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ON THE COMPLETION MONAD VIA THE YONEDA EMBEDDING IN QUASI-UNIFORM SPACES

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Dedicated to Eraldo Giuli on the occasion of his seventieth birthday

Abstract: Making use of the presentation of quasi-uniform spaces as generalised enriched categories, and employing in particular the calculus of modules, we define the Yoneda embedding, prove a (weak) Yoneda Lemma, and apply them to describe the Cauchy completion monad for quasi-uniform spaces.

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0. Introduction

Following our road map of exploring Lawvere's motto that "fundamental structures are themselves categories" [11], in this paper we focus on the study of quasi-uniform spaces as generalised enriched categories over $\mathsf{2}$. Analogously to the interpretation of a *(pre)ordered set* $(X, \leq)$ as an enriched category over $\mathsf{2} = \{0 < 1\}$, that is as a relation $a : X \relbar\joinrel\mapsto X$ such that

$$1_X \leq a, \qquad\qquad a \cdot a \leq a,$$

and of a *monotone map* $f : (X, \leq) \to (Y, \leq')$ as a functor $f : (X, a) \to (Y, b)$ such that

$$f \cdot a \leq b \cdot f,$$

a *quasi-uniform space* (X, U) is a prorelation $A : X \relbar\joinrel\mapsto X$, that is a down-directed set of relations $X \relbar\joinrel\mapsto X$, such that

$$\forall a \in A \;:\; 1_X \leq a, \qquad\qquad \forall a \in A \;\exists b \in A \;:\; b \cdot b \leq a,$$

and a *uniformly continuous map* $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that

$$\forall b \in B \;\exists a \in A \;:\; f \cdot a \leq b \cdot f.$$

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This amounts to describing the category QUnif of quasi-uniform spaces and uniformly continuous maps as a category of lax proalgebras – or $(\mathrm{Id}, 2)$-proalgebras – in the sense of [2, 5]. This approach allows us to define categorical concepts and apply categorical techniques in the study of quasi-uniformities. (An interesting survey on the general theory of quasi-uniform spaces can be found in [10].)

In [3] we showed that the notion of Lawvere complete enriched category can be extended to this setting and that it recovers Cauchy completeness for quasi-uniform spaces. Here we construct a Yoneda embedding for quasi-uniform spaces, which allows us to construct the Cauchy completion in the language of modules.

1. Quasi-uniformities as lax proalgebras

Throughout we will consider the category $\mathsf{ProMat}(2)$, or ProRel, of *prorelations*, having sets as objects and prorelations $R : X \rightarrow\!\!\!\!\!\mapsto Y$ as morphisms, where a prorelation $R : X \rightarrow\!\!\!\!\!\mapsto Y$ is a down-directed up-set of relations $X \rightarrow\!\!\!\!\!\mapsto Y$. Composition is defined componentwise. ProRel is in fact a 2-category when we consider 2-cells given by relational order as follows: for $R, S \in \mathsf{ProRel}(X, Y)$,

$$R \leq S \text{ if } \forall s \in S\ \exists r \in R\ :\ r \leq s,$$

which means exactly that $S \subseteq R$. Any relation $r : X \rightarrow\!\!\!\!\!\mapsto Y$, and in particular any map, can be seen as a prorelation, $R := \uparrow r$, hence there are 2-functors

$$\mathsf{Set} \longrightarrow \mathsf{Rel} \longrightarrow \mathsf{ProRel},$$

that leave objects unchanged. If X is a set, *a quasi-uniform space* $X = (X, A)$ is given by a prorelation $A : X \rightarrow\!\!\!\!\!\mapsto X$ so that

$$1_X \leq A \qquad\qquad \text{and} \qquad\qquad A \cdot A \leq A,$$

while a map $f : X \to Y$ is a uniformly continuous map $f : (X, A) \to (Y, B)$ if

$$f \cdot A \leq B \cdot f;$$

that is:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Given two quasi-uniform spaces (X, A) and (Y, B), a prorelation $\Phi : X \to\!\!\!\!\!+\ Y$ is said to be a *promodule* $\Phi : (X, A) \to\!\!\!\!\!\circ\ (Y, B)$ if

$$\Phi \cdot A \leq \Phi \qquad \text{and} \qquad B \cdot \Phi \leq \Phi.$$

For each quasi-uniform space (X, A), $A : (X, A) \to\!\!\!\!\!\circ\ (X, A)$ is easily seen to be a promodule. Since the composition of promodules – as prorelations – is a promodule, and A acts as an identity for this composition, we can consider the 2-category ProMod of quasi-uniform spaces and promodules.

Each uniformly continuous map $f : (X, A) \to (Y, B)$ defines a pair of bimodules $f_* : X \to\!\!\!\!\!\circ\ Y$ and $f^* : Y \to\!\!\!\!\!\circ\ X$, where f_* is given by the composite

$$X \xrightarrow{f} Y \xrightarrow{B} Y$$

and f^* by

$$Y \xrightarrow{B} Y \xrightarrow{f^\circ} X\ ;$$

here f° is the opposite relation of f. Clearly $(1_X)_* = A = (1_X)^*$, and, given also $g : Y \to Z$, then $(g \cdot f)_* = g_* \cdot f_*$ and $(g \cdot f)^* = f^* \cdot g^*$. Therefore these constructions define functors

$$(-)_* : \mathsf{QUnif} \to \mathsf{ProMod} \qquad \text{and} \qquad (-)^* : \mathsf{QUnif}^{\mathrm{op}} \to \mathsf{ProMod},$$

where $X_* = X = X^*$. Furthermore, via the 2-categorical structure of ProMod, f_* is left adjoint to f^*, that is

$$A \leq f^* \cdot f_* \qquad \text{and} \qquad f_* \cdot f^* \leq B.$$

In particular, when $x \in X$, the uniformly continuous map $1 \to X$, $* \mapsto x$, defines two adjoint promodules

$$(1 \xrightarrow{x_*} X) \dashv (X \xrightarrow{x^*} 1)\ .$$

We call a uniformly continuous map $f : (X, A) \to (Y, B)$ *fully faithful* if $f^* \cdot f_* = A$, and *fully dense* if $f_* \cdot f^* = B$. These two categorical notions have indeed a topological flavour. Here the *topology associated to each quasi-uniform space* (X, A) is the topology induced by the (symmetric) uniformity defined by A. That is, for $x \in X$ and $M \subseteq X$,

$$x \in \overline{M} \ \text{ if } \ \forall a \in A\ \exists y \in M\ :\ x\, a\, y\, a\, x.$$

In the language of promodules, x is in the closure of M if and only if the adjunction $x_* \dashv x^*$ on X restricts to an adjunction on M. It is worthwhile to mention here that, for instance, the b-closure of a topological space or the topology of a metric space admit formally the same description, see [7] for details. We say that a uniformly continuous map $f : (X, A) \to (Y, B)$ is *topologically dense* if it is dense for the topology described above, that is $\overline{f(X)} = Y$.

Proposition. *Let $f : (X, A) \to (Y, B)$ be a uniformly continuous map.*

(1) *f is fully faithful if, and only if, $A = f^\circ \cdot B \cdot f$, that is*

$$\forall a \in A \; \exists b \in B \; : \; a \geq f^\circ \cdot b \cdot f.$$

This means that A is the initial quasi-uniformity for $f : X \to (Y, B)$.

(2) *f is fully dense if, and only if,*

$$\forall b \in B \; \exists b_0 \in B \; : \; b_0 \leq b \cdot f \cdot f^\circ \cdot b.$$

(3) *f is topologically dense if, and only if,*

$$\forall b \in B \; : \; 1_Y \leq b \cdot f \cdot f^\circ \cdot b.$$

(4) *The following conditions are equivalent:*
   (i) *f is fully dense;*
   (ii) *f is topologically dense.*

Proof: Since one always has $A \leq f^\circ \cdot B \cdot f$, (1) is obvious because the condition stated means exactly that $A \geq f^\circ \cdot B \cdot f$.

(2): Analogously, f is fully dense if, and only if, $B \leq f_* \cdot f^*$; that is $B \leq B \cdot f \cdot f^\circ \cdot B$, which means

$$\forall b \in B \; \exists b_0 \in B \; : \; b_0 \leq b \cdot f \cdot f^\circ \cdot b.$$

(3): By definition of topological closure, $\overline{f(X)} = Y$ if, and only if,

$$\forall y \in Y \; \forall b \in B \; \exists x \in X \; : \; f(x)\, b\, y \text{ and } y\, b\, f(x),$$

or, equivalently,

$$\forall y \in Y \; \forall b \in B \; : \; y\, (b \cdot f \cdot f^\circ \cdot b)\, y,$$

which means that $1_Y \leq b \cdot f \cdot f^\circ \cdot b$ for every $b \in B$.

(4) (i) $\Rightarrow$ (ii) follows directly from $1_Y \leq b$ for every $b \in B$. Conversely, let f be topologically dense and $b \in B$. Let $b_0 \in B$ be such that $b_0 \cdot b_0 \leq b$. Then

$$1_Y \leq b_0 \cdot f \cdot f^\circ \cdot b_0 \; \Rightarrow \; b_0 \leq b_0 \cdot b_0 \cdot f \cdot f^\circ \cdot b_0 \leq b \cdot f \cdot f^\circ \cdot b. \qquad \blacksquare$$

2. The Yoneda embedding

Our first goal is to describe a Yoneda-like embedding for quasi-uniform spaces.

Let 1 be the quasi-uniform structure on the singleton $\{*\}$. For any quasi-uniform space $X = (X, A)$, let

$$PX := \{\Psi : X \xrightarrow{\circ} 1 \mid \Psi \text{ is a promodule}\}.$$

We equip PX with the quasi-uniformity

$$\widetilde{A} = \{\widetilde{a} \mid a \in A\},$$

where, for $\Psi_1, \Psi_2 \in PX$,

$$\Psi_1 \, \widetilde{a} \, \Psi_2 \text{ if } \Psi_1 \leq \Psi \cdot a :$$

$$\begin{tikzcd} X \arrow[r, "a", "\shortmid"{marking}] \arrow[dr, "\Psi_1"', "\shortmid"{marking}] & X \arrow[d, "\Psi_2", "\shortmid"{marking}] \arrow[dl, phantom, "\leq" near start] \\ & 1 \end{tikzcd}$$

It is easily checked that, for any $a, b \in A$,

$$1_{PX} = \widetilde{1_X}, \qquad a \leq b \;\Rightarrow\; \widetilde{a} \leq \widetilde{b}, \qquad \widetilde{b} \cdot \widetilde{a} \leq \widetilde{b \cdot a}$$

and, consequently, $\widetilde{A}$ is a quasi-uniformity on PX. Among the elements of PX we have the right-adjoint promodules defined by elements x of X,

$$X \xrightarrow[\circ]{x^*} 1 \,,$$

which play a key role in the sequel.

Proposition (Yoneda embedding)**.** *If (X, A) is a quasi-uniform space, the assignment $x \mapsto x^*$, for $x \in X$, defines a map $y_X : X \to PX$.*

(1) $y_X : (X, A) \to (PX, \widetilde{A})$ *is a uniformly continuous map.*
(2) $y_X : (X, A) \to (PX, \widetilde{A})$ *is fully faithful.*

Proof: (1): We want to show that $y_X \cdot A \leq \widetilde{A} \cdot y_X$; that is

$$\forall a \in A \; \exists a_0 \in A \;:\; y_X \cdot a_0 \leq \widetilde{a} \cdot y_X,$$

or, equivalently,

$$\forall a \in A \; \forall x, y \in X \;:\; x \, a_0 \, y \;\Rightarrow\; x^* \, \widetilde{a} \, y^*.$$

We recall that $x^* \,\widetilde{a}\, y^*$ means $x^\circ \cdot A \leq y^\circ \cdot A \cdot a$. For $a \in A$ let $a_0 \in A$ be such that $a_0 \cdot a_0 \leq a$. Let $x, y \in X$ with $x \, a_0 \, y$. For every $b \in A$, next we show that $x^\circ \cdot a_0 \leq y^\circ \cdot b \cdot a$, i.e.

$$\forall z \in Z \ : \ z \, a_0 \, x \ \Rightarrow \ z \, (b \cdot a) \, y,$$

and then the result follows. Let $z \in X$ with $z \, a_0 \, x$. Then from $z \, a_0 \, x \, a_0 \, y \, b \, y$ it follows that $z \, (b \cdot a) \, y$ as claimed.

(2): Following Proposition 1, we want to show that

$$\forall a \in A \ \exists \widetilde{a_0} \in \widetilde{A} \ : \ \forall x, y \in X \ x^* \, \widetilde{a_0} \, y^* \ \Rightarrow \ x \, a \, y.$$

Let $a_0 \in A$ be such that $a_0 \cdot a_0 \leq a$, and let $x, y \in X$ with $x^* \, \widetilde{a_0} \, y^*$, that is $x^\circ \cdot A \leq y^\circ \cdot A \cdot a_0$. Then there exists $b \in A$ with $x^\circ \cdot b \leq y^\circ \cdot a_0 \cdot a_0$, hence $x \, b \, x \ \Rightarrow \ x \, (a_0 \cdot a_0) \, y \ \Rightarrow \ x \, a \, y$. ∎

In general y_X is not fully dense. Our next goal is to compute $\overline{y_X(X)}$ in PX.

Theorem (Yoneda Lemma). *For every $\Psi \in PX$, in the following diagram:*

$$\begin{tikzcd} X \arrow[r, "(y_X)_*"] \arrow[dr, "\Psi"'] & PX \arrow[d, "\Psi^*"] \\ & 1 \end{tikzcd}$$

(1) $\Psi \geq \Psi^* \cdot (y_X)_*$;
(2) *if* $\Psi \in \overline{y_X(X)}$, *then also* $\Psi \leq \Psi^* \cdot (y_X)_*$.

Proof: (1): Since $\Psi^* \cdot (y_X)_* = \Psi^\circ \cdot \widetilde{A} \cdot \widetilde{A} \cdot y_X = \Psi^\circ \cdot \widetilde{A} \cdot y_X$, we want to show that, for every $\psi : X \relbar\joinrel\mapsto 1 \in \Psi$, there exists $a \in A$ such that, for all $x \in X$, $\psi^\circ \, \widetilde{a} \, x^* \ \Rightarrow \ x \, \psi \, *$. Let $\psi \in \Psi$. Since $\Psi \cdot A \leq \Psi$, there exists $\psi' \in \Psi$ and $a \in A$ with $\psi' \cdot a \leq \psi$. For such $a \in A$, consider $x \in X$ with $\Psi^\circ \, \widetilde{a} \, x^*$, that is $x^\circ \cdot A \leq \Psi^\circ \cdot a$. Then there exists $a' \in A$ such that $z \, a' \, x \ \Rightarrow \ z \, (\psi' \cdot a) \, *$. Since $x \, a' \, x$ we may conclude that $x \, (\psi' \cdot a) \, *$ and so $x \, \psi \, *$.

(2): We want to show that $\Psi \leq \Psi^\circ \cdot \widetilde{A} \cdot y_X$, that is

$$\forall a \in A \ \exists \psi \in \Psi \ \forall x \in X \ x \, \psi \, * \ \Rightarrow \ x^* \, \widetilde{a} \, \Psi.$$

Fix $a \in A$ and let $b \in A$ with $b \cdot b \leq a$. Since y_X is uniformly continuous, there is $a_0 \in A$ such that, if $x \, (a_0 \cdot a_0) \, y$, then $x^* \, \widetilde{b} \, y^*$. We choose such an

$a_0 \in A$ with the extra property of being less or equal to b. The condition $\Psi \in \overline{y_X(X)}$ assures the existence of $x_0 \in X$ such that

$$\Psi\, \widetilde{a_0}\, x_0^*\, \widetilde{a_0}\, \Psi.$$

From $\Psi\, \widetilde{a_0}\, x_0^*$, i.e. $\Psi \le x_0^\circ \cdot A \cdot a_0$, there exists $\psi \in \Psi$ such that $\psi \le x_0^\circ \cdot a_0 \cdot a_0$. Hence,

$$x\,\psi\,* \;\Rightarrow\; x\,(a_0 \cdot a_0)\,x_0 \;\Rightarrow\; x^*\,\widetilde{b}\,x_0^*.$$

Together with $x_0^*\,\widetilde{a_0}\,\psi$, hence $x_0^*\,\widetilde{b}\,\psi$, we conclude that $x^*\,(\widetilde{b} \cdot \widetilde{b})\,\psi$ and then $x^*\,\widetilde{a}\,\psi$. ∎

We remark that, for an enriched category X, the equality $\Psi^* \cdot (y_X)_* = \Psi$ is valid for every module Ψ, and in fact this is just an alternative way of stating the Yoneda Lemma. For quasi-uniform spaces we are only able to prove this equality in the case Ψ is a right adjoint.

Corollary. *If $\Psi \in PX$, then the following conditions are equivalent:*

(i) $\Psi \in \overline{y_X(X)}$;
(ii) Ψ *is right-adjoint.*

Proof: (i) $\Rightarrow$ (ii): If $\Psi \in \overline{y_X(X)}$, then $\Psi = \Psi^* \cdot (y_X)_*$, hence it can be written – through the (co)restriction to $\overline{y_X(X)}$ – as the composition of a right adjoint $\Psi^* : \overline{y_X(X)} \,{-\!\!\circ\!\!\rightarrow}\, 1$ and an equivalence $(y_X)_* : X \,{-\!\!\circ\!\!\rightarrow}\, \overline{y_X(X)}$. Therefore it is a right adjoint.

(ii) $\Rightarrow$ (i): Assume that $\Phi \dashv \Psi$, that is $1 \le \Psi \cdot \Phi$ and $\Phi \cdot \Psi \le A$. We want to show that, for every $a \in A$, there exists $x \in X$ with

$$\Psi\,\widetilde{a}\,x^*\,\widetilde{a}\,\Psi.$$

Let $a \in A$. From $\Phi \cdot \Psi \le A$ it follows that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that $\phi \cdot \psi \le a$. The inequalities $A \cdot \Phi \le \Phi$ and $\Psi \cdot A \le \Psi$ guarantee the existence of $\phi' \in \Phi$, $\psi' \in \Psi$, $a', a'' \in A$ such that $a' \cdot \phi' \le \phi$ and $\psi' \cdot a'' \le \psi$. From $1 \le \Psi \cdot \Phi$ there exists $x \in X$ such that $x\,\psi'\,*\,\phi'\,x$, or, equivalently, $x \le \phi'$ and $x^\circ \le \psi'$. Then

$$a' \cdot x \cdot \psi' \le a' \cdot \phi' \cdot \psi' \le a' \cdot \phi' \cdot \psi' \cdot a'' \le a$$

gives $x_* \cdot \Psi \le a$, hence $\Psi \le x^* \cdot x_* \cdot \Psi \le x^* \cdot a$, and so $\Psi\,\widetilde{a}\,x^*$, and

$$\phi' \cdot x^\circ \cdot a'' \le \phi' \cdot \psi' \cdot a'' \le a' \cdot \phi' \cdot \psi' \cdot a'' \le a$$

gives $\Phi \cdot x^* \le a$, hence $x^* \le \Psi \cdot \Phi \cdot x^* \le \Psi \cdot a$, and so $x^*\,\widetilde{a}\,\Psi$. ∎

A quasi-uniform space (X, A) is said to be *separated* if, for any $x, y \in X$, $x = y$ provided that, for every $a \in A$, $x\,a\,y$ and $y\,a\,x$. This condition can be stated using promodules as follows.

Lemma. *For a quasi-uniform space (X, A), the following assertions are equivalent:*

(i) *X is separated.*
(ii) *For $x, y \in X$, $x = y$ if $\{-\,a\,x \mid a \in A\} = \{-\,a\,y \mid a \in A\}$.*
(iii) *For all uniformly continuous maps $f, g : Y \to X$, if $f^* = g^*$, then $f = g$.*
(iv) *For all uniformly continuous maps $f, g : Y \to X$, if $f_* = g_*$, then $f = g$.*

Proof: It is easily checked that (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). The proof is complete once we show that (i) $\Leftrightarrow$ (ii). Let $x, y \in X$ with $x \neq y$. If X is separated, there is $a \in A$ such that either $\neg(x\,a\,y)$ or $\neg(y\,a\,x)$, which implies (ii). Assuming (ii),

$$\exists a \in A\; \forall a' \in A\; \exists z \in X\ :\ (z\,a\,x\ \wedge\ \neg(z\,a'\,y)) \text{ or } (z\,a\,y\ \wedge\ \neg(z\,a'\,x)).$$

If, for all $a \in A$, $x\,a\,y$ and $y\,a\,x$, we can conclude that, for any $a \in A$, for $a' = a \cdot a$ and for any $z \in X$ as in the condition above, if $z\,a\,x$ then $z\,(a \cdot a)\,y$ and, if $z\,a\,y$ then $z\,(a \cdot a)\,x$, contradicting our hypothesis. ∎

Observing that condition (ii) means injectivity of $\mathfrak{y}_X$, we obtain:

Corollary. *For a quasi-uniform space (X, A), the following assertions are equivalent:*

(i) *X is separated.*
(ii) *$\mathfrak{y}_X$ is injective.*

A quasi-uniform space (X, A) is *Cauchy complete* if every Cauchy filter in X converges, or, equivalently, if every minimal Cauchy filter in X is the neighbourhood filter of a point. As shown in [3], given a quasi-uniform space (X, A) and a pair of prorelations $(\Phi : 1 \relbar\!\!\mapsto X, \Psi : X \relbar\!\!\mapsto 1)$, Φ and Ψ are adjoint promodules, with $\Phi \dashv \Psi$, if, and only if,

$$\mathfrak{f} = (\{\{x \in X \mid x\,\psi\,*\},\ \psi \in \Psi\},\ \{\{x \in X \mid *\ \varphi\,x\},\ \varphi \in \Phi\})$$

is a minimal Cauchy filter in X. Indeed, the condition $1 \le \Psi\cdot\Phi$ means exactly that $\mathfrak{f}$ is a filter, $\Phi \cdot \Psi \le A$ guarantees that $\mathfrak{f}$ fulfils the Cauchy condition, while the promodule conditions state that $\mathfrak{f}$ is minimal. Furthermore, $\mathfrak{f}$ is the neighbourhood filter of $y \in X$ if, and only if, $\Phi = y_*$ and $\Psi = y^*$. Hence one has the following

Theorem ([3])**.** *For a quasi-uniform space X, the following assertions are equivalent:*

(i) *X is Cauchy complete.*
(ii) $y_X(X) = \overline{y_X(X)}$.

3. The monad $\mathbb{R}$

We have already seen that every uniformly continuous map $f : X \to Y$ between quasi-uniform spaces $X = (X, A)$ and $Y = (Y, B)$ defines an adjoint pair of promodules $f_* \dashv f^*$, and the contravariant functor $(-)^* : \mathsf{QUnif} \to \mathsf{ProMod}$ co-restricts to a contravariant functor

$$(-)^* : \mathsf{QUnif} \to \mathsf{ProMod}_{\mathrm{ra}}$$

into the category of quasi-uniform spaces and right adjoint modules. In this section we will show that this functor has an adjoint, and consequently induces a monad on both QUnif and $\mathsf{ProMod}_{\mathrm{ra}}$.

To do so, we consider now, for any quasi-uniform space (X, A), the subspace RX of PX defined by the right adjoint promodules $\Psi : X \mathrel{-\!\!\circ\!\!\to} 1$.

In general, for a right adjoint promodule $\Phi : (X, A) \mathrel{-\!\!\circ\!\!\to} (Y, B)$ we denote its left adjoint by $\widehat{\Phi}$. Recall that one has

$$B \leq \Phi \cdot \widehat{\Phi} \qquad\qquad \text{and} \qquad\qquad \widehat{\Phi} \cdot \Phi \leq A.$$

The quasi-uniformity on RX is of course the restriction of the quasi-uniformity on PX, but there is an alternative way to describe it as we show now. For $\Psi_1, \Psi_2 \in RX$ and $a \in A$, we write $\Psi_1 \mathring{a} \Psi_2$ whenever $\widehat{\Psi_2} \cdot \Psi_1 \leq a$.

$$\begin{tikzcd} X \arrow[rr, "a", "|" marking] \arrow[dr, "\Psi_1"', "|" marking] & & X \\ & 1 \arrow[ur, "\widehat{\Psi_2}"', "|" marking] \arrow[u, phantom, "\leq"] & \end{tikzcd}$$

Certainly, if $\widehat{\Psi_2} \cdot \Psi_1 \leq a$, then $\Psi_1 \leq \Psi_2 \cdot \widehat{\Psi_2} \cdot \Psi_1 \leq \Psi_2 \cdot a$, that is, $\Psi_1 \tilde{a} \Psi_2$. On the other hand, if $\Psi_1 \leq \Psi_2 \cdot a$, then $\widehat{\Psi_2} \cdot \Psi_1 \leq \widehat{\Psi_2} \cdot \Psi_2 \cdot a \leq A \cdot a \leq a \cdot a$. We conclude that $\widetilde{A} = \mathring{A}$, where $\mathring{A} = \{\mathring{a} \mid a \in A\}$.

Proposition. *For every right adjoint promodule $\Phi : (X, A) \mathrel{-\!\!\circ\!\!\to} (Y, B)$, the map*

$$R\Phi : RY \to RX,\ \Psi \mapsto \Psi \cdot \Phi$$

is uniformly continuous.

Proof: Let $a \in A$, and choose $\varphi \in \Phi$ and $\widehat{\varphi} \in \widehat{\Phi}$ with $\widehat{\varphi} \cdot \varphi \leq a$. Furthermore, take $b \in B$ and $\varphi' \in \Phi$ with $b \cdot \varphi' \leq \varphi$. Let $\Psi_1, \Psi_2 \in RY$ with $\Psi_1 \,\mathring{b}\, \Psi_2$. Hence, there exist $\psi_1 \in \Psi_1$ and $\widehat{\psi_2} \in \widehat{\Psi_2}$ with $\widehat{\psi_2} \cdot \psi_1 \leq b$. Then

$$\widehat{\varphi} \cdot \widehat{\psi_2} \cdot \psi_1 \cdot \varphi' \leq \widehat{\varphi} \cdot b \cdot \varphi' \leq \widehat{\varphi} \cdot \varphi \leq a,$$

that is, $(\Psi_1 \cdot \Phi) \,\mathring{a}\, (\Psi_2 \cdot \Phi)$. ∎

Clearly, $\Phi \mapsto R\Phi$ defines a contravariant functor $R : \mathsf{ProMod}_{\mathrm{ra}} \to \mathsf{QUnif}$.

Theorem. *The contravariant functors* $R : \mathsf{ProMod}_{\mathrm{ra}} \to \mathsf{QUnif}$ *and* $(-)^* : \mathsf{QUnif} \to \mathsf{ProMod}_{\mathrm{ra}}$ *define a dual adjunction, where the units are given by* $y_X : X \to RX$ *and* $(y_X)_* : X \,{-\!\!\circ\!\!\to}\, RX$*, respectively.*

Proof: We check first naturality of the families $(y_X : X \to RX)_X$ and $((y_X)_* : X \,{-\!\!\circ\!\!\to}\, RX)_X$. This is easy in the first case: for every uniformly continuous map $f : X \to Y$, the diagram

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "f"'] & RX \arrow[d, "R(f^*)=-\cdot f^*"] \\ Y \arrow[r, "y_Y"'] & RY \end{tikzcd}$$

commutes since $f(x)^* = x^* \cdot f^*$. Now to prove naturality of $((y_X)_*)_X$, for $\Phi : X \,{-\!\!\circ\!\!\to}\, Y$ a right adjoint promodule, we wish to show that $(y_Y)_* \cdot \Phi = (R\Phi)^* \cdot (y_X)_*$,

$$\begin{tikzcd} X \arrow[r, "(y_X)_*", "\circ" marking] \arrow[d, "\Phi"', "\circ" marking] & RX \arrow[d, "(R\Phi)^*", "\circ" marking] \\ Y \arrow[r, "(y_Y)_*"', "\circ" marking] & RY \end{tikzcd}$$

which is equivalent to $\Phi = (y_Y)^* \cdot (R\Phi)^* \cdot (y_X)_*$ since $(y_Y)_*$ is fully faithful and fully dense. Note that

$$(y_Y)^* \cdot (R\Phi)^* \cdot (y_X)_* = \{(-)^* \,\widetilde{a}\, ((-)^* \cdot \Phi) \mid a \in A\},$$

and that, for $x \in X$ and $y \in Y$,

$$\begin{aligned} x^* \,\widetilde{a}\, (y^* \cdot \Phi) \;&\iff\; x^* \leq y^* \cdot \Phi \cdot a \iff x^\circ \cdot A \leq y^\circ \cdot B \cdot \Phi \cdot a = y^\circ \cdot \Phi \cdot a \\ &\iff\; \forall \varphi \in \Phi \;\exists a' \in A \;:\; -a'\,x \leq -(\varphi' \cdot a)\,y. \end{aligned}$$

To show that $\Phi \geq (y_Y)^* \cdot (R\Phi)^* \cdot (y_X)_*$, that is

$$\forall \varphi \in \Phi\ \exists a \in A\ \forall x \in X\ \forall y \in Y\ :\ (x^* \,\widetilde{a}\, (y^* \cdot \Phi)\ \Rightarrow\ x \,\varphi\, y),$$

let $\varphi \in \Phi$. Since $\Phi \cdot A \leq \Phi$, we can find $a \in A$ and $\varphi' \in \Phi$ with $\varphi' \cdot a \leq \varphi$. Let $x \in X$ and $y \in Y$ with $x^* \,\widetilde{a}\, (y^* \cdot \Phi)$. Hence, there is some $a' \in A$ with $-\, a' \, x \leq -\, (\varphi' \cdot a)\, y$, and, since $x \, a' \, x$, one obtains $x \, (\varphi' \cdot a)\, y$ and finally $x \,\varphi\, y$.

Now, to show that

$$\forall a \in A\ \exists \varphi \in \Phi\ \forall x \in X\ \forall y \in Y\ :\ (x \,\varphi\, y\ \Rightarrow\ x^* \,\widetilde{a}\, (y^* \cdot \Phi)),$$

we fix $a \in A$, and take $\widehat{\varphi} \in \widehat{\Phi}$ and $\varphi' \in \Phi$ with $\widehat{\varphi} \cdot \varphi' \leq a$, and then take $\varphi \in \Phi$ and $a' \in A$ with $\varphi \cdot a' \leq \varphi'$. For $x \in X$ and $y \in Y$ with $x \,\varphi\, y$, we show that $\widehat{\Phi} \cdot y_* \cdot x^* \leq a$, which then implies that $x^* \leq y^* \cdot \Phi \cdot a$. To this end, let $z, z' \in X$ with $z \, a' \, x$ and $y \,\widehat{\varphi}\, z'$. Then $z \,\varphi'\, y$ and therefore $z \,(\widehat{\varphi} \cdot \varphi')\, z'$, hence $z \, a \, z'$.

Finally, one has

$$X \xrightarrow{(y_X)_*} RX \xrightarrow{(y_X)^*} X = X \xrightarrow{A} X$$

since y_X is fully faithful, and the equality

$$RX \xrightarrow{y_{RX}} RRX \xrightarrow{-\cdot (y_X)_*} RX = RX \xrightarrow{1_{RX}} RX$$

follows from the Yoneda Lemma. ■

The adjunction described above induces a monad $\mathbb{R} = (R, y, m)$ on QUnif. Here the functor $R : \mathsf{QUnif} \to \mathsf{QUnif}$ sends a quasi-uniform space X to the space RX of right adjoint promodules $\psi : X \to 1$, and a uniformly continuous map $f : X \to Y$ to $Rf := Rf^* : RX \to RY, \psi \mapsto \psi \cdot f^*$. The unit $y_X : X \to RX$ is the Yoneda embedding, and the multiplication $m_X : RRX \to RX$ sends $\Psi \in RRX$ to $\Psi \cdot (y_X)_*$. The monad $\mathbb{R}$ is idempotent since $(y_X)_* : X \to RX$ is an isomorphism in $\mathsf{ProMod}_{\mathrm{ra}}$, and the category $\mathsf{QUnif}^{\mathbb{R}}$ of Eilenberg-Moore algebras is the full subcategory of QUnif defined by those quasi-uniform spaces X where $y_X : X \to RX$ is bijective, that is, X is Cauchy complete and separated. It also follows at once that $\mathsf{QUnif}^{\mathbb{R}}$ is an embedding-firm epireflective subcategory of $\mathsf{QUnif}_{\mathrm{sep}}$ in the sense of [1]: for every separated quasi-uniform space X, the reflection map $y_X : X \to RX$ is a fully dense embedding; and, for any fully dense embedding $f : X \to Y$, where Y is in $\mathsf{QUnif}^{\mathbb{R}}$, the extension $f' : RX \to Y$ of f along $y_X : X \to RX$ is an isomorphism in $\mathsf{QUnif}^{\mathbb{R}}$ since f^* is an isomorphism in $\mathsf{ProMod}_{\mathrm{ra}}$.

Final Remarks. (1) We observe that the monad $\mathbb{R}$ does not coincide with Salbany's completion monad [12], but with his separated completion monad.

(2) A different description of quasi-uniform spaces as enriched categories is due to Schmitt [13]. While we use down-sets of relations instead of single relations, in [13] the quantale where the enrichment takes place depends on the quasi-uniform space. Nevertheless, individually each quasi-uniform space is a quantale-enriched category and to be Cauchy complete as an enriched category turns out to coincide with being Cauchy complete as a quasi-uniform space.

(3) Our proofs of the Yoneda Lemma and of the construction of the monad $\mathbb{R}$ depend essentially on the fact that we restrict our work to right adjoint promodules. It would be interesting to prove these results for other choices of promodules, in the spirit of [8, 9] for enriched categories and of [6, 4] for $(\mathbb{T}, \mathsf{V})$-categories, but our arguments cannot be easily translated to that setting.

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