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ORDER-PRESERVING REFLECTORS AND INJECTIVITY

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Dedicated to Eraldo Giuli on his seventieth birthday

Abstract: We investigate a Galois connection in poset enriched categories between subcategories and classes of morphisms, given by means of the concept of right-Kan injectivity, and, specially, we study its relationship with a certain kind of subcategories, the KZ-reflective subcategories. A number of well-known properties concerning orthogonality and full reflectivity can be seen as a particular case of the ones of right-Kan injectivity and KZ-reflectivity. On the other hand, many examples of injectivity in poset enriched categories encountered in the literature are closely related to the above connection. We give several examples and show that some known subcategories of the category of T_0-topological spaces are right-Kan injective hulls of a finite subcategory.

1. Introduction

In the realm of poset enriched categories there are several studies on injectivity (in particular, in the category of T_0 topological spaces and in the category of locales) and, dually, on projectivity (as, for example, in the categoy of frames and in the category of quantales). Some of that work can be found in [2, 3, 6, 7, 8, 9, 12, 13, 14, 17, 19] and in references there. In this paper we deal with a special type of injectivity, which, in fact, is associated with many of the injectivity occurrences investigated in the above mentioned literature: the right-Kan injectivity. In a poset enriched category, an object Z is said to be right-Kan injective with respect to a morphism $f : X \to Y$ if, for every $g : X \to Z$, there is a morphism $g/f : Y \to Z$ such that $g/f \cdot f = g$ and g/f is the supremum of all morphisms $t : Y \to Z$ such that $tf \leq g$. In a series of papers, Escardó, also with Flagg, observed that several injectivity situations are instances of a general pattern: in a poset enriched category, the objects injective with respect to T-embeddings, for T a KZ-monad over the category, are just the T-algebras of the monad. More precisely, a monad $\mathbb{T} = (T, \eta, \mu)$ over a poset enriched category $\mathcal{X}$ is said to be of Kock-Zöberlein

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type, briefly, a KZ-monad, if T is locally monotone, i.e., the restriction of T to every hom-set is order-preserving, and $\eta_{TX} \leq T\eta_X$ for every object X. (Indeed, this is a particular case of a Kock-Zöberlein doctrine, see [15].) A morphism $f : X \to Y$ of $\mathcal{X}$ is called a T-embedding if Tf has a reflective left-adjoint, that is, there exists $(Tf)^* : TY \to TX$ such that $(TF)^* \cdot Tf = 1_{TX}$ and $1_{TY} \leq Tf \cdot (Tf)^*$. In [7] it is shown that the Eilenberg-Moore algebras of a KZ-monad $\mathbb{T}$ coincide with the objects of $\mathcal{X}$ injective w.r.t. T-embeddings, and, moreover, they are also precisely the objects of $\mathcal{X}$ right-Kan injective w.r.t. T-embeddings. It is clear that the category of the Eilenberg-Moore algebras of a KZ-monad over $\mathcal{X}$ is a reflective subcategory of $\mathcal{X}$ whose reflector F is locally monotone and fulfils the inequalities $\eta_{FX} \leq F\eta_X$ for η the corresponding unit. In the present paper, the subcategories of $\mathcal{X}$ endowed with a reflector with these properties are called KZ-reflective.

Here we consider the notion of right-Kan injectivity between morphisms too: a morphism $k : Z \to W$ is right-Kan injective w.r.t. $f : X \to Y$ if Z and W are so and, moreover, $k(g/f) = (kg)/f$ for all morphisms $g : X \to Z$. In this way, the objects and morphisms which are right-Kan injective w.r.t. a given subclass $\mathcal{H}$ of $\mathrm{Mor}(\mathcal{X})$ constitute a subcategory of $\mathcal{X}$, denoted by $\mathcal{H}_\lrcorner$, and we obtain a Galois connection between classes of morphisms and subcategories. When $\mathcal{X}$ is an arbitrary category seen as an enriched poset category via the equality partial order, right-Kan injectivity just means orthogonality, and a subcategory is KZ-reflective iff it is reflective and full. There are many papers exploring the relationship between orthogonality and full reflectivity (see, for instance, [11] and [1], and references there, and also [4]). We show that right-Kan injectivity mantains the good behaviour of orthogonality. Particularly, this is clear in what concerns limits. In fact, let a limit cone $(L \xrightarrow{l_i} X_i)_{i\in I}$ be said jointly order-monic provided that the inequalities $l_i f \leq l_i g$, $i \in I$, imply that $f \leq g$. It is worth noting that in several everyday poset enriched categories all limits are jointly order-monic (see 2.8). We prove that every subcategory of the form $\mathcal{H}_\lrcorner$, for $\mathcal{H}$ a class of morphisms, is closed under jointly order-monic limits, and every KZ-reflective subcategory closed under coreflective right adjoints (see 2.11) is of that form, hence, closed under those limits. Moreover, the categories of Eilenberg-Moore of a KZ-monad over $\mathcal{X}$ coincide with the KZ-reflective subcategories of $\mathcal{X}$ closed under coreflective right adjoints. As a byproduct,

$f : X \to Y$ is said to be a *G-quotient* if the morphism Gf has a reflective right adjoint in $\mathcal{A}$.

In [2], B. Banaschewski introduced the notion of $\mathcal{K}$-flat morphism in the category **Frm** of frames, which gives a general approach to the study of projectivity in **Frm**, unifying several previously known results. Let **SLat** denote the category whose objects are meet-semilattices with a top, and whose morphisms are maps preserving the meet and the top. Let $\mathcal{K}$ be a subcategory of **SLat** in which **Frm** is reflective, with reflector F and such that, for every reflection $\eta_A : A \to FA$, the frame FA is generated by the image of η_A. A frame homomorphism $h : L \to M$ is *$\mathcal{K}$-flat* if it is onto and its right adjoint $h_* : M \to L$ (which exists in **SLat**) belongs to $\mathcal{K}$. As it is shown in [2], when $\mathcal{K}$ is a category under the above described conditions, F is locally monotone and the comonad $H : \mathbf{Frm} \to \mathbf{Frm}$ induced by the adjunction between **Frm** and $\mathcal{K}$ is of Kock-Zöberlein type. Furthermore, it is easy to see that the $\mathcal{K}$-flat morphisms are exactly the H-quotients, or, equivalently, the frame homomorphisms which are F-quotients. To show that, first observe that, given a $\mathcal{K}$-flat frame homomorphism $f : L \to N$, the adjunction $f \dashv f_*$ is reflective: For every $y \in N$, the sobrejectivity of f assures the existence of some x such that $f(x) = y$, and then we have that $(f \cdot f_*)(y) = (f \cdot f_*)(f(x)) = f((f_* \cdot f)(x)) \geq f(x) = y$; thus $f \cdot f_* = 1_N$. Now it is clear that $Ff \dashv_R Ff_*$. Conversely, a reflective adjunction $Ff \dashv_R (Ff)_*$, with $f : L \to N$ in **Frm**, implies that $f \dashv_R \varepsilon_L \cdot (Ff)_* \cdot \eta_N$, where η and ε are the unit and counit.

B. Banaschewski showed that several examples of projectivity in **Frm** are instances of $\mathcal{K}$-projectivity for a convenient category $\mathcal{K}$. These examples encompass several cases of projectivity previously studied (in [14], [3], [8], [17] and more) and, in particular, the frame counterpart of examples of 4.1. From 3.7, we know that the $\mathcal{K}$-flat morphisms are not only a class of morphisms w.r.t. which the corresponding category of co-algebras is right-Kan projective, but, moreover, that it is the largest one with such property.

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$g_*(G) \cup g_*(H)$ for all G, $H \in \Omega X$. Let then G and H be open sets of X and define $h : X \to A$ by

$$h(x) = \begin{cases} 1 & \text{if } x \in G \cap H \\ a & \text{if } x \in G \setminus G \cap H \\ b & \text{if } x \in H \setminus G \cap H \\ 0 & \text{if } x \notin G \cup H \end{cases}$$

Then, from Remark 4.4, we know that $h/g : Y \to A$ is given by

$$h/g(y) = \begin{cases} 1 & \text{if } y \in g_*(G \cap H) \\ a & \text{if } y \in g_*(G) \setminus g_*(G \cap H) \\ b & \text{if } y \in g_*(H) \setminus g_*(G \cap H) \\ 0 & \text{if } y \notin g_*(G) \cup g_*(H) \end{cases}$$

Consequently,

$$(k \cdot (h/g))(y) = \begin{cases} 1 & \text{if } y \in g_*(G) \cup g_*(H) \\ 0 & \text{if } y \notin g_*(G) \cup g_*(H) \end{cases} \tag{4.2}$$

But, by 4.4, $(kh)/g = \chi_{g_*(G \cup H)}$, and we know that $(kh)/g = k \cdot (h/g)$, since k is right-Kan injective w.r.t. g. Hence, by (4.2), $\chi_{g_*(G\cup H)} = \chi_{g_*(G)\cup g_*(H)}$, thus $g_*(G \cup H) = g_*(G) \cup g_*(H)$, as requested. □

Remark 4.7. Let $\mathcal{X}$ be a poset enriched category. The dual category $\mathcal{X}^{\mathrm{op}}$ may be seen as a poset enriched category with the order given by $f^{\mathrm{op}} \leq g^{\mathrm{op}}$ iff $f \leq g$. This way, the dual notions of right-Kan injectivity, KZ-reflectivity and KZ-monad are clear. In particular:

- Given a morphism $f : X \to Y$ and an object A in $\mathcal{X}$, we say that A is *right-Kan-projective* w.r.t. f if it is right-Kan injective w.r.t. f^{op} in $\mathcal{X}^{\mathrm{op}}$. That is, for every morphism $g : A \to Y$, there exists $g' : A \to X$ such that $f \cdot g' = g$ and $f \cdot t \leq g \Rightarrow t \leq g'$, for all possible morphisms t.
- A co-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, with right adjoint G and co-unit ε, is said to be *KZ-co-reflective* in $\mathcal{X}$ provided that G is locally monotone and the inequality $\varepsilon_{GX} \leq G\varepsilon_X$ is fulfilled for all $X \in \mathcal{X}$ (equivalently, $\eta_{GA} \geq G\eta_A$ for all $A \in \mathcal{A}$, see Remark 3.3).

Analogously, we obtain the definition of a *KZ-comonad*. For the dual of the concept of F-embedding, we use the term G-quotient; that is, if $G : \mathcal{X} \to \mathcal{A}$ is a locally monotone functor between poset enriched categories, a morphism

we complete Escardó's result on the relationship between T-algebras and T-embeddings: Let $\mathcal{A}$ be the Eilenberg-Moore category of a KZ-monad T; we show that the class $\mathcal{E}$ of T-embeddings is the largest one such that $\mathcal{A} = \mathcal{E}_{\lrcorner}$.

A characterization of the KZ-reflective subcategories in a poset enriched category $\mathcal{X}$, which is very useful to achieve some of the results of this paper, is given in 3.4: they are exactly those subcategories $\mathcal{A}$ of $\mathcal{X}$ such that, for every $X \in \mathcal{X}$, there is a morphism $\eta_X : X \to \overline{X}$ with $\overline{X}$ in $\mathcal{A}$ satisfying the conditions:

(i) $\mathcal{A} \subseteq \{\eta_X \mid X \in \mathcal{X}\}_{\lrcorner}$ and, for every morphism $g : X \to A$ with $A \in \mathcal{A}$, g/η_X belongs to $\mathcal{A}$;

(ii) for every $f : \overline{X} \to A$ in $\mathcal{A}$ and $g : \overline{X} \to A$ in $\mathcal{X}$, if $g\eta_X \leq f\eta_X$ then $g \leq f$.

In the last section, we show that some KZ-reflective subcategories of the category $\mathbf{Top}_0$ of T_0-topological spaces and continuous maps are right-Kan injective hulls of finite subcategories. This is the case of the category of continuous lattices and maps which preserve directed supremums and infimums, and of the category of continuous Scott domains and maps which preserve directed supremums and non empty infimums, both of them regarded as subcategories of $\mathbf{Top}_0$ via the Scott topology. It is also the case of the category of stably compact spaces and stable continuous maps.

2. Right-Kan injectivity

Throughout we work in a *poset enriched* category $\mathcal{X}$: the hom sets of $\mathcal{X}$ are endowed with a partial order for which the composition is monotone, i.e., if $f, g : A \to B$ are morphisms such that $f \leq g$ then $jfh \leq jgh$ whenever the compositions are defined. Of course, the category **Pos** of posets and monotone maps, as well as several subcategories of **Pos**, in particular the category **Frm** of frames and frame homomorphisms, are poset enriched via the pointwise order. Also the category $\mathbf{Top}_0$ of T_0 topological spaces and continuous maps is so: take the pointwise specialization order.

In a poset enriched category, a morphism $r : X \to Y$ is said to be *right adjoint* to the morphism $l : Y \to X$ (and l is said to be *left adjoint* to r) if $lr \leq 1_X$ and $1_Y \leq rl$. This forms an *adjunction*, denoted by $l \dashv r$. This adjunction is said to be *reflective* if $lr = 1_X$ (notation: $l \dashv_R r$), and *coreflective* if $1_Y = rl$ (notation: $l \dashv_C r$).

Definition 2.1. Given a morphism $X \xrightarrow{f} Y$ and an object A, we say that A is *right-Kan-injective w.r.t.* f, symbolically $A \,\lrcorner\, f$, provided that, for every morphism $g : X \to A$, there exists $g' : Y \to A$ such that:

(1) $g'f = g$
(2) $tf \leq g \implies t \leq g'$, for each morphism $t : Y \to A$

When such morphism g' exists, we denote it by g/f.

A morphism $h : A \to B$ is said to be *right-Kan-injective w.r.t.* $f : X \to Y$, briefly $h \,\lrcorner\, f$, if A and B are both right-Kan injective w.r.t. f and, for every $g : X \to A$, we have

$$(hg)/f = h(g/f).$$

Remark 2.2. Recall that an object A is injective w.r.t. a morphism f (respectively, orthogonal to f) if the map $\hom(f, A) : \hom(Y, A) \to \hom(X, A)$ is surjective (respectively, bijective). We have the following properties:

1. An object A is right-Kan injective w.r.t. a morphism $f : X \to Y$ iff A is injective w.r.t. f and every morphism $g : X \to A$ admits a right Kan extension along the morphism f (that is, there is $g' : Y \to A$ such that $g'f \leq g$ and g' fulfils condition 2 of 2.1). To show the sufficiency, let $\bar{g} : Y \to A$ be such that $\bar{g}f = g$ and let $g' : Y \to A$ be the right Kan extension of g along f. Then $\bar{g} \leq g'$, so $g = \bar{g}f \leq g'f$; since $g'f \leq g$, we get $g'f = g$. Consequently, $g' = g/f$.
2. Another equivalent way of defining the right-Kan injectivity of an object A w.r.t. a morphism $f : X \to Y$ is the following: $A \,\lrcorner\, f$ iff $\hom(f, A)$ has a reflective right adjoint $(\hom(f, A))_*$ in **Pos**. Furthermore, if it is the case, it holds that $(\hom(f; A))_*(g) = g/f$ for every $g : X \to A$. To see that, let $A \,\lrcorner\, f$, and define $(\hom(f, A))_* : \hom(X, A) \to \hom(Y, A)$ in that way. Then, it is order-preserving, since, for g, $g' \in \hom(X, A)$, with $g \leq g'$, we have that $g = g/f \cdot f \leq g'$ implies, by definition of g'/f, that $g/f \leq g'/f$. Now, on one hand, for every $k \in \hom(Y, A)$, we have that $(\hom(f, A))_* \cdot \hom(f, A)(k) = (\hom(f, A))_*(kf) = (kf)/f \geq k = \mathrm{id}_{\hom(Y,A)}(k)$, where the inequality derives from 2 of Definition 2.1. On the other hand, for every $g \in \hom(X, A)$, we get $\hom(f, A) \cdot (\hom(f, A))_*(g) = \hom(f, A)(g/f) = (g/f) \cdot f = g = \mathrm{id}_{\hom(X,A)}(g)$. Therefore $\hom(f, A) \vdash_R (\hom(f, A))_*$. Conversely, suppose that $\hom(f, A)$ has a reflective right adjoint $(\hom(f, A))_*$. Then, for every $g : X \to A$, we have that $(\hom(f, A))_*(g) \cdot f =$

spaces is a dense embedding iff the inclusion $\mathbf{2} \xhookrightarrow{f} \mathbf{3}$ *is right-Kan injective w.r.t. it.*

Proof By using an argument similar to the one used at the beginning of the proof of Lemma 4.3, we conclude that $\mathbf{ScottDI} = \mathcal{D}_{\lrcorner}$, where $\mathcal{D}$ is the class of all dense embeddings. Moreover, since the morphism $\mathbf{2} \xhookrightarrow{f} \mathbf{3}$ belongs to **ScottDI**, we know that $\mathcal{D} \subseteq \mathcal{A}^{\lrcorner}$. So we only have to show that $\mathcal{A}^{\lrcorner} \subseteq \mathcal{D}$. Let $g : X \to Y$ be a morphism in $\mathcal{A}^{\lrcorner}$. Then, in particular, $\mathbf{2} \,\lrcorner\, g$; hence, by 4.3, g is an embedding. Now, for the inclusion $\mathbf{2} \xhookrightarrow{f} \mathbf{3}$ and the map $\chi_\varnothing : X \to \mathbf{2}$, we have that

$$(f \cdot \chi_\varnothing)/g = f \cdot (\chi_\varnothing/g) = f \cdot \chi_{g_*(\varnothing)},$$

taking into account that $f \,\lrcorner\, g$ and the description of $\chi_\emptyset/g$ given in Remark 4.4. Thus, the image of $(f \cdot \chi_\varnothing)/g$ does not contain the point 2, and, consequently, from the characterization of $(f \cdot \chi_\varnothing)/g$ given by (4.1) in Remark 4.4, we know that no point y of Y belongs to $g_*\left((f \cdot \chi_\varnothing)^{-1}(\{2\})\right)$, i.e., $g_*\left((f \cdot \chi_\varnothing)^{-1}(\{2\})\right) = \varnothing$. But $(f \cdot \chi_\varnothing)^{-1}(\{2\})$ is clearly empty, then we have $g_*(\varnothing) = \varnothing$, that is, g is a dense embedding. □

Let A be the poset with underlying set $\{0, a, b, 1\}$, where a and b are non-comparable and 0 and 1 are the bottom and the top elements, respectively. And let $k : A \to \mathbf{2}$ be the map which takes 0 to 0 and all the other elements of A to 1.

Proposition 4.6. *The subcategory* **SComp** *of* $\mathbf{Top}_0$ *coincides with* $\left(\mathcal{A}^{\lrcorner}\right)_{\lrcorner}$, *where* $\mathcal{A}$ *is the category whose only non-identity morphisms are* $k : A \to \mathbf{2}$ *and* $f : \mathbf{2} \hookrightarrow \mathbf{3}$. *Furthermore, a continuous map between* T_0 *topological spaces is a flat embedding iff the maps* k *and* f *are right-Kan injective w.r.t. it.*

Proof Of course every finite space of $\mathbf{Top}_0$ is stably compact. Moreover, every continuous map between finite T_0-spaces is trivially stable. Thus, the above morphism k belongs to **SComp**. Now, arguing as at the beginning of the proof of Lemma 4.3, we see that the only thing to prove is that $\mathcal{A}^{\lrcorner} \subseteq \mathcal{F}$ for $\mathcal{F}$ the class of all flat embeddings.

Let $g : X \to Y$ belong to $\mathcal{A}^{\lrcorner}$. Then, since $\mathbf{2} \,\lrcorner\, g$, g is an embedding. We are going to show that, moreover, it must be flat, i.e., that $g_*(G \cup H) =$

Proof We know that **ContI** is the category of algebras of the filter monad T over $\mathbf{Top}_0$ and that the T-embeddings are the embeddings (see 4.1 and 4.2). Furthermore, from 3.13 and 3.14, we have that the T-embeddings are just the F-embeddings for F the left adjoint of the inclusion of **ContI** into $\mathbf{Top}_0$. Consequently, by 3.7 and 3.9, we conclude that $\mathbf{ContI} = \mathcal{E}_{\lrcorner}$, where $\mathcal{E}$ is the class of all embeddings. Therefore, we only have to show that $\mathcal{S}^{\lrcorner} = \mathcal{E}$.

Since the Sierpiński space is in **ContI**, it is clear that $\mathcal{E} \subseteq \mathcal{S}^{\lrcorner}$. The other way round is also immediate: if $g : X \to Y$ belongs to $\mathcal{S}^{\lrcorner}$, then, in particular, 2 is injective w.r.t. g, and it is known, and easy to prove, that, then, g is an embedding. $\square$

Remark 4.4. Let $g : X \to Y$ be an embedding (in $\mathbf{Top}_0$). Then, it is easily seen that, for every map $\chi_U : X \to \mathbf{2}$, with U an open set of X, we have that $\chi_U/g = \chi_{g_*(U)}$.

More generally, let $h : X \to Z$ be a morphism of $\mathbf{Top}_0$ with Z a finite continuous lattice. In this case the topology of Z is generated by all sets of the form $z\uparrow = \{w \in Z : z \le w\}$. We are going to show that the map h/g is defined by

$$h/g(y) = \bigvee \{z \in Z : y \in g_*(h^{-1}(z\uparrow))\}. \tag{4.1}$$

First of all, it is continuous, with $(h/g)^{-1}(z\uparrow) = g_*(h^{-1}(z\uparrow))$, for all $z \in Z$. To conclude that $h/g \cdot g(x) = h(x)$ for every $x \in X$, we observe that $h^{-1}(h(x)\uparrow) = g^{-1}g_*(h^{-1}(h(x)\uparrow))$, because g is an embedding (see Remark 4.2); consequently, $g(x) \in g_*(h^{-1}(h(x)\uparrow))$. Moreover, $h(x)$ is the supremum of all z for which $g(x) \in g_*(h^{-1}(z\uparrow))$, since

$$g(x) \in g_*(h^{-1}(z\uparrow)) \Leftrightarrow x \in g^{-1}g_*(h^{-1}(z\uparrow)) \Leftrightarrow x \in h^{-1}(z\uparrow) \Leftrightarrow h(x) \ge z.$$

Let now $k : Y \to Z$ be a morphism such that $kg \le h$. In order to show that $k \le h/g$, it suffices to verify that, for every $y \in Y$, $y \in g_*(h^{-1}(k(y)\uparrow))$, that is, that there is some $V \in \Omega Y$ such that $y \in V$ and $g^{-1}(V) \subseteq h^{-1}(k(y)\uparrow)$. The set $V = k^{-1}(k(y)\uparrow)$ fulfils this requirement, since $x \in g^{-1}(V)$ is equivalent to $x \in (kg)^{-1}(k(y)\uparrow)$, and this implies that $x \in h^{-1}(k(y)\uparrow)$, because $kg \le h$.

Proposition 4.5. *The subcategory* **ScottDI** *of* $\mathbf{Top}_0$ *coincides with* $(\mathcal{A}^{\lrcorner})_{\lrcorner}$, *where* $\mathcal{A}$ *is the two-objects category whose only non-identity morphism is the inclusion* $\mathbf{2} \xhookrightarrow{f} \mathbf{3}$. *Moreover, a continuous map between* T_0 *topological*

$\hom(f,A)((\hom(f,A))_*(g)) = g$. And, given $k : Y \to A$ such that $kf \le g$, we obtain $kf \le g \Leftrightarrow \hom(f,A)(k) \le g \Rightarrow (\hom(f,A))_* \cdot \hom(f,a)(k) \le (\hom(f,A))_*(g) \Rightarrow k \le (\hom(f,A))_*(g)$. Hence $(\hom(f,A))_*(g) = g/f$.

(3) It is immediate from Definition 2.1 that if $\mathcal{X}$ is an arbitrary category, regarded as being enriched with the trivial ordering (i.e., equality), an object A is orthogonal to a morphism f iff it is right-Kan-injective w.r.t. f.

(4) Let $\mathcal{H}$ be a class of morphisms of $\mathcal{X}$, and let $\mathcal{C}$ consist of all objects and morphisms of $\mathcal{X}$ which are right-Kan injective w.r.t. f for all $f \in \mathcal{H}$. Then it is easy to see that $\mathcal{C}$ is a subcategory of $\mathcal{X}$.

Notations 2.3. Let $\mathcal{H} \subseteq \mathrm{Mor}(\mathcal{X})$. We will denote by

$$\mathcal{H}_{\lrcorner}$$

the subcategory of all objects and morphisms of $\mathcal{X}$ which are right-Kan injective w.r.t. f for all $f \in \mathcal{H}$.

Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, we denote by

$$\mathcal{A}^{\lrcorner}$$

the class of all morphisms f of $\mathcal{X}$ such that all objects and morphisms of $\mathcal{A}$ are right-Kan injective w.r.t. f.

Remark 2.4. The pair of maps $\left((\_)_{\lrcorner}, (\_)^{\lrcorner}\right)$ establishes a (contra-variant) Galois connection between the classes of $\mathcal{X}$-morphisms and the subcategories of $\mathcal{X}$.

Remark 2.5. If all morphisms of $\mathcal{H}$ are epimorphisms then the subcategory $\mathcal{H}_{\lrcorner}$ is full. In fact, in that case, given objects A and B in $\mathcal{H}_{\lrcorner}$ and an $\mathcal{X}$-morphism $f : A \to B$, then, for every $h : X \to Y \in \mathcal{H}$ and $g : X \to A$, we have that the equality $((fg)/h)h = fg = f(g/h)h$ implies $(fg)/h = f(g/h)$.

Examples 2.6. In Section 4 we will provide several examples of $\mathcal{H}_{\lrcorner}$ and $\mathcal{A}^{\lrcorner}$. Here we describe two simple ones.

(1) Let $X = \{0\}$ and $Y = \{0,1\}$ be ordered by the natural order, let $h : X \to Y$ be the inclusion map, and let $\mathcal{H}$ consist of just h. Then, in the category **Pos**, the subcategory $\mathcal{H}_{\lrcorner}$ has, as objects, the posets A for which every upper set $x\uparrow = \{z \in A \mid x \le z\}$ has a supremum, and, as morphisms, the order-preserving maps $f : A \to B$ such that

$f(\sup(x\uparrow)) = \sup(f(x)\uparrow)$, for every $x \in A$. It is clear that all those objects A belong to $\mathcal{H}_{\lrcorner}$: given a morphism $g : X \to A$, the morphism g/h is defined by $(g/h)(0) = g(0)$ and $(g/h)(1) = \sup(g(0)\uparrow)$. The other way round, let A be a poset belonging to $\mathcal{H}_{\lrcorner}$, let $x \in A$, and define $g : X \to A$ by $g(0) = x$. Then $g/h(1) = \sup(x\uparrow)$. The characterization of the morphisms of $\mathcal{H}_{\lrcorner}$ is also easily verified.

(2) Consider now **Pos** enriched with the pointwise dual order $\geq$. In this case, for the morphism h as above, it is easy to see that $\mathcal{H}_{\lrcorner}$ coincides with **Pos**, and, for each morphism f with the same domain as h, f/h is a constant map.

In the next propositions we will enumerate some properties of $\mathcal{H}_{\lrcorner}$ and $\mathcal{A}^{\lrcorner}$. First we need some definitions.

Definition 2.7. We say that a family of morphisms $\left(X \xrightarrow{f_i} X_i\right)_{i\in I}$ is *jointly order-monic* if the inequalities $f_i \cdot g \leq f_i \cdot h$, for all $i \in I$, imply $g \leq h$.

Dually, a family $\left(X \xrightarrow{f_i} X_i\right)_{i\in I}$ is *jointly order-epic* if $g \leq h$ whenever $g \cdot f_i \leq h \cdot f_i$, $i \in I$.

In particular, a morphism $X \xrightarrow{f} Y$ is said to be *order-monic* (respectively, *order-epic*) if $fg \leq fh$ implies $g \leq h$ (respectively, $gf \leq hf$ implies $g \leq h$).

We say that a limit is jointly order-monic if the corresponding cone limit is so. Analogously we speak of jointly order-epic colimits.

Examples 2.8. (1) In the category **Pos** a family of morphisms $\left(X \xrightarrow{f_i} X_i\right)_{i\in I}$ is jointly order-monic iff, for every $x, x' \in X$, $x \leq x'$ whenever $f_i(x) \leq f_i(x')$ for all i. Thus, jointly order-monic families of morphisms are also jointly monic (i.e., $f_i(x) = f_i(x')$ for all i implies that $x = x'$). Clearly, in this category, limits are jointly order-monic. On the other hand, families of morphisms $\left(X_i \xrightarrow{f_i} X\right)_{i\in I}$ which are jointly surjective (i.e., $X = \cup_{i\in I} f_i[X_i]$) are also jointly order-epic, and, hence, they comprehend colimit cocones.

Several everyday subcategories of **Pos** are closed under limits in **Pos**, including **Frm**, and, then, they have also jointly order-monic limits.

and, as it was observed by Escardó ([6]), this monad is of Kock-Zöberlein type. The category **ScottDI** was proved to be the category of algebras of the proper filter monad by Wyler [22], and this monad was showed to be of Kock-Zöberlein type by Escardó and Flagg in [9]. At last, we know from Simmons [20] and Wyler [23] that **SComp** is the category of algebras of the prime filter monad, and, again from [9], that this monad is also a KZ-monad.

In [9] the authors present other examples of categories of algebras of KZ-monads over $\mathbf{Top}_0$, which are, consequently, KZ-reflective subcategories of $\mathbf{Top}_0$.

In what follows, given a space X we denote its lattice of open sets by ΩX. Given a continuous map $f : X \to Y$, the frame homomorphism $f^{-1} : \Omega Y \to \Omega X$ preserves all joins and, hence, has a right adjoint, denoted by $f_* : \Omega X \to \Omega Y$, in the category **SLat** of meet-semilattices with a top and maps which preserve the meet and the top. The map f_* is given by

$$f_*(U) = \bigcup \left\{V : f^{-1}(V) \subseteq U\right\},\ U \in \Omega X.$$

Remark 4.2. In [9], Escardó and Flagg proved that for the filter and the proper filter monads T over $\mathbf{Top}_0$ the T-embeddings are the embeddings and the dense embeddings, respectively (see also [19]). It is easy to see that a map f in $\mathbf{Top}_0$ is an embedding iff the map f^{-1} is surjective ([13]), and it is a dense embedding iff, moreover, $f_*(\emptyset) = \emptyset$ ([9]). Moreover, f is an embedding iff the adjunction $f^{-1} \vdash f_*$ is reflective, i.e., $f^{-1}f_* = 1_{\Omega X}$. These characterizations will be used in the following.

Also in [9], it was proven that, when T is the prime filter monad, the T-embeddings are the *flat embeddings*, that is, the maps of $\mathbf{Top}_0$ such that f^{-1} is surjective and its right adjoint f_* preserves finite unions.

Next we show that the three categories listed in 4.1 are the right-Kan injective hull of a finite subcategory of $\mathbf{Top}_0$.

Let **2** and **3** denote the chains $0 < 1$ and $0 < 1 < 2$, respectively. In particular **2**, as an object of $\mathbf{Top}_0$, is the Sierpiński space.

Lemma 4.3. *For $\mathcal{S}$ the subcategory of $\mathbf{Top}_0$ consisting of the Sierpiński space **2** and the identity on **2**, it holds that $\left(\mathcal{S}^{\lrcorner}\right)_{\lrcorner} = \mathbf{ContI}$. Moreover, a continuous map between T_0 topological spaces is an embedding iff **2** is right-Kan injective w.r.t. it.*

T-embedding is an F-embedding and, therefore, the class of T-embeddings coincides with the one of F-embeddings.

4. Right-Kan injective hulls of finite subcategories of $\mathbf{Top}_0$

In this section we give some examples of KZ-reflective subcategories of $\mathbf{Top}_0$, based on results of [9] and references there, and we prove that they are the right-Kan injective hull of a finite subcategory of $\mathbf{Top}_0$, that is, they are of the form $\left(\mathcal{A}^{\lrcorner}\right)_{\lrcorner}$ for a finite subcategory $\mathcal{A}$. As a byproduct, we obtain new characterizations of embeddings, dense embeddings and flat embeddings in $\mathbf{Top}_0$. Moreover, in Remark 4.7, we consider the dual notions of KZ-reflective subcategory and F-embedding and relate them to results of [2].

Examples 4.1. The subcategories of $\mathbf{Top}_0$ described in the following are KZ-reflective:

(1) **ContI** denotes the category of continuous lattices and maps which preserve directed supremums and infimums. It is known that, considering every continuous lattice endowed with the Scott topology, **ContI** becomes a subcategory of $\mathbf{Top}_0$ ([12]).
(2) **ScottDI** is the category of continuous Scott domains and maps which preserve directed supremums and non empty infimums. **ScottDI** is a subcategory of $\mathbf{Top}_0$, again via the Scott topology ([12]).
(3) **SComp** denotes the subcategory of $\mathbf{Top}_0$ consisting of all stably compact spaces and stable continuous maps. Thus, the objects of **SComp** are those spaces which are sober, locally compact and whose family of all saturated compact sets is closed under finite intersection ([13], [21]). (A set A of a T_0 space X is saturated if it coincides with the upper set $A\uparrow$ for X equipped with the specialization order.) The morphisms of **SComp** are the *stable* continuous maps, that is, morphisms $f : X \to Y$ of $\mathbf{Top}_0$ such that for every pair of open sets U and V and a compact set K in Y such that $U \subseteq K \subseteq V$, there exists a compact K' in X such that $f^{-1}(X) \subseteq K' \subseteq f^{-1}(Y)$.

The KZ-reflectivity of the above three subcategories follows immediately from the fact that they are categories of algebras of KZ-monads (see Theorem 3.13). Concerning **ContI**, it is known from Day [5] and Wyler [22] that it coincides with the category of Einlenberg-Moore algebras of the filter monad,

(2) Similarly, in $\mathbf{Top}_0$, enriched with the usual order (that is, with the pointwise specialization order), every initial family of morphisms – and, in particular, every limit – is jointly order-monic, and, then, also jointly monic. As before, jointly surjective families – and, in particular, colimits – are jointly order-epic.
(3) Let **SLat** denote the category whose objects are meet-semilattices and whose morphisms are maps preserving meets of finite sets (including the meet 1 of the empty set). In this category, coequalizers are always surjective, then order-epic. On the other hand, recall that the injections of the coproduct of a family of objects A_i, $i \in I$, in $\mathbf{SLat}_0$, is given by $A_j \xrightarrow{\gamma_j} \Pi'_{i\in I}A_i$, where $\Pi'_{i\in I}A_i = \{(a_i)_{i\in I} \mid a_i \neq$ 1 only for a finite number of i's$\}$ and $\gamma_j(a) = (a_i)_{i\in I}$ with $a_j = a$ and $a_i = 1$ for all $i \neq j$ (see, for instance, [18]). It is easily seen that the family $(\gamma_i)_{i\in I}$ is jointly order-epic. Therefore, in **SLat**, colimits are jointly order-epic.
(4) In **Frm**, colimits are also jointly order-epic. For coequalizers it follows immediately, since they are surjective. In order to show that also coproducts are jointly order-epic, we recall briefly a description of them, whose details may be found, for instance, in [18] (see also [**?**]). It is well-know that the inclusion functor of **Frm** into **SLat** is a right adjoint and, for each object S of **SLat**, the universal morphism is given by $S \xrightarrow{\lambda_S} \mathcal{D}(S)$, where $\mathcal{D}(S)$ is the set of the lower subsets of S with the inclusion order, and $\lambda_S(s) =\downarrow s$ for each $s \in S$. Moreover, as it is going to be shown in the second example of Examples 3.5, this reflection is a KZ-reflection in the sense of Definition 3.1; by Theorem 3.4, this implies that the universal morphism λ_S sastifies the implication $g \cdot \lambda_S \leq f \cdot \lambda_S \Rightarrow f \leq g$ for every pair of morphisms $f, g : \mathcal{D}S \to L$ in **Frm**. The injections of the coproduct of a family of objects A_i, $i \in I$, in **Frm**, are of the form

$$A_j \xrightarrow{\gamma_j} \Pi'_{i\in I}A_i = S \xrightarrow{\lambda_S} \mathcal{D}(S) \xrightarrow{\nu} \mathcal{D}(S)/R$$

where the γ_j morphisms are the injections of the coproduct in **SLat** and ν is a certain onto frame homomorphism. Let now $f, g : \mathcal{D}(S)/R \to L$ be two morphisms in **Frm** such that

$$g \cdot (\nu \cdot \lambda_S \cdot \gamma_i) \leq f \cdot (\nu \cdot \lambda_S \cdot \gamma_i) \text{ for all } i \in I.$$

Then, as coproducts in **SLat** are jointly order-epic, we obtain $g \cdot (\nu \cdot \lambda_S) \leq f \cdot (\nu \cdot \lambda_S)$. From this inequality, since the reflection of **SLat** into **Frm** is KZ, we get $g \cdot \nu \leq f \cdot \nu$. Finally, being surjective, ν is order-epic, and it follows that $g \leq f$.

In the next proposition we collect some properties of the classes $\mathcal{A}^{\lrcorner}$ for $\mathcal{A}$ a subcategory. In particular we are going to see that these classes are stable under jointly order-epic pushouts and wide pushouts. Recall that a class $\mathcal{H}$ of morphisms is said to be *stable under pushouts* if when a pair of morphisms (f', g') is the pushout of (f, g) with $f \in \mathcal{H}$, also $f' \in \mathcal{H}$. And $\mathcal{H}$ is *stable under wide pushouts* provided that the wide pushout $X \xrightarrow{f} Y = (X \xrightarrow{f_i} X_i \xrightarrow{t_i} Y)$ of a family of morphisms $(X \xrightarrow{f_i} X_i)_{i \in I}$ belongs to $\mathcal{H}$ whenever all f_i do.

In the Proposition 2.10 we will see that subcategories of the form $\mathcal{A} = \mathcal{H}_{\lrcorner}$ are *closed under jointly order-monic limits*. That is, every jointly order-monic limit cone in $\mathcal{X}$ of a composition functor $I \xrightarrow{D} \mathcal{A} \xhookrightarrow{I} \mathcal{X}$, with I the inclusion of $\mathcal{A}$ into $\mathcal{X}$, is a limit cone in $\mathcal{A}$. This means that the limit cone is formed by morphisms of $\mathcal{A}$ and, moreover, that, every other cone of ID has the unique factorizing morphism in $\mathcal{A}$.

Proposition 2.9. *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. Then $\mathcal{A}^{\lrcorner}$ has the following properties:*

(1) $Iso(\mathcal{X}) \subseteq \mathcal{A}^{\lrcorner}$.
(2) *$\mathcal{A}^{\lrcorner}$ is closed under composition. Moreover, if $f : X \to Y$ and $g : Y \to Z$ belong to $\mathcal{A}^{\lrcorner}$ and $h : X \to A$ is a morphism with codomain in $\mathcal{A}$ then $h/(gf) = (h/f)/g$.*
(3) *If $X \xrightarrow{f} Y$ and $r, s : X \to A$ are morphisms such that $f \in \mathcal{A}^{\lrcorner}$ and A is an object of $\mathcal{A}$, then $r \leq s \implies r/f \leq s/f$.*
(4) *$\mathcal{A}^{\lrcorner}$ is stable under those pushouts and wide pushouts which are jointly order-epic.*

Proof 1. It is obvious. In particular, if $f : X \to Y$ is an isomorphism then, for every $g : X \to A$, it holds that $gf^{-1} = g/f$.

2. For morphisms f, g and h as in the statement 2, we have that:

$$((h/f)/g) \cdot (g \cdot f) = (((h/f)/g) \cdot g) \cdot f = (h/f) \cdot f = h.$$

and, on the other hand,

$$\begin{aligned} m_Y \cdot Tg &= r' \cdot m_B \cdot Tl' \cdot Tr' \cdot Tf \cdot (Tm_A \cdot T\eta_A) \cdot Tl, && \text{since } m_A\eta_A = 1_A \\ &\geq r' \cdot m_B \cdot Tl' \cdot T(r' \cdot f \cdot m_A) \cdot \eta_{TA} \cdot Tl, && \text{because } T\eta_A \geq \eta_{TA} \\ &= r' \cdot m_B \cdot Tl' \cdot \eta_Y \cdot r' \cdot f \cdot m_A \cdot Tl \\ &= m_Y \cdot \eta_Y \cdot r' \cdot f \cdot m_A \cdot Tl \\ &= r' \cdot f \cdot m_A \cdot Tl = g \cdot r \cdot m_A \cdot Tl = g \cdot m_X. \end{aligned}$$

Consequently, $m_Y Tg = g m_X$, i.e., $g \in \mathcal{X}^{\mathbb{T}}$.

Conversely, let $\mathcal{A}$ be a KZ-reflective subcategory of $\mathcal{X}$ closed under coreflective right adjoints, with U the inclusion functor and F the left adjoint of U. Then the corresponding monad $\mathbb{T}$, with $T = UF$, is of the Kock-Zöberlein type. In fact, T is locally monotone because F is so, and $\eta_{TX} = \eta_{UFX} = \eta_{FX} \leq F\eta_X = UF\eta_X$. It remains to show that the comparison functor

$$\mathcal{A} \xrightarrow{K} \mathcal{X}^{\mathbb{T}}$$

given by $KA = (A, \varepsilon_A)$ and $Kf = f$ for every object A and every morphism f of $\mathcal{A}$ (where ε is the counit of the adjunction) is an isomorphism of categories. K is clearly injective on objects and on morphisms. Let (A, m_A) be a $\mathbb{T}$-algebra. Thus, as mentioned in Remark 3.12, $\eta_A \dashv_C m_A$. Since $\mathcal{A}$ is closed under coreflective right-adjoints, this assures that $A \in \mathcal{A}$. But then, from Remark 3.8, we know that $\eta_A \dashv_C \varepsilon_A$. Hence, $m_A = \varepsilon_A$ and $K(A) = (A, m_A)$. Finally, given a morphism $KA = (A, m_A) \xrightarrow{g} (B, m_B) = KB$ in $\mathcal{X}^{\mathbb{T}}$, again the fact that $\mathcal{A}$ is closed under coreflective right adjoints, combined with the equality $gm_A = m_B Fg$, implies that $g \in \mathcal{A}$. $\square$

Remark 3.14. Let $\mathcal{A}$ be a KZ-reflective subcategory of $\mathcal{X}$ closed under coreflective right adjoints, let U and F be the corresponding inclusion and reflector functors, respectively, and let $T = UF$. Then T is the endofunctor part of a KZ-monad and $\mathcal{A}$ is the category of algebras for that monad. It is clear, from the Definition 3.6, that every F-embedding is a T-embedding. On the other hand, as mentioned in Remark 3.12, the T-algebras, that is, the objects of $\mathcal{A}$, are precisely those objects of $\mathcal{X}$ which are right-Kan injective w.r.t. T-embeddings. From 4.3.4 of [7] it also follows that the T-morphisms (in our case, the morphisms of $\mathcal{A}$) are right-Kan injective w.r.t. T-embeddings. But, by Proposition 3.7, the largest class of morphisms w.r.t. which all objects and morphisms of $\mathcal{A}$ are right-Kan-injective are the F-embeddings. Thus, every

Remark 3.12. Recall from [7] and [9] that a *Kock-Zöberlein monad* (shortly, *KZ-monad*) on a poset enriched category $\mathcal{X}$ is a monad $\mathbb{T} = (T, \eta, \mu) : \mathcal{X} \to \mathcal{X}$ such that T is locally monotone and $\eta_{TX} \leq T\eta_X$ for all objects X. A KZ-monad is a special case of the notion of Kock-Zöberlein doctrine introduced by Anders Kock in [15]. In [7] Martín H. Escardó has observed that, for each object X of $\mathcal{X}$ there is at most one T-algebra structure map $m_X : TX \to X$ associated to X, and that, if it is the case, $\eta_X \dashv_C m_X$. So we can identify each T-algebra with its underlying object. Moreover, Escardó has shown that the Eilenberg-Moore algebras of a KZ-monad are precisely the objects of $\mathcal{X}$ which are injective w.r.t. all T-embeddings (see Definition 3.6), and that they also coincide with those objects of $\mathcal{X}$ which are right-Kan injective w.r.t. T-embeddings.

The next theorem establishes the relationship between KZ-reflective subcategories and KZ-monads.

Theorem 3.13. *The KZ-reflective subcategories of $\mathcal{X}$ closed under coreflective right adjoints coincide, up to isomorphism of categories, with the categories of $\mathbb{T}$-algebras for $\mathbb{T}$ a KZ-monad over $\mathcal{X}$.*

Proof It is clear that if $\mathbb{T} = (T, \eta, \mu) : \mathcal{X} \to \mathcal{X}$ is a KZ-monad (see remark above) then $\mathcal{X}^{\mathbb{T}}$ is a KZ-reflective subcategory of $\mathcal{X}$. Next we show that, furthermore, $\mathcal{X}^{\mathbb{T}}$ is closed under coreflective right adjoints in the sense of 2.11. Consider the commutative diagram (2.4) of the Definition 2.11 , with $f \in \mathcal{X}^{\mathbb{T}}$, $l \dashv_C r$ and $l' \dashv_C r'$, and let m_A and m_B be the corresponding structure maps of A and B. Thus, r and r' are retractions with right inverses l and l', respectively; then, it easily follows that X and Y belong to $\mathcal{X}^{\mathbb{T}}$ with $m_X = rm_ATl$ and $m_Y = r'm_BTl'$ (see [6]). It remains to show that g is a $\mathbb{T}$-morphism, that is, that $m_YTg = gm_X$. First, observe that

$$m_Y \cdot Tg = (r' \cdot m_B \cdot Tl') \cdot Tg \cdot (Tr \cdot Tl) = r' \cdot m_B \cdot Tl' \cdot Tr' \cdot Tf \cdot Tl. \quad (3.2)$$

Now, departing from the equality (3.2), we get that, on one hand,

$$\begin{aligned} m_Y \cdot Tg &\leq r' \cdot m_B \cdot Tf \cdot Tl, && \text{because } l'r' \leq 1_B \text{ and } T \text{ is locally monotone} \\ &= r' \cdot f \cdot m_A \cdot Tl, && \text{since } f \in \mathcal{X}^{\mathbb{T}} \\ &= g \cdot r \cdot m_A \cdot Tl = g \cdot m_X; \end{aligned}$$

Moreover, if $j : Z \to A$ is such that $j(gf) \leq h$, then $jg \leq h/f$, and, consequently, $j \leq (h/f)/g$. Hence, $(h/f)/g = h/(gf)$. Now, taking into account this property of the composition, it is immediate that also every morphism of $\mathcal{A}$ is right-Kan injective w.r.t. gf.

3. It is immediate from Definition 2.1.

4. Let $f : X \to Y$ belong to $\mathcal{A}^{\lrcorner}$ and let the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ Z \arrow[r, "f'"'] & W \end{tikzcd}$$

represent a pushout such that the pair (f', g') is jointly order-epic. We want to show that, then, $f' \in \mathcal{A}^{\lrcorner}$. Given $h : Z \to A$, with $A \in \mathcal{A}$, we have

$$(h \cdot g)/f \cdot f = h \cdot g.$$

Then there exists $t : W \to A$ such that $t \cdot f' = h$ and $t \cdot g' = (h \cdot g)/f$. We show that

$$t = h/f'. \quad (2.1)$$

Let $k : W \to A$ be such that $k \cdot f' \leq h$. Then

$$k \cdot f' \leq t \cdot f'. \quad (2.2)$$

On the other hand, from the following implications

$$\begin{aligned} k \cdot f' \leq h &\Rightarrow k \cdot f' \cdot g \leq h \cdot g \\ &\Rightarrow k \cdot g' \cdot f \leq h \cdot g \\ &\Rightarrow k \cdot g' \leq (h \cdot g)/f = t \cdot g'. \end{aligned}$$

we obtain that

$$k \cdot g' \leq t \cdot g'. \quad (2.3)$$

Since the pushout (f', g') is jointly order-epic, (2.2) and (2.3) implies that $k \leq t$. Thus every object of $\mathcal{A}$ is right-Kan injective w.r.t. f'.

Concerning the right-Kan-injectivity of the morphisms of $\mathcal{A}$ w.r.t. f', let $a : A \to B$ be a morphism of $\mathcal{A}$ and $h : Z \to A$. Put $t = h/f'$, as in (2.1),

and $u = (a \cdot h)/f'$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\
Z \arrow[r, "f'"] \arrow[d, "h"'] & W \arrow[d, "u"] \arrow[dl, "t"] \\
A \arrow[r, "a"'] & B
\end{tikzcd}$$

We know that, as it holds for t in (2.1), u is the morphism such that $u \cdot f' = a \cdot h$ and $u \cdot g' = (a \cdot h \cdot g)/f$. We want to show that $at = u$. On one hand, $(at)f' = ah = uf'$; on the other hand, $(at)g' = a(hg)/f = (ahg)/f = ug'$, by using the fact that $f \in \mathcal{A}^{\lrcorner}$. Consequently, by the pushout universal property, $at = u$, i.e., $a \cdot (h/f') = (a \cdot h)/f'$.

The proof that $\mathcal{A}^{\lrcorner}$ is stable under jointly order-epic wide pushouts uses a technique similar to the one used for pushouts: Let $\left(X \xrightarrow{f_i} X_i\right)_{i \in I}$ be a family of morphisms in $\mathcal{A}^{\lrcorner}$ and let the diagram

$$\begin{tikzcd}
X \arrow[r, "f_i"] \arrow[dr, "f"'] & X_i \arrow[d, "t_i"] \\
 & Y
\end{tikzcd}$$

represent a jointly order-epic wide pushout. Given a morphism $g : X \to A$, with $A \in \mathcal{A}$, let $t : Y \to A$ be the unique morphism such that $tf = g$ and $tt_i = g/f_i$, for all $i \in I$. It is easy to see that $t = g/f$, thus A is right-Kan-injective w.r.t. f. Furthermore, given a morphism $a : A \to B$ in $\mathcal{A}$, analogously to the case of pushouts, we obtain that $at = (ag)/f$, i.e., $a(g/f) = (ag)/f$. Therefore $f \in \mathcal{A}^{\lrcorner}$. □

Proposition 2.10. *Every subcategory of the form $\mathcal{A} = \mathcal{H}_{\lrcorner}$, for $\mathcal{H} \subseteq Mor(\mathcal{X})$, is closed under jointly order-monic limits.*

Proof Let $(l_i : X \to A_i)_{i \in I}$ be a jointly order-monic limit cone in $\mathcal{X}$ with all connecting morphisms $m : A_i \to A_j$ in $\mathcal{A}$.

coreflective right adjoint of η_X. Since $\mathcal{A}$ is closed under coreflective right adjoints, $X \in \mathrm{Obj}(\mathcal{A})$ and $x \in \mathrm{Mor}(\mathcal{A})$ (see Remark 2.12).

Let now $f : X \to Y$ belong to $(\mathcal{A}^{\lrcorner})_{\lrcorner}$. As we have just seen, the objects X and Y belong to $\mathcal{A}$, and the morphisms $x = 1_X/\eta_X$ and $y = 1_Y/\eta_Y$ are coreflective right adjoints and also belong to $\mathcal{A}$. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"'] \arrow[rr, bend left, "1_X"] \arrow[d, "f"'] & FX \arrow[r, "x"'] \arrow[d, "Ff"] & X \arrow[d, "f"] \\
Y \arrow[r, "\eta_Y"] \arrow[rr, bend right, "1_Y"'] & FY \arrow[r, "y"] & Y
\end{tikzcd}$$

We show that the right square is commutative:

$$\begin{aligned}
y \cdot Ff &= y \cdot ((\eta_Y \cdot f)/\eta_X), && \text{by Theorem 3.4} \\
&= (y \cdot \eta_Y \cdot f)/\eta_X, && \text{since } y \in \mathcal{A} \text{ and } \eta_X \in \mathcal{A}^{\lrcorner} \\
&= f/\eta_X, && \text{since } y \cdot \eta_Y = 1_Y \\
&= (f \cdot 1_X)/\eta_X && \\
&= f \cdot (1_X/\eta_X), && \text{since } f \in \mathcal{A} \text{ and } \eta_X \in \mathcal{A}^{\lrcorner} \\
&= f \cdot x
\end{aligned}$$

Since $\mathcal{A}$ is closed under coreflective right adjoints, we conclude that $f \in \mathrm{Mor}(\mathcal{A})$.

Now, from Proposition 2.10, it turns out that $\mathcal{A}$ is closed under jointly order-monic limits. □

Remark 3.10. Let $\mathcal{A}$ be a KZ-reflective subcategory closed under coreflective right adjoints. From the proof of 3.9, it follows that $\mathcal{A} = \{\eta_X,\ X \in \mathcal{X}\}_{\lrcorner}$. Consequently, taking into account Remark 2.5, if the reflections are all epimorphisms then the subcategory $\mathcal{A}$ is full.

Remark 3.11. From the above theorem, we know that in several everyday poset enriched categories, where limits are jointly order-monic (see 2.8), KZ-reflective subcategories which are closed under coreflective right-adjoints are closed under all limits.

Let $\mathcal{X}$ be an arbitrary category enriched with the trivial order $(=)$. In this case, the above theorem states the well-known fact that every full and isomorphism-closed reflective subcategory $\mathcal{A}$ of $\mathcal{X}$ coincides with its orthogonal hull $(\mathcal{A}^{\perp})_{\perp}$ and is closed under limits.

Let now $k : Y \to A$ be such that $k \cdot f \leq g$, and let $\overline{k} : FY \to A$ be the $\mathcal{A}$-morphism which fulfils the equality $\overline{k} \cdot \eta_Y = k$. We show that then $k \leq \varepsilon_A \cdot Fg \cdot l \cdot \eta_Y$:

$$\begin{array}{lll} k \cdot f \leq g & \Rightarrow k \cdot f \leq \varepsilon_A \cdot \eta_A \cdot g & \\ & \Rightarrow \overline{k} \cdot \eta_Y \cdot f \leq \varepsilon_A \cdot Fg \cdot \eta_X & \\ & \Rightarrow \overline{k} \cdot Ff \cdot \eta_X \leq \varepsilon_A \cdot Fg \cdot \eta_X & \\ & \Rightarrow \overline{k} \cdot Ff \leq \varepsilon_A \cdot Fg, & \text{by (ii) of Theorem 3.4} \\ & \Rightarrow \overline{k} \leq \overline{k} \cdot Ff \cdot l \leq \varepsilon_A \cdot Fg \cdot l, & \text{because } l \text{ is a left-adjoint of } Ff \\ & \Rightarrow k = \overline{k} \cdot \eta_Y \leq \varepsilon_A \cdot Fg \cdot l \cdot \eta_Y. & \end{array}$$

It remains to show that the morphisms of $\mathcal{A}$ are also right-Kan injective w.r.t. $f : X \to Y$. Let $b : A \to B$ be a morphism of $\mathcal{A}$. Then

$$\begin{array}{lll} b \cdot (g/f) & = b \cdot \varepsilon_A \cdot Fg \cdot l \cdot n_Y, & \text{by (3.1)} \\ & = \varepsilon_B \cdot Fb \cdot Fg \cdot l \cdot n_Y = (b \cdot g)/f, & \text{again by (3.1).} \end{array}$$

We have just characterized the class $\mathcal{A}^{\lrcorner}$ for $\mathcal{A}$ a KZ-reflective subcategory. We are going to see that, whenever $\mathcal{A}$ is KZ-reflective and closed under coreflective right adjoints, then it coincides with its *right-Kan injective hull* $(\mathcal{A}^{\lrcorner})_{\lrcorner}$.

Remark 3.8. If $\mathcal{A}$ is a KZ-reflective subcategory and $A \in \mathcal{A}$, then the reflection morphism η_A is a coreflective left-adjoint. Indeed, we know that $1_{FA} = \eta_A/\eta_A$, from Theorem 3.4, and $(\eta_A \cdot \varepsilon_A) \cdot \eta_A = \eta_A$. Consequently $\eta_A \cdot \varepsilon_A \leq 1_{FA}$. Since $\varepsilon_A \cdot \eta_A = 1_A$, it follows that ε_A is a coreflective right-adjoint of η_A.

Theorem 3.9. If $\mathcal{A}$ is a KZ-reflective subcategory closed under coreflective right adjoints then

$$\mathcal{A} = (\mathcal{A}^{\lrcorner})_{\lrcorner}$$

and, consequently, $\mathcal{A}$ is closed under jointly order-monic limits.

Proof Of course, $\mathcal{A} \subseteq (\mathcal{A}^{\lrcorner})_{\lrcorner}$, so we need just to prove the converse inclusion. Denote by F the corresponding left adjoint functor from $\mathcal{X}$ to $\mathcal{A}$. We know, from (i) of Theorem 3.4, that $\{\eta_X,\, X \in \mathcal{X}\} \subseteq \mathcal{A}^{\lrcorner}$.

In order to prove the inclusion $(\mathcal{A}^{\lrcorner})_{\lrcorner} \subseteq \mathcal{A}$ for objects, let $X \in (\mathcal{A}^{\lrcorner})_{\lrcorner}$. Then, since $\eta_X \in \mathcal{A}^{\lrcorner}$, there exists a morphism $x = 1_X/\eta_X : FX \to X$ such that $x \cdot \eta_X = 1_X$. Moreover, the equality $(\eta_X \cdot x) \cdot \eta_X = \eta_X$ assures that $\eta_X \cdot x \leq \eta_X/\eta_X = 1_{FX}$, by Theorem 3.4. Thus $\eta_X \dashv_C x$, i.e., x is a

(1) First we show that $X \in \mathcal{A}$. In fact, given $h : Z \to W$ in $\mathcal{H}$, and $g : Z \to X$,

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "g"'] & W \arrow[d, "(l_i g)/h"] \\ X \arrow[r, "l_i"] & A_i \end{tikzcd}$$

the family

$$((l_i \cdot g)/h : W \to A_i)_{i \in I}$$

forms a cone, since, for every connecting morphism $m : A_i \to A_j$ we have that:

$$m \cdot (l_i \cdot g)/h = (m \cdot l_i \cdot g)/h = (l_j \cdot g)/h.$$

Hence, there is a unique morphism $\overline{g} : W \to X$ such that $l_i \cdot \overline{g} = (l_i \cdot g)/h$ $(i \in I)$. The equalities $l_i \overline{g} h = ((l_i g)/h)h = l_i g$ imply that $\overline{g} h = g$. Furthermore, if $k : W \to X$ is a morphism such that $kh \leq g$, we obtain $l_i kh \leq l_i g$, so $l_i k \leq (l_i g)/h = l_i \overline{g}$; consequently, since the given limit is jointly order-monic, $k \leq \overline{g}$. Thus, $\overline{g} = g/h$.

(2) The projections l_i belong to $\mathcal{A}$. This is immediate from (1) where we saw that, for each morphism $g : Z \to X$, $l_i \cdot (g/h) = l_i \cdot \overline{g} = (l_i g)/h$.

(3) In order to conclude that the cone $(l_i : X \to A_i)_{i \in I}$ is a limit in $\mathcal{A}$, let $(d_i : B \to A_i)_{i \in I}$ be a cone in $\mathcal{A}$ for the given diagram. Then there is a unique morphism $b : B \to X$ in $\mathcal{X}$ satisfying the equalities $l_i b = d_i$. It remains to show that b belongs to $\mathcal{A}$. Let $h : Z \to W$ be a morphism of $\mathcal{H}$, and consider a morphism $t : Z \to B$:

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "t"'] & W \arrow[d, "(bt)/h"] \arrow[dl, "t/h"] \\ B \arrow[r, "b"'] \arrow[dr, "d_i"'] & X \arrow[d, "l_i"] \\ & A_i \end{tikzcd}$$

Then, for every $i \in I$,

$$\begin{array}{lll} l_i b(t/h) & = d_i(t/h) & \\ & = (d_i t)/h, & \text{because } d_i \text{ belongs to } \mathcal{A} \\ & = (l_i bt)/h & \\ & = l_i((bt)/h), & \text{since, by (2), } l_i \text{ belongs to } \mathcal{A}. \end{array}$$

Therefore, $b(t/h) = (bt)/h$, i.e., $b \in \mathcal{A}$. □

We finish this section by showing that the subcategories of the fom $\mathcal{H}_{\lrcorner}$ are closed under a certain kind of retracts. This will be useful in the following.

Definition 2.11. A subcategory $\mathcal{A}$ of the category $\mathcal{X}$ is said to be *closed under corefletcive right adjoints* if, whenever $r : A \to X$ and $r' : B \to Y$ are coreflective right adjoint morphisms, $f : A \to B$ is a morphism of $\mathcal{A}$ and $g : X \to Y$ is a morphism which makes the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "r"'] & B \arrow[d, "r'"] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \tag{2.4}$$

commutative, then g is a morphism of $\mathcal{A}$.

Remark 2.12. If $\mathcal{A}$ is a subcategory closed under coreflective right adjoints, then every coreflective right adjoint morphism $r : A \to X$ with domain in $\mathcal{A}$ belongs to $\mathcal{A}$: in the above diagram, put $f := 1_A$, $r := 1_A$, $r' := r$ and $g := r$.

Proposition 2.13. *For every $\mathcal{H} \subseteq Mor(\mathcal{X})$, the subcategory $\mathcal{H}_{\lrcorner}$ is* closed under coreflective right adjoints.

Proof Put $\mathcal{A} = \mathcal{H}_{\lrcorner}$ and consider the commutative diagram (2.4) with $f \in \mathcal{A}$, and r and r' coreflective right adjoints morphisms. Let $l : X \to A$ be the left adjoint of $r : A \to X$, so $r \cdot l = 1_X$ and $l \cdot r \leq 1_A$. First we show that if $A \in \mathcal{H}_{\lrcorner}$ also $X \in \mathcal{H}_{\lrcorner}$. Let $j : Z \to W$ belong to $\mathcal{H}$. Given $a : Z \to X$, it is easy to see that

$$a/j = r \cdot ((l \cdot a)/j). \tag{2.5}$$

In fact we have that $(r \cdot ((l \cdot a)/j)) \cdot j = r \cdot l \cdot a = a$; and, moreover, $k \cdot j \leq a \Rightarrow (l \cdot k) \cdot j \leq l \cdot a \Rightarrow l \cdot k \leq (l \cdot a)/j \Rightarrow r \cdot l \cdot k \leq r \cdot ((l \cdot a)/j) \Rightarrow k \leq r \cdot ((l \cdot a)/j)$. Consequently, X belongs to $\mathcal{A}$ and the same happpens to Y.

Now, we show that r is right-Kan-injective w.r.t. $\mathcal{H}$. Given $j : Z \to W$ in $\mathcal{H}$, consider a morphism $d : Z \to A$. The inequaltity $r \cdot (d/j) \leq (r \cdot d)/j$ holds by definition of $(r \cdot d)/j$. Conversely,

$$\begin{aligned} (r \cdot d)/j &= r \cdot ((l \cdot r \cdot d)/j), && \text{using the property (2.5)} \\ &\leq r \cdot (d/j), && \text{by 3 of 2.9, since } l \cdot r \leq 1_A. \end{aligned}$$

Thus, r (and, analogously, r') belongs to $\mathcal{H}_{\lrcorner}$.

Definition 3.6. Let $F : \mathcal{X} \to \mathcal{A}$ be a locally monotone functor between poset enriched categories. A morphism f of $\mathcal{X}$ is said to be an F*-embedding* if Ff has a reflective left-adjoint morphism in $\mathcal{A}$.

Proposition 3.7. *If $\mathcal{A}$ is a KZ-reflective subcategory of $\mathcal{X}$, with left adjoint F, then $\mathcal{A}^{\lrcorner}$ is just the class of all F-embeddings.*

Proof

Let $f : X \to Y$ belong to $\mathcal{A}^{\lrcorner}$. Consider the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "\eta_X"'] & & Y \arrow[d, "\eta_Y"] \arrow[lld, "a"'] \\ FX \arrow[rr, "Ff"'] & & FY \arrow[dl, "Fa"] \\ & F^2X \arrow[ul, "\varepsilon_{FX}"] & \end{tikzcd}$$

where $a = \eta_X/f$. We show that $\varepsilon_{FX} \cdot Fa$ is a reflective left-adjoint of Ff, i.e., that

$$(\varepsilon_{FX} \cdot Fa) \cdot Ff = 1_{FX} \text{ and } 1_{FY} \leq Ff \cdot (\varepsilon_{FX} \cdot Fa).$$

The equality is clear:

$$(\varepsilon_{FX} \cdot Fa) \cdot Ff = \varepsilon_{FX} \cdot F(a \cdot f) = \varepsilon_{FX} \cdot F\eta_X = 1_{FX}.$$

Concerning the inequality, first we observe that, since $f \in \mathcal{A}^{\lrcorner}$, $(Ff \cdot \eta_X)/f = Ff(\eta_X/f) = Ff \cdot a$; now, since $\eta_Y \cdot f = Ff \cdot \eta_X$, by the definition of $(Ff \cdot \eta_X)/f$, we have that $\eta_Y \leq Ff \cdot a$, and, consequently,

$$1_{FY} \cdot \eta_Y \leq Ff \cdot a = Ff \cdot \varepsilon_{FX} \cdot \eta_{FX} \cdot a = Ff \cdot \varepsilon_{FX} \cdot Fa \cdot \eta_Y.$$

Then, by (ii) of Theorem 3.4, we obtain $1_{FY} \leq Ff \cdot \varepsilon_{FX} \cdot Fa$.

Conversely, let $f : X \to Y$ be such that Ff has the morphism $l : FY \to FX$ as a reflective left-adjoint, that is, $l \cdot Ff = 1_{FX}$ and $1_{FY} \leq Ff \cdot l$. We want to show that $f \in \mathcal{A}^{\lrcorner}$.

Let $g : X \to A$ be a morphism with codomain in $\mathcal{A}$. We are going to see that g/f exists and is given by

$$g/f = \varepsilon_A \cdot Fg \cdot l \cdot \eta_Y. \tag{3.1}$$

In fact,

$$(\varepsilon_A \cdot Fg \cdot l \cdot \eta_Y) \cdot f = \varepsilon_A \cdot Fg \cdot l \cdot Ff \cdot \eta_X = \varepsilon_A \cdot Fg \cdot \eta_X = \varepsilon_A \cdot \eta_A \cdot g = g.$$

straight computation shows that the morphism η_X given by the embedding of X into $\overline{X}$ fulfils (i) and (ii). Moreover, for every $f : X \to A$ with codomain in $\mathcal{A}$, the morphism f/η_X is defined by $(f/\eta_X)(x) = x$, for $x \in X$, and $(f/\eta_X)(C)$ is the supremum of the connected component containing $f[C]$, for every connected component C of X.

(2) Let **SLat** be the category of meet-semilattices (see 2.8) enriched with the pointwise dual order $\geq$. Then **Frm** is a KZ-reflective subcategory of **SLat**. Indeed, it is well-known that **Frm** is reflective in **SLat**, with the reflection of every $S \in$ **SLat** done by $\lambda_S : S \to \mathcal{D}(S)$, where $\mathcal{D}(S) = (\{U \subseteq S \,|\, U = \downarrow U\}, \subseteq)$ and $\lambda_S(s) = \downarrow s$ for each $s \in S$. Moreover, if $g : S \to L$ is a morphism of **SLat** with codomain in **Frm**, the unique morphism $\overline{g} : \mathcal{D}(S) \to L$ of **Frm** such that $\overline{g}\lambda_S = g$ is defined by $\overline{g}(U) = \sup(g[U])$. To conclude that the reflection is KZ, we show that λ_S verifies the conditions of 3.4 (for the ordering $\geq$). We begin by (ii).

Let then $k, f : \mathcal{D}(S) \to L$ be morphisms such that f is a frame homomorphism and $k\lambda_S \geq f\lambda_S$. Using the symbol $\vee$ to denote the supremum, for every $U \in \mathcal{D}(S)$, we have that

$$k(U) = k\left(\bigcup_{u\in U}(\downarrow u)\right) \geq \bigvee_{u\in U} k(\downarrow u) = \bigvee_{u\in U} k\lambda_S(u)$$

$$\geq \bigvee_{u\in U} f\lambda_S(u) = \bigvee_{u\in U} f(\downarrow u) = f(\bigcup_{u\in U} \downarrow u) = f(U),$$

the last but one equality holding because f belongs to **Frm**.

Moreover, **Frm** is right-Kan injective w.r.t. every λ_S. Indeed, if $g : S \to L$ is a morphism with codomain in **Frm**, taking into account that $\overline{g}\lambda_S = g$ with $\overline{g} \in$ **Frm**, and the property (ii), we conclude that $\overline{g} = g/\lambda_S$. The right-Kan injectivity of the morphisms of **Frm** is also easy: if $f : L \to M$ is a frame homomorphism, then we have that $f(\bigvee g[U]) = \bigvee fg[U]$, that is, $f(g/\lambda_S)(U) = (fg/\lambda_S)(U)$.

Other examples of KZ-reflective subcategories will be described in Section 4.

Next we go further on the relationship between KZ-reflectivity and right-Kan-injectivity.

Finally, we show that g is right-Kan injective w.r.t. $\mathcal{H}$. Given $u : Z \to X$, we have:

$$\begin{aligned}
g\cdot(u/j) &= g\cdot(r\cdot((l\cdot u)/j)), && \text{by (2.5)}\\
&= g\cdot r\cdot((l\cdot u)/j), && \text{because } r \in \mathcal{H}_{\lrcorner}\\
&= r'\cdot f\cdot((l\cdot u)/j)\\
&= (r'\cdot f\cdot l\cdot u)/j, && \text{because } r'\cdot f \in \mathrm{Mor}(\mathcal{H}_{\lrcorner})\\
&= (g\cdot r\cdot l\cdot u)/j\\
&= (g\cdot u)/j.
\end{aligned}$$

□

Remark 2.14. From the proof of 2.13 it follows that, moreover, in what concerns objects, $\mathcal{H}_{\lrcorner}$ is closed under arbitrary retracts, that is, if $r : A \to X$ is a retract with A in $\mathcal{H}_{\lrcorner}$ then X belongs to $\mathcal{H}_{\lrcorner}$ too.

3. KZ-reflective subcategories

A functor $F : \mathcal{X} \to \mathcal{Y}$ between poset enriched categories is said to be *locally monotone* if, for all morphisms f and g with common domain and codomain, $f \leq g$ implies $Ff \leq Fg$.

Definition 3.1. A subcategory $\mathcal{A}$ of $\mathcal{X}$ is said to be *KZ-reflective* in $\mathcal{X}$ provided that the inclusion of $\mathcal{A}$ in $\mathcal{X}$ has a left-adjoint F such that:

(1) F is locally monotone;
(2) $\eta_{FX} \leq F\eta_X$, for all objects X of $\mathcal{X}$.

Remark 3.2. Let $\mathcal{X}$ be an arbitrary category and consider it enriched with the trivial order, i.e., equality. Then for subcategories $\mathcal{A}$ of $\mathcal{X}$ (closed under isomorphisms) to be KZ-reflective just means to be reflective and full. Indeed, in this case, the equality $\eta_F = F\eta$ implies that, for every object A of $\mathcal{A}$, η_A is an isomorphism, since, together with $\varepsilon_A\eta_A = 1_A$ (for ε the counit) we have that $\eta_A\varepsilon_A = F\varepsilon_A\cdot\eta_{FA} = F\varepsilon_A F\eta_A = F(\varepsilon_A\eta_A) = 1_{FA}$. Now, given a morphism $f : A \to B$ with A and B in $\mathcal{A}$, $f = \eta_B^{-1}\cdot Ff\cdot\eta_A$, thus it belongs to $\mathcal{A}$.

Remark 3.3. Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{X}$ with left adjoint F, unit η and counit ε. Then the condition 2 of Definition 3.1 is equivalent to

$2'$. $\varepsilon_{FA} \geq F\varepsilon_A$, for all objects A of $\mathcal{A}$.

In fact, given 2., we have that $F\varepsilon_A = F\varepsilon_A\cdot\varepsilon_{F^2A}\cdot\eta_{F^2A} \leq F\varepsilon_A\cdot\varepsilon_{F^2A}\cdot F^2\eta_A = F\varepsilon_A\cdot F\eta_A\cdot\varepsilon_{FA} = \varepsilon_{FA}$. The other way round is dual.

The next theorem provides a characterization of the KZ-reflective subcategories of $\mathcal{X}$.

Theorem 3.4. *A subcategory $\mathcal{A}$ of $\mathcal{X}$ is KZ-reflective if and only if, for every object X of $\mathcal{X}$, there is an object $\overline{X}$ in $\mathcal{A}$ and a morphism $\eta_X : X \to \overline{X}$ such that*

(i) η_X belongs to $\mathcal{A}^{\lrcorner}$ and, for every morphism $g : X \to A$ with $A \in \mathcal{A}$, g/η_X belongs to $\mathcal{A}$;

(ii) for every $f : \overline{X} \to A$ in $\mathcal{A}$ and $g : \overline{X} \to A$ in $\mathcal{X}$, if $g\eta_X \leq f\eta_X$ then $g \leq f$.

In that case, the corresponding reflector is given on objects by $FX = \overline{X}$, and on morphisms by $F(X \xrightarrow{f} Y) = (\eta_Y f)/\eta_X$.

Proof Let $\mathcal{A}$ be a KZ-reflective subcategory of $\mathcal{X}$, with left adjoint F and unit η. We show that, then, (i) and (ii) are fulfilled with $FX = \overline{X}$. First we observe that the property

(ii)$'$ $g\eta_X \leq f\eta_X \;\Rightarrow\; g \leq f$, for every pair of morphisms $f, g : FX \to A$ of $\mathcal{A}$,

which is a weaker version of (ii), follows from the following obvious implications:

$$\begin{aligned} g \cdot \eta_X \leq f \cdot \eta_X &\Rightarrow \varepsilon_A \cdot Fg \cdot F\eta_X \leq \varepsilon_A \cdot Ff \cdot F\eta_X \\ &\Rightarrow g \cdot \varepsilon_{FX} \cdot F\eta_X \leq f \cdot \varepsilon_{FX} \cdot F\eta_X \\ &\Rightarrow g \leq f. \end{aligned}$$

(i) For $X \in \mathrm{Obj}(\mathcal{X})$, $A \in \mathrm{Obj}(\mathcal{A})$ and $g : X \to A$, let $\overline{g}$ be the unique $\mathcal{A}$-morphism such that $g = \overline{g} \cdot \eta_X$. We want to show that $\overline{g} = \bigvee\{ FX \xrightarrow{t} A : t \cdot \eta_X \leq g\}$. For morphisms $t : FX \to A$ and $\overline{t} : F^2X \to A$ in $\mathcal{X}$ and $\mathcal{A}$, respectively, such that $t \cdot \eta_X \leq g$ and $\overline{t} \cdot \eta_{FX} = t$, we have:

$$\begin{aligned} (\overline{t} \cdot F\eta_X) \cdot \eta_X &= \overline{t} \cdot (F\eta_X \cdot \eta_X) \\ &= \overline{t} \cdot (\eta_{FX} \cdot \eta_X) \\ &= t \cdot \eta_X \leq g = \overline{g} \cdot \eta_X \end{aligned}$$

i.e.,

$$(\overline{t} \cdot F\eta_X) \cdot \eta_X \leq \overline{g} \cdot \eta_X.$$

Consequently, by (ii)$'$, $\overline{t} \cdot F\eta_X \leq \overline{g}$. Hence, $t = \overline{t} \cdot \eta_{FX} \leq \overline{t} \cdot F\eta_X \leq \overline{g}$.

To show the right-Kan injectivity of η_X w.r.t. the morphisms of $\mathcal{A}$, consider morphisms $g : X \to A$ and $f : A \to B$, with f in $\mathcal{A}$. We have, from above, and using the same notation, that

$$f(g/\eta_X)\eta_X = f \cdot \overline{g} \cdot \eta_X = fg = \overline{fg} \cdot \eta_X = ((fg)/\eta_X)\eta_X$$

with both morphisms $f(g/\eta_X)$ and $(fg)/\eta_X$ in $\mathcal{A}$; by (ii)$'$, this implies that $f(g/\eta_X) = (fg)/\eta_X$.

(ii) It is immediate from (i): Let f and g be under the assumed conditions. Then, the equality $f\eta_X = ((f\eta_X)/\eta_X) \cdot \eta_X$, with f and $(f\eta_X)/\eta_X$ belonging to $\mathcal{A}$, implies $f = (f\eta_X)/\eta_X$. Then, by definition of $(f\eta_X)/\eta_X$, we get the inequality $g \leq f$.

Concerning the sufficiency, define $F : \mathcal{X} \to \mathcal{A}$ by $FX = \overline{X}$ and $F(X \xrightarrow{f} Y) = (\eta_Y f)/\eta_X$. It is easy to see that F is a functor:

(a) For every X, $F1_X = (\eta_X 1_X)/\eta_X = \eta_X/\eta_X$, by definition. But $1_{\overline{X}}\eta_X = \eta_X$ and, moreover, for every $g : \overline{X} \to \overline{X}$ such that $g\eta_X \leq 1_{\overline{X}}\eta_X$, it holds that $g \leq 1_{\overline{X}}$, by (ii). Consequently, $1_{\overline{X}} = \eta_X/\eta_X$ and, thus, $1_{FX} = F1_X$.

(b) Given a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, taking into account that the morphism $((\eta_Z g)/\eta_Y)$ belongs to $\mathcal{A}$, we have:

$$FgFf = ((\eta_Z g)/\eta_Y)\cdot((\eta_Y f)/\eta_X) = (((\eta_Z g)/\eta_Y)\cdot\eta_Y)\cdot f)/\eta_X = (\eta_Z gf)/\eta_X = F(gf).$$

The local monotonicity of F follows from (ii): Given $f, g : X \to Y$ such that $f \leq g$, we have that $Ff \cdot \eta_X = \eta_Y f \leq \eta_Y g = Fg \cdot \eta_X$, and, as Fg belongs to $\mathcal{A}$, this implies that $Ff \leq Fg$.

In order to conclude that F is a left adjoint of the inclusion functor of $\mathcal{A}$ into $\mathcal{X}$, it suffices to show that, for every $f : X \to A$, with A in $\mathcal{A}$, f/η_X is the unique morphism in $\mathcal{A}$ such that $(f/\eta_X) \cdot \eta_X = f$. Let $f' : FX = \overline{X} \to A$ be an $\mathcal{A}$-morphism such that $f'\eta_X = f$. Then, by (ii), we have simultaneously that $f' \leq f/\eta_X$ and $f/\eta_X \leq f'$, so $f' = f/\eta_X$.

The inequality $\eta_{FX} \leq F\eta_X$ follows immediately from the equality $\eta_{FX} \cdot \eta_X = F\eta_X \cdot \eta_X$ and property (ii), taking into account that the morphism $F\eta_X$ belongs to $\mathcal{A}$. $\square$

Examples 3.5. (1) In **Pos**, let $\mathcal{H} = \{h\}$ be as in Example 2.6.1. Using 3.4, we can conclude that $\mathcal{H}_{\lrcorner}$ is KZ-reflective. To see that, we observe first that, as it is easy to verify, if C is a connected component of a poset A, then it has a supremum iff all upper sets $x\uparrow$ with $x \in C$ do, and, in this case, all these supremums coincide with the supremum of C. Now, for every poset X, consider $\overline{X} = X \cup \{C, C$ is a connected component of $X\}$ with the partial order generated by the one inherited from X together with $x \leq C$ if $x \in C$ for every $x \in X$ and every connected component C. Then, a