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RESOLUTIONS, HIGHER EXTENSIONS AND THE RELATIVE MAL'TSEV AXIOM

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Dedicated to James Gray and Tamar Janelidze on the occasion of their wedding

ABSTRACT: We study how the concept of higher-dimensional extension which comes from categorical Galois theory relates to simplicial resolutions. For instance, an augmented simplicial object is a resolution if and only if its truncation in every dimension gives a higher extension, in which sense *resolutions are infinite-dimensional extensions* or *higher extensions are finite-dimensional resolutions.* We also relate certain stability conditions of extensions to the Kan property for simplicial objects. This gives a new proof of the fact that a regular category is Mal'tsev if and only if every simplicial object is Kan, using a relative setting of extensions.

KEYWORDS: higher extension; simplicial resolution; Mal'tsev condition; relative homological algebra; arithmetical category.
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Introduction

The concept of *higher-dimensional extension* first appeared in the approach to non-abelian homological algebra based on categorical Galois theory in semi-abelian categories. In that context *centrality* of higher extensions plays a very important role, but we do not treat this aspect in the current paper. We rather focus on stability conditions of the higher extensions themselves. Initially we work in a setting where all split epimorphisms are extensions, but in the last section we slightly change our conditions to include examples such as T. Janelidze's *relative homological* and *relative semi-abelian categories* [35, 36, 37].

A major point of this article is that certain properties of simplicial objects and simplicial resolutions are actually properties of the induced cubes and higher extensions. As a consequence, some proofs (see, for instance, Proposition 3.11) which are rather technical when considered from the simplicial point of view become almost trivial when higher extensions are used instead.

Background on higher (central) extensions. Already the article [8] of R. Brown and G. J. Ellis on higher Hopf formulae for groups was based on a notion of higher extension of groups. Following G. Janelidze's ideas set out in [29, 30] and extending the theory from [33], *higher-dimensional central extensions* were introduced alongside a general categorical concept of *higher-dimensional extension* in [17] to study homology in semi-abelian categories. The theory presented there allows for an interpretation of the canonical comonadic homology objects induced by the reflection of a semi-abelian variety to a subvariety in terms of (higher) Hopf formulae, generalising those obtained in [8] to contexts beyond the case of abelianisation of groups. For instance, if B is a loop and $B \cong A/K$ a projective presentation of B, then

$$\mathrm{H}_2(B, \mathsf{gp}) \cong \frac{K \cap [A, A, A]}{[K, A, A]},$$

where $\mathrm{H}_2(B, \mathsf{gp})$ is the second homology of B relative to the category of groups (i.e., with coefficients in the reflector $\mathsf{gp}\colon \mathsf{Loop} \to \mathsf{Gp}$) and the brackets on the right hand side are associators [18].

The article [17] gives calculations of the homology objects for groups vs. abelian groups, rings vs. zero rings, precrossed modules vs. crossed modules, Lie algebras vs. modules, groups vs. groups of a certain nilpotency or solvability class, etc., in all dimensions. This approach to homology was extended to cover other examples [15, 16] and several theoretical perspectives were explored:

slightly different approaches [13, 11, 32], links with higher-dimensional and relative commutator theory [12, 14, 18, 20], first steps towards an interpretation of cohomology [25, 44], the characterisation of higher central extensions [19], and satellites [21, 23].

This gives an indication of the importance of higher central extensions in non-abelian homological algebra, in particular in homology and cohomology of non-abelian algebraic objects. However, they could not exist without higher extensions themselves, and in this paper we examine certain stability conditions that higher extensions may have. This leads to strong results on simplicial objects, which of course also play an important role in the study of homology.

Higher extensions. Classically, one-dimensional extensions are just regular epimorphisms in a regular category $\mathcal{A}$, which, in the varietal case, are exactly the surjections. Denoting by $\mathcal{E}$ the class of extensions in $\mathcal{A}$, a **double extension** is a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

in $\mathcal{A}$ where the morphisms a, b, f_1, f_0 and the universally induced morphism $\langle a, f_1\rangle\colon A_1 \to A_0 \times_{B_0} B_1$ to the pullback of b and f_0 are in $\mathcal{E}$. We denote the class of double extensions thus obtained by $\mathcal{E}^1$. Of course this definition does not depend on the exact nature of one-dimensional extensions, so it can be used for any (reasonable) class of morphisms $\mathcal{E}$. In particular, it can be iterated to give n-fold extensions for any $n \geq 2$: then all the arrows in the induced diagram are $(n-1)$-fold extensions. We write $\mathsf{Ext}\mathcal{A}$ for the full subcategory of the category of arrows $\mathsf{Arr}\mathcal{A}$ in $\mathcal{A}$ determined by the extensions, and similarly $\mathsf{Ext}^n\mathcal{A}$ for the full subcategory of the category of n-fold arrows $\mathsf{Arr}^n\mathcal{A} = \mathsf{Arr}\mathsf{Arr}^{n-1}\mathcal{A}$ determined by the n-fold extensions. We denote the class of $(n+1)$-fold extensions by $\mathcal{E}^n$. By treating extensions axiomatically, as described below, we can deal with the pairs $(\mathsf{Ext}^n\mathcal{A}, \mathcal{E}^n)$ just like the "base case" $(\mathcal{A}, \mathcal{E})$, since such a pair is just another example of a category with a class of extensions. This makes the statements and proofs of many results much easier, and also clarifies in which other situations the results may hold.

Axioms for extensions. Treating extensions axiomatically (rather than ad hoc, as in [17]) has the following advantage. Because the set of axioms is such that it "goes up" to higher dimensions, as first formulated by T. Everaert in [11] and [13], it allows a simultaneous treatment of extensions in all dimensions without having to remember which dimension is currently needed.

The main list of axioms for a class of extensions $\mathcal{E}$ in a category $\mathcal{A}$ considered in this paper is the following.

(E1) $\mathcal{E}$ contains all isomorphisms;
(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
(E3) $\mathcal{E}$ is closed under composition;
(E4) if $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$ (right cancellation);
(E5) the **$\mathcal{E}$-Mal'tsev axiom**: any split epimorphism of extensions

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[d, "a"'] & B_1 \arrow[l, shift left] \arrow[d, "b"] \\ A_0 \arrow[r, shift left, "f_0"] & B_0 \arrow[l, shift left] \end{tikzcd}$$

in $\mathcal{A}$ is a double extension.

These axioms come in slightly different flavours and are not all treated at once. The first three, (E1)–(E3), go up to higher dimensions without help of the others and already imply the important fact that *higher extensions are symmetric*. Axioms (E1)–(E5) are the setting of Section 3. In fact, (E5) is equivalent to (E4) applied to $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ and implies axiom (E4) for $(\mathcal{A}, \mathcal{E})$. In Section 4 we weaken (E4) in such a way that not all split epimorphisms need to be in the class $\mathcal{E}$, and add an axiom ensuring a certain stability of split epimorphisms which do fall into $\mathcal{E}$. The axiom (E5) in its absolute form comes from D. Bourn's [6]; see also [3].

Resolutions vs. extensions. In Section 2 we assume that the pair $(\mathcal{A}, \mathcal{E})$ satisfies axioms (E1)–(E3). We compare higher extensions satisfying these axioms to *simplicial $\mathcal{E}$-resolutions*, which are augmented simplicial objects in which all comparison morphisms to the simplicial kernels are morphisms in $\mathcal{E}$. Truncating an augmented simplicial object induces higher dimensional arrows, and we prove in Theorem 2.17 that the augmented simplicial object is an $\mathcal{E}$-resolution if and only if each of these truncations gives rise to a higher dimensional extension. In this sense

resolutions are infinite-dimensional extensions
or
higher extensions are finite-dimensional resolutions.

This is, in fact, also how they are used in practice, for example in [8] or [17].

The Kan property and Mal'tsev conditions. In Section 3 we work with a pair $(\mathcal{A}, \mathcal{E})$ satisfying (E1)–(E5). In fact, under (E1)–(E4) we prove that

(E5) holds $\Leftrightarrow$ *every simplicial object in* $\mathcal{A}$ *is* $\mathcal{E}$*-Kan*

(Theorem 3.13). This justifies calling (E5) the relative Mal'tsev axiom, as it is well known that a regular category $\mathcal{A}$ is Mal'tsev if and only if every simplicial object in $\mathcal{A}$ is Kan [9, Theorem 4.2]. As a first indication on the usefulness of the relative Kan property we prove that

contractible + $\mathcal{E}$*-Kan* $\Rightarrow$ $\mathcal{E}$*-resolution*

for augmented $\mathcal{E}$-simplicial objects (Proposition 3.9). Here an $\mathcal{E}$-simplicial object is one in which all faces ∂_i are extensions, and such an object is $\mathcal{E}$-Kan when all comparison morphisms to the universal horn objects are in $\mathcal{E}$.

Weaker conditions on extensions. Axioms (E1) and (E4) together imply that all split epimorphisms are extensions. However, this is not the case in all examples of interest. In Section 4 we substitute (E4) by the weaker axiom

(E4$^-$) if $f \in \mathcal{E}$ and $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$.

In this setting we have to slightly change the statement of (E5) and consider another axiom ensuring a certain stability of those split epimorphisms which do belong to $\mathcal{E}$. Thus we have

(E5$^-$) any split epimorphism of extensions

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[d, "a"'] & B_1 \arrow[l, shift left] \arrow[d, "b"] \\ A_0 \arrow[r, shift left, "f_0"] & B_0 \arrow[l, shift left] \end{tikzcd}$$

in $\mathcal{A}$ with f_1 and f_0 in $\mathcal{E}$ is a double extension;

(E6) given a split epimorphism of extensions

$$\begin{tikzcd} \mathrm{R}[a] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "r"'] & A_1 \arrow[r, "a"] \arrow[d, shift right, "f_1"'] & A_0 \arrow[d, shift right, "f_0"'] \\ \mathrm{R}[b] \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift right] & B_1 \arrow[r, "b"'] \arrow[u, shift right] & B_0 \arrow[u, shift right] \end{tikzcd}$$

with f_1 and f_0 in $\mathcal{E}$, taking kernel pairs of a and b gives an extension r.

An example for such a weaker setting is given by T. Janelidze's *relative homological* and *relative semi-abelian categories* [35, 37].

1. Axioms for extensions

We treat the concept of *higher-dimensional extension* [11, 13, 17] in an axiomatic manner, recalling the basic definitions and proving fundamental properties: symmetry, and the axioms of extensions going up to higher dimensions (Proposition 1.6).

Higher-dimensional arrows. To understand higher extensions, we must first define what we mean by a *higher-dimensional arrow*. As these play a very important role throughout the paper, we shall take some time to really understand these objects.

To set up a convenient numbering system for our higher-dimensional arrows, we consider the natural numbers by their standard (von Neumann) construction and write $0 = \emptyset$ and $n = \{0, \ldots, n-1\}$ for $n \geq 1$. We write 2^n for the powerset of n. Recall that 2^n is a category with an arrow $S \to T$ for each inclusion $S \subseteq T$ of subsets S, $T \subseteq n$. Clearly $2^1 = 2$, the category generated by a single morphism $0 \to 1$, is an obvious "template" for an arrow in a category.

Definition 1.1. The category $\mathsf{Arr}^n\mathcal{A}$ consists of **n-dimensional arrows** in $\mathcal{A}$: $\mathsf{Arr}^0\mathcal{A} = \mathcal{A}$, $\mathsf{Arr}^1\mathcal{A} = \mathsf{Arr}\mathcal{A}$ is the category of arrows $\mathsf{Fun}(2^{\mathrm{op}}, \mathcal{A}) = \mathcal{A}^{2^{\mathrm{op}}}$, and $\mathsf{Arr}^{n+1}\mathcal{A} = \mathsf{Arr}\mathsf{Arr}^n\mathcal{A}$.

Example 1.2. A zero-fold arrow is just an object of $\mathcal{A}$, a one-fold arrow is given by an arrow in $\mathcal{A}$, while a two-fold arrow A is a commutative square in $\mathcal{A}$ with a specified direction:

$$\begin{tikzcd} A_2 \arrow[r, "a_1"] \arrow[d, "a_0"'] \arrow[dr, phantom, "\Rightarrow"] & A_1 \arrow[d, "a_0^1"] \\ A_{\{1\}} \arrow[r, "a_0^{\{1\}}"'] & A_0. \end{tikzcd} \tag{A}$$

This particular numbering of the objects and arrows will become clear below, after Definition 1.3. Similarly an n-fold arrow is a commutative n-cube in $\mathcal{A}$ with specified directions. By definition a morphism (a natural transformation) between n-fold arrows is also an $(n+1)$-fold arrow.

Notice that, by induction, we have an isomorphism

$$\mathsf{Arr}^n\mathcal{A} \cong \mathcal{A}^{2^{\mathrm{op}}\times\cdots\times 2^{\mathrm{op}}}.$$

However, in the step which says that "a functor $2^{\mathrm{op}} \to \mathcal{A}^{2^{\mathrm{op}}}$ corresponds to a functor $2^{\mathrm{op}} \times 2^{\mathrm{op}} \to \mathcal{A}$" (and the higher versions of this) we may easily lose sight of the direction of the arrow, as $2^{\mathrm{op}} \times 2^{\mathrm{op}}$ is of course symmetric. This leads to the concept of the *n-cube* corresponding to an n-fold arrow, which we shall make more precise and connect to the issue above. Later we shall see that distinguishing between a cube and an arrow with directions is often not as important for our purposes as it may first seem.

Definition 1.3. Let $n \geq 0$. We define an **n-cube** in $\mathcal{A}$ to be a functor

$$A\colon (2^n)^{\mathrm{op}} \to \mathcal{A}.$$

A morphism between n-cubes A and B in $\mathcal{A}$ is a natural transformation $f\colon A \to B$. We write $\mathsf{Cub}^n\mathcal{A}$ for the corresponding category.

Thus an n-cube is a diagram of a specified shape in $\mathcal{A}$. Clearly a zero-cube is just an object of $\mathcal{A}$ and a one-cube is a morphism in $\mathcal{A}$, so we have $\mathsf{Cub}^0\mathcal{A} = \mathsf{Arr}^0\mathcal{A} = \mathcal{A}$ and $\mathsf{Cub}^1\mathcal{A} = \mathsf{Arr}\mathcal{A}$. A two-cube is a commutative diagram as above, but (a priori) without a specified direction.

Notice that $2 \times 2 \cong 2^2$ and similarly $2 \times 2^n \cong 2^{n+1}$, but these isomorphisms are not unique. Roughly speaking, the extra 1 can be inserted either "at the bottom" or "at the top" or even "somewhere in the middle", and this determines how the new object is numbered. From the existence of these isomorphisms we see that we can view every n-fold arrow as an n-cube, by replacing the directions with a specific numbering, and that the two categories $\mathsf{Arr}^n\mathcal{A}$ and $\mathsf{Cub}^n\mathcal{A}$ are isomorphic—but there are several possible isomorphisms which reflect the different ways a direction may correspond to the numbering of the objects. Also, a morphism between n-cubes can be viewed as an $(n+1)$-cube. Conversely an $(n+1)$-cube can be considered as an arrow between n-cubes in $n+1$ different ways.

Having chosen one of the isomorphisms $\mathsf{Arr}^n\mathcal{A} \to \mathsf{Cub}^n\mathcal{A}$ mentioned above, we may number an n-fold arrow by viewing it as an n-cube. If A is an n-fold arrow and S and T are subsets of n such that $S \subseteq T$, we write A_S for the image $A(S)$ of S by the functor A and $a_S^T \colon A_T \to A_S$ for the image $A(S \subseteq T)$ of $S \subseteq T$. If $f \colon A \to B$ is a morphism between n-fold arrows, we write $f_S \colon A_S \to B_S$ for the S-component of the natural transformation f. Moreover, in order to simplify our notations, we write a_i instead of $a^n_{n \setminus \{i\}}$, for $0 \leq i \leq n-1$. (See the picture of a double extension (**A**) above for an example.)

Convention 1.4. As mentioned above, there are several different isomorphisms between $\mathsf{Cub}^n\mathcal{A}$ and $\mathsf{Arr}^n\mathcal{A}$. We now describe one of these and we shall use this one throughout the paper. Given an n-cube $A \colon (2^n)^{\mathrm{op}} \to \mathcal{A}$, we see that each edge or one-fold arrow in A is of the form $A_{S \cup \{i\}} \to A_S$ for some $i \in n$ and some subset $S \subset n$ not containing i. All edges of this form with the same i are "parallel" in the n-cube. Thus for each k-cube inside A, we choose the direction to be that which corresponds to the largest such i. As an example, consider the following cube.

$$\begin{tikzcd}
 & A_{\{2,0\}} \arrow[rr] \arrow[dd, dotted] & & A_1 \arrow[dd] \\
A_3 \arrow[ur] \arrow[rr] \arrow[dd] & & A_2 \arrow[ur] \arrow[dd] & \\
 & A_{\{2\}} \arrow[rr, dotted] & & A_0 \\
A_{\{1,2\}} \arrow[ur, dotted] \arrow[rr] & & A_{\{1\}} \arrow[ur] &
\end{tikzcd}$$

Going from left to right is the direction of "leaving out 2", from front to back is "leaving out 1" and from top to bottom is "leaving out 0". Therefore the right and left square go from front to back, the front, back, top and bottom squares all go from left to right, and the whole cube also goes from left to right.

Proposition 1.16 will show us that remembering the specified directions of an n-fold arrow is often not necessary, so that we are mostly happy to use n-cubes and n-fold arrows synonymously without specifying the isomorphism between them.

Extensions. We now consider a class of morphisms $\mathcal{E}$ in a category $\mathcal{A}$ satisfying the following axioms:

(E1) $\mathcal{E}$ contains all isomorphisms;
(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
(E3) $\mathcal{E}$ is closed under composition.

Given such a class $\mathcal{E}$, we write $\mathcal{E}^1$ for the class of arrows $(f_1, f_0)\colon a \to b$

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & B_1 \arrow[d, "b"] \\
A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

in $\mathsf{Arr}\mathcal{A}$ such that all arrows in the induced diagram

$$\begin{tikzcd}
A_1 \arrow[dr, dotted] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "a"'] & & \\
& P \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, "b"] \\
& A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

are in $\mathcal{E}$. We write $\mathsf{Ext}\mathcal{A}$ for the full subcategory of $\mathsf{Arr}\mathcal{A}$ determined by the arrows in $\mathcal{E}$.

Remark 1.5. The pullback in the diagram above exists as we assume that b and f_0 are in $\mathcal{E}$, and (E1) ensures that there is no ambiguity in the choice of pullback.

Proposition 1.6. *Let $\mathcal{A}$ be a category and $\mathcal{E}$ a class of arrows in $\mathcal{A}$. If $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3), then $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ satisfies the same conditions.*

Proof: Mutatis mutandis the proof of [17, Proposition 3.5] may be copied. ■

Remark 1.7. Pullbacks of double extensions in $\mathsf{Ext}\mathcal{A}$ are computed degree-wise as in $\mathsf{Arr}\mathcal{A}$.

Remark 1.8. Notice that these axioms have a slightly different appearance to their corresponding ones in [13]: there it is important to keep track of the objects which can occur as domains or codomains of extensions. The class of these objects is called $\mathcal{E}^-$ and does not occur here because using $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ instead of $(\mathsf{Arr}\mathcal{A}, \mathcal{E}^1)$ automatically restricts us to the right domains and codomains. In [13] this extra care is needed because the construction of the (higher) centralisation functors depends on the categories $\mathsf{Arr}^n\mathcal{A}$ being semi-abelian (for $n \geq 0$). Note that, while $\mathcal{A}$ being semi-abelian implies that $\mathsf{Arr}\mathcal{A}$ is semi-abelian, in general $\mathsf{Ext}\mathcal{A}$ does not keep this property (cf. Proposition 4.11).

As we shall not be considering (higher) central extensions here, we can work directly with $\mathsf{Ext}\mathcal{A}$ and so dispense with the $\mathcal{E}^-$.

Definition 1.9. If $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3) then an element of $\mathcal{E}$ is called a **(one-fold) extension** of $\mathcal{A}$ and an element of $\mathcal{E}^1$ a **two-fold extension** or **double extension** of $\mathcal{A}$. We also write $\mathcal{E}^0$ for $\mathcal{E}$. By induction, we obtain a class of arrows $\mathcal{E}^n = (\mathcal{E}^{n-1})^1$ in $\mathsf{Arr}^n\mathcal{A}$ and a full subcategory $\mathsf{Ext}^n\mathcal{A}$ of $\mathsf{Arr}^n\mathcal{A}$ (determined by the elements of $\mathcal{E}^{n-1}$) for all $n \geq 2$. An object of $\mathsf{Ext}^n\mathcal{A}$ (= an element of $\mathcal{E}^{n-1}$) is called an **n-fold extension** of $\mathcal{A}$. We shall sometimes talk about **n-fold $\mathcal{E}$-extensions** or simply of **extensions**.

Remark 1.10. Notice that, when $\mathcal{A}$ has finite products, (E2) and (E3) imply that the product $f \times g$ of two extensions f and g is also an extension (to see this, observe that $f \times g = (f \times 1)\circ(1 \times g)$).

Example 1.11 (Regular epimorphisms)**.** If $\mathcal{A}$ is a regular category (finitely complete with coequalisers of kernel pairs and pullback-stable regular epimorphisms, see [2]) and $\mathcal{E}$ is the class of regular epimorphisms in $\mathcal{A}$, then the pair $(\mathcal{A}, \mathcal{E})$ satisfies conditions (E1)–(E3). Indeed, any isomorphism is a regular epimorphism, and regular epimorphisms are pullback-stable and closed under composition. The higher extensions obtained here are the ones considered in [17].

Example 1.12 (Projective classes)**.** Let $\mathcal{A}$ be a finitely complete category. Recall that a **projective class** on $\mathcal{A}$ is a pair $(\mathcal{P}, \mathcal{E})$, where $\mathcal{P}$ is a class of objects of $\mathcal{A}$ and $\mathcal{E}$ a class of morphisms of $\mathcal{A}$, such that $\mathcal{P}$ consists of all $\mathcal{E}$-projectives P, $\mathcal{E}$ consists of all $\mathcal{P}$-epimorphisms f, and $\mathcal{A}$ has enough $\mathcal{E}$-projectives.

$$\begin{tikzcd} & A \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists"] \arrow[r, "\forall"'] & B \end{tikzcd}$$

It is easily seen that $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3).

Clearly, when $\mathcal{A}$ is a regular category with enough regular projective objects and $\mathcal{E}$ is the class of regular epimorphisms in $\mathcal{A}$, we regain Example 1.11.

An extreme case is given by taking $\mathcal{P}$ to be the class of all objects of $\mathcal{A}$, so that $\mathcal{E}$ consists of all split epimorphisms; see also Example 3.16.

Example 1.13 (Effective descent morphisms)**.** Let $\mathcal{A}$ be a category with pullbacks and let $\mathcal{E}$ be the class of effective descent morphisms in $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{E})$

satisfies (E1)–(E3), and also the axiom (E4), which we will meet in Section 3; see e.g. [34].

Example 1.14 (Etale maps)**.** Recall that an étale map is the same as a local homeomorphism of topological spaces: a continuous map $f\colon A \to B$ such that for any element $a \in A$ there is an open neighbourhood U of a such that $f(U)$ is open in B and the restriction of f to a map $U \to f(U)$ is a homeomorphism. Taking $\mathcal{A}$ to be the category Top of topological spaces and $\mathcal{E}$ the class of étale maps, it is well known and easily verified that (E1)–(E3) hold for $(\mathcal{A}, \mathcal{E})$; see e.g. [41].

Example 1.15 (Topological groups)**.** Let GpTop be the category of topological groups. Since this category is regular, Example 1.11 implies that $(\mathsf{GpTop}, \mathcal{E})$ satisfies (E1)–(E3) when $\mathcal{E}$ is the class of all regular epimorphisms.

Another choice of $\mathcal{E}$ would be the class of morphisms which are split as morphisms in the category of topological spaces. It is easy to check that $(\mathcal{A}, \mathcal{E})$ satisfies the axioms (E1)–(E3). Similarly the category of topological groups together with all morphisms which are split as morphisms of groups satisfies (E1)–(E3). These two examples have been considered important elsewhere in the literature, see for instance [26] and [48] (cf. also [35, Example 3.3.3]). In fact, both these examples also satisfy the axiom (E4). More examples of this kind are the category of rings together with morphisms which are split in the category of abelian groups Ab and the category of R-modules for a ring R, with morphisms which are split in Ab.

There is an alternative way of looking at extensions which is inspired by [8] and [10].

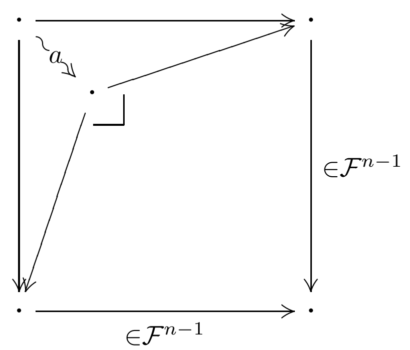
Proposition 1.16. *Given any n-fold arrow A, the following are equivalent:*

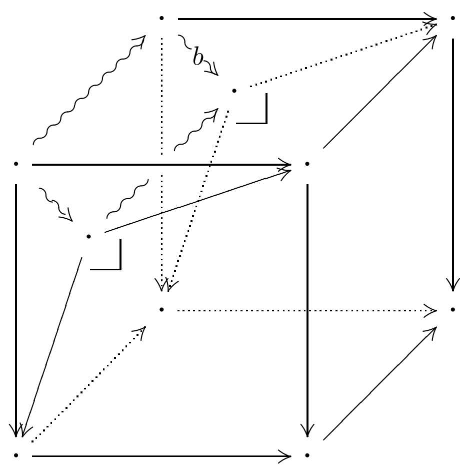
(i) *A is an extension;*
(ii) *for all $\emptyset \neq I \subseteq n$, the limit $\lim_{J \subsetneq I} A_J$ exists and the induced morphism $A_I \to \lim_{J \subsetneq I} A_J$ is in $\mathcal{E}$.*

Proof: We fix an isomorphism between $\mathsf{Arr}^n\mathcal{A}$ and $\mathsf{Cub}^n\mathcal{A}$, for instance the one described in Convention 1.4. For $n \geq 1$ let us denote the class of all n-cubes A in $\mathcal{A}$ that satisfy Condition (ii) by $\mathcal{F}^{n-1}$. Note that the classes $\mathcal{F}^n$ may also be defined inductively as follows. The class $\mathcal{F}^0$ is $\mathcal{E}$. Now suppose the class $\mathcal{F}^{n-1}$ is defined. Then $\mathcal{F}^n$ consists of all $(n+1)$-cubes A such that, when considered as an arrow between n-cubes in any of the $n+1$ possible directions, the codomain

n-cube is in $\mathcal{F}^{n-1}$, and moreover the limit $\lim_{J \subsetneq n+1} A_J$ exists and the induced morphism $A_{n+1} \to \lim_{J \subsetneq n+1} A_J$ is in $\mathcal{E}$.

We are to show for all n that $\mathcal{E}^n$ consists of all $(n+1)$-fold arrows A of which the corresponding $(n+1)$-cube is in $\mathcal{F}^n$. Using the fixed isomorphism between $\mathsf{Arr}^{n+1}\mathcal{A}$ and $\mathsf{Cub}^{n+1}\mathcal{A}$, we can denote this by $\mathcal{E}^n = \mathcal{F}^n$. For $n \in \{0,1\}$ we clearly have $\mathcal{E}^n = \mathcal{F}^n$. Now consider $n \geq 2$ and suppose that $\mathcal{E}^{i-1} = \mathcal{F}^{i-1}$ for all $i \leq n$. Let A be an $(n+1)$-fold arrow which is in $\mathcal{F}^n$. Then A is a square in $\mathsf{Ext}^{n-1}\mathcal{A}$ as below.





As A is in $\mathcal{F}^n$, both the right and bottom arrow of the square are elements of $\mathcal{F}^{n-1} = \mathcal{E}^{n-1}$, so their pullback exists. We have to check that the comparison morphism a is also in $\mathcal{E}^{n-1} = \mathcal{F}^{n-1}$. For this we must first check that all possible codomains of the $(n-1)$-cube a are in $\mathcal{F}^{n-2}$. For the direction given in the square this is clear. So consider any other direction, and extend the square to a cube in that direction as on the right hand side above. Then, as A is an element of $\mathcal{F}^n$, the back square of the cube is in $\mathcal{F}^{n-1} = \mathcal{E}^{n-1}$, so the factorisation b to the pullback, which is also the chosen codomain of a, is in $\mathcal{E}^{n-2} = \mathcal{F}^{n-2}$.

Secondly, it is easy to see that $\lim_{J \subsetneq n+1} A_J$ is the same as $\lim_{J \subsetneq n} a_J$, since a is the comparison to a pullback. Therefore a is in $\mathcal{F}^{n-1} = \mathcal{E}^{n-1}$ and A is in $\mathcal{E}^n$.

Conversely, suppose A is in $\mathcal{E}^n$. Then A is again a square in $\mathsf{Ext}^{n-1}\mathcal{A}$ as before. This time we know that the right and bottom arrows are in $\mathcal{E}^{n-1} = \mathcal{F}^{n-1}$, but we must also show it for any other direction. Pick such a direction and extend the square to a cube as before. We are to show that the back square is in $\mathcal{F}^{n-1}$. We already know this of the right and bottom squares, so the right and bottom arrow of the back square are in $\mathcal{E}^{n-2}$. Now the comparison a is in $\mathcal{E}^{n-1} = \mathcal{F}^{n-1}$, so its codomain b is in $\mathcal{E}^{n-2}$, which shows that the back square is in $\mathcal{E}^{n-1} = \mathcal{F}^{n-1}$.

Finally, the limits $\lim_{J\subsetneq n+1} A_J$ and $\lim_{J\subsetneq n} a_J$ are again the same, so as a is in $\mathcal{F}^{n-1}$, the $(n+1)$-fold arrow A is in $\mathcal{F}^n$, as desired. ∎

Remark 1.17. The condition $I \neq \emptyset$ in (ii) just means that we do not demand A_0 to have global support, that is, we do not demand the unique morphism $A_0 \to 1$ to the terminal object 1 to be an extension.

Remark 1.18. This proves that, in the case of surjective group homomorphisms (which is an instance of Example 1.11), our higher extensions coincide with the *exact cubes* considered in [10]; see also [8].

Depending on which is more convenient, from now on we shall use either of these characterisations of extensions.

Remark 1.19. Proposition 1.16 implies that, for an n-fold arrow A, to be an extension is rightfully a property of the corresponding n-cube of A. The independence of the chosen isomorphism between $\mathsf{Arr}^n\mathcal{A}$ and $\mathsf{Cub}^n\mathcal{A}$ means that this property is preserved by all functors $\mathsf{Cub}^n\mathcal{A} \to \mathsf{Cub}^n\mathcal{A}$ induced by an automorphism of 2^n. Therefore we may sometimes say that an n-cube *is* an extension, and the distinction between the different isomorphisms between the categories $\mathsf{Arr}^n\mathcal{A}$ and $\mathsf{Cub}^n\mathcal{A}$ becomes less important.

2. Resolutions and extensions

In this section we analyse the concept of simplicial $\mathcal{E}$-resolution in terms of n-fold $\mathcal{E}$-extensions. Our main result in this section is Theorem 2.17 which states that an augmented semi-simplicial object $\mathbb{A}$ is an $\mathcal{E}$-resolution if and only if the induced n-fold arrows $\mathsf{arr}_n\mathbb{A}$ are n-fold extensions for all $n \geq 1$. From now on we assume that the pair $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3).

$\mathcal{E}$-resolutions. We start by giving the necessary definitions leading up to that of an $\mathcal{E}$-resolution.

Definition 2.1 (Augmented semi-simplicial objects)**.** Let $\mathcal{A}$ be a category. The category $\mathsf{S}_s^+\mathcal{A}$ of **(augmented) semi-simplicial objects** in $\mathcal{A}$ and morphisms between them is the functor category $\mathsf{Fun}((\Delta_s^+)^{\mathrm{op}}, \mathcal{A})$, where Δ_s^+ is the augmented semi-simplicial category. Its objects are the finite ordinals $n \geq 0$ and its morphisms are (compositions of) monotone maps $n \to n+1$. For a functor

$$\mathbb{A}\colon (\Delta_s^+)^{\mathrm{op}} \to \mathcal{A},$$

we denote the objects $\mathbb{A}(n)$ by A_{n-1}, and the image of the inclusion $n \to n+1$ which does not reach i by ∂_i, so that an augmented semi-simplicial object $\mathbb{A}$

corresponds to the following data: a sequence of objects $(A_n)_{n\geq -1}$ with face operators (or faces) $(\partial_i\colon A_n \to A_{n-1})_{0\leq i\leq n}$ for $0 \leq n$,

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; A_2 \; \substack{\xrightarrow{\partial_0}\\\xrightarrow{\partial_1}\\\xrightarrow[\partial_2]{}} \; A_1 \; \substack{\xrightarrow{\partial_0}\\\xrightarrow[\partial_1]{}} \; A_0 \xrightarrow{\partial_0} A_{-1}$$

subject to the identities

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$$

for $i < j$. The morphism $\partial_0\colon A_0 \to A_{-1}$ is called the **augmentation** of $\mathbb{A}$.

Definition 2.2 (Augmented quasi-simplicial objects)**.** In addition to face operators, an **(augmented) quasi-simplicial object** $\mathbb{A}$ in $\mathcal{A}$ has degeneracy operators (or degeneracies) $(\sigma_i\colon A_n \to A_{n+1})_{0\leq i\leq n}$ for $0 \leq n$, subject to the identities

$$\partial_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases}$$

The augmented quasi-simplicial objects in $\mathcal{A}$ with the natural augmented quasi-simplicial morphisms between them form a category $\mathsf{S}_q^+\mathcal{A}$ which may be seen as a functor category $\mathsf{Fun}((\Delta_q^+)^{\mathrm{op}}, \mathcal{A})$.

Definition 2.3. If, in addition to the above, an augmented quasi-simplicial object $\mathbb{A}$ satisfies

$$\sigma_i \circ \sigma_j = \sigma_{j+1} \circ \sigma_i$$

for all $i \leq j$, we recover the usual definition of **(augmented) simplicial object**.

Example 2.4 (Comonadic resolutions and Tierney-Vogel resolutions)**.** Given a comonad $\mathbb{G} = (G, \epsilon, \delta)$ on a category $\mathcal{A}$, each object A in $\mathcal{A}$ can be extended to an augmented simplicial object $\mathbb{A} = \mathbb{G}A$ by setting $A_{-1} = A$ and $A_i = G^{i+1}A$ for $i \geq 0$, with faces $\partial_i = G^i\epsilon_{G^{n-i}A}\colon G^{n+1}A \to G^nA$ and degeneracies $\sigma_i = G^i\delta_{G^{n-i}A}\colon G^{n+1}A \to G^{n+2}A$ (see e.g. [1]). This gives a genuine augmented simplicial object.

Tierney-Vogel resolutions [46, 47] of an object on the other hand only give rise to an augmented quasi-simplicial object. Such a resolution is obtained in a category with a projective class $(\mathcal{P}, \mathcal{E})$ by covering $A = A_{-1}$ by a projective object, then successively taking simplicial kernels (see Definition 2.7 below) and covering these by a projective object again.

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[rr, shift left=3] \arrow[rr, shift left=1] \arrow[rr, shift right=1] \arrow[rr, shift right=3] \arrow[dr, "\mathcal{E}\ni"'] & & P_2 \arrow[rr, shift left=2, "\in\mathcal{P}"] \arrow[rr] \arrow[rr, shift right=2] \arrow[dr, "\mathcal{E}\ni"'] & & P_1 \arrow[rr, shift left=1, "\in\mathcal{P}"] \arrow[rr, shift right=1] \arrow[dr, "\mathcal{E}\ni"'] & & P_0 \arrow[rr, "\in\mathcal{P}", "\in\mathcal{E}"'] & & A_{-1} \\
& K_3 \arrow[ur, shift left=3] \arrow[ur, shift left=1] \arrow[ur, shift right=1] \arrow[ur, shift right=3] & & K_2 \arrow[ur, shift left=2] \arrow[ur] \arrow[ur, shift right=2] & & K_1 \arrow[ur, shift left=1] \arrow[ur, shift right=1] & & &
\end{tikzcd}
$$

This does induce degeneracies which commute with the face operators as demanded, but they may not commute with each other as required for a simplicial object.

Definition 2.5 (Contractibility)**.** An augmented semi-simplicial object $\mathbb{A}$ is **contractible** when there is a sequence of morphisms $(\sigma_{-1}\colon A_{n-1} \to A_n)_{0\leq n}$ such that

$$\partial_0\circ\sigma_{-1} = 1 \qquad \text{and} \qquad \partial_i\circ\sigma_{-1} = \sigma_{-1}\circ\partial_{i-1}$$

for all $i \geq 1$.

Example 2.6. Given a comonad $\mathbb{G}$ on a category $\mathcal{A}$, any augmented simplicial object of the form $\mathbb{G}GA$ is contractible.

Definition 2.7 (Simplicial kernels)**.** Let

$$(f_i\colon X \to Y)_{0\leq i\leq n}$$

be a sequence of $n+1$ morphisms in the category $\mathcal{A}$. A **simplicial kernel** of $(f_0, \ldots, f_n)$ is a sequence

$$(k_i\colon K \to X)_{0\leq i\leq n+1}$$

of $n+2$ morphisms in $\mathcal{A}$ satisfying $f_i k_j = f_{j-1} k_i$ for $0 \leq i < j \leq n+1$, which is universal with respect to this property. In other words, it is the limit for a certain diagram in $\mathcal{A}$.

For example, the simplicial kernel of one morphism is just its kernel pair. When simplicial kernels of a particular augmented semi-simplicial object $\mathbb{A}$ exist, we can factor $\mathbb{A}$ through its simplicial kernels as follows.

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[rr, shift left=3] \arrow[rr, shift left=1] \arrow[rr, shift right=1] \arrow[rr, shift right=3] \arrow[dr] & & A_2 \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] \arrow[dr] & & A_1 \arrow[rr, shift left=1] \arrow[rr, shift right=1] \arrow[dr] & & A_0 \arrow[rr] & & A_{-1} \\
& K_3\mathbb{A} \arrow[ur, shift left=3] \arrow[ur, shift left=1] \arrow[ur, shift right=1] \arrow[ur, shift right=3] & & K_2\mathbb{A} \arrow[ur, shift left=2] \arrow[ur] \arrow[ur, shift right=2] & & K_1\mathbb{A} \arrow[ur, shift left=1] \arrow[ur, shift right=1] & & &
\end{tikzcd}
$$

Here the $K_{n+1}\mathbb{A}$ are the simplicial kernels of the morphisms $(\partial_i)_i\colon A_n \to A_{n-1}$. We may also sometimes write $K_0\mathbb{A} = A_{-1}$.

Definition 2.8. If all faces ∂_i of an (augmented) semi-simplicial object $\mathbb{A}$ are in $\mathcal{E}$, we call $\mathbb{A}$ an **(augmented) $\mathcal{E}$-semi-simplicial object**.

Definition 2.9. An (augmented) semi-simplicial object $\mathbb{A}$ is said to be **$\mathcal{E}$-exact at** A_{n-1} when the simplicial kernel $K_n\mathbb{A}$ exists and the factorisation $A_n \to K_n\mathbb{A}$ is in $\mathcal{E}$.

An augmented semi-simplicial object $\mathbb{A}$ is called an **$\mathcal{E}$-resolution (of A_{-1})** when $\mathbb{A}$ is $\mathcal{E}$-exact at A_n for all $n \geq -1$.

Remark 2.10. An $\mathcal{E}$-resolution is always an augmented $\mathcal{E}$-semi-simplicial object.

Notation 2.11. Let $\mathbb{A}$ be an augmented semi-simplicial object in $\mathcal{A}$. We can form another augmented semi-simplicial object $\mathbb{A}^-$ by setting

$$A^-_{n-1} = A_n \qquad \text{and} \qquad \partial^-_i = \partial_{i+1} \colon A_{n+1} \to A_n,$$

for $n \geq 0$ and $0 \leq i \leq n$. This is the augmented semi-simplicial object obtained from $\mathbb{A}$ by leaving out A_{-1} and all $\partial_0 \colon A_n \to A_{n-1}$. Observe that $\partial = (\partial_0)_n$ defines a morphism from $\mathbb{A}^-$ to $\mathbb{A}$.

When $\mathbb{A}$ is a (quasi)-simplicial object, the degeneracy operators can be shifted in the same way to give a (quasi)-simplicial object $\mathbb{A}^-$ and a morphism $\partial \colon \mathbb{A}^- \to \mathbb{A}$ of (quasi)-simplicial objects.

Remark 2.12. Note that $\mathbb{A}^-$ is contractible when $\mathbb{A}$ is an augmented quasi-simplicial object, and that an augmented semi-simplicial object $\mathbb{A}$ is contractible if and only if $\partial \colon \mathbb{A}^- \to \mathbb{A}$ is a split epimorphism of augmented semi-simplicial objects.

Remark 2.13. We may also view the morphism $\partial \colon \mathbb{A}^- \to \mathbb{A}$ as an augmented semi-simplicial object of arrows, say $\mathbb{B}$, with the $\partial_0 \colon A_{n+1} \to A_n$ forming the objects B_n. Notice that, when we view ∂ as a morphism of semi-simplicial objects in $\mathcal{A}$, the direction of a square goes parallel to the ∂_0 as in the left diagram below, depicting the morphism $\partial^-_i \to \partial_i$, whereas if we view it as a semi-simplicial object of arrows, the direction goes from one ∂_0 to the next as in the right diagram, displaying the ∂_i of the semi-simplicial object $\mathbb{B}$.

$$\begin{tikzcd} A_{n+1} \arrow[r, "\partial_{i+1}=\partial_i^-"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "\partial_0"] \\ A_n \arrow[r, "\partial_i"'] & A_{n-1} \end{tikzcd} \qquad \begin{tikzcd} A_{n+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "\partial_0=B_n"'] \arrow[dr, phantom, "\Rightarrow"] & A_n \arrow[d, "\partial_0=B_{n-1}"] \\ A_n \arrow[r, "\partial_i"'] & A_{n-1} \end{tikzcd}$$

Truncations and higher arrows. If an augmented semi-simplicial object $\mathbb{A}$ in $\mathcal{A}$ is truncated at level n, it corresponds to an $(n+1)$-fold arrow in $\mathcal{A}$ as follows. Truncation at level zero automatically gives a morphism $\partial_0\colon A_0 \to A_{-1}$. When we truncate at level one, we can use $\partial\colon \mathbb{A}^- \to \mathbb{A}$ to view all the remaining information as an augmented semi-simplicial object of morphisms $\mathbb{B}$, truncated at level zero.

$$\begin{tikzcd} A_1 \arrow[r, "\partial_1"] \arrow[d, "\partial_0=B_0"'] \arrow[dr, phantom, "\Rightarrow"] & A_0 \arrow[d, "\partial_0=B_{-1}"] \\ A_0 \arrow[r, "\partial_0"'] & A_{-1} \end{tikzcd}$$

This can clearly be viewed as a double arrow. Similarly an augmented semi-simplicial object truncated at level n corresponds to an augmented semi-simplicial object of morphisms truncated at level $n-1$, which by induction corresponds to an n-fold arrow of morphisms in $\mathcal{A}$, which in turn can be viewed as an $(n+1)$-fold arrow of objects in $\mathcal{A}$.

Definition 2.14. The above determines a functor $\mathsf{arr}_n\colon \mathsf{S}^+_s\mathcal{A} \to \mathsf{Arr}^n\mathcal{A}$ for any $n \geq 1$. We also consider

$$\mathsf{arr}_0\colon \mathsf{S}^+_s\mathcal{A} \to \mathsf{Arr}^0\mathcal{A} = \mathcal{A}\colon \mathbb{A} \mapsto A_{-1}.$$

This description may be illustrated by the commutative square

$$\begin{tikzcd} \mathsf{S}^+_s\mathcal{A} \arrow[r, "\mathsf{arr}_{n+1}"] \arrow[d] & \mathsf{Arr}^{n+1}\mathcal{A} \arrow[d] \\ \mathsf{S}^+_s\mathsf{Arr}\mathcal{A} \arrow[r, "\mathsf{arr}_n"'] & \mathsf{Arr}^n\mathsf{Arr}\mathcal{A} \end{tikzcd} \tag{B}$$

in which the left downward arrow sends $\mathbb{A}$ to $\partial\colon \mathbb{A}^- \to \mathbb{A}$ viewed as an augmented semi-simplicial object of arrows $\mathbb{B}$ as in Remark 2.13. The right downward arrow is the following isomorphism. We know that $\mathsf{Arr}^{n+1}\mathcal{A} \cong \mathcal{A}^{(2^{n+1})^{\mathrm{op}}}$ and $\mathsf{Arr}^n\mathsf{Arr}\mathcal{A} \cong \mathcal{A}^{(2^n)^{\mathrm{op}}\times 2^{\mathrm{op}}}$ (using the isomorphism fixed in Convention 1.4), so it is enough to describe the isomorphism between 2^{n+1} and $2^n \times 2$. Given a set $S \subset n$, we write S^{+1} for the set obtained from S by shifting all elements up by one, that is, we have $i \in S$ if and only if $i+1 \in S^{+1}$. Using this notation, we choose the isomorphism which sends a set $(S, 0) \in 2^n \times 2$ to $S^{+1} \in 2^{n+1}$ and $(S, 1) \in 2^n \times 2$ to $S^{+1} \cup \{0\} \in 2^{n+1}$.

There is another way of obtaining the functors arr_n which may be described as follows. For any $n \geq 0$, let

$$F_n \colon 2^n \to \Delta_s^+$$

be the functor which maps a set $S \subseteq n$ to the associated ordinal $|S|$, and an inclusion $S \subseteq T$ to the corresponding order-preserving map $|S| \to |T|$: if $T = \{x_0 < x_1 < \cdots < x_{|T|-1}\}$ and $S = \{x_{i_0} < x_{i_1} < \cdots < x_{i_{|S|-1}}\}$ then the map $|S| \to |T|$ sends k to i_k. We again fix the isomorphism $\mathsf{Arr}^n\mathcal{A} \cong \mathsf{Cub}^n\mathcal{A}$ as described in Convention 1.4.

Lemma 2.15. *For any $n \geq 0$, the functor $\mathsf{arr}_n \colon \mathsf{S}_s^+\mathcal{A} \to \mathsf{Arr}^n\mathcal{A}$ is equal to*

$$\mathsf{Fun}(F_n^{\mathrm{op}}, -) \colon \mathsf{Fun}((\Delta_s^+)^{\mathrm{op}}, \mathcal{A}) \to \mathsf{Fun}((2^n)^{\mathrm{op}}, \mathcal{A}).$$

Proof: As arr_{n+1} is defined inductively by the square (**B**) above and arr_0 clearly coincides with $\mathsf{Fun}(F_0^{\mathrm{op}}, -)$, it is enough to check that the square

$$\begin{tikzcd} \mathsf{S}_s^+\mathcal{A} \arrow[r, "\mathsf{Fun}(F_{n+1}^{\mathrm{op}}{,}-)"] \arrow[d] & \mathsf{Arr}^{n+1}\mathcal{A} \arrow[d] \\ \mathsf{S}_s^+\mathsf{Arr}\mathcal{A} \arrow[r, "\mathsf{Fun}(F_n^{\mathrm{op}}{,}-)"'] & \mathsf{Arr}^n\mathsf{Arr}\mathcal{A} \end{tikzcd}$$

commutes. ■

Lemma 2.16. *Let $\mathbb{A}$ be an augmented semi-simplicial object, $n \geq 1$ and $\mathsf{arr}_n\mathbb{A}$ the induced n-fold arrow. As mentioned above, the corresponding n-cube may be considered as an arrow between $(n-1)$-cubes in n different ways. The codomains of all of these arrows determine the same $(n-1)$-cube.*

Proof: A subset S of n determines the full subcategory 2^S of 2^n. If $|S| = n-1$, the restriction of $\mathsf{arr}_n\mathbb{A}$ to 2^S is one of the codomains considered in the statement of the lemma. Given two subsets S and T of n such that $|S| = |T| = n-1$, the subcategories 2^S and 2^T are mapped by the functor F_n to one and the same subcategory of Δ_s^+. Thus, using the alternative description of the functor arr_{n-1} from Lemma 2.15, we see that for any augmented semi-simplicial object $\mathbb{A}$, the two induced restrictions of $\mathsf{arr}_n\mathcal{A}$ to the $(n-1)$-cubes determined by S and T are equal to each other. ■

This brings us to the main result of this section.

Theorem 2.17. *An augmented semi-simplicial object $\mathbb{A}$ is an $\mathcal{E}$-resolution if and only if $\mathsf{arr}_n\mathbb{A}$ is an n-fold extension for all $n \geq 1$.*

Proof: If $\mathbb{A}$ is an $\mathcal{E}$-resolution, then $\mathsf{arr}_1\mathbb{A} = \partial_0 \colon A_0 \to A_{-1}$ is an extension by definition, and conversely $\mathsf{arr}_1\mathbb{A}$ being an extension implies that $\mathbb{A}$ is $\mathcal{E}$-exact at A_{-1}, which is the first condition for $\mathbb{A}$ to be an $\mathcal{E}$-resolution. For $n \geq 2$, consider the full subcategory $\mathcal{D}$ of 2^n determined by all sets $S \subseteq n$ with $n-2 \leq |S| \leq n-1$. It is easy to see that $\mathcal{D}$ is initial in the full subcategory of $(2^n)^{\mathrm{op}}$ containing all objects except n. It follows that, for any n-fold arrow $A \colon (2^n)^{\mathrm{op}} \to \mathcal{A}$,

$$\lim_{J \subsetneq n} A_J = \lim_{J \in |\mathcal{D}|} A_J.$$

If now $A = \mathsf{arr}_n\mathbb{A}$ for an augmented semi-simplicial object $\mathbb{A}$, the subdiagram is exactly the diagram which determines $K_n\mathbb{A}$.

$$\begin{tikzcd}
 & A_1 \arrow[rr, "\partial_1"] \arrow[dd, "\partial_0"] & & A_0 \arrow[dd, dotted] \\
L \arrow[ur, dashed, "l_1"] \arrow[rr, dashed, "l_2"'] \arrow[ddd, dashed, "l_0"'] & & A_1 \arrow[ur, "\partial_1"'] \arrow[dd, "\partial_0"] & \\
 & A_0 \arrow[rr, dotted] & & A_{-1} \\
A_1 \arrow[ur, "\partial_0"'] \arrow[rr, "\partial_1"'] & & A_0 \arrow[ur, dotted] &
\end{tikzcd}$$

This shows that if either limit exists, then the other exists and they are the same. This automatically proves one of the implications, using the condition for extensions given in (ii) of Proposition 1.16. For the other, we must also show that each codomain of $\mathsf{arr}_n\mathbb{A}$ is an $(n-1)$-extension. Lemma 2.16 shows that checking one codomain suffices. The canonical codomain of the n-fold arrow $\mathsf{arr}_n\mathbb{A}$ is $\mathsf{arr}_{n-1}\mathbb{A}$ and as such is an extension by induction. ■

This makes clear that we can view (semi)-simplicial resolutions as "infinite dimensional extensions", and a higher extension as a finite-dimensional resolution.

Remark 2.18. Theorem 2.17 shows, in particular, that one can use the n-truncation of a canonical simplicial resolution $\mathbb{G}A$ of an object A as an n-fold projective presentation of A (a special kind of higher extension) in order to compute the higher Hopf formulae which give the homology of A (as e.g. in the article [17]).

Corollary 2.19. *An augmented semi-simplicial object* $\mathbb{A}$ *is an* $\mathcal{E}$*-resolution if and only if the augmented semi-simplicial object of arrows* $\partial\colon \mathbb{A}^{-} \to \mathbb{A}$ *is an* $\mathcal{E}^1$*-resolution.*

Proof: For $n \geq 0$, the $(n+1)$-truncation $\mathsf{arr}_{n+1}\mathbb{A}$ of the augmented semi-simplicial object of objects $\mathbb{A}$ is an $(n+1)$-fold $\mathcal{E}$-extension in $\mathcal{A}$ precisely when the n-truncation $\mathsf{arr}_n\mathbb{B}$ of the augmented semi-simplicial object of arrows $\mathbb{B} = \partial\colon \mathbb{A}^{-} \to \mathbb{A}$ is an n-fold $\mathcal{E}^1$-extension in $\mathsf{Ext}\mathcal{A}$. ■

3. The relative Mal'tsev axiom

We now investigate a relative version of the Kan property for simplicial objects and its connections to properties of higher extensions. The main condition on higher extensions in this context is a relative Mal'tsev axiom, which is equivalent to two other important conditions.

Throughout this section, we consider the following axioms on $(\mathcal{A}, \mathcal{E})$:

(E1) $\mathcal{E}$ contains all isomorphisms;
(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
(E3) $\mathcal{E}$ is closed under composition;
(E4) if $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;
(E5) the **$\mathcal{E}$-Mal'tsev axiom**: any split epimorphism of extensions

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[d, "a"'] & B_1 \arrow[l, shift left] \arrow[d, "b"] \\ A_0 \arrow[r, shift left, "f_0"] & B_0 \arrow[l, shift left] \end{tikzcd}$$

in $\mathcal{A}$ is a double extension.

Notice that (E1) and (E4) together imply that all split epimorphisms are in $\mathcal{E}$.

Remark 3.1. Axioms (E1)–(E4) say exactly that our class $\mathcal{E}$ generates a Grothendieck topology (or Grothendieck coverage) on $\mathcal{A}$, see for instance [40, Definition C2.1.8].

Axiom (E5). We will first show that (E5) is equivalent to two other conditions, connecting the class of extensions $\mathcal{E}$ and the corresponding class $\mathcal{E}^1$ of double extensions. To do this, we make use of the following lemma.

Lemma 3.2. *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E4). Consider a diagram*

$$\begin{tikzcd}
\mathrm{R}[f_1] \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] \arrow[d, "r"'] & A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\
\mathrm{R}[f_0] \arrow[r, shift left, "\pi'_0"] \arrow[r, shift right, "\pi'_1"'] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

in $\mathcal{A}$ with a, b, f_1 and f_0 in $\mathcal{E}$ and $\mathrm{R}[f_1]$ and $\mathrm{R}[f_0]$ the kernel pairs of f_1 and f_0. Either (hence both) of the left hand side commutative squares is in $\mathcal{E}^1$ if and only if the right hand side square is in $\mathcal{E}^1$.

Proof: The right-to-left implication follows from (E2) for the class $\mathcal{E}^1$. Now suppose that the square $a \circ \pi_0 = \pi'_0 \circ r$ is in $\mathcal{E}^1$. The diagram induces the following commutative cube and the right hand side commutative comparison square to the pullback.

$$\begin{tikzcd}
& A_1 \arrow[rr, "f_1"] \arrow[dd, dotted] & & B_1 \arrow[dd] \\
\mathrm{R}[f_1] \arrow[ur] \arrow[rr, crossing over] \arrow[dd, "r"'] & & A_1 \arrow[ur, "f_1"'] & \\
& A_0 \arrow[rr, dotted, "f_0"] & & B_0 \\
\mathrm{R}[f_0] \arrow[ur, dotted] \arrow[rr] & & A_0 \arrow[ur, "f_0"'] \arrow[from=uu, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{R}[f_1] \arrow[r, "\pi_1"] \arrow[d, "\langle r,\pi_0\rangle"'] & A_1 \arrow[d, "\langle a,f_1\rangle"] \\
\mathrm{R}[f_0] \times_{A_0} A_1 \arrow[r, "\pi_1 \times_{f_0} f_1"'] & A_0 \times_{B_0} B_1
\end{tikzcd}$$

In the square, the morphism $\langle r, \pi_0\rangle$ is in $\mathcal{E}$ by assumption. Furthermore, the morphism $\pi_1 \times_{f_0} f_1$ is in $\mathcal{E}$ as a pullback of the extension f_1. It follows by (E3) that $\langle a, f_1\rangle \circ \pi_1$ is an extension, and so (E4) implies that $\langle a, f_1\rangle$ is in $\mathcal{E}$. ∎

Proposition 3.3. *Let $\mathcal{A}$ be a category and $\mathcal{E}$ a class of arrows in $\mathcal{A}$ which satisfies the axioms (E1)–(E4). The following conditions are equivalent:*

(i) *(E4) holds for the class $\mathcal{E}^1$, that is, if $g\circ f \in \mathcal{E}^1$ then $g \in \mathcal{E}^1$;*
(ii) *(E5) holds, that is, all split epimorphisms of extensions are in $\mathcal{E}^1$;*
(iii) *every split epimorphism of split epimorphisms, i.e., every diagram*

$$\begin{tikzcd} A_1 \arrow[r, shift right, "f_1"'] \arrow[d, shift right, "a"'] & B_1 \arrow[l, shift right, "\overline{f_1}"'] \arrow[d, shift right, "b"'] \\ A_0 \arrow[r, shift right, "f_0"'] \arrow[u, shift right, "\overline{a}"'] & B_0, \arrow[l, shift right, "\overline{f_0}"'] \arrow[u, shift right, "\overline{b}"'] \end{tikzcd} \tag{C}$$

such that $f_0 a = b f_1$, $\overline{f_0} b = a \overline{f_1}$, $\overline{b} f_0 = f_1 \overline{a}$, $\overline{a}\overline{f_0} = \overline{f_1}\overline{b}$ and $f_0 \overline{f_0} = 1_{B_0}$, $f_1 \overline{f_1} = 1_{B_1}$, $a\overline{a} = 1_{A_0}$, $b\overline{b} = 1_{B_0}$ is a double extension;
(iv) *consider the diagram*

$$\begin{tikzcd} \mathrm{R}[f_1] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "r"'] & A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[d, "b"] \\ \mathrm{R}[f_0] \arrow[r, shift left] \arrow[r, shift right] & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \tag{D}$$

in $\mathcal{A}$ with a, b, f_1 and f_0 in $\mathcal{E}$; the arrow r is in $\mathcal{E}$ if and only if the right hand side square is in $\mathcal{E}^1$.

Proof: Since all isomorphisms of extensions are double extensions, we see that (i) implies (ii). Clearly (iii) is a special case of (ii). Now suppose that (iii) holds and consider a diagram (**D**) as in (iv). Lemma 3.2 automatically gives one direction, that is, if the right hand square is a double extension, then r is in $\mathcal{E}$. Conversely, taking kernel pairs vertically of the left hand side square gives us a square as in (iii). By assumption this square is a double extension. Using Lemma 3.2 twice we see that all squares in the diagram are double extensions.

Finally, suppose that (iv) holds and consider the morphisms f and g in $\mathsf{Ext}\mathcal{A}$ as in the diagram below.

$$\begin{tikzcd} \mathrm{R}[a] \arrow[r, "r"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{R}[b] \arrow[r, "s"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{R}[c] \arrow[d, shift left] \arrow[d, shift right] \\ A_1 \arrow[r, "f_1"] \arrow[d, "a"'] & B_1 \arrow[r, "g_1"] \arrow[d, "b"] & C_1 \arrow[d, "c"] \\ A_0 \arrow[r, "f_0"'] & B_0 \arrow[r, "g_0"'] & C_0 \end{tikzcd}$$

Assume that the composite $g \circ f$ is a double extension. Then by assumption $s \circ r$ is in $\mathcal{E}$. Axiom (E4) implies that s is in $\mathcal{E}$, so (iv) implies that g is in $\mathcal{E}^1$. ■

The axioms (E1)–(E5) "go up":

Proposition 3.4. *Let $\mathcal{A}$ be a category and $\mathcal{E}$ a class of arrows in $\mathcal{A}$. If $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E5) then $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ satisfies the same conditions.*

Proof: By Proposition 3.3, Axiom (E5) for $(\mathcal{A}, \mathcal{E})$ is equivalent to Axiom (E4) for $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$. To see that Axiom (E5) for $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ holds, consider a split epimorphism of double extensions in $\mathcal{A}$ such as the following left hand side cube and recall Remark 1.19.

$$\begin{tikzcd}
& C \arrow[rr, shift left] \arrow[dd, dotted] & & D \arrow[ll, shift left] \arrow[dd] \\
A \arrow[ur] \arrow[rr, shift left] \arrow[dd] & & B \arrow[ll, shift left] \arrow[ur] \arrow[dd] & \\
& C' \arrow[rr, shift left, dotted] & & D' \arrow[ll, shift left, dotted] \\
A' \arrow[ur, dotted] \arrow[rr, shift left] & & B' \arrow[ll, shift left] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, shift right] \arrow[d] & B \arrow[l, shift right] \arrow[d] \\
A' \times_{C'} C \arrow[r, shift right] & B' \times_{D'} D \arrow[l, shift right]
\end{tikzcd}$$

The arrows pointing to the right are split epimorphisms. By assumption, the cube's left and right hand side squares are double extensions; Axiom (E5) for $(\mathcal{A}, \mathcal{E})$ implies, moreover, that the front, back, top and bottom squares are also double extensions. Hence the induced right hand side comparison square to the pullback exists. It is a double extension by Axiom (E5) for $(\mathcal{A}, \mathcal{E})$. ■

The $\mathcal{E}$-Kan property. The Kan property is well known for simplicial sets and simplicial groups and was used in [9] to extend the characterisation of the Mal'tsev property in terms of simplicial objects from varieties to regular categories. We slightly adapt the definition to obtain a relative notion of $\mathcal{E}$-Kan simplicial objects.

Definition 3.5. Let $\mathbb{A}$ be a semi-simplicial object and consider $n \geq 2$ and $0 \leq k \leq n$. The **object of** (n, k)**-horns in** $\mathbb{A}$ is an object $A(n, k)$ together with arrows $a_i \colon A(n, k) \to A_{n-1}$ for $i \in \{0, \ldots, n\} \setminus \{k\}$ satisfying

$$\partial_i \circ a_j = \partial_{j-1} \circ a_i \text{ for all } i < j \text{ with } i,\, j \neq k$$

which is universal with respect to this property. We also define $A(1, 0) = A(1, 1) = A_0$.

A semi-simplicial object is **$\mathcal{E}$-Kan** when all $A(n,k)$ exist and all comparison morphisms $A_n \to A(n,k)$ are in $\mathcal{E}$. In particular, the comparison morphisms to the $(1,k)$-horns are $\partial_0 \colon A_1 \to A(1,0)$ and $\partial_1 \colon A_1 \to A(1,1)$.

For simplicity, we assume that $\mathcal{A}$ has a terminal object so that every semi-simplicial object has a canonical augmentation. In fact this augmentation is only needed to allow a formulation in terms of cubes.

Proposition 3.6. *Let $\mathbb{A}$ be a semi-simplicial object and $\mathsf{arr}_{n+1}\mathbb{A}$ the $(n+1)$-cube induced by (the canonical augmentation of) $\mathbb{A}$ for some $n \geq 1$. Then $\mathbb{A}$ satisfies the $\mathcal{E}$-Kan property at level n (i.e., for all (n,k)-horns) if and only if the domains of all arrows of n-cubes in $\mathsf{arr}_{n+1}\mathbb{A}$ (i.e., in all possible directions) are extensions.*

Proof: A domain of any arrow of n-cubes in $\mathsf{arr}_{n+1}\mathbb{A}$ is given by the n-subcube which involves all faces $\partial_i \colon A_n \to A_{n-1}$ except for one particular ∂_k. In the same way as in the proof of Theorem 2.17, we see that the limit of the subdiagram of this n-cube without the initial object A_n is exactly the (n,k)-horn object. Therefore, by induction on n, the $\mathcal{E}$-Kan property holds for the (n,k)-horn object if and only if that particular cube is an extension. ■

Using Theorem 2.17 this gives us in particular

Corollary 3.7. *Let $(\mathcal{A},\mathcal{E})$ satisfy (E1)–(E4). For any $\mathcal{E}$-semi-simplicial object $\mathbb{A}$ which is $\mathcal{E}$-Kan, the associated augmented semi-simplicial object $\mathbb{A}^-$ is an $\mathcal{E}$-resolution.* ■

As a first illustration of what the relative Kan property is useful for, we show that a contractible augmented $\mathcal{E}$-semi-simplicial object $\mathbb{A}$ which is also $\mathcal{E}$-Kan is always an $\mathcal{E}$-resolution. For this we make an observation about the existence of simplicial kernels.

Lemma 3.8. *If $\mathbb{A}$ is a resolution up to level n then $K_{n+1}\mathbb{A}$ exists.*

Proof: This follows from Lemma 2.16 and the following property of higher cubes, which is proved inductively as in Proposition 1.16: if all codomains in an $(n+2)$-cube A are extensions, then the limit $\lim_{J \subsetneq n+2} A_J$ exists. ■

Proposition 3.9. *Let $(\mathcal{A},\mathcal{E})$ satisfy (E1)–(E4). An augmented $\mathcal{E}$-semi-simplicial object in $(\mathcal{A},\mathcal{E})$ which is contractible and satisfies the $\mathcal{E}$-Kan property is an $\mathcal{E}$-resolution.*

Proof: As $\mathbb{A}$ is an $\mathcal{E}$-semi-simplicial object, in particular the morphism

$$\partial_0 \colon A_0 \to A_{-1} = K_0\mathbb{A}$$

is in $\mathcal{E}$, so $\mathbb{A}$ is an $\mathcal{E}$-resolution at level 0.

Now let $\mathbb{A}$ be a resolution up to level n. By Lemma 3.8, we can assume inductively that the simplicial kernel $K_{n+1}\mathbb{A}$ exists. So we can consider the diagram

$$\begin{tikzcd}
A_{n+2} \arrow[r] \arrow[d, "\partial_0"'] & A(n+2,0) \arrow[r, shift left=2, "a_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "a_{n+2}"'] \arrow[d, shift right=1, "r"'] & A_{n+1} \arrow[r, shift left=2, "\partial_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\partial_{n+1}"'] \arrow[d, shift right=1, "\partial_0"'] & A_n \arrow[d, shift right=1, "\partial_0"'] \\
A_{n+1} \arrow[r, "\langle \partial_0, \dots, \partial_{n+1} \rangle"'] & K_{n+1}\mathbb{A} \arrow[r, shift left=2, "k_0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "k_{n+1}"'] \arrow[u, shift right=1] & A_n \arrow[r, shift left=2, "\partial_0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\partial_n"'] \arrow[u, shift right=1, "\sigma_{-1}"'] & A_{n-1} \arrow[u, shift right=1, "\sigma_{-1}"']
\end{tikzcd}$$

in which $\mathrm{K}_{n+1}\mathbb{A}$ and $A(n+2, 0)$ are the simplicial kernels of the given morphisms. As $\partial \colon \mathbb{A}^- \to \mathbb{A}$ is a split epimorphism of augmented $\mathcal{E}$-semi-simplicial objects by Remark 2.12, the induced morphism r between the limits is split epic, and thus an extension. In fact, $r = K_n\mathbb{B}$, since $\mathbb{B}$ is a resolution up to level $n-1$, so this simplicial kernel can be constructed by going to cubes as in Lemma 3.8. It is pointwise because pullbacks of double extensions are pointwise in $\mathsf{Ext}\mathcal{A}$. The comparison morphism $A_{n+2} \to A(n+2, 0)$ is an extension as $\mathbb{A}$ is $\mathcal{E}$-Kan, so the composite $\partial_0 \circ \langle \partial_0, \dots, \partial_{n+1} \rangle$ is an extension by (E3). Therefore $\langle \partial_0, \dots, \partial_{n+1} \rangle$ is an extension by the cancellation property (E4). ■

Now we prove that, with a small extra assumption, the relative Mal'tsev axiom (E5) is equivalent to *every quasi*-simplicial object being $\mathcal{E}$-Kan.

Remark 3.10. Notice that any (quasi)-simplicial object is automatically an $\mathcal{E}$-semi-simplicial object, as all split epimorphisms are in $\mathcal{E}$. However, this does not automatically extend to *augmented* (quasi)-simplicial objects.

Proposition 3.11. *If $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E5) then every quasi-simplicial object in $\mathcal{A}$ satisfies the $\mathcal{E}$-Kan property.*

Proof: For every quasi-simplicial object $\mathbb{A}$, the $\mathcal{E}$-Kan property for $A(1, k)$ just says that ∂_0 and $\partial_1 \colon A_1 \to A_0$ are in $\mathcal{E}$, which is automatically satisfied thanks to (E1) and (E4), which imply that all split epimorphisms are in $\mathcal{E}$.

Now assume that the $\mathcal{E}$-Kan property holds up to level n for all pairs $(\mathcal{A}, \mathcal{E})$ which satisfy Axiom (E1)–(E5). Let $\mathbb{A}$ be a quasi-simplicial object in $\mathcal{A}$

and $\mathbb{B} = \partial\colon \mathbb{A}^- \to \mathbb{A}$ the induced quasi-simplicial object in $\mathsf{Ext}\mathcal{A}$. Axiom (E5) for $(\mathcal{A}, \mathcal{E})$ ensures that $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ also satisfies (E4) and (E5) (Proposition 3.4). So by assumption, $\mathbb{B}$ is $\mathcal{E}^1$-Kan up to level n. By Proposition 3.6 this means that the domains of the $(n+1)$-cube $\mathsf{arr}_{n+1}\mathbb{B}$ in $\mathsf{Ext}\mathcal{A}$ are n-fold $\mathcal{E}^1$-extensions. Hence in the $(n+2)$-cube $\mathsf{arr}_{n+2}\mathbb{A}$ in $\mathcal{A}$, certain domains are $(n+1)$-fold $\mathcal{E}$-extensions. This almost shows that $\mathbb{A}$ is $\mathcal{E}$-Kan at level $n+1$: the property holds for all domains but one. The missing case follows by symmetry. ■

We can also prove a converse of Proposition 3.11, however we now need $\mathcal{A}$ to have all simplicial kernels so that truncation of simplicial objects has a right adjoint.

Proposition 3.12. *Let $\mathcal{A}$ be a category with simplicial kernels and $\mathcal{E}$ a class of morphisms in $\mathcal{A}$ which satisfies (E1)–(E4). If every simplicial object in $\mathcal{A}$ has the $\mathcal{E}$-Kan property then $(\mathcal{A}, \mathcal{E})$ satisfies (E5).*

Proof: We have to prove that every split epimorphism of split epimorphisms in $\mathcal{A}$ is a double extension. We may reduce the situation to a (truncated) contractible augmented $\mathcal{E}$-simplicial object

$$\begin{tikzcd} A_1 \arrow[r, shift left=3, "\partial_0"] \arrow[r, shift right=3, "\partial_1"'] & A_0 \arrow[l, shift right=6, "\sigma_{-1}"'] \arrow[l, "\sigma_0"] \arrow[r, shift right, "\partial_0"'] & A_{-1}. \arrow[l, shift right, "\sigma_{-1}"'] \end{tikzcd} \qquad (\mathbf{G})$$

Consider the following split epimorphism of split epimorphisms (any of the four possible squares commutes, and the arrows pointing down or right are the split epimorphisms).

$$\begin{tikzcd} A \arrow[r, shift right, "f"'] \arrow[d, shift right, "a"'] & B \arrow[l, shift right, "\overline{f}"'] \arrow[d, shift right, "b"'] \\ A' \arrow[u, shift right, "\overline{a}"'] \arrow[r, shift right, "f'"'] & B' \arrow[l, shift right, "\overline{f'}"'] \arrow[u, shift right, "\overline{b}"'] \end{tikzcd} \qquad (\mathbf{H})$$

Write $A_{-1} = B'$ and $A_0 = A$,

$$\partial_0 = f' \circ a = b \circ f\colon A_0 \to A_{-1}$$

and $\sigma_{-1} = \overline{a} \circ \overline{f'} = \overline{f} \circ \overline{b}\colon A_{-1} \to A_0$; then already $\partial_0 \circ \sigma_{-1} = 1_{A_{-1}}$. Now consider the extension $a \times_{1_{B'}} f$, which is defined by pulling back the double extension $(f' \circ a, f')\colon a \to 1_{B'}$ along the double extension $(f' \circ a, b)\colon f \to 1_{B'}$.

Hence we can form the following pullback, which defines the morphisms ∂_0 and $\partial_1 \colon A_1 \to A_0$.

$$\begin{tikzcd} A_1 \arrow[r, "p"] \arrow[d, "\langle \partial_0, \partial_1 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_0 \arrow[d, "\langle a, f \rangle"] \\ A_0 \times_{A_{-1}} A_0 \arrow[r, "a \times_{1_{B'}} f"'] & A' \times_{B'} B \end{tikzcd}$$

We see that

$$\partial_0 \circ \partial_0 = f' \circ a \circ \partial_0 = f' \circ a \circ p = b \circ f \circ p = b \circ f \circ \partial_1 = \partial_0 \circ \partial_1.$$

Write $\sigma_0 \colon A_0 \to A_1$ for the arrow universally induced by the equality

$$(a \times_{1_{B'}} f) \circ \langle 1_{A_0}, 1_{A_0} \rangle = \langle a, f \rangle \circ 1_{A_0};$$

then $\partial_0 \circ \sigma_0 = \partial_1 \circ \sigma_0 = 1_{A_0}$. Finally, let $\sigma_{-1} \colon A_0 \to A_1$ be the arrow universally induced by the equality

$$(a \times_{1_{B'}} f) \circ \langle 1_{A_0}, \overline{a} \circ \overline{f'} \circ f' \circ a \rangle = \langle a, f \rangle \circ (\overline{a} \circ a).$$

Then $\partial_0 \circ \sigma_{-1} = 1_{A_0}$ and $\partial_1 \circ \sigma_{-1} = \overline{a} \circ \overline{f'} \circ f' \circ a = \sigma_{-1} \circ \partial_0$. As both ∂_0 and ∂_1 are split epimorphisms, they are in $\mathcal{E}$.

The diagram (**G**) thus defined can be extended to a contractible augmented $\mathcal{E}$-simplicial object $\mathbb{A}$ by constructing successive simplicial kernels, which exist by assumption. This contractible augmented $\mathcal{E}$-simplicial object is $\mathcal{E}$-Kan, so by Proposition 3.9 it is an $\mathcal{E}$-resolution. In particular, the induced comparison morphism $\langle \partial_0, \partial_1 \rangle \colon A_1 \to K_1 \mathbb{A}$ is in $\mathcal{E}$. Using (E4) on the square defining $\langle \partial_0, \partial_1 \rangle$, we see that $\langle a, f \rangle$ is also in $\mathcal{E}$, which means that the split epimorphism of split epimorphisms (**H**) is a double extension. This proves that $(\mathcal{A}, \mathcal{E})$ satisfies (E5). ∎

Theorem 3.13. *Let $\mathcal{A}$ be a category with simplicial kernels and $\mathcal{E}$ a class of morphisms in $\mathcal{A}$ satisfying (E1)–(E4). Then the following are equivalent:*

- *$(\mathcal{A}, \mathcal{E})$ satisfies (E5);*
- *every quasi-simplicial object in $\mathcal{A}$ is $\mathcal{E}$-Kan;*
- *every simplicial object in $\mathcal{A}$ is $\mathcal{E}$-Kan.* ∎

Some examples. We start with an obvious example: regular Mal'tsev categories.

Example 3.14 (Regular Mal'tsev categories)**.** It is shown in [6] that when $\mathcal{A}$ is finitely complete with coequalisers of effective equivalence relations and $\mathcal{E}$ is the class of regular epimorphisms, the pair $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E5) if and only if $\mathcal{A}$ is regular Mal'tsev. Alternatively, this result follows from the above together with [9, Theorem 4.2].

More generally, when $\mathcal{A}$ is finitely complete, it was shown in [3] that $\mathcal{A}$ is Mal'tsev (i.e., every reflexive relation in $\mathcal{A}$ is an equivalence relation) if and only if Condition (iii) of Proposition 3.3 holds for $\mathcal{E}$ the class of strong (= extremal) epimorphisms. Given a pair $(\mathcal{A}, \mathcal{E})$ which satisfies (E1)–(E5), this implies that $\mathcal{A}$ is Mal'tsev as soon as $\mathcal{E}$ is contained in the class of strong epimorphisms.

Example 3.15 (Higher extensions)**.** Proposition 3.4 implies that $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$, the category of extensions (regular epimorphisms) in a regular Mal'tsev category $\mathcal{A}$ together with the double extensions, also satisfies (E1)–(E5), as do all other $(\mathsf{Ext}^n\mathcal{A}, \mathcal{E}^n)$.

Example 3.16 (Naturally Mal'tsev categories)**.** By a result in [3], a category is **naturally Mal'tsev** [39] when, given a split epimorphism of split epimorphisms as in Diagram (**C**), if it is a (downward) pullback of split epimorphisms, then it is an (upward) pushout of split monomorphisms. If now $\mathcal{A}$ is a naturally Mal'tsev category and $\mathcal{E}$ is its class of split epimorphisms, then it is easily seen that Condition (iii) in Proposition 3.3 holds. It is then obvious that $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E5). However, we do not know whether the opposite implication also holds, and the axiom (E5) implies that $\mathcal{A}$ is naturally Mal'tsev.

Now we give two examples where $\mathcal{E}$ need not be contained in the class of regular epimorphisms of $\mathcal{A}$.

Example 3.17 (Weakly Mal'tsev categories)**.** A category is called **weakly Mal'tsev** [42] when it has pullbacks of split epimorphisms and the following property holds: in any split epimorphism of split epimorphisms such as Diagram (**C**) which is a (downwards) pullback, the splittings $\overline{a}$ and $\overline{f_1}$ are jointly epic.

Let $\mathcal{A}$ be a category and $\mathcal{E}$ a class of epimorphisms in $\mathcal{A}$ such that (E1)–(E5) hold. Then $\mathcal{A}$ is weakly Mal'tsev as soon as $\mathcal{A}$ has either pushouts of split monomorphisms or equalisers. Indeed, in the first case, consider the diagram

$$\begin{tikzcd}
P \arrow[rrd, bend left, shift left] \arrow[rd, dotted] \arrow[rdd, bend right, shift right] & & \\
& A_1 \arrow[r, shift right, "f_1"'] \arrow[d, shift right, "a"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B_1 \arrow[l, shift right, "\overline{f_1}"'] \arrow[d, shift right, "b"'] \arrow[llu, bend right, shift right, "\widetilde{f_0}"'] \\
& A_0 \arrow[r, shift right, "f_0"'] \arrow[u, shift right, "\overline{a}"'] \arrow[luu, bend left, shift left, "\widetilde{b}"] & B_0 \arrow[l, shift right, "\overline{f_0}"'] \arrow[u, shift right, "\overline{b}"']
\end{tikzcd}$$

in which the square is a pullback of f_0 and b and P is a pushout of $\overline{f_0}$ and $\overline{b}$. Then $\widetilde{b}$ and $\widetilde{f_0}$ are jointly (strongly) epic, and by Proposition 3.3 the dotted comparison morphism is also an epimorphism. It follows that the splittings $\overline{a}$ and $\overline{f_1}$ in the pullback are jointly epic.

In the second case, given two parallel morphisms which coequalise $\overline{a}$ and $\overline{f_1}$, their equaliser $P \to A_1$ induces a diagram such as above. Then this morphism is both epic and regular monic, so that the two given parallel morphisms are equal to each other.

Conversely, for any pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is a weakly Mal'tsev category and $\mathcal{E}$ is the class of all epimorphims, the conditions (E1), (E3) and (E4) hold, but for (E2) we need epimorphisms in $\mathcal{A}$ to be pullback-stable. In this case Proposition 3.3 tells us that $(\mathcal{A}, \mathcal{E})$ satisfies (E5). A concrete situation where this occurs is given in Example 3.23.

Example 3.18 (All morphisms as extensions)**.** For any category with pullbacks, a trivial example is obtained by taking $\mathcal{E}$ to be the class of all morphisms.

The following two examples satisfy a stronger axiom, cf. [4, 11, 13, 35].

(E5$^+$) Given a diagram in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}[a] \arrow[r] \arrow[d, "k"'] & A_1 \arrow[r, "a"] \arrow[d, "f"'] & A_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{K}[b] \arrow[r] & B \arrow[r, "b"'] & A_0 \arrow[r] & 0
\end{tikzcd}$$

with short exact rows and a and b in $\mathcal{E}$, if $k \in \mathcal{E}$ then also $f \in \mathcal{E}$.

Notice that Axiom (E2) ensures the existence of kernels of extensions. Axiom (E5$^+$) implies (E5): consider a split epimorphism of extensions as in (E5). Take kernels of a and b to obtain a split epimorphism of short $\mathcal{E}$-exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}[a] \arrow[r, "\mathrm{Ker}\, a"] \arrow[d, shift right, "k"'] & A_1 \arrow[r, "a"] \arrow[d, shift right, "f_1"'] & A_0 \arrow[r] \arrow[d, shift right, "f_0"'] & 0 \\ 0 \arrow[r] & \mathrm{K}[b] \arrow[r, "\mathrm{Ker}\, b"'] \arrow[u, shift right] & B_1 \arrow[r, "b"'] \arrow[u, shift right] & B_0 \arrow[r] \arrow[u, shift right] & 0 \end{tikzcd}$$

As k is a split epimorphism and thus in $\mathcal{E}$, (E5$^+$) implies that the right hand square is a double extension.

Example 3.19 (Topological groups 1)**.** Example 1.15 of topological groups and morphisms split in the category of topological spaces satisfies (E1)–(E4), as commented earlier. This example also satisfies the axiom (E5$^+$) and hence (E5). Consider a diagram in GpTop as in (E5$^+$), and assume that in Top the morphism k is split by a continuous map $u\colon \mathrm{K}[b] \to \mathrm{K}[a]$, a is split by s and b is split by $t = f \circ s$. Any element β in the domain of an extension $b\colon B \to A_0$ can be written as a product of an element κ of the kernel $K[b]$ with an element $tb(\beta)$ in the image of the splitting t, because $\beta = \beta \cdot (tb(\beta))^{-1} \cdot tb(\beta)$. We show that the morphism $f\colon A_1 \to B$ is also split in Top. A splitting $B \to A_1$ is given by the composite

$$\begin{array}{ccccc} B & \longrightarrow & \mathrm{K}[b] \times A_0 & \longrightarrow & A_1 \\ \beta & \longmapsto & (\beta \cdot (tb(\beta))^{-1}, b(\beta)) & \longmapsto & u(\beta \cdot (tb(\beta))^{-1}) \cdot sb(\beta) \end{array}$$

which is easily seen to be continuous.

Example 3.20 (Rings and modules)**.** The category of rings together with morphisms split in abelian groups and the category of R-modules with morphisms split in Ab also satisfy the axioms (E1)–(E4) and (E5$^+$), and thus (E5).

Let $\mathcal{A}$ be a category with pullbacks, $(\mathcal{B}, \mathcal{F})$ a pair which satisfies (E1)–(E5) and $U\colon \mathcal{A} \to \mathcal{B}$ a pullback-preserving functor. Then the class of morphisms in $\mathcal{A}$ given by $\mathcal{E} = U^{-1}\mathcal{F}$ gives a pair $(\mathcal{A}, \mathcal{E})$ which also satisfies (E1)–(E5). The following examples are instances of this situation.

Example 3.21 (Topological groups 2)**.** Using the above, the category of topological groups may be equipped with another class of extensions, different from the one considered in Example 3.19, but such that (E1)–(E5) still hold: let U be the forgetful functor $\mathsf{GpTop} \to \mathsf{Gp}$ and take $\mathcal{E} = U^{-1}\mathcal{F}$ with $\mathcal{F}$ the class of all regular epimorphisms in Gp.

Example 3.22 (Reflective subcategories)**.** Another instance of this occurs when U is the inclusion of a reflective subcategory; hence any class of extensions satisfying (E1)–(E5) restricts to any reflective subcategory where it still satisfies (E1)–(E5).

Example 3.23 (Weakly Mal'tsev but not Mal'tsev)**.** Finally let $\mathcal{A}$ be the category of sets equipped simultaneously with a group structure and a topology, and morphisms which are continuous group homomorphisms. (We are not assuming any compatibility between the group structure and the topology as in the case of GpTop.) Consider the forgetful functor to Gp; then the class of extensions $\mathcal{E}$ induced by the regular epimorphisms of groups, i.e., the continuous surjective homomorphisms in $\mathcal{A}$, satisfies the conditions (E1)–(E5). On the other hand, $\mathcal{A}$ is not a Mal'tsev category in the absolute sense (though it is weakly Mal'tsev). The regular epimorphisms in $\mathcal{A}$ are in particular quotients (inducing the final topology on the codomain) so that not every extension is a regular epi. As a counterexample to the absolute Mal'tsev property, consider the group of integers $\mathbb{Z}$ with the indiscrete topology. Then $\mathbb{Z} \times \mathbb{Z}$ also carries the indiscrete topology, while $\mathbb{Z} + \mathbb{Z}$ carries the final topology for the (algebraic) coproduct inclusions. Now the universally induced comparison morphism $\mathbb{Z} + \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ to the pullback in the split epimorphism of regular epimorphisms

$$\begin{tikzcd}
\mathbb{Z} + \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

is not a regular epimorphism, as the topology on $\mathbb{Z} \times \mathbb{Z}$ is different from the induced quotient topology. To see this, it suffices to note that the singleton $\{(1,1)\}$ is not open in $\mathbb{Z} \times \mathbb{Z}$, whereas its inverse image along $\mathbb{Z} + \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is open in $\mathbb{Z} + \mathbb{Z}$.

4. When split epimorphisms need not be extensions

In the previous two sections we have assumed that all split epimorphisms are in the class $\mathcal{E}$. However, most results can be adapted to hold in a slightly weaker setting, where we assume a weak cancellation property instead of (E4) and a suitable stability of those split epimorphisms which are in $\mathcal{E}$. We sketch this situation in the current section, but leave some details to the reader.

We first give those axioms which change slightly as well as the new one ensuring that split epimorphisms behave "nicely": we want the subcategory of $\mathsf{Arr}\mathcal{A}$ which is determined by the extensions that are split epimorphisms to be closed under pullbacks in the category $\mathsf{Arr}\mathcal{A}$. In fact, we only assume this closedness for certain pullbacks, and demand that $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3) as well as the axioms

(E4$^-$) if $f \in \mathcal{E}$ and $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;

(E5$^-$) any split epimorphism of extensions

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[d, "a"'] & B_1 \arrow[l, shift left] \arrow[d, "b"] \\ A_0 \arrow[r, shift left, "f_0"] & B_0 \arrow[l, shift left] \end{tikzcd}$$

in $\mathcal{A}$ with f_1 and f_0 in $\mathcal{E}$ is a double extension;

(E6) given a split epimorphism of extensions

$$\begin{tikzcd} \mathrm{R}[a] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "r"'] & A_1 \arrow[r, "a"] \arrow[d, shift right, "f_1"'] & A_0 \arrow[d, shift right, "f_0"'] \\ \mathrm{R}[b] \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift right] & B_1 \arrow[r, "b"'] \arrow[u, shift right] & B_0 \arrow[u, shift right] \end{tikzcd}$$

with f_1 and f_0 in $\mathcal{E}$, taking kernel pairs of a and b gives an extension r.

Proposition 4.1. *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E4$^-$). Then $\mathcal{E}$ contains all split epimorphisms if and only if (E4) holds.*

Proof: By (E1), one of the implications is obvious. To prove the other, let $g \circ f$ be in $\mathcal{E}$. Pulling back induces the following commutative diagram.

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, shift left, "\overline{\pi_0}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{R}[g] \arrow[r, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_0"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[l, shift right] \arrow[d, "g"] \\ A \arrow[r, "f"'] \arrow[u, shift left] & B \arrow[r, "g"'] \arrow[u, shift left] & C \end{tikzcd}$$

The split epimorphism $\overline{\pi_0}$ is in $\mathcal{E}$ by assumption. Furthermore, the composite $\pi_1 \circ \overline{f}$ is in $\mathcal{E}$ by (E2). Now (E4$^-$) implies that g is in $\mathcal{E}$. ■

Clearly, when $\mathcal{E}$ contains all split epimorphisms, (E5$^-$) is equivalent to (E5). Furthermore, under (E1)–(E3), Axiom (E5$^-$) implies (E6), but we shall also need (E6) in situations where (E5$^-$) is not assumed.

Given this new axiom (E6), almost all the results of the previous two sections go through, and we obtain new examples. In particular, **relative homological categories** as defined by T. Janelidze [35] form an example. These are pairs $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ has finite products and $\mathcal{E}$ is a class of normal epimorphisms, which satisfy our axioms (E1)–(E3), (E4$^-$) and (E5$^+$), as well as the axiom

(F) if a morphism f in $\mathcal{A}$ factors as $f = e \circ m$ with m a monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m' \circ e'$ with m' a monomorphism and $e' \in \mathcal{E}$.

Axiom (F) implies that split epimorphisms behave nicely, as demonstrated by the following lemma.

Lemma 4.2. *If $\mathcal{A}$ has finite products, $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3) and (F), then (E6) holds.*

Proof: The morphism $(f_1 \times f_1) \circ \langle \pi_0, \pi_1 \rangle$ factors as a monomorphism followed by an extension, so by (F) it admits a factorisation $\langle r_0, r_1 \rangle \circ e$, where (R, r_0, r_1) is a relation on B_1 and e is in $\mathcal{E}$. Since e is an epimorphism by assumption, we have $b \circ r_0 = b \circ r_1$, and R is contained in $\mathrm{R}[b]$. Now r being a split epi implies that $R = \mathrm{R}[b]$. ■

In this weaker setting, Proposition 3.3 does not quite hold with equivalence between all four points. Instead we have:

Proposition 4.3. *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3), (E4$^-$) and (E6). Consider the following statements:*

(i) *(E4$^-$) holds for $\mathcal{E}^1$, that is, if $g \circ f \in \mathcal{E}^1$ and $f \in \mathcal{E}^1$ then $g \in \mathcal{E}^1$;*
(ii) *Axiom (E5$^-$) holds;*
(iii) *every split epimorphism of split epimorphisms* (**C**) *with a, b, f_1 and f_0 in $\mathcal{E}$ is a double extension;*
(iv) *given a diagram* (**D**) *in $\mathcal{A}$ with a, b, f_1 and f_0 in $\mathcal{E}$, the arrow r is in $\mathcal{E}$ if and only if the right hand side square is in $\mathcal{E}^1$.*

Then (ii), (iii), and (iv) are equivalent and imply (i).

Proof: The axiom (E6) is used going from (iii) to (iv) and from (iv) to (ii). ■

It can be seen that (E1)–(E4$^-$) and (E5$^-$) still go up to higher dimensions together.

Theorem 4.4. *Let $\mathcal{A}$ have simplicial kernels and $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E4$^-$) and (E6). Then (E5$^-$) holds if and only if every $\mathcal{E}$-quasi-simplicial object in $\mathcal{A}$ is $\mathcal{E}$-Kan.*

Proof: The proofs of Proposition 3.11 and Proposition 3.12 may easily be adapted to this setting. However, we have to show that ∂_0 and ∂_1 which are constructed in the proof of Proposition 3.12 are also extensions. We may decompose the diagram defining, say, ∂_0, as

$$\begin{tikzcd}
A_1 \arrow[r, "\overline{r}"] \arrow[d, "\langle \partial_0, \partial_1 \rangle"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q \arrow[r] \arrow[d, "\overline{\langle a, f \rangle}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A_0 \arrow[d, "\langle a, f \rangle"] \\
A_0 \times_{A_{-1}} A_0 \arrow[r, "r"] \arrow[d, "\pi_0"'] & P \arrow[r] \arrow[d, "\overline{\pi_{A'}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A' \times_{B'} B \arrow[d, "\pi_{A'}"] \\
A_0 \arrow[r, equal] & A_0 \arrow[r, "a"'] & A'.
\end{tikzcd}$$

The induced morphism r is an extension (since the bottom rectangle is a double extension), hence so is $\overline{r}$. The composite $\overline{\pi_{A'}} \circ \overline{\langle a, f \rangle}$ is also an extension, as a pullback of $a = \pi_{A'} \circ \langle a, f \rangle$. Hence $\partial_0 = \pi_0 \circ \langle \partial_0, \partial_1 \rangle$ is an extension by (E3). Similarly, so is ∂_1. ■

Relative Mal'tsev categories in the sense of T. Janelidze. Classically, Mal'tsev categories are defined using properties of relations. Therefore we now connect our relative Mal'tsev condition (E5$^-$) to the conditions on $\mathcal{E}$-relations studied by T. Janelidze [37, 36]. For this, we use a context given in T. Janelidze's Condition 2.1 in [37], that is, we assume that $\mathcal{A}$ has finite products, $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ satisfies axioms (E1)–(E3), (E4$^-$) and (F). In [22] such a pair $(\mathcal{A}, \mathcal{E})$ is called a **relative regular category**. For a more detailed explanation see [37] and [22].

Definition 4.5 ($\mathcal{E}$-relations)**.** Given two objects A and B in $\mathcal{A}$, an **$\mathcal{E}$-relation from A to B** is a subobject of $A \times B$ such that for any representing monomorphism $\langle r_0, r_1 \rangle \colon R \to A \times B$, the morphisms $r_0 \colon R \to A$ and $r_1 \colon R \to B$ are in $\mathcal{E}$.

Using the axioms given, such $\mathcal{E}$-relations can be composed and this composition is associative. The usual definitions and calculations of relations apply. This setting allows us to copy proofs and methods from [9] to a relative situation. Many of these things were proved by T. Janelidze in her thesis [36]. In particular, we can easily adapt the proof of the fact that a regular category is Mal'tsev (i.e., every reflexive relation is an equivalence relation) if and only if every simplicial object in the category is Kan from [9] to a relative context.

Proposition 4.6 (cf. [9, Theorem 4.2])**.** *If $\mathcal{A}$ has finite products and simplicial kernels, $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E4$^-$) and (F), then every reflexive $\mathcal{E}$-relation in $(\mathcal{A}, \mathcal{E})$ is an equivalence $\mathcal{E}$-relation if and only if every $\mathcal{E}$-simplicial object is $\mathcal{E}$-Kan.* ■

This gives us:

Theorem 4.7. *If $\mathcal{A}$ has finite products and simplicial kernels, $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E4$^-$) and (F), then the following are equivalent:*

- *every reflexive $\mathcal{E}$-relation in $(\mathcal{A}, \mathcal{E})$ is an equivalence $\mathcal{E}$-relation;*
- *Axiom (E5$^-$);*
- *every $\mathcal{E}$-quasi-simplicial object is $\mathcal{E}$-Kan;*
- *every $\mathcal{E}$-simplicial object is $\mathcal{E}$-Kan.* ■

This suggests a definition of *relative Mal'tsev categories* to fit into the context of relative homological and relative semi-abelian categories.

Definition 4.8. A **relative Mal'tsev category** is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is a category with finite products and simplicial kernels and $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ such that the following axioms hold:

(E1) $\mathcal{E}$ contains all isomorphisms;
(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
(E3) $\mathcal{E}$ is closed under composition;
(E4$^-$) if $f \in \mathcal{E}$ and $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;
(E5$^-$) any split epimorphism of extensions

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[d, "a"'] & B_1 \arrow[l, shift left] \arrow[d, "b"] \\ A_0 \arrow[r, shift left, "f_0"] & B_0 \arrow[l, shift left] \end{tikzcd}$$

in $\mathcal{A}$ with f_1 and f_0 in $\mathcal{E}$ is a double extension;
(F) if a morphism f in $\mathcal{A}$ factors as $f = e \circ m$ with m a monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m' \circ e'$ with m' a monomorphism and $e' \in \mathcal{E}$.

On the one hand, if we have all split epimorphisms in $\mathcal{E}$, (E5$^-$) is equivalent to (E5) and (E4$^-$) is equivalent to (E4). On the other hand, using Axiom (F) instead of (E6) allows the comparison to the relational situation above and makes the category relatively regular [22]. Thus a similar definition with (E6) instead of (F) could be used to get a "non-regular" context. In such a context D. Bourn's approach from [3] could be followed instead of the regular approach in [9].

On the axiom (F). We explain under which conditions, in the absolute case, Axiom (F) goes up to higher dimensions. Here $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$.

Remark 4.9. Notice that a morphism $f = (f_1, f_0)\colon a \to b$ between extensions a and b is a monomorphism in $\mathsf{Ext}\mathcal{A}$ if and only if f_1 is a monomorphism. In particular, there are no restrictions on f_0. When $\mathcal{A}$ is regular, pushouts of regular epimorphisms are exactly the regular epimorphisms in $\mathsf{Ext}\mathcal{A}$.

Proposition 4.10. *Let $\mathcal{A}$ be a regular category and $\mathcal{E}$ the class of all regular epimorphisms in $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is exact Mal'tsev;*
(ii) *the pushout of an extension by an extension exists and is a double extension;*
(iii) $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$ *satisfies (F).*

Proof: The equivalence of (i) and (ii) was proved by A. Carboni, G. M. Kelly and M. C. Pedicchio in [9]. Assuming (ii), any morphism $f\colon a \to b$ in $\mathsf{Ext}\mathcal{A}$ factors as a double extension followed by a monomorphism as follows.

$$\begin{tikzcd} A_1 \arrow[r, "e"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & I \arrow[r, "m"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & B_1 \arrow[d, "b"] \\ A_0 \arrow[r] & P \arrow[r] & B_0 \end{tikzcd}$$

Here $f_1 = m \circ e$ is the regular epi-mono factorisation of f_1 and the left hand square is the pushout of e by a. Note that the former exists because $\mathcal{A}$ is regular and the latter by assumption. Hence, (ii) implies (iii). To see that (iii) implies (ii), consider extensions f and g and the morphism of extensions

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d] \\ C \arrow[r] & 1 \end{tikzcd}$$

where 1 is the terminal object. This square can be factored as a monomorphism (in the category of extensions) followed by a double extension as follows.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "g"'] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d] \\ C \arrow[r] & 1 \arrow[r, equal] & 1 \end{tikzcd}$$

The assumption implies that the square can also be factored as a double extension followed by a monomorphism.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] \arrow[dr, phantom, "\Rightarrow"] & I \arrow[r, "m"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "b"] \\ C \arrow[r] & I' \arrow[r] & 1 \end{tikzcd}$$

But this means in particular that m is a monomorphism. Hence, it is an isomorphism, since it is also a regular epimorphism (as f is). It follows that the pushout of f by g exists (it is given by the left hand square) and is a double extension, as desired. ■

Let us now investigate under which circumstances Axiom (F) goes up to $(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$. Clearly, as soon as $(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$ satisfies (F), the same will be true for $(\mathsf{Ext}\mathcal{A}, \mathcal{E}^1)$. Hence, by Proposition 4.10, a necessary condition for $(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$ to satisfy (F) is that $\mathcal{A}$ is exact Mal'tsev. Observe that, in this case, $\mathsf{Ext}\mathcal{A}$ is regular: regular epimorphisms in $\mathsf{Ext}\mathcal{A}$ are double extensions, which we know are pullback-stable. Hence, we can apply Proposition 4.10 to $\mathsf{Ext}\mathcal{A}$ and find, in particular, that $(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$ satisfies (F) if and only if $\mathsf{Ext}\mathcal{A}$ is exact Mal'tsev.

Now, recall from [43] that an exact Mal'tsev category is **arithmetical** if every internal groupoid is an equivalence relation. Examples of arithmetical categories are the dual of the category of pointed sets, more generally, the dual of the category of pointed objects in any topos, and also the categories of von Neumann regular rings, Boolean rings and Heyting semi-lattices. It was proved in [5] that an exact Mal'tsev category is arithmetical if and only if the category

$\mathsf{Equiv}\mathcal{A}$ of internal equivalence relations in $\mathcal{A}$ is exact. In this case $\mathsf{Equiv}\mathcal{A}$ is in fact again arithmetical and, in particular, exact Mal'tsev. Since, moreover, there is a category equivalence $\mathsf{Equiv}\mathcal{A} \simeq \mathsf{Ext}\mathcal{A}$ because $\mathcal{A}$ is exact, we find:

Proposition 4.11. *Let $\mathcal{A}$ be an exact Mal'tsev category and $\mathcal{E}$ the class of all regular epimorphisms in $\mathcal{A}$. The following are equivalent:*

(i) *$\mathcal{A}$ is arithmetical;*
(ii) *$\mathsf{Ext}\mathcal{A}$ is arithmetical;*
(iii) *$\mathsf{Ext}\mathcal{A}$ is exact Mal'tsev;*
(iv) *any pushout of a double extension by a double extension exists (in the category $\mathsf{Ext}\mathcal{A}$) and is a three-fold extension;*
(v) *$(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$ satisfies (F).* ■

Remark 4.12. Notice that Proposition 4.11 also implies that axiom (F) is satisfied by $(\mathsf{Ext}^n\mathcal{A}, \mathcal{E}^n)$ for *every n* as soon as the category $\mathcal{A}$ is arithmetical. Conversely, the category $\mathcal{A}$ is arithmetical as soon as there exists an $n \geq 2$ such that (F) holds for $(\mathsf{Ext}^n\mathcal{A}, \mathcal{E}^n)$.

Remark 4.13. While in Propositions 4.10 and 4.11 we considered only the absolute case, it is certainly possible to formulate and prove relative versions of these results. The reader is invited to verify, in particular, that the proof of the result from [5] we used for Proposition 4.11 can indeed be relativised.

Since being arithmetical is a rather restrictive property for a (Mal'tsev) category to have, we can conclude this analysis by saying that Axiom (F) "hardly ever" goes up to $(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$ or higher.

More examples. We end this article with several examples and counterexamples.

Example 4.14 (Relative homological categories)**.** Also in the weaker situation, using ($\mathrm{E4}^-$) and (F), the axiom ($\mathrm{E5}^+$) considered in Example 3.19 implies ($\mathrm{E5}^-$). Hence relative homological and relative semi-abelian categories as defined by T. Janelidze [35, 37] are relatively Mal'tsev, but generally need not satisfy the stronger (E4) and (E5). An example for a relative semi-abelian category is a semi-abelian category $\mathcal{A}$ with $\mathcal{E}$ being the class of central extensions in the sense of Huq, closed under composition [36, Proposition 5.3.2]. That is, any morphism in $\mathcal{E}$ is the composition of regular epimorphisms $f\colon A \to B$ with $[\mathrm{K}[f], A] = 0$, where $[\mathrm{K}[f], A]$ is the commutator of $\mathrm{K}[f]$ and A in the sense of Huq [27].

When $\mathcal{E}$ is a class of regular epimorphisms in a regular Mal'tsev category $\mathcal{A}$ satisfying (E1)–(E2), then it is easy to check that (E3), (E4$^-$) and (E5$^-$) hold as soon as the following **two out of three property** is satisfied: given a composite $g \circ f$ of regular epimorphisms $f \colon A \to B$ and $g \colon B \to C$, if any two of $g \circ f$, f and g lie in $\mathcal{E}$, then so does the third. We shall make use of this fact when considering the following two examples, which are given by categorical Galois theory [28, 31]. Notice that this uses the regular Mal'tsev property to show that, in the square given in (E5$^-$), the comparison to the pullback is already a regular epimorphism, and then the two out of three property shows it is in fact in $\mathcal{E}$.

Example 4.15 (Trivial extensions)**.** Let $\mathcal{B}$ be a full and replete reflective subcategory of a regular Mal'tsev category $\mathcal{A}$. Write $I \colon \mathcal{A} \to \mathcal{B}$ for the left adjoint of the inclusion functor. Assume that I preserves regular epimorphisms and is *admissible* (with respect to regular epimorphisms) [31]. For instance, $\mathcal{B}$ could be a Birkhoff subcategory of $\mathcal{A}$ (a full reflective subcategory closed under subobjects and regular quotients) if $\mathcal{A}$ is also Barr exact (see [33]).

Recall that a **trivial extension** (with respect to I) is a regular epimorphism f such that the commutative square induced by the unit $\eta \colon 1_{\mathcal{A}} \Rightarrow I$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ IA \arrow[r, "If"'] & IB \end{tikzcd} \qquad (\mathbf{I})$$

is a pullback. With $\mathcal{E}$ the class of all trivial extensions, we claim that $(\mathcal{A}, \mathcal{E})$ satisfies conditions (E1)–(E4$^-$) and (E5$^-$). (The stronger axiom (E4) need not hold as in general not every split epimorphism is a trivial extension: for instance, when $\mathcal{A}$ is pointed, a morphism $A \to 0$ is a trivial extension if and only if A is in $\mathcal{B}$.) Indeed, the validity of (E1) is clear while (E2) follows from the admissibility of I. Hence, it suffices to prove the two out of three property, of which only one implication is not immediate. To see that $g \colon B \to C$ is a trivial extension as soon as $f \colon A \to B$ and $g \circ f$ are, it suffices to note that, since If is a pullback-stable regular epimorphism, the change of base functor $(If)^* \colon (\mathcal{E} \downarrow IB) \to (\mathcal{E} \downarrow IA)$ is conservative [34]. Here we have written $(\mathcal{E} \downarrow X)$ for the full subcategory of the slice category $(\mathcal{A} \downarrow X)$ determined by $\mathcal{E}$.

When $\mathcal{A}$ is Barr exact and $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$, then $(\mathcal{A}, \mathcal{E})$ also satisfies (F). Indeed, condition (F) is easily inferred from the fact that in this case the square (**I**) is a pushout, hence a regular pushout (a double extension) for any regular epimorphism f [9, 33]. When $\mathcal{A}$ is semi-abelian, T. Janelidze explained in her talk [38] that $(\mathcal{A}, \mathcal{E})$ is in fact relatively semi-abelian.

Example 4.16 (Torsion theories)**.** Let $\mathcal{A}$ be a homological category in which every regular epimorphism is effective for descent (for instance, $\mathcal{A}$ could be semi-abelian) and let $\mathcal{F}$ be a torsion-free subcategory of $\mathcal{A}$ (a full regular epi-reflective subcategory of $\mathcal{A}$ such that the associated radical $T\colon \mathcal{A} \to \mathcal{A}$ is idempotent, see [7]). Then the reflector $F\colon \mathcal{A} \to \mathcal{F}$ is admissible with respect to *all* morphisms, so that, in particular, the previous example applies. Thus we find that $(\mathcal{A}, \mathcal{E})$ satisfies conditions (E1)–(E4^{-}) and (E5^{-}), for $\mathcal{E}$ the class of all trivial extensions.

Let us now write $\mathcal{E}^*$ for the class of (regular epi)morphisms $f\colon A \to B$ that are "locally in $\mathcal{E}$", in the sense that there exists an effective descent morphism $p\colon E \to B$ in $\mathcal{A}$ such that the pullback $p^*(f)\colon E \times_B A \to E$ is in $\mathcal{E}$. The morphisms in $\mathcal{E}^*$ are usually called **coverings** or **central extensions**. While the pair $(\mathcal{A}, \mathcal{E}^*)$ satisfies conditions (E1) and (E2) because $(\mathcal{A}, \mathcal{E})$ does, $\mathcal{E}^*$ is in general not closed under composition. However, it was shown in [15] that $\mathcal{E}^*$ *is* composition-closed as soon as the reflector F is **protoadditive** [16, 15]: F preserves split short exact sequences. Let us briefly recall the argument. First of all, it was shown in [15] that the central extensions with respect to F (which we shall, from now on, assume to be protoadditive) are exactly those regular epimorphisms $f\colon A \to B$ whose kernel $K[f]$ is in $\mathcal{F}$. Now, let $f\colon A \to B$ and $g\colon B \to C$ be regular epimorphisms. Then we have a short exact sequence

$$0 \longrightarrow K[f] \longrightarrow K[g \circ f] \longrightarrow K[g] \longrightarrow 0$$

and we see that $g \circ f$ is a central extension as soon as f and g are, since the torsion-free subcategory $\mathcal{F}$ is closed under extensions (which means that when $K[f] \in \mathcal{F}$ and $K[g] \in \mathcal{F}$ then $K[g \circ f] \in \mathcal{F}$) [7]. Furthermore, since $\mathcal{F}$ is a (regular epi)-reflective subcategory of $\mathcal{A}$, $\mathcal{F}$ is closed under subobjects, and so f is a central extension as soon as $g \circ f$ is. If we assume that $\mathcal{F}$ is, moreover, closed under regular quotients (i.e., $\mathcal{F}$ is a Birkhoff subcategory of $\mathcal{A}$) then g is a central extension as soon as $g \circ f$ is, and we may conclude that $\mathcal{E}^*$ satisfies the two out of three property. Once again using that $\mathcal{F}$ is closed under subjects in $\mathcal{A}$, it is easily verified that $(\mathcal{A}, \mathcal{E}^*)$ also satisfies Axiom (F). (Note that the same two out of three property can be used to show that $(\mathcal{A}, \mathcal{E}^*)$ is, in fact, relatively homological.)

Examples of such an $\mathcal{A}$ and $\mathcal{F}$ are given, for instance, by taking $\mathcal{A}$ to be the category of compact Hausdorff groups and $\mathcal{F}$ the subcategory of profinite groups [15], or $\mathcal{A}$ to be the category of internal groupoids in a semi-abelian category and $\mathcal{F}$ the subcategory of discrete groupoids [16]. Since a reflector into an epi-reflective subcategory of an abelian category is necessarily (proto)additive, any cohereditary torsion theory (meaning that $\mathcal{F}$ is closed under quotients) in an abelian category $\mathcal{A}$ provides an example as well. However, there are no non-trivial examples in the categories of groups or of abelian groups, as follows from Proposition 5.5 in [45].

Example 4.17 (Internal groupoids—see [24] for the absolute case)**.** Let the pair $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E4$^-$), (E5$^-$) and (F). Denote by $\mathsf{Gpd}_{\mathcal{E}}\mathcal{A}$ the category of **internal $\mathcal{E}$-groupoids** in $\mathcal{A}$: groupoids G in $\mathcal{A}$ with the property that all split epimorphisms occurring in the diagram of G are in $\mathcal{E}$. Write $\overline{\mathcal{E}}$ for the class of degree-wise $\mathcal{E}$-extensions. Then $(\mathsf{Gpd}_{\mathcal{E}}\mathcal{A}, \overline{\mathcal{E}})$ is relatively Mal'tsev. Indeed, to see that axioms (E2) and (E5$^-$) are satisfied, observe that pullbacks along morphisms in $\overline{\mathcal{E}}$ are degree-wise pullbacks in $\mathcal{A}$. For Axiom (F) note that products are computed degree-wise as well, and that $\mathsf{Gpd}_{\mathcal{E}}\mathcal{A}$ is closed in $\mathsf{RG}_{\mathcal{E}}\mathcal{A}$—the category of "reflexive $\mathcal{E}$-graphs" in $\mathcal{A}$—under "$\overline{\mathcal{E}}$-quotients", as a consequence of the relative Mal'tsev condition for $(\mathcal{A}, \mathcal{E})$.

Example 4.18 (Regular pullback squares)**.** This is an example of a pair $(\mathcal{A}, \mathcal{E})$ which satisfies (E1)–(E4$^-$) and (E5$^-$), but where not every split epimorphism is an extension, nor does (F) hold. (Remember however that (E5$^-$) implies (E6).) We take $\mathcal{A}$ to be the category $\mathsf{ExtGp}_{\mathsf{tf}}$ of extensions (regular epimorphisms) in the category of torsion-free groups. The class $\mathcal{E}$ consists of regular pullback squares, i.e., pullbacks of regular epimorphisms. It is easy to find a split epimorphism of extensions which is not a pullback, and it is also easy to see that (E1)–(E4$^-$) and (E5$^-$) hold using that $\mathsf{Gp}_{\mathsf{tf}}$ is regular Mal'tsev. We give a counterexample for Axiom (F); it is based on the fact that pushouts in $\mathsf{Gp}_{\mathsf{tf}}$ are different from pushouts in Gp and may not be regular pushouts. They are constructed by reflecting the pushout in Gp into the subcategory $\mathsf{Gp}_{\mathsf{tf}}$.

An example of a pushout in $\mathsf{Gp}_{\mathsf{tf}}$ which is not a pushout in Gp is the square

$$\begin{tikzcd} \mathbb{Z} \times_{\mathbb{Z}_2} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & 0. \end{tikzcd} \tag{J}$$

($\mathbb{Z}_2$ is torsion while $\mathbb{Z}$ is torsion-free.) The diagram

$$\begin{tikzcd}
\mathbb{Z} \times_{\mathbb{Z}_2} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \times \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

now displays a monomorphism composed with an $\mathcal{E}$-extension which cannot be written as an $\mathcal{E}$-extension composed with a monomorphism, as the square (**J**) is not in $\mathcal{E}$.

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