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FOCK SPACES, LANDAU OPERATORS AND THE TIME-HARMONIC MAXWELL EQUATIONS

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Dedicated to the 70^{th} birthday of Richard Delanghe

Abstract: We investigate the representations of the solutions to Maxwell's equations based on the combination of hypercomplex function-theoretical methods with quantum mechanical methods. Our approach provides us with a characterization for the solutions to the time-harmonic Maxwell system in terms of series expansions involving spherical harmonics resp. spherical monogenics. Also, a thorough investigation for the series representation of the solutions in terms of eigenfunctions of Landau operators that encode $n-$dimensional spinless electrons is given.

This new insight should lead to important investigations in the study of regularity and hypo-ellipticity of the solutions to Schrödinger equations with natural applications in relativistic quantum mechanics confining massive spinor fields.

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1. Introduction

1.1. The Scope of Problems. The problem of constructing exact solutions for the time-harmonic Maxwell equations plays an important role in investigations of relativistic particles (bosons and fermions) in electromagnetic fields in terms of relativistic quantum mechanics.

For a domain $\Omega \subset \mathbb{R}^3$, and for an electromagnetic field with electric and magnetic components $\mathbf{E} : \Omega \to \mathbb{R}^3$ and $\mathbf{H} : \Omega \to \mathbb{R}^3$, respectively, the time-harmonic Maxwell equations with complex electric conductivity $\sigma := \sigma^* - i\omega\varepsilon$, dielectric constant ε, magnetic permeability μ and medium electrical conductivity σ^*, are described in terms of the following coupled system of equations:

$$\left\{\begin{array}{rcl} \mathrm{rot}\,\mathbf{H} & = & \sigma\mathbf{E}, \\ \mathrm{rot}\,\mathbf{E} & = & i\omega\mu\mathbf{H}, \\ \mathrm{div}\,\mathbf{H} & = & 0, \\ \mathrm{div}\,\mathbf{E} & = & 0. \end{array}\right. \tag{1}$$

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As is well-known, the above system of equations may be formulated in terms of differential forms or integral representations. In particular, the electrical and magnetic fields, $\mathbf{E} : \Omega \to \mathbb{R}^3$ and $\mathbf{H} : \Omega \to \mathbb{R}^3$ respectively, are also solutions of homogeneous Helmholtz equations with respect to the square of the medium wave number $\lambda^2 := i\omega\mu\sigma^* + \omega^2\mu\varepsilon = i\omega\mu\sigma \in \mathbb{C}$:

$$\left\{ \begin{array}{rcl} \Delta\mathbf{E} + \lambda^2\mathbf{E} & = & 0 \\ \Delta\mathbf{H} + \lambda^2\mathbf{H} & = & 0 \end{array} \right., \tag{2}$$

For further details see [35, 22, 25].

In the case where λ is a pure imaginary value, say $\lambda = i\alpha$ ($\alpha \in \mathbb{R}$), it should be noticed that the equation (2) has also another important physical meaning. If we assign $\alpha = \frac{mc}{\hbar}$, where m represents the mass of a particle, c the speed of light and $\hbar$ the Planck number, then this equation, which then is nothing else than the Klein-Gordon equation, correctly describes the spin-less pion. See also [21] for more details on this particular case.

The formulation of (1) and (2) (c.f. [22], Chapter 2) in terms of the Dirac operator $D = -\mathrm{div} + \mathrm{rot}$, allows us to describe solutions of both systems in terms of displacements of D, say $D \mp \lambda I$, in the skew-field of quaternions.

Indeed, the vector-fields $\mathbf{E}$ and $\mathbf{H}$ that satisfy (1) coincide exactly with the solutions to the equation $(D - \lambda I)f = 0$ in that domain.

In view of the factorization of the Helmholtz operator $\Delta + \lambda^2 I$ in the form $\Delta + \lambda^2 I = -(D - \lambda I)(D + \lambda I)$, we can express the solutions $\mathbf{E}$ and $\mathbf{H}$ of (2) in terms of the function $\mathbf{F} = (D + \lambda I)\,[\mathbf{E} + i\mathbf{H}]$. That function belongs to $\ker(D - \lambda I)$ and in turn allows us to re-express the electric and magnetic components for (1).

An alternative geometric structure to describe (1) and (2) in terms of massive spinor fields is the setting of Clifford algebras (c.f. [22], Chapter 3). Clifford algebras (see Section 2.1) endow finite dimensional quadratic vector spaces with an additional multiplication operation which shall be understood in the language of differential forms as the combined action in terms of wedge and contraction operators (c.f. [20], page 18).

When passing from the coordinate vector variable $x = \sum_{j=1}^{n} x_j \mathbf{e}_j$ to polar coordinates:

$$x = r\theta \quad \text{with } r = |x| \quad \text{and } \theta = \frac{1}{r}x,$$

the normal derivative along the unit vector $\widehat{n} = \frac{x}{r}$ given by $\partial_{\widehat{n}} = r\frac{\partial}{\partial r}$ coincides with the so-called Euler operator E (see Subsection 2.2) while D is given by

$$D = \theta\left(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma\right).$$

Here, Γ is the so-called Gamma operator (see Subsection 2.2) or spherical Dirac operator.

The Laplace operator Δ corresponds to the decomposition that involves the action of the classical Laplace-Beltrami operator $\Delta_{LB} = ((n-2)I - \Gamma)\Gamma$:

$$\Delta = -D^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}I + \frac{1}{r^2}\Delta_{LB}.$$

As explained in [34] and [8] the part of the solutions to the equation $(D - \lambda I)f = 0$ that is regular inside a ball centered around the origin can explicitly be expressed in terms of finite sums of homogeneous polynomials from $\ker D$ multiplied with particular Bessel J-functions of integer resp. half-integer parameter. Outside the ball we get a similar representation, but the Bessel J functions are then replaced by the corresponding Bessel Y functions of the same parameter.

Following [31, 23] and others, the spherical analogue of the operator $D-\lambda I$ is the operator $\Gamma - \lambda I$. Its kernel consists of the solutions to the spherical time-harmonic Maxwell operator and can be written in terms of sums over standard hypergeometric functions ${}_2F_1$ multiplied with homogeneous polynomials in $\ker D$.

Furthermore, in the recent work [9] it has been shown that the solutions to the time harmonic Maxwell system on a sphere of radius $R > 0$ coincides with the null solutions to the radial type operator $D - \lambda I - \frac{1}{R}E$. The case described in the earlier works [31, 23] arises as particular subcase. When $R \to +\infty$, the system $(D - \lambda I - \frac{1}{R}E)f = 0$ simplifies to $(D - \lambda I)f = 0$ where we are dealing with the null solutions to the time-harmonic Maxwell operator in the Euclidean flat space.

Apart from the development of function theoretical tools to compute analytically the solutions to the time-harmonic Maxwell equations (1) and the homogeneous Helmholtz equations (2), there is also considerable interest in the study of the spectra of Landau operators from the border view of physics and mathematics.

While the study of Landau operators has its roots in the construction of coherent states for relativistic Klein-Gordon and Dirac equations (see [7] and references given there), the surge of interest in these operators arose in the study of the possible occurrence of orbital electromagnetism [19] as well as in the study of pseudo-differential operators on modulation spaces [13] and quantum representations of Gabor-windowed Fourier analysis [5].

An extended survey confining relativistic quantum mechanics and the construction of coherent states can be found in [18] (see Chapter 4) and [27] (see Chapter 19), respectively; for a deep understanding of the link between Gabor-windowed Fourier analysis and the Heisenberg-Weyl group or of the Weyl transform with Hermite series expansions we may refer to [30].

Recently, in [36] the authors proposed a meaningful formulation of Landau operators in the quaternionic field that yields from the sub-Laplacian that arises in the quaternionic version of the Weyl-Heisenberg group.

From the border view of Clifford algebras, the extension of the above construction that describes the topological laws underlying the time-harmonic Maxwell equations (1) shall be defined as follows:

Let us consider the following Hamiltonian operator with mass m, frequency ω and potential energy $\frac{m\omega^2}{2}|x|^2$ that encodes the Helmholtz operator $\Delta+\lambda^2 I$:

$$\tilde{\mathcal{H}}_\lambda = -\frac{1}{2m}(\Delta+\lambda^2 I) + \frac{m\omega^2}{2}|x|^2 I.$$

We define the Landau operator $\mathcal{H}_\lambda$ as the superposition of the Hamiltonian operator $\tilde{\mathcal{H}}_\lambda$ by a symmetric gauge term $\mathcal{L}_\lambda$ which involves the spherical Dirac operator Γ:

$$\mathcal{H}_\lambda = \tilde{\mathcal{H}}_\lambda + \frac{m\omega^2}{2}\mathcal{L}_\lambda. \tag{3}$$

Along this paper $\mathcal{L}_\lambda$ corresponds to the special choice

$$\mathcal{L}_\lambda = -\frac{2\lambda}{n}\left(xI - \left(I+\frac{2\lambda}{n}\Gamma\right)\Gamma\right)$$

that shall be understood as a certain sort of electromagnetic counterpart for the Laplace-Betrami operator Δ_{LB} defined above.

Here we would like to stress that when restricted to dimension 2 or 4, the spherical Dirac operator Γ is equivalent to the orbital angular momentum operators that appear in [19, 36].

1.2. Motivation and Main Results. The Fock space formalism (c.f. [16]) and the Bargmann-Fock representation (c.f. [4]) proposed by Fock (1932) and Bargmann (1961) popularized by Newman and Shapiro [26] for the spaces of analytic functions, by Perelomov and Wünsche [27, 33] in terms of coherent states and by Folland and Thangavelu [17, 30] in the context of harmonic analysis and special function theory were fully developed in the most various contexts like the theory of generalized Bargmann spaces/poly-Fock spaces [32, 3] and Gabor-Window Fourier analysis [1, 2, 5] and pseudo-differential operators [13].

One of the goals of this paper is to emphasize the great potential of the quantum mechanical formalism in Clifford analysis for the study of the solutions of the time-harmonic Maxwell equations (1), namely the Bargmann-Fock representation of (poly-)monogenic functions and the construction of the eigenspaces for the Hamiltonian operator

$$\mathcal{H} = -\frac{1}{2m}\Delta + \frac{m\omega^2}{2}|x|^2 I \tag{4}$$

as dense linear subspaces of the Clifford-valued Schwartz space $\mathcal{S}(\mathbb{R}^n; \mathbb{R}_{0,n})$ that encode the symmetries of $\mathfrak{osp}(1|2)$.

When restricted to the algebra of Clifford-valued polynomials $\mathcal{P}$, these spaces have turned out to be an adequate setting for the construction of series representations in terms of Clifford-Hermite functions/polynomials.

Although Clifford-Hermite functions/polynomials expansions and their applications are well-known in harmonic/Fourier/wavelet analysis (c.f. [28, 6, 11, 12]) they have not received a lot of attention in quantum mechanics after the approach of Cnops and Kisil [10] on which the representation of nilpotent Lie groups like $SL_2(\mathbb{R})$ arise in the hypercomplex formulation.

Let us get some motivation from the representation of nilpotent Lie groups (see [27], Chapter 10). If one considers operators of the type $\exp\left(\frac{\lambda}{n}(D - X)\right)$ as the Clifford extension of the displacement/Heisenberg-Weyl operators (c.f [33, 13]) for the Heisenberg group $\mathbb{H}^n$, we will get an intriguing link that allows us to describe the solutions of the PDE system

$$\left(D - \lambda I + \frac{2\lambda}{n}\Gamma\right) f_\lambda = g_\lambda, \text{ with } g_\lambda \in \ker\left(D - \lambda I + \frac{2\lambda}{n}\Gamma\right)^s$$

as a generating-type function involving Clifford-Hermite polynomials.

The most interesting output of this paper is the link between the Landau operator $\mathcal{H}_\lambda$ standard Hamiltonian operator $\mathcal{H}$ that follows from the covariant action $x \leftarrow xI - \lambda I + \frac{2\lambda}{n}\Gamma$ on the spherical potential $\frac{m\omega^2}{2}|x|^2 I$ and the (apparently) new connection between the series expansions of $D - \lambda I + \frac{2\lambda}{n}\Gamma$ and the eigenfunctions for the Landau operator $\mathcal{H}_\lambda$.

1.3. Organization of the paper. Along this paper we have tried to present a self-contained exposition. The outline is as follows: In Section 2 we will start to recollect some basic features of Clifford algebras and Clifford analysis; we may refer for example to [20, 14] in which an extended survey is presented.

Section 3 will be devoted to the study of the spectra of the Hamiltonian (4) in terms of Clifford algebra-valued functions. We will start to review some basic facts regarding Hermite functions in terms of Weyl-Heisenberg symmetries (c.f. [17, 30, 18]). Afterwards, using the $\mathfrak{osp}(1|2)$ symmetries encoded in the Clifford-valued operators, we will get a fully description for the eigenspaces of (4) in terms of $\mathcal{H}_0 = \frac{1}{2}(-\Delta + |x|^2 I)$ by means of $\mathfrak{osp}(1|2)$ symmetries that comprise the function spaces spanned by Clifford-Hermite polynomials and Clifford-Hermite functions giving in this way a quantum mechanical interpretation for the results obtained in [28].

Finally, in Section 4 we will study the structure of the solutions of the time-harmonic Maxwell equations (1) subjected to an orbital angular momentum action encoded in the spherical Dirac operator Γ and its interplay with the theory of spherical monogenics and with the eigenfunctions underlying the Landau operator $\mathcal{H}_\lambda = \widetilde{\mathcal{H}}_\lambda + \mathcal{L}_\lambda$ (see equation (3)).

2. The Clifford analysis setting

2.1. Clifford Algebras. Let $\mathbb{R}^n$ be endowed with the non-degenerate bilinear symmetric form $\mathcal{B}(\cdot,\cdot)$ of signature (p,q), with $p+q=n$ and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthogonal basis of $\mathbb{R}^n$ satisfying

$$\begin{array}{rcll} \mathcal{B}(\mathbf{e}_j, \mathbf{e}_j) & = & -1, & j = 1, \ldots, p \\ \mathcal{B}(\mathbf{e}_j, \mathbf{e}_j) & = & 1, & j = p+1, \ldots, n \\ \mathcal{B}(\mathbf{e}_j, \mathbf{e}_k) & = & 0, & j \neq k. \end{array}$$

We denote by $\mathbb{R}_{p,q}$ the real Clifford algebra of signature (p,q) generated by the identity $\mathbf{1}$ and the standard basis elements $\mathbf{e}_j$ modulo the relations

$$\{\mathbf{e}_j, \mathbf{e}_k\} = -2\mathcal{B}(\mathbf{e}_j, \mathbf{e}_k). \tag{5}$$

Here $\{\mathbf{a}, \mathbf{b}\} := \mathbf{ab} + \mathbf{ba}$ denotes the anti-commutator between $\mathbf{a}$ and $\mathbf{b}$.

The elements of $\mathbb{R}_{p,q}$ are called Clifford numbers. The relation (5) is the so-called Kronecker factorization. For $\underline{\mathbf{e}} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ and for each $\alpha \in \{0,1\}^n$, we set $\underline{\mathbf{e}}^\alpha = \mathbf{e}_1^{\alpha_1}\mathbf{e}_2^{\alpha_2}\ldots\mathbf{e}_n^{\alpha_n}$ as the canonical basis for $\mathbb{R}_{p,q}$. For $\alpha = \underline{0}$, we put $\underline{\mathbf{e}}^{\underline{0}} = \mathbf{1}$.

An element $\mathbf{a} \in \mathbb{R}_{p,q}$ is called an $r-$vector if $\mathbf{a}$ may be written as a sum of elements of the form $a_\alpha \underline{\mathbf{e}}^\alpha$, with $|\alpha| = r$ (i.e. α has r non-vanishing indices). The space of all $r-$vectors is denoted by $\mathbb{R}^r_{p,q}$ and $[\cdot]_r : \mathbb{R}_{p,q} \to \mathbb{R}^r_{p,q}$ stands for the projection operator of $\mathbb{R}_{p,q}$ onto $\mathbb{R}^r_{p,q}$ defined as $[\mathbf{a}]_r = \sum_{|\alpha|=r} a_\alpha \underline{\mathbf{e}}^\alpha$.

This leads to the identification of $\mathbb{R}$ with the subspace $\mathbb{R}^0_{p,q}$ (scalars), $\mathbb{R}^n$ with $\mathbb{R}^1_{p,q}$, (Clifford vectors of signature (p,q)) and of the space of volume-forms with the subspace $\mathbb{R}^n_{p,q}$ generated by the single $n-$vector $\mathbf{e}_1 \ldots \mathbf{e}_n$ (the so-called pseudoscalar). Moreover, every element $\mathbf{a} \in \mathbb{R}_{p,q}$ may be decomposed in a unique way as a finite sum of the form $\mathbf{a} = \sum_{r=0}^n [\mathbf{a}]_r$ and hence

$$\mathbb{R}_{p,q} = \sum_{r=0}^n \oplus \mathbb{R}^r_{p,q}.$$

We would like to stress that $\mathbb{R}_{p,q}$ is in fact an algebra of radial-type $R(\mathcal{S})$ generated by $\mathcal{S} = \mathbb{R}^1_{p,q}$:

$$[\{x,y\}, z] = 0 \quad \text{for any } x, y, z \in \mathcal{S}.$$

This means that there actually is no a priori defined linear space to which the vector variables $x = \sum_{j=1}^n x_j \mathbf{e}_j \in \mathcal{S}$ belong. Nevertheless, by only using (6) one can already deduce many properties but we shall not explore them here. Further details can be found in [29] and additionally in [20] on page 17 to get the characterization of $\mathbb{R}_{p,q}$ in terms of $R(\mathcal{S})$ modulo the center of $\mathbb{R}_{p,q}$.

For $x \in \mathbb{R}^1_{p,q}$ and $\mathbf{a}^r \in \mathbb{R}^r_{p,q}$, the *inner* product and the *wedge* product on $\mathbb{R}_{p,q}$ are defined by

$$\begin{aligned} x \bullet \mathbf{a}^r &= [x\mathbf{a}^r]_{r-1} = \tfrac{1}{2}\left(x\mathbf{a}^r - (-1)^r \mathbf{a}^r x\right), \\ x \wedge \mathbf{a}^r &= [x\mathbf{a}^r]_{r+1} = \tfrac{1}{2}\left(x\mathbf{a}^r + (-1)^r \mathbf{a}^r x\right). \end{aligned} \tag{6}$$

The geometric Clifford product is given by $x\mathbf{a}^r = x \bullet \mathbf{a}^r + x \wedge \mathbf{a}^r$.

Geometrically speaking, $x \in \mathbb{R}^1_{p,q}$ is orthogonal (respectively, parallel) to $y \in \mathbb{R}^1_{p,q}$ if $x \bullet y = 0$ (respectively, $x \wedge y = 0$). Therefore, the orthogonality between x and y leads to $xy = -yx$ while the commutativity between x and y (i.e. $xy = yx$) occurs if x is parallel to y. By (6), $x^2 = x \bullet x$ is a real number for any x. Hence, x is invertible if and only if $x^2 \neq 0$. In this case the inverse x^{-1} is given by $x^{-1} = \frac{x}{x^2}$ with $x^2 = -\sum_{j=1}^{n} x_j^2$. If $x^2 = 0$ then either x is zero or it is a zero divisor and hence not invertible.

There are essentially two linear anti-automorphisms (reversion and conjugation) and a linear automorphism (main involution) acting on $\mathbb{R}_{p,q}$.

-: The *main involution* is defined by

$$\mathbf{e}_j` = -\mathbf{e}_j, \quad 1` = 1, \quad (j = 1, \ldots, n), \quad (ab)` = a`b`, \quad \forall\, a, b \in \mathbb{R}_{p,q};$$

-: the *reversion* is defined by

$$\mathbf{e}_j^* = \mathbf{e}_j, \quad 1^* = 1, \quad (i = 1, \ldots, n), \quad (ab)^* = b^* a^*, \quad \forall\, a, b \in \mathbb{R}_{p,q};$$

-: the *conjugation* is defined by

$$\mathbf{e}_j^\dagger = -\mathbf{e}_j, \quad \mathbf{1}^\dagger = \mathbf{1}, \quad (j = 1, \ldots, n), \quad (ab)^\dagger = b^\dagger a^\dagger, \quad \forall\, a, b \in \mathbb{R}_{p,q}.$$

We stress that *conjugation* can be obtained as a composition between *main involution* and *reversion* i.e. $x^\dagger = (x')^* = (x^*)'$, $\forall x \in \mathbb{R}_{p,q}$. From the definition we can derive the action on the basis elements $\underline{\mathbf{e}}^\alpha$ by the rules:

$$(\underline{\mathbf{e}}^\alpha)` = (-1)^{|\alpha|}\underline{\mathbf{e}}^\alpha, \qquad (\underline{\mathbf{e}}^\alpha)^* = (-1)^{\frac{|\alpha|(|\alpha|-1)}{2}}\underline{\mathbf{e}}^\alpha, \qquad (\underline{\mathbf{e}}^\alpha)^\dagger = (-1)^{\frac{|\alpha|(|\alpha|+1)}{2}}\underline{\mathbf{e}}^\alpha.$$

In particular, if x is a vector, then we obtain $x^\dagger = x` = -x$ and $x^* = x$.

The $\dagger-$conjugation leads to the Clifford algebra-valued inner product and its associated norm on $\mathbb{R}_{p,q}$ given by $(\mathbf{a}, \mathbf{b}) = [\mathbf{a}^\dagger \mathbf{b}]_0$ and $|\mathbf{a}|^2 = (\mathbf{a}, \mathbf{a})$, respectively. Notice that when $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathbb{R}_{p,q}^1$, their inner product and associated norm reduces to the classical inner product and norm on the ambient space $\mathbb{R}^n$, respectively.

2.2. Orthosymplectic Lie algebra representation of Clifford-valued operators. Let us restrict ourselves to the real Clifford algebra of signature (n, n), $\mathbb{R}_{n,n}$, in particular in its realization as the algebra of endomorphisms $\mathrm{End}(\mathbb{R}_{0,n})$. Here and elsewhere, we will consider Clifford-valued functions on $\mathbb{R}_{0,n}$, i.e., functions which can be decomposed in terms of $\mathbb{R}_{0,n}^r-$valued functions:

$$f(x) = \sum_{j=0}^{r} [f(x)]_r, \ \text{ with } [f(x)]_r = \sum_{|\alpha|=r} f_\alpha(x)\mathbf{e}^\alpha.$$

Let us observe that, from (6) for any $\mathbf{a} \in \mathbb{R}_{0,n}$, the *inner* product and the *wedge* product, $\mathbf{e}_j \bullet \mathbf{a}$ and $\mathbf{e}_j \wedge \mathbf{a}$, respectively, correspond to

$$\mathbf{e}_j \bullet \mathbf{a} = \frac{1}{2}\left(\mathbf{e}_j\mathbf{a} - \mathbf{a}^*\mathbf{e}_j\right), \quad \mathbf{e}_j \wedge \mathbf{a} = \tfrac{1}{2}\left(\mathbf{e}_j\mathbf{a} + \mathbf{a}^*\mathbf{e}_j\right), \tag{7}$$

where $*$ is the main involution defined in the preceding section. This suggests to take the basic endomorphisms $\xi_j : \mathbf{a} \mapsto \mathbf{e}_j\mathbf{a}$ and $\xi_{j+n} : F(\underline{x}) \mapsto \mathbf{a}^*\mathbf{e}_j$ acting on $\mathbb{R}_{0,n}$.

It is clear that ξ_j and ξ_{j+n} correspond one-to-one to the generators of the algebra $\mathbb{R}_{n,n}$ since $\xi_j(\xi_j\mathbf{a}) = \mathbf{e}_j^2\mathbf{a} = -\mathbf{a}$, $\xi_{j+n}(\xi_{j+n}\mathbf{a}) = (\mathbf{a}^*\mathbf{e}_j)^*\mathbf{e}_j = \mathbf{a}\mathbf{e}_j^*\mathbf{e}_j = \mathbf{a}$ and

$$\xi_j(\xi_k\mathbf{a}) + \xi_k(\xi_j\mathbf{a}) = 0, \quad \text{for } j,k = 1,\ldots,2n \text{ with } j \neq k.$$

Moreover, the operator actions $\mathbf{e}_j \wedge (\cdot) = \frac{1}{2}(\xi_j - \xi_{j+n})$ and $\mathbf{e}_j \bullet (\cdot) = \frac{1}{2}(\xi_j + \xi_{j+n})$ naturally give rise to a new basis for $\mathrm{End}(\mathbb{R}_{0,n})$, the so-called Witt basis for $\mathbb{R}_{n,n}$ satisfying the relations below:

$$\begin{array}{ll} \text{Grassmann identities:} & \{\mathbf{e}_j \wedge (\cdot), \mathbf{e}_k \wedge (\cdot)\} = 0 = \{\mathbf{e}_j \bullet (\cdot), \mathbf{e}_k \bullet (\cdot)\}, \\ \text{duality identities:} & \qquad\qquad \{\mathbf{e}_j \bullet (\cdot), \mathbf{e}_k \wedge (\cdot)\} = \delta_{jk} I. \end{array}$$

Next, we set $\mathcal{A}_n$ to be the algebra of differential operators given in terms of left endomorphisms $X_j : f(x) \mapsto x_j f(x)$, $\partial_{X_j} : f(x) \mapsto \frac{\partial f}{\partial x_j}(x)$ and $\xi_j : f(x) \mapsto \mathbf{e}_j f(x)$:

$$\mathcal{A}_n = \mathrm{span}\{X_j, \partial_{X_j}, \xi_j \ : \ j = 1, 2, \ldots, n\}.$$

It is straightforward to see that $X_1, X_2, \ldots, X_j, \partial_{X_1}, \partial_{X_2}, \ldots, \partial_{X_n}, I$, where I stands for the identity operator, are the generators of the $(2n+1)-$dimensional Heisenberg-Weyl Lie algebra $\mathfrak{h}_n$:

$$[X_j, X_k] = 0 \quad [\partial_{X_j}, \partial_{X_k}] = 0, \quad [\partial_{X_j}, X_k] = \delta_{jk} I. \tag{8}$$

This is a simple consequence of the Leibniz rule underlying $\frac{\partial}{\partial x_j}$ and the mutual commutativity between the coordinates x_j and partial derivatives $\frac{\partial}{\partial x_j}$.

We define the Dirac operator D and the vector multiplication operator X on $\mathbb{R}_{0,n}$ as the left endomorphisms

$$D = \sum_{j=1}^{n} \xi_j \partial_{X_j}, \quad \text{and } X = \textstyle\sum_{j=1}^{n} \xi_j X_j.$$

Clearly, these operators are elements of $\mathcal{A}_n$. In terms of D and X, $\mathcal{A}_n$ is equivalent to $\mathcal{A}_n = \mathrm{span}\{X, D, \xi_j \ : \ j = 1, 2, \ldots, n\}$. Moreover, the Laplace operator Δ and the square of X are also elements of $\mathcal{A}_n$ since $\Delta = -D^2$ and $X^2 = -|x|^2 I$.

Using the wedge product $\wedge$ and the dot product $\bullet$ introduced above, we further introduce the so-called Euler and Gamma operator as follows:

$$\begin{array}{c} E = X \bullet D = \sum_{j=1}^{n} X_j \partial_{X_j}, \\ \Gamma = X \wedge D = -\sum_{k=1}^{n} \sum_{j<k} \xi_j \xi_k (X_j \partial_{X_k} - X_k \partial_{X_j}) \end{array} \tag{9}$$

The preceding lemma, in which we consider combinations between X, D, E, Δ and Γ, will be important in the sequel. It establishes that E and Γ are also elements of $\mathcal{A}_n$.

Lemma 2.1. *The operators X, D, E, Δ and Γ satisfy the following relations:*

$$\begin{array}{ccc} \{X, D\} = -2E - nI, & [E, D] = -D, & [E, X] = X \\ [\Delta, X] = 2D, & XD = -E - \Gamma, & [E, \Gamma] = 0 \\ \Gamma = E + nI + DX, & \{\Gamma, X\} = -(n+1)X, & \{\Gamma, D\} = -(n+1)D, \\ \left[E, X^2\right] = 2X^2, & \left[\Gamma, X^2\right] = 0, & [\Gamma, \Delta] = 0. \end{array} \tag{10}$$

The proof of that statement can be found in [14], Chapter II.

The relation $\left[\Gamma, X^2\right] = 0$ implies that $\Gamma f(|x|) = 0$ while the relation $[\Gamma, \Delta] = 0$ implies that $\Gamma(\ker \Delta) \subset \ker \Delta$.

Moreover, a fully description of $\mathcal{A}_n$ as a representation of the orthosymplectic Lie algebra $\mathfrak{osp}(1|2) = \mathfrak{osp}(1|2)^{\text{even}} \bigoplus_{[\cdot,\cdot]} \mathfrak{osp}(1|2)^{\text{odd}}$ naturally follows by taking the generators P^+, P^-, Q, R^+, R^- as follows:

$$\begin{array}{l} P^- = -\frac{1}{2}\Delta, \quad P^+ = \frac{1}{2}X^2, \quad Q = \frac{1}{2}\left(E + \frac{n}{2}I\right) \\ R^+ = \frac{1}{2\sqrt{2}} iX, \quad R^- = \frac{1}{2\sqrt{2}} iD \end{array}$$

Recall that the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ (c.f. [15]) has three even generators $P^+, P^-, Q \in \mathfrak{osp}(1|2)^{\text{even}}$ and two odd generators $R^+, R^- \in \mathfrak{osp}(1|2)^{\text{odd}}$ satisfying the following commuting relations

$$\begin{array}{ccc} [R^+, P^+] = 0, & [R^+, P^-] = R^-, & [Q, P^+] = P^+ \\ [R^-, P^+] = R^+, & [R^-, P^-] = 0, & [Q, R^-] = -R^- \\ [P^-, P^+] = Q, & [Q, P^+] = P^+, & [Q, P^-] = -P^- \end{array} \tag{11}$$

Here we would like to stress that the even part of $\mathfrak{osp}(1|2)$ is isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, i.e. $\mathfrak{osp}(1|2)^{\text{even}} \cong \mathfrak{sl}_2(\mathbb{R})$.

3. The Quantum Harmonic Oscillator

3.1. Weyl-Heisenberg symmetries and Hermite Expansions revisited. In the sequel, we will consider the action of Hamiltonian operator $\mathcal{H}$ defined in (4) on the configuration space $L_2(\mathbb{R}^n)$. The usage of the dilation operator $S_a : f(x) \mapsto f(ax)$ allows us to re-scale the operator $\mathcal{H}$ in terms of the standard Hamiltonian $\mathcal{H}_0 = \frac{1}{2}\left(-\Delta + |x|^2 I\right)$ by getting rid of the constants m and ω (c.f. [18], pages 53-56):

$$S^{-1}_{\sqrt{m\omega}} \mathcal{H} S_{\sqrt{m\omega}} = \frac{1}{m\omega}\mathcal{H}_0 \tag{12}$$

This shows that the eigenfunctions for the Hamiltonian operator (4) can be described explicitly in terms of eigenfunctions of $\mathcal{H}_0$ using the Fock space formalism in phase space ([16]; [17], pages 47–49).

We embed the Fock space over $L_2(\mathbb{R}^n)$, say $\mathcal{F}(L_2(\mathbb{R}^n))$, as a dense linear subspace generated from the Gaussian window $\phi(x) = (\pi)^{-\frac{n}{4}} \exp(-\frac{|x|^2}{2})$ and by the $2n+1$ elements $A_1^+, A_2^+, \ldots, A_n^+, A_1^-, A_2^-, \ldots, A_n^-, I$, with

$$A_j^+ = \frac{1}{\sqrt{2}}\left(X_j - \partial_{X_j}\right), \quad A_j^- = \tfrac{1}{\sqrt{2}}\left(X_j + \partial_{X_j}\right). \tag{13}$$

From the relations (8), it follows that the operators $A_j^{\pm}$ and I generate the Weyl-Heisenberg Lie algebra $\mathfrak{h}^n$ (c.f. [17], Chapter 1). On the other hand, straightforward computations show that the Gaussian window ϕ satisfies the following two properties on the configuration space $L_2(\mathbb{R}^n)$. We have the

- Normalization property:
  $\|\phi\|^2_{L_2(\mathbb{R}^n)} = \langle \phi, \phi \rangle_{L_2(\mathbb{R}^n)} = (\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-|x|^2)dx = 1,$
- Annihilation property:
  $A_j^- \phi(x) = \frac{1}{\sqrt{2}}(\pi)^{-\frac{n}{4}} \left(\partial_{X_j}(\exp(-\frac{|x|^2}{2})) + x_j(\exp(-\frac{|x|^2}{2}))\right) = 0.$

With the statements that we described above for $A_j^{\pm}$ and ϕ, we can describe $\mathcal{F}(L_2(\mathbb{R}^n))$ as a boson Fock space with n particle states underlying the vacuum vector (the so-called ground state) ϕ. Moreover, a combination of the Weyl-Heisenberg character of $A_j^{\pm}$ and an application of mathematical induction on $d \in \mathbb{N}$, results into the above referred commuting expression involving A_j and the $d-$th power of $A_k^{\dagger}$:

$$[A_j, (A_k^{\dagger})^d] = d(A_k^{\dagger})^{d-1}.$$

Thus, according to the second quantization approach (c.f. [16]), the infinite set of vectors $\{\phi_\alpha\}_{\alpha \in (\mathbb{N}_0)^n}$ labelled by the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that

$$\phi_\alpha(x) = \frac{1}{\sqrt{\alpha_1! \alpha_2! \ldots \alpha_n!}} (A_1^+)^{\alpha_1} (A_2^+)^{\alpha_2} \ldots (A_n^+)^{\alpha_n} \phi(x),$$

satisfy the following raising/lowering properties:

$$A_j^+\phi_\alpha = \sqrt{\alpha_j+1}\phi_{\alpha+\mathbf{v}_j}, \quad A_j^-\phi_\alpha = \sqrt{\alpha_j}\phi_{\alpha-\mathbf{v}_j}, \tag{14}$$

respectively. Here $\mathbf{v}_j$ represents the $j-$canonical vector of $\mathbb{R}^n$. This allows us to show that $\{\phi_\alpha\}_\alpha$ is an orthonormal basis for $\mathcal{F}(L_2(\mathbb{R}^n))$ (c.f. [18], page 54). On the other hand, using the Lie algebra generators, we can also show that the standard Hamiltonian $\mathcal{H}_0 = \frac{1}{2}(-\Delta + |x|^2 I)$ and the number operator

$$E^{+-} = \sum_{j=1}^{n} A_j^+ A_j^- \tag{15}$$

are interrelated by $\mathcal{H}_0 = E^{+-} + \frac{n}{2}I$. On the basis of this property together with the relations (14) we can show that $\{\phi_\alpha\}_\alpha$ is a set of eigenvectors for $\mathcal{H}_0$ corresponding to the eigenvalues $\epsilon_n(\alpha) = \sum_{j=1}^n \alpha_j + \frac{n}{2}$. Property (12) implies that $S_{\sqrt{m\omega}}\phi_\alpha(x) = \phi_\alpha(\sqrt{m\omega}x)$ are solutions of the eigenvalue problem

$$\mathcal{H}f(x) = \frac{\epsilon_n(\alpha)}{m\omega}f(x).$$

The relation with the Weyl-Heisenberg Lie group $\mathbb{H}^n$ can be recast in terms of displacement operators by an exponentiation map $\rho(t,\omega) : \mathfrak{h}^n \to \mathbb{H}^n$ as follows (c.f. [17], Chapter 1; [30], Section 1.2). Recall that the Heisenberg group $\mathbb{H}^n$ is the non-commutative group represented on $\mathbb{R} \times \mathbb{C}^n$ in which the group multiplication is defined by

$$(s,\mathbf{w}) * (t,\mathbf{z}) = (s+t+i\Im(\overline{\mathbf{w}}\cdot\mathbf{z}), \mathbf{w}+\mathbf{z}).$$

The multiplicative inverse element then has the form $(s,\mathbf{w})^{-1} = (-s,-\mathbf{w})$.

For $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, with $w_j \in \mathbb{C}$, we set $\mathbf{A}^\pm = (A_1^\pm, A_2^\pm, \ldots, A_n^\pm)$ and we define formally the inner product $\mathbf{w}\cdot\mathbf{A}^\pm$ as $\mathbf{w}\cdot\mathbf{A}^\pm = \sum_{j=1}^n w_j A_j^\pm$ and set $d\rho(t,\mathbf{w}) = itI + \mathbf{w}\cdot\mathbf{A}^+ - \overline{\mathbf{w}}\cdot\mathbf{A}^+ \in \mathfrak{h}^n$, $\rho(t,\mathbf{w}) = \exp\left(d\rho(t,\mathbf{w})\right) \in \mathbb{H}^n$.

It is easy to see that $\rho(t,\mathbf{w}) = e^{it}\rho(0,\mathbf{w})$. On the other hand, the Baker-Campbell-Haussdorf formula ([27], page 9) tells us that

$$\exp(U)\exp(V) = \exp\left(\frac{1}{2}[U,V]\right)\exp(U+V) \tag{16}$$

whenever $[U,[U,V]] = 0 = [V,[U,V]]$. From this formula we may infer that the mapping

$$\mathbf{w} \mapsto \rho(0,\mathbf{w}) = \exp\left(\mathbf{w}\cdot\mathbf{A}^+ - \overline{\mathbf{w}}\cdot\mathbf{A}^+\right)$$

is a projective representation on $\mathbb{H}^n$. Indeed for any $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$, the relation $[\mathbf{w} \cdot \mathbf{A}^-, \mathbf{z} \cdot \mathbf{A}^+] = \mathbf{w} \cdot \mathbf{z} I$ leads to

$$\rho(0, \mathbf{w})\rho(0, \mathbf{z}) = \exp\left(\frac{1}{2}\left(\mathbf{w}\overline{\mathbf{z}} - \overline{\mathbf{w}}\mathbf{z}\right) I\right) \rho(0, \mathbf{w} + \mathbf{z}) = e^{-i\Im(\overline{\mathbf{w}}\cdot\mathbf{z})}\rho(0, \mathbf{w} + \mathbf{z})$$

and hence, the homomorphism properties described below on $\mathbb{H}^n$ follow straightforwardly:

$$\rho((s, \mathbf{w}) * (t, \mathbf{z})) = \rho(s, \mathbf{w})\rho(t, \mathbf{z}), \text{ and } \rho((s, \mathbf{w})^{-1}) = \rho(s, \mathbf{w})^{-1}.$$

So, we can now introduce Perelomov coherent states (c.f. [27, 33]) as the coherent states generated by the action of the operator $\rho(0, \mathbf{w})$ on the ground state $\phi(t)$, i.e.

$$\Phi_{\mathbf{w}} = \rho(0, \mathbf{w})\phi.$$

By straightforward computations using the Baker-Campbell-Haussdorf formula (16) and the identity $\exp(-\overline{\mathbf{w}} \cdot \mathbf{A}^-)\phi = \phi$, we can establish that

$$\Phi_{\mathbf{w}} = \exp\left(-\frac{1}{2}|\mathbf{w}|^2\right) \exp(\mathbf{w}\cdot\mathbf{A}^+) \exp(-\overline{\mathbf{w}}\cdot\mathbf{A}^-)\phi = \exp\left(-\frac{1}{2}|\mathbf{w}|^2\right) \sum_{\alpha} \frac{1}{\alpha!}\mathbf{w}^{\alpha}\phi_{\alpha}.$$

Here, we would like to remark that $\Phi_{\mathbf{w}}(t)$ is the so-called Bargmann kernel on $L_2(\mathbb{R}^n)$ that allows us to express the Bargmann transform

$$\mathcal{B} = \exp\left(\frac{1}{2}|\mathbf{w}|^2\right) \langle \cdot, \Phi_{\overline{\mathbf{w}}}\rangle_{L_2(\mathbb{R}^n)}$$

in the theory of Bargmann-Fock spaces of analytic functions (c.f. [4]).

Summarizing, the flexibility of this approach based on the group-theoretical backdrop allows us to extend the construction of Perelomov coherent states to more general coherent states by means of the action of $\rho(0, \mathbf{w})$ on the states ϕ_γ (c.f. [27], Chapter 2). These operators are known in literature as displacement operators [33] or Heisenberg-Weyl operators [13] underlying the so-called Weyl transform (c.f. [30], Section 1.1).

These produce coherent states $\Phi_{\mathbf{w},\gamma} = \rho(0, \mathbf{w})\phi_\gamma$ that allow us to express the so-called true poly-Bargmann transforms

$$\mathcal{B}^\gamma = \exp(\frac{1}{2}|\mathbf{w}|^2)\langle \cdot, \Phi_{\overline{\mathbf{w}},\gamma}\rangle_{L_2(\mathbb{R}^n)}$$

in the theory of Poly-Fock spaces or generalized Bargmann spaces. It also sheds some light on its interplay with the Gabor's windowed Fourier transform underlying Hermite windows. However, we shall not explore these relations here in depth. We refer for instance to [3, 32, 1, 2, 5] and the references given therein to get a more complete overview on that topic.

3.2. $\mathfrak{osp}(1|2)$ symmetries *vs* Spectra of the Harmonic Oscillator. In the sequel, we will consider the ladder operators $D^{\pm}$ that belong to $\mathcal{A}_n$ (see Subsection 2.2):

$$\begin{array}{rcccl} D^+ & = & \frac{1}{\sqrt{2}}(X - D) & = & \sum_{j=1}^n \xi_j A_j^+ \\ D^- & = & \frac{1}{\sqrt{2}}(X + D) & = & \sum_{j=1}^n \xi_j A_j^- \end{array} \tag{17}$$

where $A_j^{\pm}$ are defined by (13).

From the definition, it follows that $D^+, D^-, \mathcal{H}_0 \in \mathrm{End}(\mathcal{S}(\mathbb{R}^n; \mathbb{R}_{0,n}))$ satisfy the anti-commuting relation

$$\{D^+, D^-\} = -2\mathcal{H}_0, \tag{18}$$

or equivalently, $\{D^+, D^-\} = -2E^{+-} - nI$, in terms of the number operator E^{+-} given by (15).

In order to improve the Fock space formalism, we will take:

- The $\mathbb{R}_{0,n}-$Hilbert module $L_2(\mathbb{R}^n; \mathbb{R}_{0,n}) = L_2(\mathbb{R}^n) \bigotimes \mathbb{R}_{0,n}$ endowed with the bilinear form
$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)^\dagger g(x) dx,$$
where dx stands for the Lebesgue measure over $\mathbb{R}^n$;
- The Clifford algebra-valued Schwartz spaces
$$\mathcal{S}(\mathbb{R}^n; \mathbb{R}_{0,n}) := \mathcal{S}(\mathbb{R}^n) \bigotimes \mathbb{R}_{0,n}$$
on which $A_j^{\pm}$ and $D^{\pm}$ act as left endomorphisms;
- The Clifford algebra-valued polynomial space $\mathcal{P} = \mathbb{R}[\underline{x}] \bigoplus \mathbb{R}_{0,n}$, where $\mathbb{R}[\underline{x}]$ denotes the real-valued polynomial space over $\mathbb{R}^n$.

The following lemma from which one can deduce the Rodrigues's formula in special function theory provides us with the necessary motivation for the construction of eigenspaces for the Hamiltonian operator (4) in terms of Clifford algebra-valued functions:

Lemma 3.1. *The operators*

$$A_j^+, A_j^- \in End(\mathcal{S}(\mathbb{R}^n;\mathbb{R}_{0,n})) \quad \textit{and} \quad D^+, D^- \in End(\mathcal{S}(\mathbb{R}^n;\mathbb{R}_{0,n}))$$

can be represented by

$$A_j^+ = -\frac{1}{\sqrt{2}}\exp\left(-\tfrac{1}{2}X^2\right)\partial_{X_j}\exp\left(\tfrac{1}{2}X^2\right), \quad A_j^- = \frac{1}{\sqrt{2}}\exp\left(\tfrac{1}{2}X^2\right)\partial_{X_j}\exp\left(-\tfrac{1}{2}X^2\right). \tag{19}$$

$$D^+ = -\frac{1}{\sqrt{2}}\exp\left(-\tfrac{1}{2}X^2\right)D\exp\left(\tfrac{1}{2}X^2\right), \quad D^- = \frac{1}{\sqrt{2}}\exp\left(\tfrac{1}{2}X^2\right)D\exp\left(-\tfrac{1}{2}X^2\right). \tag{20}$$

Proof: Recall that $X_1, X_2, \ldots, X_n, \partial_{X_1}, \partial_{X_2}, \ldots, \partial_{X_n}$ are the canonical generators of the Heisenberg-Weyl algebra $\mathfrak{h}^n$ and $X^2 = -\sum_{k=1}^n X_k^2 I$.

Then we have

$$\left[\partial_{X_j}, \frac{1}{2}X^2\right] = -\sum_{k=1}^n \left[\partial_{X_j}, \frac{X_k^2}{2}\right] = -\sum_{k=1}^n \delta_{jk}\, X_k\, id = -X_j\, id,$$

and analogously $\left[\partial_{X_j}, -\frac{X^2}{2}\right] = \sum_{k=1}^n \left[\partial_{X_j}, \frac{X_k^2}{2}\right] = X_j\, id.$

Induction on $t \in \mathbb{N}$ allows us to establish that

$$\left[\partial_{X_j}, (\pm\frac{X^2}{2})^t\right] = \mp t X_j \left(\pm\frac{X^2}{2}\right)^{t-1},$$

and hence,

$$\begin{array}{rcl} \left[\partial_{X_j}, \exp\left(\pm\frac{1}{2}X^2\right)\right] & = & \sum_{t=0}^\infty \frac{1}{t!}\left[\partial_{X_j}, (\pm\frac{X^2}{2})^t\right] \\ & = & \sum_{t=0}^\infty \frac{\mp X_j}{(t-1)!}\left(\pm\frac{X^2}{2}\right)^{t-1} \\ & = & \mp X_j \exp\left(\pm\frac{1}{2}X^2\right). \end{array} \tag{21}$$

By adding the terms $\pm X_j \exp\left(\pm\frac{1}{2}X^2\right) + \exp\left(\pm\frac{1}{2}X^2\right)\partial_{X_j}$ to both sides of the above relations, we obtain

$$\begin{array}{rcl} \left(\partial_{X_j} + X_j\right)\exp\left(\frac{1}{2}X^2\right) & = & \exp\left(\frac{1}{2}X^2\right)\partial_{X_j}, \\ \left(\partial_{X_j} - X_j\right)\exp\left(-\frac{1}{2}X^2\right) & = & \exp\left(-\frac{1}{2}X^2\right)\partial_{X_j}. \end{array}$$

Finally, by multiplying on both sides of the above mentioned relations from the right with the expressions $-\frac{1}{\sqrt{2}}\exp\left(-\frac{1}{2}X^2\right)$ and $\frac{1}{\sqrt{2}}\exp\left(\frac{1}{2}X^2\right)$, respectively, and taking into account the definitions of $A_j^\pm$ (see 13), we arrive at

$$\begin{array}{rcl} A_j^+ & = & -\frac{1}{\sqrt{2}}\exp\left(-\frac{1}{2}X^2\right)\partial_{X_j}\exp\left(\frac{1}{2}X^2\right), \\ A_j^- & = & \frac{1}{\sqrt{2}}\exp\left(\frac{1}{2}X^2\right)\partial_{X_j}\exp\left(-\frac{1}{2}X^2\right). \end{array} \tag{22}$$

Using linearity arguments, the statements for the operators D^+ and D^- are then immediate when using their coordinate expressions. ■

The construction of Fock spaces on the Hilbert module $L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$ can be performed in the same flavor:

Firstly, we would like to stress that an application of the Clifford $\dagger-$conjugation combined with the Leibniz rule over $\mathbb{R}^n$ gives

$$\begin{array}{rcl}\langle D^- f, g\rangle & = & \frac{1}{\sqrt{2}}\int_{\mathbb{R}^n}((X+D)f(x))^\dagger g(x)dx \\ & = & \frac{1}{\sqrt{2}}\int_{\mathbb{R}^n} f(x)^\dagger(-X+D)g(x)dx \\ & = & -\langle f, D^+ g\rangle.\end{array} \tag{23}$$

This shows that, up to a minus sign, D^+ is the adjoint of D^- in $L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$. Hence, from (20) Lemma 3.1 D^+ is the adjoint of $\frac{1}{\sqrt{2}}D$ with respect to the $\mathbb{R}_{0,n}-$valued bilinear form:

$$\langle f, g\rangle_{\mathcal{F}} = \pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)^\dagger g(x)\exp\left(-|x|^2\right)dx. \tag{24}$$

Here we would like to stress that $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ shall be regarded as the integral representation for the Fischer inner product (c.f. [14], pp. 204-205) that extends the integral representation obtained in [4] by Bargmann. Moreover, the spaces

$$\mathcal{F} = \left\{ f\in\ker D \ : \ \langle f, f\rangle_{\mathcal{F}} = \pi^{-\frac{n}{2}}\int_{\mathbb{R}^n}|f(x)|^2\exp(-|x|^2)dx<\infty\right\}, \tag{25}$$

$$\mathcal{F}_k = \left\{ f\in\ker D^k \ : \ \langle f, f\rangle_{\mathcal{F}} = \pi^{-\frac{n}{2}}\int_{\mathbb{R}^n}|f(x)|^2\exp(-|x|^2)dx<\infty\right\}. \tag{26}$$

shall be interpreted as the monogenic counterparts for the real Bargmann spaces (also called Segal-Bargmann, Fock or Fischer space, see [10, 26]) and poly-Bargmann spaces (also called Poly-Fock or generalized Bargmann spaces, [32, 3]), respectively. This sort of spaces are proper subspaces of the so-called poly-monogenic functions with respect to the C^∞-topology (c.f. [24]).

On the other hand, from Lemma 3.1 and from the relation (20), it follows that D^- annihilates $\phi(x)f(x)$ for any $f\in kerD$:

$$\begin{aligned} D^-\left(\phi(x)f(x)\right) &= (\pi)^{-\frac{n}{4}}D^-\left(\exp\left(-\frac{|x|^2}{2}\right)f(x)\right) \\ &= (\pi)^{-\frac{n}{4}}\exp\left(\frac{X^2}{2}\right)\ Df(x) = 0. \end{aligned} \tag{27}$$

Since the Gaussian window $\phi(x) = (\pi)^{-\frac{n}{4}} \exp(-\frac{|x|^2}{2})$ satisfies $\langle \phi, \phi \rangle = 1$ (the normalization property), and the spectra of $\mathcal{H}_0$ corresponds to the increasing sequence $\{k + \frac{n}{2}\}_{k\in\mathbb{N}_0}$, the construction of the Fock states $\{\Psi_k\}_{k\in\mathbb{N}_0}$ viz $\Psi_k(x) = \frac{1}{\sqrt{c_k}}(D^+)^k\Psi(x)$, for some constant c_k, should take into account the constraints below:

- *Annihilation property:* $D^-\Psi = 0$;
- *Normalization of ground level energy:* $\langle \Psi, \Psi \rangle = 1$;
- *Raising property:* $D^+\Psi_k = \sqrt{k+1+\frac{n-1}{2}}\Psi_{k+1}$;
- *Lowering property:* $D^-\Psi_k = \sqrt{k+\frac{n-1}{2}}\Psi_{k-1}$.

We will start by proving the following statements:

Lemma 3.2. *The operators $D^-, D^+ \in End(\mathcal{S}(\mathbb{R}^n;\mathbb{R}_{0,n}))$ satisfy*

$$\left[D^-, (D^+)^k\right] = \begin{cases} -2j(D^+)^{2j-1} & , \text{if } k = 2j \\ 2(D^+)^{2j}\left(\Gamma - (\frac{n}{2}+j)I\right) & , \text{if } k = 2j+1 \end{cases} \tag{28}$$

Proof: We will use mathematical induction to prove (28). For $k = 1$, we proceed as follows:

We can split $[D^-, D^+]$ according to the definition as

$$[D^-, D^+] = \frac{1}{2}[X + D, X - D] = -[X, D]$$

From $XD = -E - \Gamma$ and $\{X, D\} = -2E - nI$ (Lemma 2.1) it follows that $[X, D] = -2\Gamma + nI$ and hence

$$[D^+, D^-] = 2\Gamma - nI = 2\left(\Gamma - \frac{n}{2}I\right). \tag{29}$$

For $k = 2$, notice that from Lemma 2.1 we have $\{\Gamma, X\} = -(n-1)X$ and $\{\Gamma, D\} = -(n-1)D$, and hence, $\{\Gamma, D^+\} = -(n-1)D^+$.

Thus, relying on $[D^-, D^+] = 2\Gamma - nI$, we obtain

$$\begin{aligned} D^-(D^+)^2 &= (2\Gamma - nI)D^+ + D^+(D^-D^+) \\ &= (2\Gamma - nI)D^+ + D^+(2\Gamma - nI) + (D^+)^2D^- \\ &= 2\left(\{\Gamma, D^+\} - nD^+\right) + (D^+)^2D^- \\ &= -2D^+ + (D^+)^2D^-. \end{aligned} \tag{30}$$

Next, we assume that (28) holds for $k \in \mathbb{N}$, i.e.

$$D^-(D^+)^k = \begin{cases} -2j(D^+)^{2j-1} + (D^+)^{2j}D^-, & \text{if } k = 2j \\ 2(D^+)^{2j}\left(\Gamma - (\frac{n}{2}+j)I\right) + (D^+)^{2j+1}D^-, & \text{if } k = 2j+1 \end{cases}.$$

Hence, the induction assumption together with the relations (29) and (30) lead to

$$\begin{aligned} D^-(D^+)^{2j+2} &= \left(D^-(D^+)^{2j+1}\right)D^+ \\ &= \left(2(D^+)^{2j}\left(\Gamma - (\frac{n}{2}+j)I\right) + (D^+)^{2j+1}D^-\right)D^+ \\ &= 2(D^+)^{2j}\left((n-1)D^+ - D^+\Gamma - (\frac{n}{2}+j)D^+\right) + (D^+)^{2j+1}(D^-D^+) \\ &= 2(D^+)^{2j+1}\left(-\Gamma - (-\frac{n}{2}+j+1)D^+\right) + \\ &\qquad\qquad +2(D^+)^{2j+1}(\Gamma - \frac{n}{2}I) + (D^+)^{2j+2}D^- \\ &= -(2j+2)(D^+)^{2j+1} + (D^+)^{2j+2}D^-, \end{aligned}$$

$$\begin{aligned} D^-(D^+)^{2j+3} &= \left(D^-(D^+)^{2j+2}\right)D^+ \\ &= \left((2j+2)(D^+)^{2j+1} + (D^+)^{2j+2}D^-\right)D^+ \\ &= (2j+2)(D^+)^{2j+2} + (D^+)^{2j+2}(D^-D^+) \\ &= -(2j+2)(D^+)^{2j+2} + (D^+)^{2j+2}\left(2\Gamma - nI + D^+D^-\right) \\ &= -2(D^+)^{2j+2}\left(\Gamma - \left(\frac{n}{2}+j+1\right)I\right) + (D^+)^{2j+3}D^-. \end{aligned}$$

This proves (28). ■

Next, for any for a starlike domain Ω with center 0, we define for each $s > 0$ the operator $I_s : C^1(\Omega; \mathbb{R}_{0,n}) \longrightarrow C^1(\Omega; \mathbb{R}_{0,n})$ by

$$I_s f(\underline{x}) = \int_0^1 f(t\underline{x})t^{s-1}dt. \tag{31}$$

Furthermore, we will write I instead of I_0 to denote the identity operator.

The next theorem proved in [24] shows that I_s is the inverse of the operator $E_s := E + sI$:

Lemma 3.3. ([24], *Lemma 3.1*) *Let* $\underline{x} \in \mathbb{R}^n$ *and* Ω *be a domain with* $\Omega \supset [0, \underline{x}]$. *If* $s > 0$ *and* $f \in C^1(\Omega; \mathbb{R}_{0,n})$, *then*

$$f(\underline{x}) = I_s E_s f(\underline{x}) = E_s I_s f(\underline{x}). \tag{32}$$

Roughly speaking, from $\{X, D\} = -2E - nI = -2(E + \frac{n}{2}I)$ (see Lemma 2.1), the mapping $f \mapsto -\frac{1}{2} I_{\frac{n}{2}} f$ shall be interpreted as a sort of right inverse for DX on the range $\ker D$.

On the other hand, from $\left[E + \frac{n}{2}I, D\right] = -D$ (see relations (11) for the elements R^+ and Q), we obtain $D\left(E + \left(\frac{n}{2} + j\right) I\right) = \left(E + \left(\frac{n}{2} + j + 1\right) I\right) D$ and hence

$$D I_{\frac{n}{2}+j} = I_{\frac{n}{2}+j+1} \left(E + \left(\frac{n}{2} + j + 1\right) I\right) D I_{\frac{n}{2}+j} = I_{\frac{n}{2}+j+1} D. \tag{33}$$

This shows that the family of maps $I_{\frac{n}{2}+j} : C^1(\Omega; \mathbb{R}_{0,n}) \longrightarrow C^1(\Omega; \mathbb{R}_{0,n})$ leave $\ker D$ invariant.

We have now the key ingredients to construct the eigenspaces for the Hamiltonian operator $\mathcal{H}_0$ in terms of Clifford algebra-valued functions:

For any $k \in \mathbb{N}$, let $U_k = T_k T_{k-1} \ldots T_1$ with

$$T_k = \begin{cases} (\frac{n-1}{4j} + 1)I & ,\text{if } k = 2j \\ \left(\frac{n-1}{4} + j + \frac{1}{2}\right) \ I_{\frac{n}{2}+j} & ,\text{if } k = 2j+1 \end{cases} \tag{34}$$

The theorem below shows that the Clifford-valued function spaces

$$\mathcal{F}_k^{+-} = \Big\{ f_k(x) = \frac{1}{\sqrt{c_k}} (D^+)^k \left(\Psi_k(x)\right) : \\ \Psi_k(x) = \phi(x) U_k(f(x)), f \in \ker D, \langle \Psi_k, \Psi_k \rangle = 1 \Big\},$$

with $c_k = \left(k + \frac{n-1}{2}\right)\left(k - 1 + \frac{n-1}{2}\right) \ldots \left(1 + \frac{n-1}{2}\right) \frac{n-1}{2}$, contain the eigenfunctions of $\mathcal{H}_0$ with eigenvalue $k + \frac{n}{2}$.

The following sequence of results will be useful for the characterization of $\mathcal{F}_k^{+-}$:

Theorem 3.1. *The functions* $f_k \in \mathcal{F}_k^{+-}$ *satisfy the following raising and lowering properties:*

$$D^+ f_k = \sqrt{k + 1 + \frac{n-1}{2}} f_{k+1}, \quad \textit{and } D^- f_k = -\sqrt{k + \tfrac{n-1}{2}} f_{k-1}.$$

Moreover, they are solutions of the eigenvalue problem $\mathcal{H}_0 f_k = \left(k + \frac{n}{2}\right) f_k$.

Proof: Let $f_k(x) = \frac{1}{\sqrt{c_k}}(D^+)^k\,(\Psi_k(x)) \in \mathcal{F}^{+-}_k$, with

$$c_k = \left(k + \frac{n-1}{2}\right)\left(k - 1 + \frac{n-1}{2}\right)\ldots\left(1 + \frac{n-1}{2}\right)\frac{n-1}{2}.$$

From the definition of $\mathcal{F}^{+-}$, the raising property $D^+ f_k = \sqrt{k+1+\frac{n-1}{2}} f_{k+1}$ follows naturally:

$$\begin{aligned} D^+ f_k(x) &= \frac{1}{\sqrt{c_k}}(D^+)^{k+1}\,(\Psi_k(x)) \\ &= \sqrt{k+1+\frac{n-1}{2}}\frac{1}{\sqrt{\left(k+1+\frac{n-1}{2}\right)c_k}}(D^+)^{k+1}\,(\Psi_k(x)) \\ &= \sqrt{k+1+\frac{n-1}{2}}\; f_{k+1}(x). \end{aligned}$$

For the proof of the lowering property, we will combine Lemma 3.2 with (27):

Observe that the family of mappings U_k defined by (34) leave $\ker D$ invariant. Thus $U_k(f(x))$ is monogenic and hence, from (27) $\Psi_k(x) = \phi(x)U_k(f(x))$ is a null solution of $D^- = \frac{1}{\sqrt{2}}(x + D)$:

$$D^-(\Psi_k(x)) = \pi^{-\frac{n}{4}} D^-\left(\exp\left(-\frac{|x|^2}{2}\right)U_k(f(x))\right) = 0.$$

Then, taking into account the recursive relation $U_{2j} = (\frac{n-1}{4j} + 1)U_{2j-1}$. An application of Lemma 3.2 then gives

$$\begin{aligned} D^- f_{2j}(x) &= -\frac{2j}{\sqrt{c_{2j}}}(D^+)^{2j-1}(\Psi_{2j}(x)) \\ &= -\sqrt{2j+\frac{n-1}{2}}\sqrt{\frac{2j+\frac{n-1}{2}}{c_{2j}}}(D^+)^{2j-1}(\phi(x)(U_{2j-1}f(x))) \\ &= -\sqrt{2j+\frac{n-1}{2}}\frac{1}{\sqrt{c_{2j-1}}}(D^+)^{2j-1}(\Psi_{2j-1}(x)) \\ &= -\sqrt{2j+\frac{n-1}{2}}\; f_{2j-1}(x). \end{aligned}$$

For $k = 2j+1$ (k odd), we show the lowering property by using the relation $\Gamma = -XD - E$ (see Lemma (2.1)), the recursive relation

$$U_{2j+1} = \left(\frac{n-1}{4} + j + \frac{1}{2}\right) \; I_{\frac{n}{2}+j} U_{2j}$$

and the property $\Gamma(\phi(x)f(x)) = \phi(x)\Gamma f(x)$ (radial character of Gamma operator).

This results into

$$\begin{aligned}
D^- f_{2j+1}(x) &= \frac{2}{\sqrt{c_{2j+1}}}(D^+)^{2j}\left(\Gamma - (\frac{n}{2}+j)I\right)(\Psi_{2j+1}(x)) \\
&= \frac{2j+1+\frac{n-1}{2}}{\sqrt{c_{2j+1}}}(D^+)^{2j}\left(\Gamma - (\frac{n}{2}+j)I\right)\left(\phi(x) I_{\frac{n}{2}+j} Q_{2j} f(x)\right) \\
&= \frac{2j+1+\frac{n-1}{2}}{\sqrt{c_{2j+1}}}(D^+)^{2j}\left(\phi(x)\left(\Gamma - (\frac{n}{2}+j)I\right) I_{\frac{n}{2}+j} Q_{2j} f(x)\right) \\
&= \sqrt{2j+1+\frac{n-1}{2}}\sqrt{\frac{2j+1+\frac{n-1}{2}}{c_{2j+1}}}(D^+)^{2j} \\
&\qquad \left(\phi(x)\left(-E - (\frac{n}{2}+j)I\right) I_{\frac{n}{2}+j} Q_{2j} f(x)\right) \\
&= -\sqrt{2j+1+\frac{n-1}{2}}\frac{1}{\sqrt{c_{2j}}}(D^+)^{2j}\left(\phi(x)\left(Q_{2j}f(x)\right)\right) \\
&= -\sqrt{2j+1+\frac{n-1}{2}}\; f_{2j}(x).
\end{aligned}$$

Thus, we have shown that $D^- f_k(x) = -\sqrt{k + \frac{n-1}{2}} f_{k-1}(x)$ holds for each $k \in \mathbb{N}$.

Finally, from equation (18) we obtain that

$$\begin{aligned}
\mathcal{H}_0 f_k &= -\frac{1}{2}(D^-(D^+ f_k) + D^+(D^- f_k)) \\
&= \frac{1}{2}\left(\left(k+1+\frac{n-1}{2}\right) f_k + \left(k + \frac{n-1}{2}\right) f_k\right) = \left(k + \frac{n}{2}\right) f_k.
\end{aligned}$$

■

We will conclude this subsection by showing the mutual orthogonality between the spaces $\mathcal{F}_k^{+-}$. This provides us with a direct decomposition of the $\mathbb{R}_{0,n}-$module $L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$.

Theorem 3.2. *The spaces $\mathcal{F}_k^{+-}$ are mutually orthonormal with respect to the bilinear form $\langle\cdot,\cdot\rangle$, i.e.*

$$\mathcal{F}_k^{+-}\perp_{\langle\cdot,\cdot\rangle}\mathcal{F}_l^{+-} \text{ for } k\neq l.$$

Each $\Psi_k\in\mathcal{F}_k^{+-}$ satisfies $\langle\Psi_k,\Psi_k\rangle=1$.

Moreover, the following direct decomposition of $L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$ holds:

$$L_2(\mathbb{R}^n;\mathbb{R}_{0,n})=\sum_{k=0}^{\infty}\oplus_{\langle\cdot,\cdot\rangle}\mathcal{F}_k^{+-}.$$

Proof: We first observe that $\langle\Psi,\Psi\rangle=1$ and $D^-(\Psi(x))=0$ hold from construction. Then, the adjoint property (23) implies that for each $k\in\mathbb{N}$, $\Psi_k(x)=\frac{1}{\sqrt{c_k}}(D^+)^k(\Psi(x))$ is orthogonal to $\Psi(x)$:

$$\langle\Psi,\Psi_k\rangle=-\frac{1}{\sqrt{c_k}}\left\langle D^-\Psi,(D^+)^{k-1}(\Psi)\right\rangle=0.$$

Moreover, if $l\geq k>0$, then the combination of (23) with the raising/lowering property for f_k (Theorem 3.1) leads to

$$\begin{aligned}\langle\Psi_l,\Psi_k\rangle &= \frac{1}{\sqrt{(l+\frac{n-1}{2})(k+\frac{n-1}{2})}}\langle D^+\Psi_{l-1},D^+\Psi_{k-1}\rangle\\ &= -\frac{1}{\sqrt{(l+\frac{n-1}{2})(k+\frac{n-1}{2})}}\langle D^-D^+\Psi_{l-1},\Psi_{k-1}\rangle\\ &= \sqrt{\frac{l+\frac{n-1}{2}}{k+\frac{n-1}{2}}}\langle\Psi_{l-1},\Psi_{k-1}\rangle\end{aligned}$$

By induction, the preceding calculation results into

$$\langle\Psi_l,\Psi_k\rangle=\sqrt{\frac{c_l}{c_k}}\langle\Psi,\Psi_{l-k}\rangle=\sqrt{\frac{c_l}{c_k}}\delta_{k,l}.$$

We have proved the mutual orthonormality between the spaces $\mathcal{F}_k^{+-}$.

The statement of the direct decomposition of $L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$ in terms of $\mathcal{F}_k^{+-}$ is then an immediate consequence following from the Fourier expansion for $f\in L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$: $f=\sum_{k=0}^{\infty}\langle f,\Psi_k\rangle\Psi_k$. ■

3.3. Series Representation in terms of Clifford-Hermite functions or polynomials. Before we proceed to the construction of series involving Clifford-Hermite functions and Clifford-Hermite polynomials, we will start to analyze the operator $\mathcal{H}_0 = \frac{1}{2}\left(-\Delta + |x|^2 I\right)$ by means of symmetries of $\mathfrak{sl}_2(\mathbb{R})$ (the even part of $\mathfrak{osp}(1|2)$).

We start with the following lemma, which relates the symmetries of $\mathcal{H}_0$ with the symmetries of the Hamiltonian $\mathcal{J}_0$:

$$\mathcal{J}_0 = -\frac{\Delta}{2} + \frac{1}{2}\left(E + \frac{n}{2}I\right), \tag{35}$$

and moreover, the symmetries of $\mathcal{J}_0$ with the symmetries of $\frac{1}{2}\left(E + \frac{n}{2}I\right)$.

Lemma 3.4. *The operators $\mathcal{H}_0$, $\mathcal{J}_0$ and $\frac{1}{2}\left(E + \frac{n}{2}I\right)$ are interrelated by*

$$\exp\left(\frac{X^2}{2}\right)\mathcal{J}_0 = \mathcal{H}_0 \exp\left(\frac{X^2}{2}\right), \tag{36}$$

$$\exp\left(-\frac{\Delta}{2}\right)\left(E + \frac{n}{2}I\right) = \mathcal{J}_0 \exp\left(-\frac{\Delta}{2}\right). \tag{37}$$

Proof: Recall that from Lemma (11) the operators $P^+ = -\frac{\Delta}{2}$, $P^- = \frac{X^2}{2}$ and $Q = \frac{1}{2}\left(E + \frac{n}{2}I\right)$ satisfy

$$[Q, P^+] = P^+, \ \ [Q, P^-] = -P^-, \ \ [P^-, P^+] = Q.$$

Proof of relation (36):

Use the relation $\left[-\frac{\Delta}{2}, \frac{X^2}{2}\right] = \frac{1}{2}\left(E + \frac{n}{2}I\right)$. By an induction argument on $k \in \mathbb{N}$ we obtain

$$\begin{aligned}
-\frac{\Delta}{2}\left(\frac{X^2}{2}\right)^k &= \left(\frac{1}{2}\left(E + \frac{n}{2}I\right) + \frac{X^2}{2}\left(-\frac{\Delta}{2}\right)\right)\left(\frac{X^2}{2}\right)^{k-1} \\
&= \frac{1}{2}\left(E + \frac{n}{2}I\right)\left(\frac{X^2}{2}\right)^{k-1} + \left(\frac{1}{2}\left(E + \frac{n}{2}I\right) + \frac{X^2}{2}\left(-\frac{\Delta}{2}\right)\right)\left(\frac{X^2}{2}\right)^{k-2} \\
&= \left(E + \frac{n}{2}I\right)\left(\frac{X^2}{2}\right)^{k-1} + \left(\frac{X^2}{2}\right)^2 (-\frac{\Delta}{2})^2 \left(\frac{X^2}{2}\right)^{k-2} \\
&= \dots \\
&= \frac{k}{2}\left(E + \frac{n}{2}I\right)\left(\frac{X^2}{2}\right)^{k-1} + \left(\frac{X^2}{2}\right)^k (-\frac{\Delta}{2})^2,
\end{aligned}$$

or equivalently, $\left[-\frac{\Delta}{2},\left(\frac{X^2}{2}\right)^k\right]=\frac{k}{2}\left(E+\frac{n}{2}I\right)\left(\frac{X^2}{2}\right)^{k-1}$.

On the other hand, applying the same order of ideas, by induction over $k\in\mathbb{N}$ one can show that $\left[\frac{1}{2}(E+\frac{n}{2}I),(\frac{X^2}{2})^k\right]=k(\frac{X^2}{2})^k=k\frac{X^2}{2}\left(\frac{X^2}{2}\right)^{k-1}$. Hence,

$$\left[\frac{1}{2}\left(E+\frac{n}{2}I\right),\exp\left(\frac{X^2}{2}\right)\right]=\frac{X^2}{2}\exp\left(\frac{X^2}{2}\right).$$

Combining the above mentioned two relations with each other, then the commuting relation

$$\left[-\frac{\Delta}{2}+\frac{1}{2}\left(E+\frac{n}{2}I\right),\exp\left(\frac{X^2}{2}\right)\right]=\left(\frac{X^2}{2}+\frac{1}{2}\left(E+\frac{n}{2}I\right)\right)\exp\left(\frac{X^2}{2}\right)$$

follows from linearity. After straightforward simplifications we immediately get

$$\left(-\frac{\Delta}{2}-\frac{X^2}{2}\right)\exp\left(\frac{X^2}{2}\right)=\exp\left(\frac{X^2}{2}\right)\left(-\frac{\Delta}{2}+\frac{1}{2}\left(E+\frac{n}{2}I\right)\right),$$

that is, $\mathcal{H}_0\exp\left(\frac{X^2}{2}\right)=\exp\left(\frac{X^2}{2}\right)\mathcal{J}_0$.

Proof of relation (37):

For the proof of (37), we use the relation $\left[\frac{1}{2}\left(E+\frac{n}{2}I\right),-\frac{\Delta}{2}\right]=\frac{\Delta}{2}$. Following the same order of ideas that we applied in the proof of (36), we get

$$\left[\frac{1}{2}\left(E+\frac{n}{2}I\right),\exp(-\frac{\Delta}{2})\right]=\frac{\Delta}{2}\exp(-\frac{\Delta}{2}).$$

This is equivalent to $\mathcal{J}_0\exp(-\frac{\Delta}{2})=\exp\left(-\frac{\Delta}{2}\right)\left(\frac{1}{2}\left(E+\frac{n}{2}I\right)\right)$. ■

Let us now restrict ourselves to the space of Clifford algebra-valued homogeneous polynomials of total degree k:

$$\mathcal{P}_k=\{f\in\mathcal{P}\ :\ P(t\underline{x})=t^kP(\underline{x}),\ \forall\ t\in\mathbb{R},\forall\ \underline{x}\in\mathbb{R}^n\}.$$

Recall that we have a direct decomposition of $\mathcal{P}$ of the form:

$$\mathcal{P}=\sum_{k=0}^{\infty}\bigoplus\mathcal{P}_k. \tag{38}$$

This follows from the fact that $\mathcal{P}_k$ are eigenspaces for the so-called Euler operator E with eigenvalue k.

On the other hand, it is clear that $\mathcal{P}_k$ is a subset of the generalized poly-Bargmann space $\mathcal{F}_{k+1}$ (see (26)). Notice that $D^{k+1}P_k(x) = 0$ is true for each k (c.f. [24]).

In the sequel, we also need to use the subspaces $\mathcal{P}_k \cap \ker \Delta$ (the so-called space of spherical harmonics of degree k) and $\mathcal{P}_k \cap \ker D$ (the so-called space of spherical monogenics of degree k). These spaces correspond to closed subspaces of $\mathcal{P}_k$ that satisfy one of the following coupled systems of equations, respectively:

$$\Delta f = 0, \quad Ef = kf \tag{39}$$

$$Df = 0, \quad Ef = kf \tag{40}$$

Decompositions $\mathcal{P}_k$ in terms of spherical harmonics resp. monogenics of lower degrees are given by the Almansi/Fischer decomposition obtained in the book [14] and extended in [24] for poly-harmonic resp. poly-monogenic functions of degree $k+1$ supported on star-like domains.

Recall that Fischer's decomposition ([14], Theorem 1.10.1) gives a direct decomposition of $\mathcal{P}_k$ in terms of spherical monogenics of lower degrees:

$$\mathcal{P}_k = \sum_{s=0}^{k} \bigoplus X^s \left(\mathcal{P}_{k-s} \cap \ker D\right)$$

while [14], Corollary 1.3.3 gives a refinement of spherical harmonics in terms of spherical monogenics:

$$\mathcal{P}_s \cap \ker \Delta = (\mathcal{P}_s \cap \ker D) \oplus X\left(\mathcal{P}_{s-1} \cap \ker D\right)$$

Moreover, each $P_k \in \mathcal{P}_k \cap \ker \Delta$ corresponds to $P_k(x) = M_k(x) + xM_{k-1}(x)$, where $M_k \in \mathcal{P}_k \cap \ker D$ and $M_{k-1} \in \mathcal{P}_{k-1} \cap \ker D$ are uniquely determined using the projection operators $\mathbb{P}$ and $\mathbb{Q}$:

$$\mathbb{P} = I + \frac{1}{2k+n-2} XD : \mathcal{P}_k \cap \ker \Delta \to \mathcal{P}_k \cap \ker D$$

$$\mathbb{Q} = -\frac{1}{2k+n-2} XD : \mathcal{P}_k \cap \ker \Delta \to X\left(\mathcal{P}_{k-1} \cap \ker D\right)$$

i.e. $M_k(x) = \mathbb{P}(P_k(x))$ and $xM_{k-1}(x) = \mathbb{Q}(P_k(x))$.

Alternatively, in terms of the integral operators I_s defined by (31) we have, $M_k(x) = (I + \frac{1}{2}XI_{\frac{n}{2}}D)P_k(x)$ and $xM_{k-1}(x) = -\frac{1}{2}XI_{\frac{n}{2}}(DP_k(x))$ (c.f. [24]).

The next theorem will give us the building blocks to construct Clifford-Hermite polynomials resp. functions in interplay with Sommen's approach from [28] :

Theorem 3.3. *For $k \in \mathbb{N}_0$, define*

$$\begin{aligned}\mathcal{P}_k^{\Delta} &= \left\{ P_k^{\Delta}(x) = \exp\left(-\frac{\Delta}{2}\right) P_k(x) \ : \ P_k \in \mathcal{P}_k \right\},\\ \mathcal{P}_k^{+-} &= \left\{ P_k^{+-}(x) = \phi(x) P_k^{\Delta}(x) \ : \ P_k^{\Delta} \in \mathcal{P}_k^{\Delta} \right\}.\end{aligned}$$

Then $\mathcal{P}_k^{\Delta}$ and $\mathcal{P}_k^{+-}$ are eigenspaces for $\mathcal{J}_0$ and $\mathcal{H}_0$ corresponding to the eigenvalue $k+\frac{n}{2}$.

Proof: Let $P_k \in \mathcal{P}_k$ and set $P_k^{\Delta}(x) = \exp\left(-\frac{\Delta}{2}\right) P_k(x)$, $P_k^{\Delta}(x) = \exp\left(-\frac{\Delta}{2}\right) P_k(x)$.

From the homogeneity of P_k, we obtain that $(E+\frac{n}{2}I)P_k(x) = (k+\frac{n}{2})P_k(x)$. Next, a direct application of Lemma 3.4 leads to

$$\mathcal{J}_0 P_k^{\Delta}(x) = \exp\left(-\frac{\Delta}{2}\right)\left(E+\frac{n}{2}I\right) P_k(x) = \left(k+\frac{n}{2}\right) P_k^{\Delta}(x),$$

$$\begin{aligned}\mathcal{H}_0 P_k^{+-}(x) &= \pi^{-\frac{n}{4}} \exp\left(\frac{X^2}{2}\right)\left(\mathcal{J}_0 P_k^{\Delta}(x)\right)\\ &= \left(k+\frac{n}{2}\right)\left(\phi(x) P_k^{\Delta}(x)\right) = \left(k+\frac{n}{2}\right) P_k^{+-}(x).\end{aligned}$$

This proves that $\mathcal{P}_k^{\Delta}$ and $\mathcal{P}_k^{+-}$ are eigenspaces for $\mathcal{J}_0$ and $\mathcal{H}_0$, respectively, with eigenvalue $k+\frac{n}{2}$. ■

Now we are able to establish a parallel between our approach and the approach obtained in [28] by Sommen:

Firstly, we recall that the Cauchy-Kowaleskaya extension (see [14], Subsection 5.1) of $f(x)$ in terms of the Cauchy-Riemann operator $\mathcal{D} = \frac{\partial}{\partial x_{n+1}} + \overline{\mathbf{e}_{n+1}} D$ corresponds to the solution of the Cauchy problem on $\mathbb{R}\times[0,\infty)$ with initial data $F(x,0)=f(x)$:

$$\begin{cases} \frac{\partial}{\partial x_{n+1}} F(x,x_{n+1}) = -\overline{\mathbf{e}_{n+1}} D F(x,x_{n+1}), & \text{if } x_{n+1} > 0 \\ F(x,0) = f(x), & \text{if } x_{n+1} = 0. \end{cases} \tag{41}$$

For $f(x) = P_k^{\Delta}(x)$, the Cauchy-Kowaleskaya extension results into the infinite series representation in terms of $M_k(x,x_{n+1}) = \exp\left(-x_{n+1}\overline{\mathbf{e}_{n+1}} D\right) P_k(x) \in$

$\ker \mathcal{D}$:

$$F(x,x_{n+1}) = \exp\left(-x_{n+1}\overline{\mathbf{e}_{n+1}}D\right) P_k^{\Delta}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j j!}\Delta^j(M_k(x,x_{n+1}))$$

The latter representation shall be understood as the series representation for the inversion of the Segal-Bargmann transform applied on the spaces of monogenic functions on $\mathbb{R}^n \times [0,\infty)$ (c.f. [10]).

On the other hand, from $I_s(P_k(x)) = \frac{1}{k+s}P_k(x)$, it is clear from the construction of U_k by (34) that $U_k(P_k(x)) = u_k P_k(x)$ for some $u_k \in \mathbb{R}$.

So, on the basis of the integral representation (24) we can say that, if we take $P_k \in \mathcal{P}_k$ (an element of the space $\mathcal{F}_{k+1}$) such that $\langle P_k, P_k\rangle_{\mathcal{F}} = \frac{1}{u_k^2}$, then it is clear that $\Psi(x) = \phi(x)U_k(P_k(x))$ is a normalized vector on $L_2(\mathbb{R}^n;\mathbb{R}_{0,n})$:

$$\langle \Psi, \Psi\rangle = \langle U_k P_k, U_k P_k\rangle_{\mathcal{F}} = u_k^2\langle P_k, P_k\rangle_{\mathcal{F}} = 1.$$

Moreover, from Theorem 3.1, 3.2 and 3.3, the characterizations that we are going to recall now for Hermite Clifford-Hermite functions resp. polynomials are in fact rather obvious (c.f. [28], Section 5):

- **Normalized Clifford-Hermite functions of degree k:** The functions of the type $\Psi_k(x) = \frac{1}{\sqrt{c_k}}(D^+)^k(\Psi(x))$ that belong to $\mathcal{F}_k^{+-}$;
- **Normalized Clifford-Hermite polynomials of degree k:** The functions of the type $\psi_k(x) = \pi^{\frac{n}{4}}e^{\frac{|x|^2}{2}}\Psi_k(x)$ are eigenfunctions of the Hamiltonian $\mathcal{J}_0$ with eigenvalue $k+\frac{n}{2}$ are mutually orthonormal in the Fock space $\mathcal{F}$ (see (25)).
- **Generating functions:** The Cauchy-Kowaleskaya extension

  $$F(x,x_{n+1}) = \exp(-x_{n+1}\overline{\mathbf{e}_{n+1}}D)\psi_k(x)$$

  obtained from (41) gives a generating function in terms of Clifford-Hermite polynomials while

  $$\phi(x)F(x,x_{n+1}) = \phi(x)\exp(-x_{n+1}\overline{\mathbf{e}_{n+1}}D)\psi_k(x)$$

  gives a generating function in terms of Clifford-Hermite functions.

In conclusion, the construction of Clifford-Hermite functions resp. polynomials obtained from the above discussed constructions yield in a combinatorial way by means of the Fock space formalism (c.f. [16]) and, contrary to [28], this approach does not require *a priori* any knowledge of Cauchy's integral formula to assure the mutual orthonormality of the Clifford-Hermite polynomials resp. functions.

4. Solutions of the time-harmonic Maxwell equations with additional angular part

4.1. Symmetries and Series Representation of Solutions. The main purpose of this section is to introduce some generalized Clifford algebra valued operators that allow us to describe the solutions of Maxwell's equations by means of the theory of spherical monogenic resp. harmonic functions.

At a first glance, we will start to prove the existence of an isomorphism between the algebras of Clifford operators $\mathcal{A}^n$ and

$$\mathcal{A}^n_\lambda = \text{span}\left\{X - \lambda + \frac{2\lambda}{n}\Gamma, D - \lambda + \frac{2\lambda}{n}\Gamma, \xi_j \ : \ j = 1, \ldots, n\right\}$$

by means of the $\exp\left(\frac{\lambda}{n}(D - X)\right)$-action:

Lemma 4.1. *The action of the operator*

$$\exp\left(\frac{\lambda}{n}(D - X)\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{n^k k!}(D - X)^k$$

on $\mathcal{S}(\mathbb{R}^n; \mathbb{R}_{0,n})$ gives rise to

$$\exp\left(\frac{\lambda}{n}(D - X)\right) \ D = \left(D + \frac{2\lambda}{n}\Gamma - \lambda I\right)\exp\left(\frac{\lambda}{n}(D - X)\right),$$
$$\exp\left(\frac{\lambda}{n}(D - X)\right) \ X = \left(X + \frac{2\lambda}{n}\Gamma - \lambda I\right)\exp\left(\frac{\lambda}{n}(D - X)\right).$$

Proof: Recall that from $XD = -E - \Gamma$ and $\{X, D\} = -2E - nI$ (Lemma 2.1), it follows that $[X, D] = -2\Gamma + nI$. Hence,

$$\left[D, \frac{\lambda}{n}(D - x)\right] = \frac{\lambda}{n}[x, D] = -\frac{2\lambda}{n}\Gamma + \lambda I = \left[X, \frac{\lambda}{n}(D - x)\right]$$

Applying mathematical induction over $k \in \mathbb{N}_0$ results into

$$\left[D, \left(\frac{\lambda}{n}(X - D)\right)^k\right] \ = \ k\left(-\frac{2\lambda}{n}\Gamma + \lambda I\right)\left(\frac{\lambda}{n}(D - X)\right)^{k-1}$$
$$\left[X, \left(\frac{\lambda}{n}(X - D)\right)^k\right] \ = \ k\left(-\frac{2\lambda}{n}\Gamma + \lambda I\right)\left(\frac{\lambda}{n}(D - X)\right)^{k-1}.$$

Therefore, from the series expansion of $\exp(\frac{\lambda}{n}(D-X))$, we obtain that

$$\begin{aligned}
\left[D, \exp\left(\frac{\lambda}{n}(D-X)\right)\right] &= \left(-\frac{2\lambda}{n}\Gamma + \lambda I\right)\exp(\frac{\lambda}{n}(D-X)) \\
\left[X, \exp\left(\frac{\lambda}{n}(D-X)\right)\right] &= \left(-\frac{2\lambda}{n}\Gamma + \lambda I\right)\exp(\frac{\lambda}{n}(D-X)).
\end{aligned}$$

After applying straightforward algebraic manipulations, we arrive at

$$\begin{aligned}
\exp\left(\frac{\lambda}{n}(D-X)\right)D &= \left(D + \frac{2\lambda}{n}\Gamma - \lambda I\right)\exp\left(\frac{\lambda}{n}(D-X)\right), \\
\exp\left(\frac{\lambda}{n}(D-X)\right)X &= \left(X + \frac{2\lambda}{n}\Gamma - \lambda I\right)\exp\left(\frac{\lambda}{n}(D-X)\right).
\end{aligned}$$

■

The above established lemmas show that the operators $D + \frac{2\lambda}{n}\Gamma - \lambda I$ and $X + \frac{2\lambda}{n}\Gamma - \lambda I$ play the same role as the standard classical operators X and D, respectively. This is due to the fact that the action of $\exp\left(\frac{\lambda}{n}(D-X)\right)$ preserves the (anti-)commutation relations.

Next, we will describe the series representation of the solutions for PDEs of the type

$$\left(D - \lambda + \frac{2\lambda}{n}\Gamma\right) f_\lambda = g_\lambda, \quad \text{with } g \in \ker\left(D - \lambda I + \tfrac{2\lambda}{n}\Gamma\right)^s. \tag{42}$$

To do so, we will apply the following multiplication rule on $\mathcal{S}(\mathbb{R}^n;\mathbb{R}_{0,n})$:

$$\exp\left(\frac{\lambda}{n}(D-X)\right) = \exp\left(-\frac{X^2}{2}\right)\exp\left(\frac{\lambda}{n}D\right)\exp\left(\frac{X^2}{2}\right).$$

This equality follows from the relations (20) and from the series expansion of $\exp\left(\frac{\lambda}{n}D\right)$ in the C^∞-topology.

The next proposition provides us with the key ingredient to compute the expressions $f_\lambda := \exp(\frac{\lambda}{n}(D-X))f$ from $\mathcal{S}(\mathbb{R}^n,\mathbb{R}_{0,n})$:

Proposition 4.1. *When acting on $\mathcal{S}(\mathbb{R}^n;\mathbb{R}_{0,n})$, the operator*

$$\exp\left(\frac{\lambda}{n}(D-X)\right) = \exp\left(\frac{X^2}{2}\right)\exp\left(\frac{\lambda}{n}D\right)\exp\left(-\frac{X^2}{2}\right)$$

is represented by the following series expansion that converges in the C^∞-topology:

$$\exp\left(\frac{\lambda}{n}(D-X)\right)=\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{n^{2k}(2k)!}\left(I+\frac{\lambda}{n(2k+1)}(D-X)\right)(2\mathcal{J}_0)^k$$

Here, $\mathcal{J}_0$ is the Hamiltonian operator defined by the equation (35).

Proof: From the definition of $\exp\left(\frac{\lambda}{n}D\right)$ we know that we can split the series described above in the way

$$\exp\left(\frac{\lambda}{n}D\right)=\cosh\left(\frac{\lambda}{n}D\right)+\sinh\left(\frac{\lambda}{n}D\right),$$

with

$$\cosh\left(\frac{\lambda}{n}D\right)=\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{n^{2k}(2k)!}(-\Delta)^k,\ \text{ and }\ \sinh\left(\tfrac{\lambda}{n}D\right)=\textstyle\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{n^{2k+1}(2k+1)!}D(-\Delta)^k.$$

From (11) we know that the operators $P^-=-\frac{\Delta}{2}$, $P^+=\frac{X^2}{2}$ and $Q=\frac{1}{2}\left(E+\frac{n}{2}I\right)$ (canonical generators of $\mathfrak{sl}_2(\mathbb{R})$) satisfy

$$[P^-,P^+]=Q,\ \ [Q,P^-]=-P^-,\ \ [Q,P^+]=P^+.$$

This leads to $\left[-\Delta,-\frac{X^2}{2}\right]=-\left(E+\frac{n}{2}I\right)$. Therefore,

$$\left[-\Delta,\exp\left(-\frac{X^2}{2}\right)\right]=-\left(E+\frac{n}{2}I\right)\exp\left(\frac{X^2}{2}\right)\tag{43}$$

follows straightforwardly from induction arguments and from the formal series expansion of $\exp\left(\frac{X^2}{2}\right)$.

This equality is equivalent to

$$\left(-\Delta+E+\frac{n}{2}I\right)\exp\left(-\frac{X^2}{2}\right)=\exp\left(-\frac{X^2}{2}\right)(-\Delta).$$

So, we may further derive that

$$\begin{aligned}\exp\left(-\frac{X^2}{2}\right)\cosh\left(\frac{\lambda}{n}D\right)\exp\left(\frac{X^2}{2}\right)&=\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{n^{2k}(2k)!}\exp\left(\frac{X^2}{2}\right)(-\Delta)^k\exp\left(\frac{X^2}{2}\right)\\&=\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{n^{2k}(2k)!}(-\Delta+E+\frac{n}{2}I)^k.\end{aligned}$$

For the computation of $\exp\left(-\frac{X^2}{2}\right)\sinh\left(\frac{\lambda}{n}D\right)\exp\left(\frac{X^2}{2}\right)$ we rely on the relation $(D-X)\exp(-\frac{X^2}{2})=\exp(-\frac{X^2}{2})D$ which follows from (20).

Standard computations yield

$$\begin{aligned}\exp\left(-\frac{X^2}{2}\right)\sinh\left(\frac{\lambda}{n}D\right)\exp\left(\frac{X^2}{2}\right)&=\\&=\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{n^{2k+1}(2k+1)!}\exp\left(\frac{X^2}{2}\right)D(-\Delta)^k\exp\left(\frac{X^2}{2}\right)\\&=\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{n^{2k+1}(2k+1)!}(D-X)\left(-\Delta+E+\frac{n}{2}I\right)^k.\end{aligned}$$

In view of (35) we have $-\Delta+E+\frac{n}{2}I=2\mathcal{J}_0$, so we can complete the proof of Proposition 4.1 by applying linearity arguments. ■

Remark 4.1. *Notice that* $\ker\left(D-\lambda I+\frac{2\lambda}{n}\Gamma\right)^0$ *coincides with the trivial subspace* $\{0\}$*. Hence, using the series representation in the* $C^\infty-$*topology for* f_λ *that is given in Proposition 4.1, we can establish that*

$$f_\lambda=\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{n^{2k}(2k)!}\left((2\mathcal{J}_0)^k f+\frac{\lambda}{n(2k+1)}(D-X)\,(2\mathcal{J}_0)^k f\right)$$

is a solution of (42) that belongs to $\ker\left(D-\lambda+\frac{2\lambda}{n}\Gamma\right)$ *in the case when* $s=0$.

Moreover, from Lemma 4.1 we can conclude that the above described series representation is a solution of (42) whenever f *belongs to* $\mathcal{F}_{s+1}$ *(see (26)).*

The next theorem provides us with a meaningful characterization for the series representation for the solutions of the PDE system (42):

Theorem 4.1. *Let* $P_{\lambda,s}(x)=\exp\left(\frac{\lambda}{n}(D-X)\right)\left(\exp(-\frac{\Delta}{2})P_s(x)\right)$*, where* $P_s\in\mathcal{P}_s$ *(i.e.* P_s *is homogeneous of total degree* s*). Then* $P_{\lambda,s}$ *is a solution of the PDE system (42) with the following properties:*

1. $P_{\lambda,s}$ *is a solution of the eigenvalue problem*

$$\Delta f(x)+\frac{\lambda}{n}(D-X)f(x)=\left(E+\left(\frac{n}{2}-2s\right)I\right)f(x)$$

2. $P_{\lambda,s}$ *is explicitly given by*

$$\begin{aligned} P_{\lambda,s}(x) &= \cosh\left(\frac{\lambda}{n}\sqrt{2s+n}\right) P_s^{\Delta}(x) + \frac{1}{\sqrt{2s+n}}\sinh\left(\frac{\lambda}{n}\sqrt{2s+n}\right)(D-X)P_s^{\Delta}(x) \\ &= \exp\left(-\frac{\Delta}{2}\right)\left(\cosh\left(\frac{\lambda}{n}\sqrt{2s+n}\right) P_s(x) - \frac{1}{\sqrt{2s+n}}\sinh\left(\frac{\lambda}{n}\sqrt{2s+n}\right) xP_s(x)\right). \end{aligned}$$

3. If $P_s \in \ker D$*, then*

$$P_{\lambda,s}(x) = \cosh\left(\sqrt{2s+n}\frac{\lambda}{n}\right) P_s(x) - \frac{1}{\sqrt{2s+n}}\sinh\left(\sqrt{2s+n}\frac{\lambda}{n}\right) x\ P_s(x)$$

is a solution of the coupled system of equations:

$$\begin{cases} (D-\lambda I)f(x) = -\frac{2\lambda}{n}\ \Gamma\left(f(x)\right), \\ E(f(x)) + \left(\frac{n}{2} - 2s\right) f(x) = -\frac{2\lambda}{n} x f(x) \end{cases}$$

Moreover, $P_{\lambda,s}(x)$ *is harmonic, i.e.* $P_{\lambda,s} \in \ker\Delta$.

Proof: We will prove the statements 1, 2 and 3 separately:
Proof of Statement 1: Notice that for $P_s \in \mathcal{P}_s$, the expression $P_s^{\Delta}(x) := \exp\left(-\frac{\Delta}{2}\right) P_s(x)$ satisfies

$$\mathcal{J}_0\left(P_s^{\Delta}(x)\right) = sP_s^{\Delta}(x),$$

where $\mathcal{J}_0$ is the Hamiltonian operator defined in (35).

First we show that $\exp\left(\frac{\lambda}{n}(D-X)\right)\mathcal{J}_0 = \left(\mathcal{J}_0 + \frac{\lambda}{n}X\right)\exp\left(\frac{\lambda}{n}(D-X)\right)$. From the relations (11), we may infer that

$$\left[\frac{1}{2\sqrt{2}}iX, -\frac{\Delta}{2}\right] = \frac{1}{2\sqrt{2}}iD, \quad \left[\tfrac{1}{2}\left(E+\tfrac{n}{2}I\right), \tfrac{1}{2\sqrt{2}}iD\right] = -\tfrac{1}{2\sqrt{2}}iD$$

$$\left[\frac{1}{2}\left(E+\frac{n}{2}I\right), \frac{1}{2\sqrt{2}}iX\right] = \frac{1}{2\sqrt{2}}iX \qquad \left[\tfrac{1}{2\sqrt{2}}iD, -\tfrac{\Delta}{2}\right] = 0$$

This in turn leads to

$$\begin{aligned}\left[\mathcal{J}_0, \frac{\lambda}{n}(D-X)\right] &= \frac{\lambda}{n}\Big(\left[-\frac{\Delta}{2}, D\right] - \left[-\frac{\Delta}{2}, X\right] + \left[\frac{1}{2}\left(E+\frac{n}{2}I\right), D\right] - \\ &\qquad - \left[\frac{1}{2}\left(E+\frac{n}{2}I\right), X\right]\Big) \\ &= \frac{\lambda}{n}\left(0 + D - D - X\right) \\ &= -\frac{\lambda}{n}X\end{aligned}$$

and hence, $\left[\mathcal{J}_0, \exp\left(\frac{\lambda}{n}(D-X)\right)\right] = -\frac{\lambda}{n}X\exp\left(\frac{\lambda}{n}(D-X)\right)$.

The latter equation is equivalent to

$$\left(\mathcal{J}_0 + \frac{\lambda}{n}X\right)\exp\left(\frac{\lambda}{n}(D-X)\right) = \exp\left(\frac{\lambda}{n}(D-X)\right)\mathcal{J}_0.$$

Therefore, for $P_{\lambda,s}(x) = \exp\left(\frac{\lambda}{n}(D-X)\right)P_s^{\Delta}(x)$ we obtain

$$\left(\mathcal{J}_0 + \frac{\lambda}{n}X\right)P_{\lambda,s}(x) = \exp\left(\frac{\lambda}{n}(D-X)\right)\left(\mathcal{J}_0 P_s^{\Delta}(x)\right) = sP_{\lambda,s}(x)$$

This is equivalent to

$$\Delta(P_{\lambda,s}(x)) - \frac{2\lambda}{n}X(P_{\lambda,s}(x)) = E(P_{\lambda,s}(x)) + \left(\frac{n}{2} - 2s\right)P_{\lambda,s}(x).$$

Proof of Statement 2: From Proposition 4.1 we know that we can express $P_{\lambda,s}(x) = \exp\left(\frac{\lambda}{n}(D-X)\right)P_s^{\Delta}(x)$ as follows

$$\begin{aligned}P_{\lambda,s}(x) &= \sum_{k=0}^{\infty}\frac{\lambda^{2k}}{n^{2k}k!}\left(\left(2\mathcal{J}_0\right)^k P_s^{\Delta}(x) + \frac{\lambda}{n(2k+1)}(D-X)\left(2\mathcal{J}_0\right)^k P_s^{\Delta}(x)\right) \\ &= \sum_{k=0}^{\infty}\frac{\lambda^{2k}(2s+n)^k}{n^{2k}k!}\left(P_s^{\Delta}(x) + \frac{\lambda}{n(2k+1)}(D-X)P_s^{\Delta}(x)\right).\end{aligned}$$

Next we apply the series expansion of the hyperbolic functions $t \mapsto \cosh(t)$ and $t \mapsto \sinh(t)$. This allows us to conclude that the latter expression is equivalent to the following expression in the $C^\infty-$topology:

$$\begin{aligned} P_{\lambda,s}(x) &= \cosh\left(\frac{\lambda}{n}\sqrt{2s+n}\right)P_s^{\Delta}(x)+ \quad (44)\\ &+\frac{1}{\sqrt{2s+n}}\sinh\left(\frac{\lambda}{n}\sqrt{2s+n}\right)(D-X)P_s^{\Delta}(x). \quad (45)\end{aligned}$$

Using the relation $[\Delta, X] = 2D$ (see Lemma 2.1) and applying induction arguments together with the series expansion for $\exp\left(-\frac{\Delta}{2}\right)$, allows us to establish the relation

$$[X, \exp\left(-\frac{\Delta}{2}\right)] = D\exp\left(-\frac{\Delta}{2}\right).$$

This is equivalent to $(D-X)\exp\left(-\frac{\Delta}{2}\right) = \exp\left(-\frac{\Delta}{2}\right)(-X)$.

Therefore, equation (44) is equivalent to

$$\begin{aligned} P_{\lambda,s}(x) = \exp\left(-\frac{\Delta}{2}\right)\Bigg(\cosh\left(\frac{\lambda}{n}\sqrt{2s+n}\right)P_s(x)-\\ -\frac{1}{\sqrt{2s+n}}\sinh\left(\frac{\lambda}{n}\sqrt{2s+n}\right)xP_s(x)\Bigg).\end{aligned}$$

This completes the proof of statement 2.

Proof of Statement 3: Since $P_s \in \ker D$, we obtain $-\Delta P_s(x) = D(DP_s(x)) = 0$. Therefore,

$$P_s^{\Delta}(x) = \exp(-\frac{\Delta}{2})P_s(x) = P_s(x).$$

Hence, the validity of the expression for $P_{\lambda,s}(x)$ in the following form

$$P_{\lambda,s}(x) = \cosh\left(\sqrt{2s+n}\frac{\lambda}{n}\right)P_s(x) - \frac{1}{\sqrt{2s+n}}\sinh\left(\sqrt{2s+n}\frac{\lambda}{n}\right)x\ P_s(x)$$

follows by inserting $P_s^{\Delta}(x) = P_s(x)$ into the equation that we previously obtained in Statement 2.

Finally, the validity of the equation

$$E(P_{\lambda,s}(x)) + \left(\frac{n}{2}-2s\right)P_{\lambda,s}(x) = -\frac{2\lambda}{n}x(P_{\lambda,s}(x))$$

follows by inserting $P_s^{\Delta}(x) = P_s(x)$ into the equation obtained in statement 1. The validity of the equation $(D-\lambda I)P_{\lambda,s}(x) = -\frac{2\lambda}{n}\Gamma P_{\lambda,s}(x)$ follows from Lemma 4.1.

Moreover, the validity of the property $P_{\lambda,s} \in \ker \Delta$ is a consequence from the property $[\Delta, X] = 2D$ and from the inclusion property $\ker D \subset \ker \Delta$. ■

4.2. Relation with Landau Operators. In this subsection we will obtain a derivation of Landau operators that describe symmetries in electromagnetism (c.f. [19]) in terms of $\mathfrak{osp}(1|2)$ symmetries. Moreover, we are able to derive series representations that involve the solutions of the time-harmonic Maxwell equations with angular part i.e. the null solutions of $D - \lambda I + \frac{2\lambda}{n}\Gamma$.

In what follows we will exhibit the relations between the spectra of $\mathcal{H}_0$ and the Landau operator $\mathcal{H}_\lambda = \widetilde{\mathcal{H}}_\lambda + \mathcal{L}_\lambda$. Without loss of generality, let us consider for unit mass and frequency, the Landau operator (3) written in terms of the operators $D \mp \lambda I$, X and Γ:

$$\mathcal{H}_\lambda = \frac{1}{2}\left((D - \lambda I)(D + \lambda I) - X^2 - \frac{2\lambda}{n}\left(X - \left(I + \frac{2\lambda}{n}\Gamma\right)\Gamma\right)\right) \tag{46}$$

We start by deriving the analogues of $D^\pm$ in terms of the $\exp\left(\frac{\lambda}{n}D\right)$ −action:

Lemma 4.2. *When acting on $End(\mathcal{S}(\mathbb{R}^n; \mathbb{R}_{0,n}))$, we obtain*

$$\exp\left(\frac{\lambda}{n}D\right) D^+ \exp\left(-\frac{\lambda}{n}D\right) = \frac{1}{\sqrt{2}}\left(X - (D + \lambda I) + \frac{2\lambda}{n}\Gamma\right), \tag{47}$$

$$\exp\left(\frac{\lambda}{n}D\right) D^- \exp\left(-\frac{\lambda}{n}D\right) = \frac{1}{\sqrt{2}}\left(X + (D - \lambda I) + \frac{2\lambda}{n}\Gamma\right). \tag{48}$$

Proof: When we follow the same lines of the proof of Lemma 4.1, then we obtain for $D^- = \frac{1}{\sqrt{2}}(X - D)$ and $D^+ = \frac{1}{\sqrt{2}}(X + D)$

$$\left[D^\mp, \exp\left(\frac{\lambda}{n}D\right)\right] = \frac{\lambda}{n\sqrt{2}}\,[x, D]\exp\left(\frac{\lambda}{n}D\right) = \left(\frac{\lambda}{\sqrt{2}}I - \frac{\sqrt{2}\lambda}{n}\Gamma\right)\exp\left(\frac{\lambda}{n}D\right).$$

A rearrangement the terms in the identity presented above leads to

$$\exp\left(\frac{\lambda}{n}D\right) D^+ = \left(D^+ - \frac{\lambda}{\sqrt{2}}I + \frac{\sqrt{2}\lambda}{n}\Gamma\right)\exp\left(\frac{\lambda}{n}D\right),$$

$$\exp\left(\frac{\lambda}{n}D\right) D^- = \left(D^- - \frac{\lambda}{\sqrt{2}}I + \frac{\sqrt{2}\lambda}{n}\Gamma\right)\exp\left(\frac{\lambda}{n}D\right).$$

Finally, the equations (47) follow after straightforward algebraic manipulations. ■

Here we would like to stress that the isomorphism $\exp\left(\frac{\lambda}{n}D\right)$ preserves the (anti-)commuting relations, and hence, the relations (11) can be lifted from X, D and E for the operators $X-\lambda I+\frac{2\lambda}{n}\Gamma$, D and $E+\frac{n}{2}I+\frac{\lambda}{n}D$, respectively. This corresponds to the proposition given below:

Proposition 4.2. *The elements P^+, P^-, Q, R^+, R^- defined by*

$$\begin{array}{lll} P^- = -\frac{1}{2}\Delta, & P^+ = \frac{1}{2}\left(X-\lambda+\frac{2\lambda}{n}\Gamma\right)^2, & Q = \frac{1}{2}\left(E+\frac{n}{2}I\right)+\frac{\lambda}{2n}D \\ R^- = \frac{1}{2\sqrt{2}}iD & R^+ = \frac{1}{2\sqrt{2}}i\left(X-\lambda+\frac{2\lambda}{n}\Gamma\right), & \end{array}$$

are the generators of the orthosymplectic Lie algebra $\mathfrak{osp}(1|2)$.

The next lemma shows that $\mathcal{H}_\lambda$ can be decomposed in terms of the ladder operators $D^\pm - \frac{\lambda}{\sqrt{2}}I + \frac{\sqrt{2}\lambda}{n}\Gamma$:

Lemma 4.3. *When acting on $End(\mathcal{S}(\mathbb{R}^n;\mathbb{R}_{0,n}))$, we have*

$$\left\{X-(D+\lambda I)+\frac{2\lambda}{n}\Gamma, X+(D-\lambda I)+\frac{2\lambda}{n}\Gamma\right\} = -2\mathcal{H}_\lambda$$

Proof: In view of the definition we can split

$$\left\{X-(D+\lambda I)+\frac{2\lambda}{n}\Gamma, X+(D-\lambda I)+\frac{2\lambda}{n}\Gamma\right\}$$

as

$$\begin{aligned} -\left\{D+\lambda I, D-\lambda I\right\} - \left\{D+\lambda I, X+\frac{2\lambda}{n}\Gamma\right\} + \\ + \left\{X+\frac{2\lambda}{n}\Gamma, D-\lambda I\right\} + \left\{X+\frac{2\lambda}{n}\Gamma, X+\frac{2\lambda}{n}\Gamma\right\}. \end{aligned} \tag{49}$$

The terms $-\{D+\lambda I, D-\lambda I\}$ and

$$-\left\{D+\lambda I, X+\frac{2\lambda}{n}\Gamma\right\} + \left\{X+\frac{2\lambda}{n}\Gamma, D-\lambda I\right\}$$

are equal to $2(\Delta+\lambda^2 I)$ and $-4\lambda\left(X+\frac{2\lambda}{n}\Gamma\right)$, respectively, while

$$\begin{aligned}\left\{X+\frac{2\lambda}{n}\Gamma, X+\frac{2\lambda}{n}\Gamma\right\} &= 2\left(X+\frac{2\lambda}{n}\Gamma\right)^2 \\ &= 2\left(X^2+\left\{X,\frac{2\lambda}{n}\Gamma\right\}+\left(\frac{2\lambda}{n}\Gamma\right)^2\right) \\ &= 2X^2-4\lambda\frac{n+1}{n}X+2\left(\frac{2\lambda}{n}\Gamma\right)^2\end{aligned}$$

follows from $\{\Gamma, X\}=-(n+1)X$ (Lemma 2.1) and from the decomposition $(S+T)^2=S^2+\{S,T\}+T^2$.

Thus, rearranging all the above expressions, equation (49) is equivalent to

$$2\left(\Delta+\lambda^2 I\right)+2X^2+\frac{4\lambda}{n}\left(X-\left(I+\frac{2\lambda}{n}\Gamma\right)\Gamma\right)=-2\mathcal{H}_\lambda.$$

This completes our proof. ■

Remark 4.2. *In the spirit of* [36], *the operators* $\frac{1}{\sqrt{2}}\left(X-(D+\lambda I)+\frac{2\lambda}{n}\Gamma\right)$ *and* $\frac{1}{\sqrt{2}}\left(X+(D-\lambda I)+\frac{2\lambda}{n}\Gamma\right)$ *may be interpreted as hypercomplex extensions of the canonical creation resp. annihilation operators on the quaternionic field which are equivalent to* D^- *resp.* D^+ *under the* $\exp\left(\frac{\lambda}{n}D\right)$ *−action.*

In the spirit of [13], $\exp\left(\frac{\lambda}{n}D\right)$ *plays the same role as the wave-packet transform encoded in the cross-Wigner distribution. On the other hand, from Proposition 4.2, the magnetic Laplacian* $\mathcal{H}_\lambda$ *written in (49) can be obtained by the covariant action* $X \leftarrow X-\lambda+\frac{2\lambda}{n}\Gamma$ *on the spherical potential* $-\frac{1}{2}X^2=\frac{1}{2}|x|^2 I$.

Let us now focus our attention on the construction of Clifford-Hermite functions resp. polynomials that we obtained in Subsection 3.3 and on the series representation for the solutions of the PDE system (42) obtained in Subsection 4.1:
From Theorem 4.1, we can draw the conclusion that f_λ may be written as a generating function in terms of Clifford-Hermite polynomials ψ_k:

$$f_\lambda(x)=\sum_{k=0}^{\infty}\frac{\lambda^k}{n^k\ k!}\psi_k(x).$$

This can be done following the same lines of the proof of statement 2 from Theorem 4.1. The latter result shall be interpreted as the Bargmann inversion formula in the space of Clifford algebra -valued polynomials $\mathcal{P}$ (c.f. [10]).

Moreover, since for each $\psi_k(x)$, the expression

$$\mathbb{P}(\psi_k(x)) = \psi_k(x) + \frac{1}{2k+n-2} x D(\psi_k(x))$$

is monogenic (c.f. [14], Corollary 1.3.3); each $f_\lambda \in \ker\left(D - \lambda + \frac{2\lambda}{n}\Gamma\right)$ may be represented in terms of the series expansion

$$f_\lambda(x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{n^k\ k!}\left(\psi_k(x) + \frac{1}{2k+n-2} x D(\psi_k(x))\right)$$

that corresponds to a generating function in terms of (monogenic) Clifford-Hermite polynomials.

Thus, by applying the multiplication formula

$$\exp\left(\frac{\lambda}{n}(D-X)\right) = \exp\left(-\frac{X^2}{2}\right)\exp\left(\frac{\lambda}{n}D\right)\exp\left(\frac{X^2}{2}\right)$$

and the characterization of Clifford Hermite functions resp. polynomials presented in Lemma 3.3, we can deduce the next theorem.

Theorem 4.2. *The following statements are true*

(1) *For each $\Psi_k \in \mathcal{F}_k^{+-}$ (see also Theorem 3.1), the series expansion*

$$\Psi_{\lambda,k}(x) := \exp\left(\frac{\lambda}{n}D\right)\Psi_k(x) = \sum_{k=0}^{\infty}\frac{\lambda^k}{n^k\ k!}D^k(\Psi_k(x))$$

satisfy the raising resp. lowering properties:

$$\begin{aligned} \frac{1}{\sqrt{2}}\left(X - (D+\lambda I) + \frac{2\lambda}{n}\Gamma\right)\Psi_{\lambda,k}(x) &= \sqrt{k+1+\frac{n-1}{2}}\Psi_{\lambda,k}(x), \\ \frac{1}{\sqrt{2}}\left(X + (D-\lambda I) + \frac{2\lambda}{n}\Gamma\right)\Psi_{\lambda,k}(x) &= -\sqrt{k+\frac{n-1}{2}}\Psi_{\lambda,k}(x). \end{aligned}$$

Moreover, $\Psi_{\lambda,k}$ is an eigenfunction for the magnetic Laplacian (46) with eigenvalue $k+\frac{n}{2}$.

(2) *For each $\Psi_k \in \mathcal{F}_k^{+-}$ such that Ψ_k is a Clifford-Hermite function of degree k (see also Theorem 4.1), the function*

$$P_{\lambda,k}(x) = \pi^{\frac{n}{4}} e^{\frac{|x|^2}{2}}\Psi_{\lambda,k}(x),$$

with $\Psi_{\lambda,k}(x) = \exp\left(\frac{\lambda}{n}D\right)\Psi_k(x)$, *gives a generating function in terms of Clifford-Hermite polynomials and corresponds to a solution of the eigenvalue problem*

$$\Delta f(x) + \frac{\lambda}{n}(D - X)f(x) = \left(E + \left(\frac{n}{2} - 2s\right) I\right) f(x)$$

(3) *For each* $\Psi_k \in \mathcal{F}_k^{+-}$ *such that* Ψ_k *is a Clifford-Hermite function of degree* k *(see also Theorem 4.1), the function*

$$P_{\lambda,k}(x) = \pi^{\frac{n}{4}}\mathbb{P}\left(e^{\frac{|x|^2}{2}}\Psi_{\lambda,k}(x)\right),$$

with $\Psi_{\lambda,k}(x) = \exp\left(\frac{\lambda}{n}D\right)\Psi_k(x)$, *corresponds to a solution of the coupled system of equations:*

$$\begin{cases} (D-\lambda I)f(x) = -\frac{2\lambda}{n}\,\Gamma\left(f(x)\right), \\ E(f(x)) + \left(\frac{n}{2} - 2k\right) f(x) = -\frac{2\lambda}{n} x f(x) \end{cases}$$

Moreover, $P_{\lambda,k}(x)$ *is harmonic, i.e.* $P_{\lambda,k} \in \ker\Delta$.

Remark 4.3. *This approach comprises Xu's approach* [35] *in the case when* $\Gamma f(x) = 0$ *(e.g.* f *is a radial function). In particular, a series representation for the null solutions of* $D - \lambda I$ *can be computed by using a generating function that encodes the eigenfunctions of the Landau operator* $\widetilde{\mathcal{H}}_\lambda = \frac{1}{2}\left((D-\lambda I)(D+\lambda I) - X^2\right)$.

Physically speaking, Xu's approach describes an electromagnetic field without the action of orbital electromagnetic sources (c.f. [7, 19]*).*

The above given results also provide us with some intriguing relations in terms of Perelomov coherent states for nilpotent Lie groups (c.f. [10], Subsection 1.3) that yield discrete series representations of the group $SU(1,1)$ being isomorphic to the symplectic group $\mathrm{Sp}(2,\mathbb{R})$. We will leave this issue for a future research topic (for further details see e.g. [27], Chapter 14).

In conclusion, the results obtained in this section correspond to the extension of De Gosson and Luef's approach [13], i.e. the link between standard Weyl calculus and Landau–Weyl calculus was obtained in the presence of $\mathfrak{osp}(1|2)-$symmetries, our recent results should also lead to important investigations in the study of the regularity and hypo-ellipticity of the solutions to Schrödinger equations in the context of the theory of modulation spaces.

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