseudomorphisms. The second exists because iso-inserters factor through the respective inserters, or in other words, pseudomorphisms are also lax morphisms.

Because $\mathscr{W}$-**Cat** $\to$ 2-**Cat** is induced by $\mathscr{W}_1(I,-) : \mathscr{W}_1 \to$ **Cat**, and representable 2-functors preserve limits, it is easy to see that T-Alg$_{\ell,1}$ is the usual 2-category of algebras and lax morphisms T_1-Alg$_\ell$; similarly, T-Alg$_1$ is T_1-Alg.

tensor product. The neutral object of this tensor product is (equivalent to) $\mathscr{V}_f^{\mathrm{op}}$, the opposite of the category of finitely presented objects of $\mathscr{V}$. The 2-monad L is related to R by $LX \cong (R(X^{\mathrm{op}}))^{\mathrm{op}}$, and is not K-Z or lax idempotent but the dual notion of *oplax idempotent.*

9. Appendix

9.1. Flexible replacement. In this section we provide the, for the most part routine, $\mathscr{W}$-enriched versions of some of results concerning the left adjoint of J. This can be done by mimicking the constructions of [29]; instead we choose a slightly less general setting in which we can deduce the results from the case of 2-monads by means of well-known and easy results of enriched category theory.

Although the 2-categories of algebras and strict morphisms T-Alg$_{\mathrm{s}}$ are of a theoretical importance, most of the examples of interesting 2-categories associated to a 2-monad appear in the form 2-categories of (sometimes pseudo or lax algebras) and lax or pseudo-morphisms. For simplicity, and because it is the case relevant to this paper, we will only consider the 2-categories T-Alg of strict algebras and pseudomorphisms.

Thanks to the classical theory of monads and algebras, we have a great deal of control over the 2-categories T-Alg$_{\mathrm{s}}$, so it is a good idea to try to transform this knowledge to the most interesting 2-categories T-Alg via the inclusion 2-functor $J : T\text{-Alg}_{\mathrm{s}} \to T\text{-Alg}$. This idea originally appears in [14] and was pushed on in [3], where conditions are given that guarantee the existence of a left adjoint to J, usually denoted by $(-)' : T\text{-Alg} \to T\text{-Alg}_{\mathrm{s}}$. Later Lack [29] gave necessary and sufficient conditions for the existence of this left adjoint, greatly clarifying the situation.

According to [29], given a 2-monad T on a 2-category $\mathscr{K}$, A' can be constructed as a *codescent object* of of the (strict) codescent data in T-Alg$_{\mathrm{s}}$

$$
\begin{tikzcd}
T^3A \arrow[r, shift left=2, "\mu_{TA}"] \arrow[r, "T\mu_A" description] \arrow[r, shift right=2, "Ta"'] & T^2A \arrow[r, shift left=2, "\mu_A"] \arrow[r, shift right=2, "a"'] & TA \arrow[l, "T\eta_A" description]
\end{tikzcd}
$$

So J has a left adjoint whenever T-Alg$_{\mathrm{s}}$ admits codescent objects, which are a special class of colimits, and in particular when $\mathscr{K}$ is complete and cocomplete and T has a rank.

Remark 9.1. An easy consequence of this description of A' is that if T preserves ϕ-colimits for a certain weight ϕ then $(-)'J$ does so too. This is

(2) $\chi_{A,B}$ is dense and each object of $A \boxtimes B$ is a finite colimit of a $\mathscr{V}$-functor $\chi_{A,B}.F$, for some $\mathscr{V}$-functor F into $A \otimes B$ (however this colimit is not necessarily $\chi_{A,B}$-absolute, *i.e.*, preserved by all the representables $(A \boxtimes B)(\chi_{A,B}(a,b), -)$, so it is not part of a density presentation of $\chi_{A,B}$).

Example 8.8. Consider the example of $\mathscr{V} = \mathbb{2}$, the category with two objects $\top$ and $\bot$ and one non-identity arrow $\bot \to \top$. The name of the objects is chosen to make the cartesian product behave like the meet $x \wedge y$, the coproduct as the meet $x \vee y$, and the internal hom $x \Rightarrow y$ as the implication of classical logic. It is known that $\mathbb{2}$-**Cat** is isomorphic to the 2-category **POrd** of partially ordered sets: the partially ordered set corresponding to a $\mathbb{2}$-category A is ob$A$ with ordering $a \leq b$ iff $A(a,b) = \top$. In fact, this partially ordered set is just the underlying category of the $\mathbb{2}$-enriched category A, and because $\top$ is a strong generator in $\mathbb{2}$ (this is $\mathbb{2}(\top, -) : \mathbb{2} \to \mathbf{Set}$ is conservative), there is no difference between $\mathbb{2}$-enriched conical limits and ordinary conical limits. Hence, A has finite coproducts when it has finite joins; coequalizers are trivial in the case of partially ordered sets. Tensor products of the form $\bot * a$ always give an initial object and the other case $\top * a \cong a$ is always trivial. From these observations we deduce that a A finitely cocomplete when it has finite joins and an initial object (a bottom element).

Given a $\mathbb{2}$-category A, $[A^{\mathrm{op}}, \mathbb{2}]$ can be identified with the partially ordered set of order-ideals of A (subsets I of A such that if $a \in I$ then any $b \leq a$ is also in I). The representable presheaf $A(-,a)$ is identified with the order-ideal $\downarrow(a) = \{b \in A : b \leq a\}$. The free completion of A under finite colimits can be identified with the partially ordered set of order-ideals of A of the form $\downarrow(a_1) \cup \cdots \cup \downarrow(a_n)$, for some finite subset $\{a_1, \cdots, a_n\}$ of A. The $\mathbb{2}$-enriched monad R on $\mathbb{2}$-**Cat** whose algebras are partially ordered sets with chosen finite joins (including bottom object) is the a monad corresponding to completion under a class of finite colimits, and hence it is K-Z and pseudo commutative. Since $\mathbb{2}$ is a partially ordered set, as observed in Remark 7.6, R is in fact *commutative*. However, R is not idempotent.

Remark 8.9. We have decided to consider finite colimits to present the theory above, but we could have chosen finite limits instead and obtained the same results. There is a pseudo-commutative 2-monad L on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen limits, and thus the induced internal homs and

$$
\begin{tikzcd}
 & {[A,C]} \arrow[r, "\mathsf{T}"] & {[TA,TC]} \arrow[ddr, "{[1,c]}"] & \\
{[A,B][B,C]} \arrow[ur, "\mathrm{comp}"] \arrow[r, "\mathsf{TT}"] & {[TA,TB][TB,TC]} \arrow[ur, "\mathrm{comp}"] \arrow[r, "{1[1,c]}"] \arrow[d, phantom, "\Downarrow 1\otimes\gamma"] & {[TA,TB][TB,C]} \arrow[dr, "\mathrm{comp}"] & \\
T\text{-}\mathrm{Alg}_\ell(A,B)T\text{-}\mathrm{Alg}_\ell(B,C) \arrow[u, "U_\ell U_\ell"] \arrow[r, "U_\ell U_\ell"] \arrow[dr, "U_\ell U_\ell"'] & {[A,B][B,C]} \arrow[r, "\mathsf{T}1"] \arrow[d, phantom, "\Downarrow \gamma\otimes 1"] & {[TA,TB][B,C]} \arrow[u, "{1[b,1]}"'] \arrow[d, "{[1,b]1}"'] & {[TA,C]} \\
 & {[A,B][B,C]} \arrow[r, "{[a,1]1}"'] & {[TA,B][B,C]} \arrow[ur, "\mathrm{comp}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {[A,A]} \arrow[dr, "\sigma_{A,A}"] & \\
I \arrow[ur, "\mathrm{id}"] \arrow[rr, "a"] \arrow[dr, "\mathrm{id}"'] & & {[TA,A]} \\
 & {[A,A]} \arrow[ur, "{[a,1]}"'] &
\end{tikzcd}
$$

FIGURE 2

4. Pseudo-closed $\mathscr{W}$-categories and pseudo-commutative $\mathscr{W}$-monads

In this section and the next we give an outline of the main constructions and results of [10], where Hyland and Power give structures on a 2-monad T that ensure that the 2-category T-Alg is pseudo-closed in a suitable sense. Here we recall the definition of pseudo-closed structures, leaving the structures on the 2-monad for the next section.

4.1. Pseudo-closed structures. Closed categories arose in the early days of category theory [9], and although in many examples a closed structure is accompanied by a monoidal structure, most of the time the former is easier to describe (*e.g.*, the category of k-modules for a commutative ring k). In the the case of the 2-categories of algebras something similar takes place: in order to construct a tensor product, if possible, it is simpler to first consider a pseudo-closed structure.

We take the definition of a pseudo-closed structure from Hyland-Power [10], changing **Cat** by $\mathscr{W}$.

$$
\begin{tikzcd}
I \arrow[r, "j_B"] \arrow[dr, "j_{[A,B]}"'] & {[B,B]} \arrow[d, "k_A"] \\
 & {[[A,B],[A,B]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A,C]} \arrow[r, "k_{A,A,C}"] \arrow[d, equal] & {[[A,A],[A,C]]} \arrow[d, "{[j_A,1]}"] \\
{[A,C]} & {[I,[A,C]]} \arrow[l, "e_{[A,C]}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[C,D]} \arrow[r, "k_{A,C,D}"] \arrow[d, "k_{B,C,D}"'] & {[[A,C],[A,D]]} \arrow[r, "k"] & {[[[A,B],[A,C]],[[A,B],[A,D]]]} \arrow[d, "{[k_{A,B,C},1]}"] \\
{[[B,C],[B,D]]} \arrow[rr, "{[1,k_{A,B,D}]}"'] & & {[[B,C],[[A,B],[A,D]]]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[A,B]} \arrow[r, "k_{I,A,B}"] \arrow[dr, "{[e_A,1]}"'] & {[[I,A],[I,B]]} \arrow[d, "{[1,e_B]}"] \\
 & {[[I,A],B]}
\end{tikzcd}
$$

FIGURE 3. Some of the axioms of a pseudo-closed 2-category.

Definition 4.1 ([10]). A *pseudo-closed $\mathscr{W}$-category* is a $\mathscr{W}$-category $\mathscr{K}$ equipped with the following data: $\mathscr{W}$-functors $V : \mathscr{K} \to \mathscr{W}$ and $[-,-] : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \to \mathscr{K}$, an object $I \in \mathscr{K}$, $\mathscr{W}$-(extraordinary) natural transformations $j_A : I \to [A,A]$, $e_A : [I,A] \to A$, $i_A : A \to [I,A]$, $k_{A,B,C} : [B,C] \to [[A,B],[A,C]]$. This data must satisfy the commutativity of the diagrams in $\mathscr{K}_1$ in Figure 3 and

- $V[-,-] = \mathscr{K}(-,-) : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \to \mathscr{W}$;
- the 1-cell $I \xrightarrow{j_A} \mathscr{K}(I,[A,A]) = V[I,[A,A]] \xrightarrow{Ve_{[A,A]}} V[A,A] = \mathscr{K}(A,A)$ is the identity of A;
- there are equivalences $i_A \dashv e_A$ in the 2-category $\mathscr{K}_1$ whose units are identity 2-cells, *i.e.*, retracts equivalences;
- the 1-cell $\mathscr{W}_1(I, V(i_A e_A)) : \mathscr{K}_1(I,A) \to \mathscr{K}_1(I,A)$ in **Cat** takes each $f : I \to A$ in $\mathscr{K}_1$ to $e_A[f,A]j_A : I \to [A,A] \to [I,A] \to A$.

When $\mathscr{W}$ is **Cat** we recover the definition of a pseudo-closed 2-category in [10]. It is also clear that if $\mathscr{K}$ is a pseudoclosed $\mathscr{W}$-category, then its underlying 2-category $\mathscr{K}_1$ is pseudo-closed.

Example 4.1. A candidate to a pseudo-closed 2-categories is, for example, the 2-category of braided strict monoidal categories, braided monoidal functors

$$
\mathrm{colim}(\psi, \mathrm{colim}(\phi, f(g \otimes h)) \xrightarrow{\cong} \mathrm{colim}(\phi, \mathrm{colim}(\psi, f(g \otimes h)) \xrightarrow{\cong}
$$
$$
\mathrm{colim}(\phi, f(g \otimes \mathrm{colim}(\psi, h))) \xrightarrow{\cong} f(\mathrm{colim}(\phi, g) \otimes \mathrm{colim}(\psi, h)). \quad (41)
$$

The first isomorphism in the composite (41) is induced by the pseudo-commutativity of R.

Remark 8.4. The fact that the free R-algebra functor $F : \mathscr{V}\text{-}\mathbf{Cat} \to R\text{-Alg}$ is strong monoidal implies, as explained in the paragraph after Theorem 6.10, that $R(X) \boxtimes R(Y)$ is equivalent to $R(X \otimes Y)$.

Example 8.5. Let k be a commutative ring and A a k-algebra, finitely presented as a k-module, and denote by $\Sigma\mathsf{A}$ the corresponding $\mathscr{V}$-category with one object. Then $R(\Sigma\mathsf{A})$ is equivalent to $\mathsf{A}\text{-Mod}_f$, the category of finitely presented A-modules. The Remark 8.4 above can be reinterpreted as the equivalence

$$
\mathsf{A}\text{-Mod}_f \boxtimes \mathsf{B}\text{-Mod}_f \simeq \mathsf{A} \otimes \mathsf{B}\text{-Mod}_f.
$$

This can be of course shown directly, and in fact is one of the most basic observations about $\boxtimes$ or any tensor product that might play its role, including Deligne's tensor product [6]. The universal functor

$$
\mathsf{A}\text{-Mod}_f \otimes \mathsf{B}\text{-Mod}_f \to \mathsf{A} \otimes \mathsf{B}\text{-Mod}_f
$$

is given by $\otimes_k$, the tensor product over k.

Remark 8.6. The last section of Kelly's book [17, 19] uses the extensive machinery developed therein to describe tensor products of Φ-cocomplete enriched categories. If A, B are two such categories, $A \boxtimes B$ is (equivalent to) the closure under Φ-colimits of the representables in $\Phi\text{-}\mathbf{Cts}[A^{\mathrm{op}}, B^{\mathrm{op}}; \mathscr{V}]$, the $\mathscr{V}$-category of $\mathscr{V}$-functors $A^{\mathrm{op}} \otimes B^{\mathrm{op}} \to \mathscr{V}$ that are Φ-continuous in each variable. In particular, when $\Phi = \mathsf{Fin}$ the class of finite colimits, we have that $A \boxtimes B$ is equivalent to the closure under finite colimits of the representables in $\mathrm{Lex}[A^{\mathrm{op}}, B^{\mathrm{op}}; \mathscr{V}]$, and $A \otimes B \to A \boxtimes B$ is *dense* in the sense of [17, 19, Chapter 5]. (Note that the inclusion functor from $\mathrm{Lex}[A^{\mathrm{op}}, B^{\mathrm{op}}; \mathscr{V}]$ to $[A^{\mathrm{op}} \otimes B^{\mathrm{op}}, \mathscr{V}]$ does *not* preserve colimits.)

Remark 8.7. Let A, B be two $\mathscr{V}$-categories with chosen finite colimits and $\chi_{A,B} : A \otimes B \to A \boxtimes B$ the corresponding universal multilinear $\mathscr{V}$-functor. Remark 8.6 above implies

(1) $\chi_{A,B}$ is fully faithful.

with chosen Φ-colimits and 1-cells Φ-cocontinuous $\mathscr{V}$-functors. [21, Theorem 6.2] shows that there is a canonical isomorphism $T_\Phi\text{-Alg} \cong \Phi\text{-}\mathbf{Cocts}_c$ that commutes with the corresponding forgetful 2-functors into $\mathscr{V}\text{-}\mathbf{Cat}$.

Although the following results hold for any class of finite colimits, for simplicity we restrict ourselves to the class of all finite colimits Fin.

Lemma 8.2. *The 2-monad R on $\mathscr{V}\text{-}\mathbf{Cat}$ whose algebras are $\mathscr{V}$-categories with chosen finite colimits is finitary. Equivalently, the forgetful 2-functor $U_\mathrm{s} : \mathsf{Fin}\text{-Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$ is finitary.*

For a proof of this Lemma see Section 9.3.

From the results and remarks of Section 6.4 we deduce:

Corollary 8.3. R_1-Alg*, and hence* $\mathsf{Fin}\text{-}\mathbf{Cocts}_c$*, are monoidal 2-categories, with the the monoidal structure induced by the canonical pseudo-closed structure. Moreover, the biadjunction $F \dashv_\mathrm{b} U : T_1\text{-Alg} \to \mathscr{V}\text{-}\mathbf{Cat}$ is monoidal.*

The tensor product $\boxtimes$ in R-Alg satisfies (35), which could be rewritten as

$$\mathrm{Rex}[A \boxtimes B, C] \simeq \mathrm{Rex}[A, \mathrm{Rex}[B, C]].$$

This universal property can be expressed in terms of the monoidal constraint (37) $\chi_{A,B} : A \otimes B \to A \boxtimes B$ that classifies multilinear maps. By Corollary 7.5 multilinear maps are just partial maps in each variable, that in the present case means simply $\mathscr{V}$-functors that are right exact in each variable. Part of the universal property of the tensor product asserts that every functor $A \otimes B \to C$ that is right exact in each variable factors as $\chi_{A,B}$ followed by a unique up to isomorphism right exact functor $A \boxtimes B \to C$.

$$\begin{tikzcd} A \otimes B \arrow[r, "\chi_{A,B}"] \arrow[dr] & A \boxtimes B \arrow[d, dotted] \arrow[dl, phantom, "\cong"] \\ & C \end{tikzcd}$$

The fact that multilinear $f : A \otimes B \to C$ are simply the partial maps, *i.e.*, the right exact $\mathscr{V}$-functors, can be rephrased as the familiar fact that if $\phi : D_\phi^{\mathrm{op}} \to \mathscr{V}$, $\psi : D_\psi^{\mathrm{op}} \to \mathscr{V}$ are finite weights and $g : D_\phi^{\mathrm{op}} \to A$, $h : D_\psi^{\mathrm{op}} \to B$ two $\mathscr{V}$-functors, then the following two isomorphisms are equal.

$$\mathrm{colim}(\psi, \mathrm{colim}(\phi, f(g \otimes h)) \xrightarrow{\cong} \mathrm{colim}(\psi, f(\mathrm{colim}(\phi, g) \otimes h))$$
$$\xrightarrow{\cong} f(\mathrm{colim}(\phi, g) \otimes \mathrm{colim}(\psi, h))$$

and monoidal transformations: given two such categories A, B, the category of braided monoidal functors $A \to B$ and monoidal transformations between them has a canonical structure of a braided strict monoidal category. This example is extensively studied, in the symmetric case, in [10]. Another possible example of a pseudo-closed 2-category is the 2-category finitely cocomplete categories, finitely cocontinuous functors and natural transformations. Here again, given two such categories A, B, finitely cocontinuous functors $A \to B$ and transformations between them form a finitely cocontinuous category. However, this 2-category is not quite pseudo-closed; to obtain a pseudo-closed structure one has to move to categories with *chosen colimits*. This example is studied in Section 8.

4.2. Pseudo-commutativities. The series [24, 25, 26] studies the structures on a strong monad (defined on a closed category) that induce a closed structure on the corresponding category of Eilenberg-Moore algebras in such a way that the associated forgetful functor preserves the closed structure. The main result is that these closed structures correspond to a property of this monad that was named *commutativity*. One basic example is the free abelian group monad on **Set**. The category of algebras for this monad is the category of abelian groups, which is manifestly closed. The commutativity on the monad is an expression of the fact that the addition in an abelian group is commutative. Hyland and Power [10] deal with the higher dimensional problem of defining pseudo-commutativity for 2-monads, and finding the right level of generality that allows for a large number of interesting examples but at the same time remains manageable.

Definition 4.2 ([10])**.** A *pseudo-commutativity* for a $\mathscr{W}$-monad $T : \mathscr{W} \to \mathscr{W}$ is an invertible modification depicted in (8) of Figure 4, satisfying the axioms resulting from replacing in [10, Definition 5] the cartesian product of **Cat** by the tensor product $\otimes$ of $\mathscr{W}$. We do not reproduce the axioms, as these will not be explicitly used.

A $\mathscr{W}$-monad equipped with a pseudo-commutativity will be called a *pseudo-commutative* $\mathscr{W}$-monad. Using the closed structure of $\mathscr{W}$ [10] expresses a pseudo-commutativity (8) as a 2 -cell (9) in Figure 4. The axioms of a pseudo-commutativity translate to the conditions in [10, Proposition 8], that we spell below for later use.

$$\begin{tikzcd} TX\otimes TY \arrow[r, "t'_{X,TY}"] \arrow[d, "t_{TX,Y}"'] & T(X\otimes TY) \arrow[r, "Tt_{X,Y}"] \arrow[d, phantom, "\Downarrow\gamma_{X,Y}"] & T^2(X\otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\ T(TX\otimes Y) \arrow[r, "Tt'_{X,Y}"'] & T^2(X\otimes Y) \arrow[r, "\mu_{X\otimes Y}"'] & T(X\otimes Y) \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}_{X,Y}"'] & T[TX,TY] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\Downarrow\bar{\gamma}_{X,Y}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\ {[X,TY]} \arrow[r, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]} \end{tikzcd} \tag{9}$$

FIGURE 4. Pseudo-commutativity.

The main example of a pseudo-commutative 2-monad in [10] is the 2-monad on **Cat** whose algebras are symmetric strict monoidal categories (see Example 2.2); in this case the modification (8) is the canonical isomorphism mediating between two lexicographic orders. We refer the reader to the detailed exposition in section 3 of the cited paper.

The following result appears as [10, Proposition 8], and we may take it as a definition of a pseudo-commutativity. Our omission of condition 1 of [10, Proposition 8], that is expressed in terms of both (8) and (9), is justified by the comments in the forenamed paper that any two of the strength conditions on $\bar{\gamma}$ implies the third (see [10, p. 161]).

Proposition 4.2 ([10])**.** *To give a pseudo-commutativity for a $\mathscr{W}$-enriched monad T is equivalent to giving a modification $\bar{\gamma}$ as in* (9) *subject to the following conditions.*

(1) $[X,\bar{\gamma}_{Y,Z}].\bar{t}_{X,[Y,Z]}$ *is the exponential transpose of* $[t,TZ].\bar{\gamma}_{X\otimes Y,Z}$.
(2) $[TX,\bar{t}_{Y,Z}].\bar{\gamma}_{X,[Y,Z]}$ *is the exponential transpose of* $[t'_{X,Y},TZ].\bar{\gamma}_{X\otimes Y,Z}$.
(3) $\bar{\gamma}_{X,Y}.\eta_{[X,Y]}$ *is an identity.*
(4) $[\eta_X,TY].\bar{\gamma}_{X,Y}$ *is an identity.*

Let Φ be a small class of colimits, by which we understand a small class of weights $\phi : D \to \mathscr{V}$. Recall from [17, Section 5.5] that the free completion of a $\mathscr{V}$-category A under Φ-colimits, denoted by ΦA, can be obtained as the closure under Φ-colimits of the representables in $[A^{\mathrm{op}},\mathscr{V}]$. The Yoneda embedding $\mathsf{y}_A : A \to \Phi A$ induces equivalences of $\mathscr{V}$-categories Φ-$\mathbf{Cocts}[\Phi A, B] \simeq [A,B]$ for all Φ-cocomplete $\mathscr{V}$-category B, with pseudoinverse given by left Kan extension along y_A. Here Φ-$\mathbf{Cocts}[C,D]$ denotes the $\mathscr{V}$-category of Φ-cocontinuous $\mathscr{V}$-functors $C \to D$; these are the enriched homs of a $\mathscr{V}$-**Cat**-category Φ-**Cocts** with objects the Φ-cocomplete small $\mathscr{V}$-categories.

Let us denote by Φ-**Colim** be the 2-category of $\mathscr{V}$-categories with chosen Φ-colimits, $\mathscr{V}$-functors strictly preserving these and $\mathscr{V}$-natural transformations. The hom $\mathscr{V}$-category Φ-$\mathbf{Colim}(A,B)$ is the full sub-$\mathscr{V}$-category of $[A,B]$ determined by the $\mathscr{V}$-functors that strictly preserve Φ-colimits. There is an obvious forgetful 2-functor $U_{\mathrm{s}} : \Phi\text{-}\mathbf{Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$. The main result of [21] is the monadicity of U_{s} (as a 2-functor) in the strong sense that is an adjunction $F_{\mathrm{s}} \dashv U_{\mathrm{s}}$ and the canonical comparison 2-functor Φ-**Colim** $\to$ T_Φ-$\mathrm{Alg}_{\mathrm{s}}$ is an isomorphism, where $T_\Phi = U_{\mathrm{s}}F_{\mathrm{s}}$. If $\eta : 1 \Rightarrow T_\Phi$ is the unit of the monad, there is an equivalence of $\mathscr{V}$-categories making the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathsf{y}_A"'] & T_\Phi A \arrow[d, "\simeq"] \\ & \Phi A \end{tikzcd}$$

Corollary 8.1. *The 2-monad T_Φ on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen Φ-colimits is pseudo-commutative. Therefore, the 2-category T_Φ-Alg is pseudo-commutative.*

Proof: Theorem 6.3 of [21] asserts that the 2-monad T_Φ is a K-Z 2-monad. The result follows from Theorem 7.2. For the last part apply Corollary 7.3. ∎

Still following [21], the canonical "inclusion" 2-functor from Φ-**Colim** to the 2-category Φ-**Cocts** of Φ-cocomplete $\mathscr{V}$-categories and Φ-cocontinuous $\mathscr{V}$-functors can be factored as

$$\Phi\text{-}\mathbf{Colim} \to \Phi\text{-}\mathbf{Cocts}_c \to \Phi\text{-}\mathbf{Cocts}$$

where the first 2-functor is bijective on objects and the second is fully faithful. In other words, the 2-category in the middle has objects $\mathscr{V}$-categories

Corollary 7.3. *If $T : \mathscr{W} \to \mathscr{W}$ is a K-Z $\mathscr{W}$-monad, then T-*Alg *has a canonical structure of a pseudo-closed $\mathscr{W}$-category. Moreover, if T has a rank, the induced pseudo-closed structure on the 2-category T_1-*Alg *has an associated monoidal structure with unit object FI and whose tensor product satisfies* (35).

Proof: It is a consequence of Theorem 7.2 together with section 6. ∎

Example 7.4. There are pseudo-commutative 2-monads which are *not* K-Z. For example, the 2-monad T on **Cat** whose algebras are the symmetric strict monoidal categories. See [10] for a detailed description of the pseudo-commutativity for this 2-monad. One of the several possible ways of seeing that this T is not lax-idempotent is to show that there can not be a 2-natural transformation $\delta_X : T\eta_X \Rightarrow \eta_{TX} : TX \to T^2X$.

We record the following easy consequence of Proposition 6.7 that will be reinterpreted in the next section as the familiar fact about colimits of functors of several variables.

Corollary 7.5. *In the case of K-Z monads there is no distinction between partial maps in each variable and multilinear maps.*

Remark 7.6. When $\mathscr{W}$ is locally a preorder any pseudo-commutativity is just a commutativity in the sense of [24, 26, 25], but the monad could still be K-Z and not an idempotent monad. See Example 8.8.

8. Categories with finite colimits

We now turn to our main example of K-Z monads, and thus pseudo-commutative 2-monads; namely, monads on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with a given class of *chosen* colimits. These monads are enriched in $\mathscr{V}$-**Cat**, which is essential in order to endow the $\mathscr{V}$-categories of pseudomorphisms with an algebra structure, as shown in the previous sections. This is just the familiar fact that given $\mathscr{V}$-categories A, B admitting colimits of a certain class Φ, Φ-cocontinuous $\mathscr{V}$-functors $A \to B$ form not only an ordinary category but a Φ-cocomplete $\mathscr{V}$-category Φ-**Cocts**$[A, B]$. This family of monads expands the examples of pseudo-commutative 2-monads provided in [10].

When the class of colimits in question is a class of finite colimits, the corresponding monad is finitary and thus we can construct a corresponding tensor product.

(5) $\bar\gamma_{X,Y} . \mu_{[X,Y]}$ *is equal to the pasting*

$$\begin{tikzcd}
T^2[X,Y] \arrow[r, "T^2(\mathsf{T})"] \arrow[d, "T\bar t"'] & T^2[TX,TY] \arrow[r, "T\bar t"] \arrow[d, phantom, "\Downarrow T\bar\gamma_{X,Y}"] & T[TX,T^2Y] \arrow[d, "{T[1,\mu_Y]}"] \\
T[X,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar t"'] & T[TX,T^2Y] \arrow[r, "{T[1,\mu_Y]}"] \arrow[d, "\bar t"'] & T[TX,TY] \arrow[d, "\bar t"] \\
{[X,T^2Y]} \arrow[d, "T"'] \arrow[r, phantom, "\Downarrow \bar\gamma_{X,TY}"] & {[TX,T^3Y]} \arrow[r, "{[1,T\mu_Y]}"] \arrow[d, "{[1,\mu_{TY}]}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[TX,T^3Y]} \arrow[r, "{[1,\mu_{TY}]}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}$$

(6) $[\mu_X, TY]\bar\gamma_{X,Y}$ *is equal to the pasting*

$$\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar t"'] & T[TX,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar t"] & T[T^2X,T^2Y] \arrow[r, "\bar t"] \arrow[d, phantom, "\Downarrow \bar\gamma_{TX,TY}"] & {[T^2X,T^3Y]} \arrow[d, "{[1,\mu_{TY}]}"] \\
{[X,TY]} \arrow[d, "\mathsf{T}"'] \arrow[r, phantom, "\bar\gamma_{X,Y} \Downarrow"] & {[TX,T^2Y]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,\mu_Y]}"] & {[T^2X,T^3Y]} \arrow[r, "{[1,\mu_{TY}]}"] \arrow[d, "{[1,T\mu_Y]}"] & {[T^2X,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]} \arrow[r, "\mathsf{T}"'] & {[T^2X,T^Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[T^2X,TY]}
\end{tikzcd}$$

Example 4.3. To illustrate the previous proposition, and for the benefit of the reader unfamiliar with [10], we exhibit the canonical pseudo-commutativity, in its form $\bar\gamma$, for the 2-monad T on **Cat** whose algebras are symmetric strict monoidal categories. An object of $T[X, Y]$ is an n-tupe $(f_1, \cdots, f_n)$ of functors $f_i : X \to Y$. The domain of the component of $\bar\gamma_{X,Y}$ corresponding to this object has as domain the functor $TX \to TY$ given on objects by

$$(x_1, \cdots, x_m) \mapsto (f_1x_1, \cdots, f_1x_m, f_2x_1, \cdots, f_2x_m, \cdots f_nx_1, \cdots, f_nx_m) \quad (10)$$

while the codomain is the functor $TX \to TY$ given on objects by

$$(x_1, \cdots, x_m) \mapsto (f_1x_1, \cdots, f_nx_1, f_1x_2, \cdots, f_nx_2, \cdots f_1x_m, \cdots, f_nx_m) \quad (11)$$

So domain and codomain are given by the two different lexicographic orderings of the objects f_ix_j. The component $((\bar\gamma_{X,Y})_{(f_1,\cdots,f_n)})_{(x_1,\cdots,x_m)}$ is the unique isomorphism between (10) and (11) induced by the symmetry of TY.

5. A characterisation of pseudo-commutativity

As mentioned at the beginning of the previous section, one of the main points of [26, 25, 24] is that there is a correspondence between the commutativity of a monad and closed structures on its Eilenberg-Moore category of algebras. In the case of pseudo-commutativities something similar happens, but the correspondence is not so clean; this reflects the fact that Definitions 4.1 and 4.2 are not as "weak" as possible but as "strict" as examples allow.

A key observation of [10] is that a pseudo-commutativity on a $\mathscr{W}$-monad T on $\mathscr{W}$ induces a pseudomorphism structure on the composite

$$\sigma_{X,B} : [X,B] \xrightarrow{\mathsf{T}} [TX,TB] \xrightarrow{[TX,b]} [TX,B] \tag{12}$$

for every T-algebra B, and these arrows form a pseudonatural transformation in the following way. Consider the 2-functors $[-,-],[T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to T\text{-}\mathrm{Alg}_1$ and observe that the 1-cells (12) are the components of a pseudonatural transformation

$$U[-,-] \Rightarrow U[T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to \mathscr{W}_1. \tag{13}$$

Indeed, if $f : B \to C$ is a 1-cell in T-Alg, the structural 2-cell σ_f corresponding to f is the 2-cell below.

$$\begin{tikzcd}
{[X,B]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,f]}"'] & {[TX,TB]} \arrow[r, "{[1,b]}"] \arrow[d, "{[1,Tf]}"'] \arrow[dr, phantom, "{[1,\bar{f}^{-1}]\ \Downarrow}"] & {[TX,B]} \arrow[d, "{[1,f]}"] \\
{[X,C]} \arrow[r, "\mathsf{T}"] & {[TX,TC]} \arrow[r, "{[1,c]}"] & {[TX,C]}
\end{tikzcd} \tag{14}$$

The pseudonatural transformation obtained by precomposing σ with

$$1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_{\mathrm{s},1} \to \mathscr{W}_1^{\mathrm{op}} \otimes T\text{-}\mathrm{Alg}_1$$

is in fact 2-natural. In other words, σ is 2-natural on *strict* morphisms.

We provide a refinement of the observations of [10] in the form of the proposition below.

Proposition 5.1. *There is a bijection between pseudo-commutativities on T and liftings of σ to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to T\text{-}\mathrm{Alg}_1$ satisfying the following conditions.*

(1) $[\eta_X, B].\sigma_{X,B} = 1_{[X,B]}$ *in* T-Alg_1.
(2) $\sigma_{TX,B}.\sigma_{X,B} = [\mu_X, B].\sigma_{X,B}$ *in* T-Alg_1.

Lemma 7.1. *Let $T : \mathscr{W} \to \mathscr{W}$ be a K-Z $\mathscr{W}$-monad Then the 1-cell*

$$\sigma_{X,B} : [X,B] \xrightarrow{\mathsf{T}} [TX,TB] \xrightarrow{[TX,b]} [TX,B]$$

is part of a coretract adjunction with right adjoint $[\eta_X, B] : [TX,B] \to [X,B]$. In particular, it is a pseudomorphism.

Proof: We have $[\eta_X,B].[TX,b].\mathsf{T} = [X,b].[\eta_X,TB].\mathsf{T} = [X,b].[X,\eta_X] = 1$ by 2-naturality of η, so indeed we can define the unit of our adjunction as the identity. Now define the counit as the following 2-cell

$$\begin{tikzcd}
& {[X,B]} \arrow[drr, bend left, "\mathsf{T}"] & & & \\
{[TX,B]} \arrow[ur, "{[\eta_X,1]}"] \arrow[r, "\mathsf{T}"] \arrow[rrr, bend right, "{[1,\eta_B]}"'] & {[T^2X,TB]} \arrow[rr, shift left=2, "{[T\eta_X,1]}"] \arrow[rr, shift right=2, "{[\eta_{TX},1]}"'] \arrow[rr, phantom, "\Downarrow"] & & {[TX,TB]} \arrow[r, "{[1,b]}"] & {[TX,B]}
\end{tikzcd}$$

where the unlabelled 2-cell is $[\delta_X, 1]$. Now we check the axioms of an adjunction. First, $[\eta_X,B].[TX,b].[\delta_X,TB].\mathsf{T} = [X,b].[\delta_X\eta_X,TB].\mathsf{T} = 1$ by (39). The other triangular identity of an adjunction follows from (39):

$$\begin{aligned}
[TX,b].[\delta_X,TB].\mathsf{T}.[TX,b].\mathsf{T} &= [\delta_X,B].[T^2X,b].[T^2X,Tb].\mathsf{T}.\mathsf{T} \\
&= [\delta_X,B].[T^2X,b].[T^2X,\mu_B].\mathsf{T}.\mathsf{T} \\
&= [\delta_X,B].[T^2X,b].[\mu_X,TB].\mathsf{T} \\
&= [\delta_X,B].[\mu_X,B].[TX,b].\mathsf{T} \\
&= 1.
\end{aligned}$$

■

Theorem 7.2. *Every K-Z $\mathscr{W}$-monad $T : \mathscr{W} \to \mathscr{W}$ is pseudo-commutative. Moreover, the pseudo-commutativity is unique.*

Proof: We have to check the conditions in Proposition 5.1. By Lemma 7.1 σ lifts to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T_1\text{-}\mathrm{Alg} \to T_1\text{-}\mathrm{Alg}$. Moreover this lifting is unique because $U_1 : T_1\text{-}\mathrm{Alg} \to \mathscr{W}_1$ is injective on 1-cells and locally fully faithful. The conditions (1) to (4) in Proposition 5.1 hold trivially, because U_1 is injective in 1-cells; in other words, these conditions hold if and only if they hold in $\mathscr{W}$. The uniqueness of the pseudo-commutativity is equivalent to the uniqueness of the pseudomorphism structure on each $\sigma_{X,B}$, which holds by the properties of U_1 already mentioned. ■

We leave the examples for the next and subsequent sections.

7. K-Z monads

With cocomplete categories as an example, Kock [27] (published in the form of [28]) and Zöberlein [36] introduced a special kind of 2-monad, with later contributions by Street [35, 34] and more recently Lack and Kelly [20].

A 2-monad (T, η, μ) on a 2-category $\mathscr{K}$ is *Kock-Zöberlein*, abbreviated *K-Z*, or *lax-idempotent*, when any 1-cell $f : A \to B$ in $\mathscr{K}$ between T-algebras has a unique structure of a lax morphism of T-algebras. This is equivalent to the condition that a 1-cell $a : TA \to A$ is a T-algebra structure if and only if there exists an adjunction $a \dashv \eta_A$ whose counit is an identity. Another equivalent condition is the existence of a modification $\delta : T\eta \Rrightarrow \eta T : T \Rightarrow T^2$ satisfying

$$\delta\eta = 1 \qquad \text{and} \qquad \mu\delta = 1. \tag{39}$$

Many more equivalent conditions are given in [20, Theorem 6.2]. If T is a K-Z monad the forgetful 2-functor $U_\ell : T\text{-Alg} \to \mathscr{K}$ is locally fully faithful.

If A, B are T-algebras, the unique lax morphism structure on a 1-cell $f : A \to B$ in $\mathscr{K}$ is given by the following 2-cell, where the unlabelled 2-cell denotes the counit of the adjunction $a \dashv \eta_A$.

$$\begin{tikzcd}
TA \arrow[r, equal, "\Uparrow"'] \arrow[d, "a"'] & TA \arrow[r, "Tf"] & TB \arrow[d, "b"] \\
A \arrow[ur, "\eta_A"'] \arrow[r, "f"'] & B \arrow[ur, "\eta_B"] \arrow[r, equal] & B
\end{tikzcd} \tag{40}$$

It follows that a 1-cell $f : A \to B$ has a (unique) structure of a pseudomorphism of T-algebras if and only if (40) is invertible. Also, the forgetful 2-functor $U : T\text{-Alg} \to \mathscr{K}$ is injective on 1-cells and locally fully faithful.

In [28] it is shown that left adjoint morphisms between algebras are pseudomorphisms. If A, B are T-algebras and $f \dashv f^* : B \to A$ is an adjunction in $\mathscr{K}$, then f^*, just as any 1-cell, is a lax morphism and hence f has a structure of an oplax (or colax) morphism of T-algebras. It follows from [20, Lemma 6.5] that the oplax structure $fa \Rightarrow bTf$ is invertible and its inverse is a pseudomorphism structure on f.

Definition 7.1. We say that a $\mathscr{W}$-monad T on $\mathscr{K}$ is a *K-Z* or *lax idempotent* $\mathscr{W}$-monad if its underlying 2-monad T_1 on the 2-category $\mathscr{K}_1$ is a K-Z 2-monad in the usual sense.

(3) *$[X, \sigma_{Y,B}] : [X, [Y, B]] \to [X, [TY, B]]$ is the exponential transpose of the 1-cell $[t_{X,Y}, B]\sigma_{X\otimes Y,B} : [X \otimes Y, B] \to [X \otimes TY, B]$ in $T\text{-Alg}_1$, for all T-algebra B.*
(4) *$\sigma_{X,[Y,TZ]} : [X, [Y, TZ]] \to [TX, [Y, TZ]]$ is the exponential transpose of the 1-cell $[t'_{X,Y}, TZ].\sigma_{X\otimes Y,TZ} : [X\otimes Y, TZ] \to [TX\otimes Y, TZ]$ in $T\text{-Alg}_1$, for all $X, Y, Z \in \mathscr{W}$.*
(5) *The composition of $1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_{s,1} \to \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1$ with σ is a 2-natural transformation.*

Observe that condition 4 is slightly different than the rest in that the T-algebra is required to be of a special form, namely a free T-algebra.

We split the proof of the proposition in lemmas.

Lemma 5.2. *Let $T : \mathscr{W} \to \mathscr{W}$ be a $\mathscr{W}$-enriched monad. There is a bijection between modifications $\bar\gamma$ as in (9) satisfying conditions 3 and 5 of Proposition 4.2 and liftings of σ to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1 \to T\text{-Alg}_1$ which composed with $\mathscr{W}_1^{\mathrm{op}} \times J$ are 2-natural.*

Proof: Given a modification $\bar\gamma$ as in (9), we can define 2-cells $\bar\sigma_{X,B}$ for $X \in \mathscr{W}$, $B \in T\text{-Alg}$ as the following composition.

$$\begin{tikzcd}
T[X,B] \arrow[rr, "T(\mathsf{T})"] \arrow[dd, "\bar t"'] & & T[TX,TB] \arrow[r, "T[1,b]"] \arrow[d, "\bar t"] & T[TX,B] \arrow[d, "\bar t"] \\
& \Downarrow\bar\gamma_{X,B} & {[TX,T^2B]} \arrow[r, "{[TX,Tb]}"] \arrow[d, "{[1,\mu_B]}"] & {[TX,TB]} \arrow[dd, "{[1,b]}"] \\
{[X,TB]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,b]}"'] & {[TX,T^2B]} \arrow[r, "{[1,\mu_B]}"] \arrow[d, "{[1,Tb]}"] & {[TX,TB]} \arrow[dr, "{[1,b]}"'] & \\
{[X,B]} \arrow[r, "\mathsf{T}"'] & {[TX,TB]} \arrow[rr, "{[1,b]}"'] & & {[TX,B]}
\end{tikzcd}$$

Each 2-cell $\bar\sigma_{X,B}$ endows $[TX, b]\mathsf{T}$ with the structure of a pseudomorphism of T-algebras: the condition involving the unit η follows from condition 3 of Proposition 4.2 and the condition involving the multiplication μ follows from condition 5 of the same proposition. With this pseudomorphism structure, the 2-cell (14) is a 2-cell in $T\text{-Alg}_1$; in other words, $(\sigma, \bar\sigma)$ is a lifting of σ to a pseudonatural transformation between the 2-functors $[-,-], [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1 \to T\text{-Alg}_1$. (If such a lifting exists, it is unique). Moreover,

the composition of $(\sigma, \bar{\sigma})$ with $1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_{s,1} \to \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1$ is a 2-natural transformation.

Conversely, we now show that any lifting $(\sigma, \bar{\sigma})$ of σ whose composition with $\mathscr{W}_1^{\mathrm{op}} \times J_1$ is 2-natural, induces a modification $\bar{\gamma}$ as in (9). Given $\bar{\sigma}_{X,B}$ define $\bar{\gamma}_{X,Y}$ by

$$\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[dr, "{T[1,\eta_Y]}"'] \arrow[dd, "\bar{t}"'] & T[TX,TY] \arrow[dr, "{T[1,T\eta_Y]}"] \arrow[drr, bend left, "1"] & & \\
& T[X,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"'] & T[TX,T^2Y] \arrow[r, "{T[1,\mu_Y]}"] & T[TX,TY] \arrow[d, "\bar{t}"] \\
{[X,TY]} \arrow[r, "{[1,T\eta_Y]}"] \arrow[dr, "1"'] & {[X,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] & \Downarrow \bar{\sigma}_{X,TY} & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
& {[X,TY]} \arrow[r, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}$$

To show that $\bar{\gamma}_{X,Y}$ is a modification, we use that $\bar{\sigma}$ is 2-natural on strict morphisms: for $f : Z \to Y$, $h : W \to X$ in $\mathscr{W}_1$,

$$\begin{aligned}
\bar{\gamma}_{X,Y}.T[h,f] = \bar{\sigma}_{X,TY}.(T[X,\eta_Y]).(T[h,f]) &= \bar{\sigma}_{X,TY}.(T[h,Tf]).(T[W,\eta_Z]) \\
&= [Th,Tf].\bar{\sigma}_{W,TZ}.(T[W,\eta_Z]) = [Th,Tf].\bar{\gamma}_{W,Z}.
\end{aligned}$$

Condition 3 of Proposition 4.2 follows easily from the unit axiom of a pseudomorphism: $\bar{\gamma}_{X,Y}.\eta_{[X,Y]} = \bar{\sigma}_{X,TY}.T[X,\eta_Y].\eta_{[X,Y]} = \bar{\sigma}_{X,TY}.\eta_{[X,TY]}.[X,\eta_Y]. = 1$. Condition 5 of the same proposition is a bit harder to prove, but routine nonetheless. We leave the verification to the reader; we only mention that the equality $[TX,\mu_Y].\bar{\sigma}_{X,T^2Y} = \bar{\sigma}_{X,TY}.T[X,\mu_Y]$ and the multiplication axiom of a pseudomorphism must be used in the verification.

These constructions are inverse of each other: there is a bijection between modifications $\bar{\gamma}$ and liftings of σ to a pseudonatural transformation $(\sigma, \bar{\sigma})$ which composed with $\mathscr{W}_1^{\mathrm{op}} \times J_1$ are 2-natural. ■

Lemma 5.3. *Assume the hypotheses of Lemma 5.2. Then*

(1) *Condition 4 of Proposition 4.2 holds if and only if* $[\eta, -].\sigma$ *is the identity pseudonatural transformation of* $[-,-]$.
(2) *Condition 6 of Proposition 4.2 holds for* $\bar{\gamma}$ *if and only if* $\sigma_{TX,B}.\sigma_{X,B} = [\mu_X, B].\sigma_{X,B}$ *for all* $X \in \mathscr{W}_1$ *and* $B \in T\text{-Alg}_1$.

is *strong monoidal*; that is, it is weak monoidal and the morphisms $F(X) \boxtimes F(Y) \to F(X \otimes Y)$, $FI \to FI$ are equivalences (the latter can be taken to be the identity). The unit $n : 1 \Rightarrow UF$ and the counit $e : FU \Rightarrow 1$ are monoidal pseudonatural transformations [5, Definition 3], and the invertible modifications $Ue.nU \cong 1$ and $eF.Fn \cong 1$ are monoidal.

Remark 6.11. The equivalences (35) show that the tensor product $A \boxtimes B$ classifies multilinear maps with domain A, B, in the sense that the 1-cell (37) induces equivalences

$$T_1\text{-Alg}(A,B;C) \simeq T_1\text{-Alg}(A \boxtimes B, C). \tag{38}$$

Next we show that the constructed tensor product pseudofunctor is 2-natural when restricted to strict morphisms of algebras. This result will be useful in a forthcoming paper, allowing us to speak of the preservation by the tensor product $\boxtimes$ of certain 2-categorical colimits in $T_1\text{-Alg}_{\mathrm{s}}$.

Theorem 6.12. *The 1-cells* (36) *are 2-natural not only in* $A, C \in T_1\text{-Alg}$ *but also in* $B \in T_1\text{-Alg}_s$.

Proof: The result is obtained by setting in Theorem 9.7: $\mathscr{P} = T_1\text{-Alg}_{\mathrm{s}}$, $\mathscr{L} = T_1\text{-Alg}$, $G(B,C) = [\![B,C]\!]$, $H(B,A) = A' \oslash B$. The isomorphisms (33) exhibit H as a left parametrised left adjoint of G. ■

Corollary 6.13. *The restriction of the tensor product pseudofunctor to strict morphisms in the second variable*

$$T_1\text{-Alg} \times T_1\text{-Alg}_{\mathrm{s}} \xrightarrow{1 \times J} T_1\text{-Alg} \times T_1\text{-Alg} \xrightarrow{\boxtimes} T_1\text{-Alg}$$

is (isomorphic to) a 2-functor.

Proof: By Theorem 6.12 above, for any strict morphism of T-algebras $f : B \to D$ we have a commutative diagram

$$\begin{tikzcd}
T_1\text{-Alg}(J(A' \oslash B), C) \arrow[r] \arrow[d, "{T_1\text{-Alg}(J(A' \oslash f),C)}"'] & T_1\text{-Alg}(A, [\![B,C]\!]) \arrow[d, "{T_1\text{-Alg}(A,[\![f,C]\!])}"] \\
T_1\text{-Alg}(J(A' \oslash D), C) \arrow[r] & T_1\text{-Alg}(A, [\![D,C]\!])
\end{tikzcd}$$

that is 2-natural on $A, C \in T_1\text{-Alg}$, where the horizontal functors are the retract equivalences (36). This means that the strict morphism $J(A' \oslash f)$ satisfies the defining condition of $A \boxtimes f$. Thus the pseudofunctor $(A \boxtimes J-)$ is isomorphic to the 2-functor $J(A' \oslash -)$, and letting A vary, $(? \boxtimes J-)$ is isomorphic to the 2-functor $J(?' \oslash -) : T_1\text{-Alg} \times T\text{-Alg}_{\mathrm{s}} \to T_1\text{-Alg}$. ■

creates limits. The isomorphism

$$U_{\mathrm{s}}[\![-,C]\!] = T\text{-Alg}(J-,C) \cong T\text{-Alg}_{\mathrm{s}}((J-)',C)$$

transforms the problem into showing that $T\text{-Alg}_s((J-)',C)$ sends colimits that are preserved by T into limits. This holds by Corollary 9.4 of the Appendix. ∎

For the rest of this section we will work upon the pseudo-closed 2-category T_1-Alg (see Remark 6.8). We do no attempt to obtain a $\mathscr{W}$-enriched version of the monoidal structure on this 2-category, which is *pseudo* or *weak* in nature.

After obtaining the functor $\oslash$, [10] constructs a tensor product $\boxtimes$ in T_1-Alg by applying [3, Theorem 5.1]. To summarise the details needed here,

$$A \boxtimes B = J(A' \oslash B) \tag{34}$$

with unit object FI the free T-algebra on the unit object of $\mathscr{W}$. The relationship between monoidal and pseudo-closed structure can be expressed as the existence of pseudonatural equivalences

$$T_1\text{-Alg}(A \boxtimes B, C) \simeq T_1\text{-Alg}(A, [\![B,C]\!]). \tag{35}$$

These equivalences are given by the composite

$$\begin{aligned} T_1\text{-Alg}(J(A'\oslash B),C) &\xrightarrow{[\![B,-]\!]} T_1\text{-Alg}([\![B,J(A'\oslash B)]\!],[\![B,C]\!]) \\ &\xrightarrow{T_1\text{-Alg}(s_{A,B},1)} T_1\text{-Alg}(A,[\![B,C]\!]) \end{aligned} \tag{36}$$

where $s_{A,B} : A \to [\![B, J(A' \oslash B)]\!]$ is the unit of the adjunction $(-)' \oslash B \dashv [\![B, J-]\!]$.

The observations above are the basic ingredients of Hyland-Power's result:

Theorem 6.10 ([10])**.** *A pseudo-commutativity on a $\mathscr{W}$-monad T on $\mathscr{W}$ induces a monoidal structure on T_1-Alg. Moreover, the biadjunction $F \dashv_{\mathrm{b}} U : T\text{-Alg} \to \mathscr{W}$ is monoidal.*

The last assertion that $F \dashv_{\mathrm{b}} U$ is monoidal means the following. Firstly, U is, in the terminology of [5], *weak monoidal*. This means that it is equipped a pseudonatural transformation

$$\chi_{A,B} : U(A) \otimes U(B) \to U(A \boxtimes B) \tag{37}$$

and a 1-cell $I \to UFI$ (in this case the unit of T) satisfying a higher version of the usual axioms of a monoidal functor. See [5, Definition 2]. Secondly, F

(3) *Condition 1 of Proposition 4.2 holds for $\bar\gamma$ if and only if the pseudomorphism $[X, \sigma_{Y,B}] : [X, [Y, B]] \to [X, [TY, B]]$ corresponds to the pseudomorphism $[t, B].\sigma_{X\otimes Y,B} : [X \otimes Y, B] \to [X \otimes TY, B]$ under the closedness structure of $\mathscr{W}_1$.*

(4) *Condition 2 of Proposition 4.2 holds for $\bar\gamma$ if and only if the pseudomorphism $\sigma_{X,[Y,TZ]} : [X, [Y, TZ]] \to [TX, [Y, TZ]]$ corresponds to the pseudomorphism $[t'_{X,Y}, TZ].\sigma_{X\otimes Y,TZ} : [X \otimes Y, TZ] \to [TX \otimes Y, TZ]$*

Proof: The proof of part 1 is obvious.

Now we show 2. Suppose that $\sigma_{TX,B}.\sigma_{X,B} = [\mu_X, B].\sigma_{X,B}$. If $\bar\gamma$ is defined as in the proof of Lemma 5.2, condition 6 of Proposition 4.2 is the equality

$$\begin{aligned} \big(\sigma_{TX,TY}.\bar\sigma_{X,TY}.T[X,\eta_Y]\big)\big([T^2X,\mu_Y].\bar\sigma_{TX,T^2Y}.(T[TX,\eta_Y]).(T\sigma_{X,TY}).(T[X,\eta_Y])\big) \\ = [\mu_X, TY].\bar\sigma_{X,TY}.T[X,\eta_Y] \end{aligned} \tag{15}$$

Using the 2-naturality of σ with respect to strict morphisms,

$$[T^2X, \mu_Y].\bar\sigma_{X,T^2Y} = \bar\sigma_{X,TY}.T[TX, \mu_Y]$$

and using this we can transform the left hand side of (15) into the pasting

$$\begin{tikzcd} T[X,Y] \arrow[r, "{T[1,\eta_Y]}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar\sigma_{X,TY}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar\sigma_{TX,TY}"] & \bullet \arrow[d] \\ & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

that is by hypothesis equal to $[\mu_X, TY].\bar\sigma_{X,TY}.T[X, \eta_Y]$.

Conversely, assuming condition 6 of Proposition 4.2, and defining $\bar\sigma$ in terms of $\bar\gamma$ as in the proof of Lemma 5.2, we have to show

$$\big([T^2X, b].\bar\gamma_{TX,B}.T[TX, b].T(\mathsf{T})\big)\big([T^2X, b].\mathsf{T}.[TX, b].\bar\gamma_{X,B}\big) = [\mu_X, B].[TX, b].\bar\gamma_{X,B}$$

for $X \in \mathscr{W}$ and a T-algebra (B, b). Using the fact that $\bar\gamma$ is a modification, one can see that the left hand side in the equality above is equal to the

pasting

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[dd] \arrow[dr, phantom, "\bar{\gamma}_{X,B}"] & T[TX,TB] \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar{\gamma}_{TX,TB}"] & \bullet \arrow[d] \\
 & {[TX,T^2B]} \arrow[r] \arrow[d, "{[1,\mu]}"] & {[T^2X,T^B]} \arrow[d, "{[T^2X,b][T^2X,Tb]}"] \\
\bullet \arrow[r] & {[TX,TB]} \arrow[r, "{[T^2X,b][T^2X,Tb]\mathsf{T}}"'] & {[T^2X,B]}
\end{tikzcd}
$$

that by hypothesis is just $[T^2X,b][\mu_X,TB]\bar{\gamma}_{X,B}$. This completes the proof of part 2.

Now we prove 3. It is not hard to show that at the level of 1-cells in $\mathscr{W}_1$, $[X,\sigma_{Y,B}]$ always corresponds to $[t,B]\sigma_{X\otimes Y,B}$, so we must only check the 2-dimensional aspect. Suppose condition 1 of Proposition 4.2 holds. The pseudomorphism structure of $[X,\sigma_{Y,B}]$ is given by the 2-cell $[X,\bar{\sigma}_{Y,B}].\bar{t}_{X,[Y,B]}$, and then we must show that its exponential transpose is the 2-cell $[t_{X,Y},B].\bar{\sigma}_{X\otimes Y,B}$. This follows trivially from our hypothesis as the former is equal to the composition $[X,[TY,b]].[X,\bar{\gamma}_{X,B}].t_{X,[Y,B]}$ and the latter is equal to

$$[X\otimes TY,b].[t_{X,Y},TB].\bar{\gamma}_{X\otimes Y,B}.$$

Conversely, if we assume that the exponential transpose of $[X,\bar{\sigma}_{Y,B}].\bar{t}_{X,[Y,B]}$ is $[t_{X,Y},B].\bar{\sigma}_{X\otimes Y,B}$, it is clear that

$$[X,\bar{\gamma}_{Y,Z}].\bar{t}_{X,[Y,Z]}=[X,\bar{\sigma}_{X,TZ}].[X,T[Y,\eta_Z]].\bar{t}_{X,[Y,Z]}$$

corresponds to

$$[t_{X,Y},TZ].\bar{\gamma}_{X\otimes Y,Z}=[t_{X,Y},TZ].\bar{\sigma}_{X\otimes Y,TZ}.T[X\otimes Y,\eta_Z].$$

Finally, to prove number 4 we first note that $\sigma_{X,[Y,TZ]}$ corresponds to the pseudomorphism $[t'_{X,Y},TZ].\sigma_{X\otimes Y,TZ}$ only means that the corresponding pseudomorphisms structures correspond to each other, or more explicitly that the 2-cells (16) and (17) below correspond to each other.

$$\bar{\sigma}_{X,[Y,TZ]} \tag{16}$$
$$[t'_{X,Y},TZ].\bar{\sigma}_{X\otimes Y,TZ} \tag{17}$$
$$[TX,\bar{t}_{Y,Z}].\bar{\gamma}_{X,[Y,Z]} \tag{18}$$
$$[t'_{X,Y},TZ].\bar{\gamma}_{X\otimes Y,Z} \tag{19}$$

components strict morphisms

$$q_A : A' \to A \tag{31}$$

is an equivalence (in T-Alg_1) and a retract. The existence of this left adjoint was further studied and clarified in [29]. Some of the fundamental facts about it are recalled in Section 9.1.

As we saw in Lemma 6.6, for each T-algebra A the $\mathscr{W}$-functor $[\![A,-]\!]$: T-Alg $\to$ T-Alg restricts to a $\mathscr{W}$-functor T-$\mathrm{Alg_s}\to T$-$\mathrm{Alg_s}$, and we have a commutative diagram

$$
\begin{tikzcd}
T\text{-}\mathrm{Alg_s} \arrow[r, "{[\![A,-]\!]}"] \arrow[d, "J"'] & T\text{-}\mathrm{Alg_s} \arrow[d, "U_{\mathrm{s}}"] \\
T\text{-}\mathrm{Alg} \arrow[r, "T\text{-}\mathrm{Alg}(A,-)"'] & \mathscr{W}
\end{tikzcd}
\tag{32}
$$

Now we show that $[\![A,-]\!]:T\text{-}\mathrm{Alg}_s\to T\text{-}\mathrm{Alg}_s$ has a left adjoint. The composition $U_s[\![A,-]\!]$ preserves cotensor products because T-$\mathrm{Alg}(A,-)$ and J do (see Lemma 9.2); as U_{s} creates cotensor products, this means that $[\![A,-]\!]$ preserves cotensor products. This implies, by a basic fact of enriched category theory, that $[\![A,-]\!]$ has a left adjoint precisely when its underlying ordinary functor has one. This observation, together with the adjoint triangle theorem [7] and the fact that U_{s} is monadic and T-$\mathrm{Alg_s}$ cocomplete (Lemma 9.6), implies that $[\![A,-]\!]$ has a left adjoint if and only if T-$\mathrm{Alg}(A,J-)$ does. And it indeed does, the left adjoint being $-*A':\mathscr{W}\to T\text{-}\mathrm{Alg_s}$. So we have a 2-functor $-\oslash A:T\text{-}\mathrm{Alg_s}\to T\text{-}\mathrm{Alg_s}$ and $\mathscr{W}$-natural isomorphisms

$$T\text{-}\mathrm{Alg_s}(-\oslash A,C)\cong T\text{-}\mathrm{Alg_s}(-,[\![A,C]\!]). \tag{33}$$

As usual, (33) combines all the $\mathscr{W}$-functors $-\oslash A$ into a $\mathscr{W}$-functor $\oslash$: $T\text{-}\mathrm{Alg_s}\otimes T\text{-}\mathrm{Alg_s}\to T\text{-}\mathrm{Alg_s}$.

Lemma 6.9. *The $\mathscr{W}$-functor $\oslash$ preserves all colimits in the first variable. It preserves ϕ-colimits on the second variable, for a weight ϕ, if T preserves ϕ-colimits.*

Proof: The first assertion is obvious as each $-\oslash A$ has a right adjoint. To prove the second assertion, observe that $A\oslash -$ preserves a certain colimit for all A if and only if for each T-algebra C the 2-functor $[\![-,C]\!]$ sends this colimit into a limit, if and only if $U_{\mathrm{s}}[\![-,C]\!]$ have the same property, since U_{s}

The internal hom $[\![-,-]\!]$ is the $\mathscr{W}$-functor described in the previous section, with unit object will be FI, the free T-algebra on the neutral object I of $\mathscr{W}$. The 1-cell $j_A : FI \to [\![A,A]\!]$ is the unique strict morphism corresponding to the identity $I \to T\text{-Alg}(A,A)$. Next we have to provide the retract equivalence $i_A \dashv e_A : [\![FI,A]\!] \to A$ in $T\text{-Alg}_1$:

$$e_A : [\![FI,A]\!] \xrightarrow{U_{FI,A}} [FI,A] \xrightarrow{[\eta_I,A]} [I,A] \xrightarrow{\cong} A \tag{29}$$

$$i_A : A \xrightarrow{\cong} [I,A] \xrightarrow{F_{I,A}} [\![FI,FA]\!] \xrightarrow{[\![FI,a]\!]} [\![FI,A]\!] \tag{30}$$

Since U_1 reflects adjoint equivalences and retract equivalences, it is enough to show that there is a retract equivalence $U_1(i_A) \dashv U_1(e_A)$ in $\mathscr{W}_1$ and that e_A is a pseudomorphism of T-algebras. The latter is trivial, e_A being a strict morphism, while the existence of the retract equivalence in $\mathscr{W}_1$ is a particular case of Corollary 9.5.

The composition $k : [\![B,C]\!] \to [\![[\![A,B]\!],[\![A,C]\!]]\!]$ will be the pseudomorphism of T-algebras associated (see Section 6.2) to the multilinear map (27) of Example 6.5.

The verification of the axioms of a pseudo-closed $\mathscr{W}$-category is left to the reader; it is mostly straightforward, and uses Corollary 9.5.

Remark 6.8. For a pseudo-commutative $\mathscr{W}$-monad T, the 2-category T_1-Alg inherits a pseudo-closed structure from T-Alg.

6.4. Tensor products. Given a pseudo-commutative 2-monad T, under a mild assumption on T [10, Theorem 14] ensures the existence of an induced tensor product on T-Alg. However, can obtain more information than simply that.

We shall assume that T is a $\mathscr{W}$-monad with a rank on $\mathscr{W}$; *e.g., T is finitary*. The 2-monad of Examples 2.1 and 2.2 are finitary; see also Lemma 8.2.

The construction of the tensor product, always following [10], proceeds in the following manner. The assumption that T has a rank ensures that the $\mathscr{W}$-category $T\text{-Alg}_s$ is cocomplete (see Lemma 9.6) and in particular $T\text{-Alg}_s$ will admit *tensor* products with objects of $\mathscr{W}$: given X in $\mathscr{V}$-**Cat** and A, B in $T\text{-Alg}_s$, there is a T-algebra $X * A$ and a $\mathscr{W}$-natural isomorphism

$$T\text{-Alg}_s(X * A, B) \cong [X, T\text{-Alg}_s(A,B)].$$

In Section 9.1 we will see that the 2-adjoint $(-)'$ to the inclusion $J : T\text{-Alg}_s \to T\text{-Alg}$ lifts to a $\mathscr{W}$-enriched adjoint, and the counit of this adjunction, with

In one direction, we assume that this is the case and we have to show that (18) is the exponential transpose of (19). By the definition of $\bar\sigma$ in terms of $\bar\gamma$ above, we have

$$\begin{aligned}
[TX,\bar t_{Y,Z}].\bar\gamma_{X,[Y,Z]} &= [TX,\bar t_{Y,Z}].\bar\sigma_{X,T[Y,Z]}.T[X,\eta_{[Y,Z]}] \\
&= \bar\sigma_{X,[Y,TZ]}.T[X,\bar t_{Y,Z}].T[X,\eta_{[Y,Z]}] \\
&= \bar\sigma_{X,[Y,TZ]}.T[X,[Y,\eta_Z]]
\end{aligned}$$

The second equality holds by the 2-pseudonaturality of σ with respect to strict morphisms of algebras and the simple observation that $\bar t_{Y,Z} : T[Y,Z] \to [Y,TZ]$ is a strict morphism, because

$$[Y,\mu_Z].\bar t_{Y,TZ}.T\bar t_{Y,Z} = \bar t_{Y,Z}.\mu_{[Y,Z]};$$

the last equality follows from the compatibility between η and the strength. On the other hand,

$$[t'_{X,Y},TZ].\bar\gamma_{X\otimes Y,Z} = [t'_{X,Y},TZ].\bar\sigma_{X\otimes Y,TZ}.T[X\otimes Y,\eta_Z]$$

from where is obvious that (18) corresponds to (19). Conversely, assume that (18) corresponds to (19), and write (16) and (17) in terms of $\bar\gamma$ respectively as

$$[TX,\mu_Y].[TX,\bar t_{Y,TZ}].\bar\gamma_{X,[Y,TZ]}$$

and

$$[t'_{X,Y},TZ].[T(X\otimes Y),\mu_Z].\bar\gamma_{X\otimes Y,TZ} = [TX\otimes Y,\mu_Y].[t'_{X,Y},T^2Z].\bar\gamma_{X\otimes Y,TZ}$$

By our assumption, these two 2-cells are exponential transpose one of the other. This concludes the proof of the lemma. ■

6. T-Alg as pseudo-closed $\mathscr{W}$-category

6.1. T-algebra on the object of pseudomorphisms. In this section we briefly explain how a pseudo-commutativity on T induces a T-algebra $[\![A,B]\!]$ with underlying $\mathscr{W}$-category $T\text{-Alg}(A,B)$, yielding the following result, which is a $\mathscr{W}$-enriched version of [10, Section 6].

Theorem 6.1. *A pseudo-commutativity on a $\mathscr{W}$-monad T on $\mathscr{W}$ induces a pseudo-closed structure on T-Alg.*

A fact we need to recall from [3] is that T_1-Alg admits certain limits and the forgetful 2-functor $U_1 : T_1\text{-Alg} \to \mathscr{W}_1$ preserves them. The limits we are referring to are products, inserters and equifers, and therefore those limits that can be constructed from these. See [3, Section 2]. Moreover, these limits can assume that the components of the projections from these limits are strict morphisms of algebras.

We want a T-algebra $[\![A, B]\!]$ with underlying object $T\text{-Alg}(A, B)$ in $\mathscr{W}_1$, and so there will be a universal invertible 2-cell

$$\begin{tikzcd} & {[A,B]} \arrow[dr, "\sigma_{A,B}"] \arrow[dd, phantom, "\Downarrow\gamma"] & \\ {[\![A,B]\!]} \arrow[ur, "U_{A,B}"] \arrow[dr, "U_{A,B}"'] & & {[TA,B]} \\ & {[A,B]} \arrow[ur, "{[a,1]}"'] & \end{tikzcd} \tag{20}$$

in $\mathscr{W}_1$ satisfying the universal property described in Section 3.2. This 2-cell can be constructed from $\sigma_{A,B}$ and $[a, B]$ by considering an iso-inserter and two equifers, and therefore, by the observations on the existence of limits in T_1-Alg of the previous paragraph, the diagram (20) itself will be a limit in T_1-Alg if $\sigma_{A,B}, [a, B] : [A, B] \to [TA, B]$ is a pseudomorphisms of T-algebras.

Both $[A, B]$ and $[TA, B]$ are T-algebras as described in section 3.1, and $[a, 1]$ is always a strict morphism of algebras. Finally, the fact that $\sigma_{A,B}$ is a pseudomorphism is precisely what happens when T is pseudo-commutative by Proposition 5.1. The components $U_{A,B}$ of the forgetful $\mathscr{W}$-functor $U : T\text{-Alg} \to \mathscr{W}$ are strict morphisms of algebras, and the T-algebra structure $T(T\text{-Alg}(A, B)) \to T\text{-Alg}(A, B)$ is the unique 1-cell in $\mathscr{W}_1$ whose postcomposition with $U_{A,B}$ equals

$$T(T\text{-Alg}(A, B)) \xrightarrow{T(U_{A,B})} T[A, B] \xrightarrow{\bar{t}} [A, TB] \xrightarrow{[A,b]} [A, B].$$

Remark 6.2. Since σ is 2-natural on *strict* morphisms of algebras, one easily deduces that $[\![A, f]\!]$ is a strict morphism of algebras for any strict morphism f. The fact that $[\![f, B]\!]$ is a strict morphism for any pseudomorphism f is easily verified.

6.2. Multilinear maps. Before describing the pseudo-closed structure on T-Alg we will briefly mention its multicategory structure, which in this case seems to arise more naturally. Later we shall use these multilinear maps to describe the composition of the pseudo-closed structure and to describe a tensor product of T-algebras.

be strict. This will happen exactly when the parametrised morphism below (T-action on the middle variable) is strict,

$$T\text{-Alg}_{\mathrm{s}}(B, C) \otimes [\![A, B]\!] \otimes UA \xrightarrow{J\otimes 1\otimes 1} T\text{-Alg}(B, C) \otimes [\![A, B]\!] \otimes UA$$
$$\xrightarrow{1\otimes \mathrm{ev}} T\text{-Alg}(B, C) \otimes B \xrightarrow{\mathrm{ev}} C$$

If we rewrite this composite as

$$T\text{-Alg}_{\mathrm{s}}(B, C) \otimes [\![A, B]\!] \otimes UA \xrightarrow{1\otimes \mathrm{ev}} T\text{-Alg}_{\mathrm{s}}(B, C) \otimes B$$
$$\xrightarrow{J\otimes 1} T\text{-Alg}(B, C) \otimes B \xrightarrow{\mathrm{ev}} C$$

which is strict by the two observations in Example 6.4. ■

The following observation will be used in Corollary 7.5 and re-interpreted in section 8 as a familiar fact about functors that are right exact in each variable.

Proposition 6.7. *If the forgetful 2-functor T_1-Alg $\to \mathscr{W}_1$ is full on invertible 2-cells, then every partial map in each variable $f : A_1 \otimes \cdots \otimes A_n \to C$ is automatically a multilinear map.*

Proof: We briefly provide the proof in the case of $n = 2$. Given a partial map in each variable $f : A \otimes B \to C$ and the corresponding pseudomorphism $h : A \to [B, C]$, the commutation condition between both left and right structures is equivalent to the 2-cell corresponding to the right structure $h_2 : c.Tf.t_{A,B} \Rightarrow f.(A \otimes b)$

$$\hat{h} : \sigma_{B,C}.h \Rightarrow [b, C].h$$

being a 2-cell in T_1-Alg (this is (24)). This condition is automatic from our assumption that the forgetful 2-functor is locally full. ■

The condition in the proposition above that the forgetful 2-functor T_1-Alg $\to \mathscr{W}_1$ be full on invertible 2-cells was shown in [20, Proposition 5.1] to be equivalent to requiring that any 1-cell in $\mathscr{W}_1$ have at most one lax morphism structure. In particular, K-Z monads satisfy this condition.

6.3. Pseudo-closed structure on T-Alg. Now we exhibit the pseudo-closed structure on T-Alg induced by a pseudo-commutativity on T, keeping the details to a minimum as this description is completely analogous to the case of 2-monads considered in [10].

and only if g factors through $T\text{-Alg}_{\mathrm{s}}(B,C)$, and f is strict in the first variable ($\bar{f}_1$ is an identity) if and only if g is a strict morphisms of T-algebras.

Example 6.4. A first but nonetheless important example of a multilinear map is the evaluation $\mathrm{ev} : [\![A,B]\!] \otimes A \to B$; by definition, this is the multilinear map associated to the identity pseudomorphism of $[\![A,B]\!]$. According to Remark 6.3 (taking the evaluation as f and the identity as g) we can deduce a couple of properties that we will need later:

(1) Since the identity is a strict morphism of T-algebras (in the notation above, $\bar{g} = 1$), then the parametrised morphism $\mathrm{ev} : [\![A,B]\!] \otimes UA \to B$ (this is, the action of T is on the first variable) is strict ($\bar{f}_1 = 1$).

(2) Since $J : T\text{-Alg}_{\mathrm{s}}(A,B) \to [\![A,B]\!]$ trivially factors through $T\text{-Alg}_{\mathrm{s}}(A,B)$, we deduce that the associated parametrised morphism is strict (action on the first variable). Moreover, this parametirsed morphism is the composite

$$T\text{-Alg}_{\mathrm{s}}(A,B) \otimes A \xrightarrow{J\otimes B} U[\![A,B]\!] \otimes A \xrightarrow{\mathrm{ev}} B.$$

Example 6.5. The main example of a multilinear map for us will be the composition

$$\mathrm{comp} : [\![B,C]\!] \otimes [\![A,B]\!] \to [\![A,C]\!] \tag{27}$$

The reason why comp is a multilinear map is explained in [10]: any multilinear map (27) corresponds to a unique multilinear map

$$[\![B,C]\!] \otimes [\![A,B]\!] \otimes A \to C;$$

comp will correspond to the composite of multilinear maps

$$[\![B,C]\!] \otimes [\![A,B]\!] \otimes A \xrightarrow{1\otimes \mathrm{ev}} [\![B,C]\!] \otimes B \xrightarrow{\mathrm{ev}} C \tag{28}$$

where ev is the multilinear map of the preceding Example 6.4.

The following lemma says that the endo-$\mathscr{W}$-functor $[\![A,-]\!]$ of T-Alg restricts to the sub-$\mathscr{W}$-category $T\text{-Alg}_{\mathrm{s}}$.

Lemma 6.6. *Let $k : [\![B,C]\!] \to [\![[\![A,B]\!],[\![A,C]\!]]\!]$ be the pseudomorphism of T-algebras associated to the composition multilinear map of Example 6.5. Then, kJ factors through $T\text{-Alg}_{\mathrm{s}}([\![A,B]\!],[\![A,C]\!])$.*

Proof: An equivalent condition to the thesis is that the parametrised morphism

$$T\text{-Alg}_{\mathrm{s}}(B,C) \otimes [\![A,B]\!] \xrightarrow{J\otimes 1} T\text{-Alg}(B,C) \otimes [\![A,B]\!] \xrightarrow{\mathrm{comp}} [\![A,C]\!]$$

Multilinear maps in the case of a pseudo-commutative 2-monad on **Cat** are explained at length in [10]. We choose, then, to keep the details to a minimum as our own contribution is only marginal.

Given T-algebras A, B and an object X of $\mathscr{W}$, a 1-cell $f : X \otimes A \to B$ is a *left parametrised morphism of T-algebras* when it is equipped with an invertible 2-cell

$$\begin{tikzcd} X \otimes TA \arrow[r, "t"] \arrow[d, "1\otimes a"'] & T(X\otimes A) \arrow[r, "f"] \arrow[d, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\ X \otimes A \arrow[rr, "f"] & & B \end{tikzcd} \tag{21}$$

satisfying the obvious equations, analogous to the axioms for a morphism of T-algebras but this time involving the strength $t_{X,A} : X \otimes TA \to T(X \otimes A)$. We say that this parametrised morphism is *strict* when the 2-cell $\bar{f}$ is an identity.

Right parametrised morphisms can be defined in the same way, now using $t' : TA \otimes X \to T(X \otimes A)$ instead of t, and combining t, t' morphisms parametrised on the two sides are easily described.

In fact, there is a universal object $T\text{-Alg}(X, A; B)$ in $\mathscr{W}$ that classifies parametrised morphisms; in particular (but not equivalently) 1-cells $I \to T\text{-Alg}(X, A; B)$ are in bijection with parametrised morphisms as described above. Indeed, we will have

$$T\text{-Alg}(X, A; B) \cong [X, T\text{-Alg}(A, B)] \tag{22}$$

and we could take this a defining the left hand side. Alternatively, we can transpose along the adjunction $(X \otimes -) \dashv [X, -]$ the universal 2-cell defining the right hand side of (22) (recall that $T\text{-Alg}(A, B)$ is defined as a certain limit, and then so is the left hand side of (22)), to obtain a universal invertible 2-cell (after using Proposition 5.1.3)

$$\begin{tikzcd} & {[X\otimes A, B]} \arrow[dr, "{[t,1].\sigma_{A,B}}"] & \\ T\text{-Alg}(X,A;B) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\cong"] & & {[X\otimes TA, B]} \\ & {[A,B]} \arrow[ur, "{[a,1]}"'] & \end{tikzcd} \tag{23}$$

satisfying two conditions that correspond to the equations in Figure 1. The equation involving the multiplication of T uses Proposition 5.1.2, while the condition involving the unit of T uses Proposition 5.1.1.

$$
\begin{tikzcd}
& T(A\otimes TB) \arrow[r, "Tt"] & T^2(A\otimes B) \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow\gamma"] & \\
TA\otimes TB \arrow[ur, "t'"] \arrow[r, "t"] \arrow[d, "a\otimes 1"'] & T(TA\otimes B) \arrow[r, "Tt'"] \arrow[d, "T(a\otimes 1)"'] & T^2(A\otimes B) \arrow[r, "\mu"] \arrow[d, "T^2 f"] & T(A\otimes B) \arrow[d, "Tf"] \\
A\otimes TB \arrow[r, "t"] \arrow[dd, "1\otimes b"'] & T(A\otimes B) \arrow[dr, "Tf"'] \arrow[r, phantom, "\Downarrow T\bar f_1"] & T^2C \arrow[r, "\mu"] \arrow[d, "Tc"] & TC \arrow[dd, "c"] \\
& \Downarrow \bar f_2 & TC \arrow[dr, "c"] & \\
A\otimes B \arrow[rrr, "f"] & & & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
TA\otimes TB \arrow[r, "t'"] \arrow[d, "1\otimes b"'] & T(A\otimes TB) \arrow[r, "Tt"] \arrow[d, "T(1\otimes b)"'] & T^2(A\otimes B) \arrow[r, "\mu"] \arrow[d, "Tf"] & T(A\otimes B) \arrow[d, "f"] \\
TA\otimes B \arrow[r, "t'"] \arrow[dd, "a\otimes 1"'] & T(A\otimes B) \arrow[dr, "Tf"'] \arrow[r, phantom, "\Downarrow T\bar f_2"] & T^2C \arrow[r, "\mu"] \arrow[d, "Tc"] & TC \arrow[dd, "c"] \\
& \Downarrow \bar f_1 & TC \arrow[dr, "c"] & \\
A\otimes B \arrow[rrr, "f"] & & & C
\end{tikzcd}
$$

FIGURE 5. Commutation axiom of a multilinear map.

A multilinear map $f : A\otimes B \to C$ will have two structures: one of a parametrised morphism $UA\otimes B \to C$ and another of a parametised morphism $A \otimes UB \to C$, and both will commute in the sense that the two pastings in Figure 5 must be equal (see also [10, p. 169]). Observe that to express this condition one requires the existence of a pseudo-commutativity on T (in the form (8)). There is a bijection between multiliner maps $A\otimes B \to C$ and pseudomorphisms $A \to [\![B, C]\!]$; if we call

$$\bar f_2 : c.(Tf).t \Rightarrow f.(A\otimes b) : A\otimes TB \to C$$
$$\bar f_1 : c.(Tf).t' \Rightarrow f.(a\otimes B) : TA\otimes B \to C$$

the left and right parametrised morphisms structures, these are related to the corresponding pseudomorphism $g : A \to [\![B, C]\!]$ in the following way.

The 1-cell g in T_1-Alg corresponds, by definition of $[\![B, C]\!]$ (see section 6.1), to an invertible 2-cell in T_1-Alg

$$\hat g : \sigma_{B,C}.U.g \Rightarrow [b, C].U.g \tag{24}$$

This means that $U.g : A \to [B, C]$ is a pseudomorphism with two-dimensional structure

$$
\begin{tikzcd}
TA \arrow[r, "T(U.g)"] \arrow[dd, "a"'] & T[B,C] \arrow[d, "\bar t"] \\
\arrow[r, phantom, "\swarrow \bar g"] & {[B,TC]} \arrow[d, "{[1,c]}"] \\
A \arrow[r, "U.g"] & {[B,C]}
\end{tikzcd} \tag{25}
$$

that satisfies a condition that states that $\hat g$ is a 2-cell in T_1-Alg; namely, a compatibility condition involving $\bar g$, $\bar\sigma_{B,C}$ and $\hat g$.

The 1-cell $U.g$ and the 2-cell (24) have as exponential transpose en $\mathscr{W}_1$ respectively the 1-cell $f : A\otimes B \to C$ and the 2-cell $\bar f_2$, while $\bar f_1$ is the exponential transpose of $\bar g$ (25). The compatibility condition referred to in the previous paragraph corresponds to the commutation condition between $\bar f_1, \bar f_2$ involving the pseudo-commutativity of T (that corresponds to $\bar\sigma$).

The details of this bijection between multilinear maps $A\otimes B \to C$ and pseudomorphisms $A \to [\![B, C]\!]$ are analogous to the case $\mathscr{W} = \mathbf{Cat}$ found in [10].

The notion of a multilinear map "in two variables" can be easily extended to allow any number of variables. Together with an obvious notion of morphism between multilinear maps, these form a category what we shall denote by $T_1\text{-Alg}(A_1, \cdots, A_n; C)$; accordingly to the paragraph above there is an isomorphic to

$$T_1\text{-Alg}(A_1, \cdots, A_{n+1}; C) \cong T_1\text{-Alg}(A_1, \cdots, A_n; [\![A_{n+1}, C]\!]) \tag{26}$$

Defining $T_1\text{-Alg}(; C) = \mathscr{W}_1(I, C)$ we obtain a closed **Cat**-enriched multicategory T_1-Alg such that the usual forgetful 2-functor in to $\mathscr{W}_1$ is a morphism of **Cat**-enriched muticategories.

Remark 6.3. Observe that on the bijection between multilinear maps $f : A\otimes B \to C$ and pseudomorphisms $g : A \to [\![B, C]\!]$ the following can be be added. The multimap f is strict in the second variable ($\bar f_2$ is an identity) if