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LOCAL HÖLDER CONTINUITY OF WEAK SOLUTIONS FOR AN ANISOTROPIC ELLIPTIC EQUATION

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ABSTRACT: We prove, following DiBenedetto's intrinsic scaling method, that the weak solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] = f & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases}$$

is locally Hölder continuous, where f is a bounded given function and $p_i \geq 2$, for any $i = 1, ..., N$.

1. Introduction

In this paper we identify a class of anisotropic second-order elliptic equations for which the Hölder continuity can be established, following DiBenedetto's method of intrinsic scaling. More precisely we prove that the weak solution u of problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] = f & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

belongs to $C^{0,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$, where f is a given function belonging to $L^\infty(\Omega)$, Ω is a smooth, bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $p_i \geq 2$ for any $i = 1, ..., N$. Without loss of generality, we can assume that the p_i's are ordered, that is

$$2 \leq p_1 \leq ... \leq p_N;$$

thus,

$$p_1 = \min_i\{p_i\} \quad \text{and} \quad p_N = \max_i\{p_i\}.$$

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Moreover, we suppose that $\overline{p} < N$, where $\overline{p}$ is the harmonic mean of the p_i's, that is

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i};$$

otherwise, the weak solution of the problem (1) is $C^{0,\alpha}_{\text{loc}}(\Omega)$ by the anisotropic Sobolev embeddings (see [28], and also [14], [26]).

The interest in various types of anisotropic operators grew in the last few years, not only for a humongous number of applications (in modeling electrorheological fluids, image processing and the theory of elasticity) but also for mathematical reasons involved: it is necessary to perform essential modifications to the classical methods in the analysis, because of the nonlinear and non homogeneous nature of the considered operators.

The regularity theory for elliptic equations and for integrals of the calculus of variations with non-standard growth was considered in many papers. We recall some of these, without hope to be complete, in which local boundedness and local Lipschitz regularity for the solutions and their gradients, are treated, [1], [11], [12], [17], [18], [21], [22], [23].

Also with respect to the local Hölder continuity for solutions of elliptic equations (and the corresponding minimization problems), in particular associated to operators having $p(x)$-growth and p,q type conditions, there are many papers, for example [2], [4], [9], [20], [25]. The qualitative theory of equations of the type (1) has not yet been developed to the same extent. There is however a recent paper about this problem. In [19], the authors establish the local Hölder continuity for solutions of equations as that in (1), assuming

$$p_1 = 2 < p_2 = ... = p_N = p$$

and

$$p_N < \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}.$$

As seen from the structure, the x_1 variable is separated from the others and so it was treated similarly to the time variable in the corresponding studies of parabolic equations. The main novelty of that note is to use the DiBenendetto intrinsic scaling method for proving Hölder continuity of weak solutions for a class of anisotropic quasi-linear elliptic equations. This method is a powerful technique introduced in the 1980's (see [6] and more recently [7], [29] and references therein) that helped understanding the local behavior of weak solutions of singular and degenerate PDEs. It was originally created for studying the evolutionary p-Laplace equation.

In this paper we also use the method of intrinsic scaling but in a simpler way and we treat the general case in which all the p_i's are different. Moreover, another novelty of this work is that we don't need to assume $p_N < \overline{p}^*$, since we are taking homogeneous Dirichlet boundary conditions. Indeed this assumption assures that the weak solution of problem (1) is bounded without further hypothesis on the p_i's, as showed in [8] (see also [27]). If we do not assume Dirichlet boundary conditions the result of this paper is still true, but we have to add the hypothesis that the solutions are bounded and hence $p_N < \overline{p}^*$. As a matter of fact there exists an example of unbounded solutions due to Marcellini (see [24] and also [13]) when no boundary assumptions are given.

The main result of this paper is the following.

Theorem 1.1. *Let u the weak solution of problem* (1) *with $f \in L^\infty(\Omega)$ and $M = \|u\|_{L^\infty(\Omega)}$. Then u is locally Hölder continuous in Ω, that is there exists a constant $\gamma > 1$ and $\alpha \in (0,1)$ depending only on the data such that for every compact $K \subset\subset \Omega$ the inequality*

$$|u(x_1) - u(x_2)| \leq \gamma M \left(\frac{\sum_{i=1}^N |x_{1,i} - x_{2,i}|^{\frac{p_i}{p_N}} M^{\frac{p_N - p_i}{p_N}}}{\operatorname{dist}(K, \partial\Omega)} \right)^\alpha$$

holds for any x_1, $x_2 \in K$.

Here $\operatorname{dist}(K, \partial\Omega)$ is the distance from K to the boundary of Ω, defined by

$$\operatorname{dist}(K, \partial\Omega) = \inf_{\substack{x \in K \\ y \in \partial\Omega}} \left(\sum_{i=1}^N |x_i - y_i|^{\frac{p_i}{p_N}} M^{\frac{p_i - p_N}{p_N}} \right).$$

The rest of the paper contains the proof of the above theorem, after a first section in which definitions, notations and some already known results are given.

In the following we will denote $\partial_i := \partial/\partial x_i$. We will write C to denote positive constants, the value of which may vary from line to line, depending on the data, that is they will be fixed in the assumptions we will make, as the dimension N, the set Ω, the exponents p_i, etc.

2. Definitions, notations and basic tools

It is well-known (see [16], and also [8] and [3]) that, for any N-vector of real numbers

$$2 \leq p_1 \leq \ldots \leq p_N$$

and for any $f \in L^\infty(\Omega)$, there exists a unique weak solution of problem (1), that is a function $u \in W_0^{1,(p_i)}(\Omega)$ such that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i v = \int_\Omega f v, \quad \forall \ v \in W_0^{1,(p_i)}(\Omega), \tag{2}$$

where $W_0^{1,(p_i)}(\Omega)$ denotes the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|v\|_{1,(p_i)} := \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}$$

or, equivalently,

$$W_0^{1,(p_i)}(\Omega) = \left\{ v \in W_0^{1,p_1}(\Omega) : \partial_i v \in L^{p_i}(\Omega), \ \ i = 1, ..., N \right\}.$$

Moreover $u \in L^\infty(\Omega)$ by the assumption on f. The same result holds under less stringent assumptions on the regularity of the given function f, namely $f \in L^m(\Omega)$, $m > N/\overline{p}$, $\overline{p}$ defined below (see [3], [8], [27]).

In [14], [26] and [28], the theory of anisotropic Sobolev spaces is developed and, in particular, the corresponding Sobolev embeddings theorems are studied. Let

$$\overline{p}^* = \frac{N\overline{p}}{N - \overline{p}}, \quad \text{for } \ \overline{p} < N \ \text{ and } \ \frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}. \tag{3}$$

In [28] it is proved that if $\overline{p} < N$, then

$$W_0^{1,(p_i)}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall \ r \in [1, \overline{p}^*].$$

This embedding is continuous and also compact if $r < \overline{p}^*$. The following Sobolev type inequality is also proved: there exists a positive constant C, depending only on Ω, such that

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \quad \forall \ r \in [1, \overline{p}^*], \tag{4}$$

for any $v \in C^1(\Omega)$. By density, (4) also holds for any $v \in W_0^{1,(p_i)}(\Omega)$. The inequality (4) also implies that

$$\|v\|_{L^r(\Omega)} \le C \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}, \quad \forall\ r \in [1, \overline{p}^*].$$

Subsequently, in [10], it is proved that the critical exponent depends on the kind of anisotropy. If the p_i's are not too far apart (i.e. the anisotropy is concentrated) the critical exponent is $\overline{p}^*$, as in [28], that is the usual critical exponent related to the harmonic mean $\overline{p}$ of the p_i's. While if the p_i's are too spread out, it coincides with the maximum of the p_i's, i.e., p_N. We also remind a Poincaré type inequality, valid for all $v \in W_0^{1,(p_i)}(\Omega)$:

$$\|v\|_{L^r(\Omega)} \le C(|\Omega|) r \|\partial_i v\|_{L^r(\Omega)}, \quad \forall\ r \ge 1,\ \forall\ i = 1, ..., N, \tag{5}$$

see [10].

We recall in this section some technical (and by now classical) tools that are essential in establishing our regularity result.

Given a continuous function $u : \Omega \to \mathbb{R}$ and two real numbers $k < l$, let

$$A_{l,S} := \{x \in S : u(x) > l\} \tag{6}$$

$$B_{k,S} := \{x \in S : u(x) < k\} \tag{7}$$

$$A_{k,S} \cap B_{l,S} := \{x \in S : k < u(x) < l\} \tag{8}$$

for $S \subseteq \Omega$. Moreover $|S|$ is the measure of the set S.

Lemma 2.1 (De Giorgi). *Let $u \in W^{1,1}(K_\rho(x_0)) \cap C(K_\rho(x_0))$ with $\rho > 0$, $x_0 \in \mathbb{R}^N$, $K_\rho(x_0)$ an arbitrary sphere of radius ρ and centre x_0 and $l > k \in \mathbb{R}$. There exists a constant C, depending only on N (and thus independent of ρ, x_0, u, k and l), such that*

$$(l-k)|A_{l,K_\rho(x_0)}| \le C \frac{\rho^{N+1}}{|B_{k,K_\rho(x_0)}|} \int_{A_{k,K_\rho(x_0)} \cap B_{l,K_\rho(x_0)}} |\nabla u|.$$

Proof: See [5]. (See also [15], Lemma 3.4 and 3.5, pp. 54-56). ∎

Remark 2.2. The conclusion of this lemma remains valid for functions $u \in W^{1,1}(\Omega) \cap C(\Omega)$, provided Ω is a convex region of diameter 2ρ. We will use it in the case Ω is a parallelepiped. We also underline that the continuity is not essential for the result to hold. For a function merely in $W^{1,1}(\Omega)$, we define the previous sets through any representative in the equivalence class. It can be shown that the conclusion of the lemma is independent of this choice.

The next lemma concerns the geometric convergence of some sequences of real numbers.

Lemma 2.3. *Suppose that a sequence y_h, for $h = 0, 1, 2, ...$, of nonnegative real numbers, satisfies the recurrence relation*

$$y_{h+1} \leq C\, b^h y_h^{1+\varepsilon}, \quad h = 0, 1, 2, ...$$

where C, ε and b are positive constants and $b > 1$. Then

$$y_h \leq C^{\frac{(1+\varepsilon)^h - 1}{\varepsilon}}\, b^{\frac{(1+\varepsilon)^h - 1}{\varepsilon^2} - \frac{h}{\varepsilon}} y_0^{(1+\varepsilon)^h}, \quad h = 0, 1, 2, ...$$

In particular, if $y_0 \leq \theta = C^{-\frac{1}{\varepsilon}}\, b^{-\frac{1}{\varepsilon^2}}$ then $y_h \leq \theta b^{-\frac{h}{\varepsilon}}$ and consequently $y_h \to 0$ as $h \to +\infty$.

Proof: See Lemma 4.7, p. 66 of [15]. ■

Before starting with the main result of this paper, we want to give some notations that we will use in the following pages. Let Q_ρ^N be the cube in $\mathbb{R}^N$ of side 2ρ and center at the origin, whose sides are parallel to the coordinate axes, defined by

$$Q_\rho^N = (-\rho, \rho) \times ... \times (-\rho, \rho) = \prod_{i=1}^{N} (-\rho, \rho). \tag{9}$$

Let also $Q_{\rho,a,(p_i)}$ be the parallelepiped in $\mathbb{R}^N$, whose sides are parallel to the coordinate axes, that is

$$\begin{aligned} Q_{\rho,a,(p_i)} &= (-a^{\frac{p_1 - p_N}{p_1}} \rho^{\frac{p_N}{p_1}}, \rho^{\frac{p_N}{p_1}} a^{\frac{p_1 - p_N}{p_1}}) \times ... \\ &\times (-a^{\frac{p_{N-1} - p_N}{p_{N-1}}} \rho^{\frac{p_N}{p_{N-1}}}, \rho^{\frac{p_N}{p_{N-1}}} a^{\frac{p_{N-1} - p_N}{p_{N-1}}}) \times (-\rho, \rho) \\ &= \prod_{i=1}^{N} (-a^{\frac{p_i - p_N}{p_i}} \rho^{\frac{p_N}{p_i}}, \rho^{\frac{p_N}{p_i}} a^{\frac{p_i - p_N}{p_i}}), \end{aligned} \tag{10}$$

for some $a > 0$. We have

$$\begin{aligned} |Q_{\rho,a,(p_i)}| &= 2a^{\frac{p_1 - p_N}{p_1}} \rho^{\frac{p_N}{p_1}} \times ... \times 2a^{\frac{p_{N-1} - p_N}{p_{N-1}}} \rho^{\frac{p_N}{p_{N-1}}} \times 2\rho \\ &= 2^N a^{\sum_{i=1}^N \frac{p_i - p_N}{p_i}} \rho^{\sum_{i=1}^N \frac{p_N}{p_i}} = 2^N a^{N - \frac{N p_N}{\overline{p}}} \rho^{\frac{N p_N}{\overline{p}}}, \end{aligned} \tag{11}$$

with $\overline{p}$ defined in (3).

Remark 2.4. We note that if $p_i = p$ for any i, that is if we consider the isotropic problem, $Q_{\rho,a,(p_i)} = Q_\rho^N$, the usual cube in $\mathbb{R}^N$ defined in (9). Moreover $Q_{\rho,a(p_i)} \subset Q_\rho^N$ if, and only if,

$$a^{\frac{p_i-p_N}{p_i}} \rho^{\frac{p_N}{p_i}} < \rho, \qquad \forall\ i = 1, ..., N,$$

that is $a > \rho$.

We also denote

$$\frac{1}{2} Q_{\rho,a,(p_i)} = \prod_{i=1}^{N} \left(-\frac{1}{2} \rho^{\frac{p_N}{p_i}} a^{\frac{p_i-p_N}{p_i}}, \frac{1}{2} \rho^{\frac{p_N}{p_i}} a^{\frac{p_i-p_N}{p_i}} \right). \tag{12}$$

3. Energy estimates

As it is well known, the building blocks of the method of intrinsic scaling are a priori estimates for weak solutions. Once established the energy estimates we can forget the equation and the problem becomes a problem in analysis. So in this section we prove integral inequalities on the level sets that measure the behavior of the weak solution near its infimum and its supremum in the interior of an appropriate parallelepiped. Consider two cubes $Q_{\rho'}^N \subset Q_\rho^N \subset \Omega$ and let

$$\xi = \prod_{i=1}^{N} \xi_i^{p_i}, \quad \text{with} \quad 0 \leq \xi_i = \xi(x_i) \leq 1, \quad \forall\ i = 1, ..., N,$$

be a nonnegative cutoff function, belonging to $C_0^1(Q_\rho^N)$, that vanishes outside of a set Q_ρ^N, is equal to unity in $Q_{\rho'}^N$ and

$$|\xi_i'| \leq \frac{C}{\rho - \rho'}, \quad \forall\ i = 1, ..., N.$$

We also define

$$\overline{\xi}_i = \prod_{j=1, j \neq i}^{N} \xi_j^{p_j}.$$

Proposition 3.1. *Let u be the weak solution of problem* (1) *and $k \in \mathbb{R}$. There exists a constant $C > 0$ depending only on the data such that*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i (u-k)_-|^{p_i} \xi \leq C \left[\sum_{j=1}^{N} \int_\Omega |(u-k)_-|^{p_j} |\xi_j'|^{p_j} \overline{\xi}_j + |B_{k,Q_\rho^N}| \right]. \tag{13}$$

Moreover for any $i = 1, ..., N$

$$\int_\Omega |\partial_i (u-k)_-|^{p_i} \xi \le C \left[\sum_{j=1}^N \int_\Omega |(u-k)_-|^{p_j} |\xi_j'|^{p_j} \overline{\xi}_j + |B_{k,Q_\rho^N}| \right], \tag{14}$$

where B_{k,Q_ρ^N} *is defined in* (7), Q_ρ^N *in* (9) *and* ξ *as above.*

Proof: We use, as a test function in (2), $\phi_- = -(u-k)_- \xi$, where

$$(u-k)_- = (k-u)_+ = \max\{k-u, 0\}.$$

We note that $\partial_i \phi_- = \xi \partial_i [-(u-k)_-] - (u-k)_- p_i \xi_i^{p_i-1} \xi_i' \overline{\xi}_i$, so we get

$$\begin{aligned} \sum_{i=1}^N \int_\Omega |\partial_i (u-k)_-|^{p_i} \xi \ &\le \ p_N \sum_{i=1}^N \int_\Omega |\partial_i (u-k)_-|^{p_i-1} |(u-k)_-| \xi_i^{p_i-1} |\xi_i'| \overline{\xi}_i \\ &+ \ \int_\Omega |f| |(u-k)_-| \xi. \end{aligned}$$

We apply the ε-version of Young's inequality to

$$|\partial_i (u-k)_-|^{p_i-1} |(u-k)_-| \xi_i^{p_i-1} |\xi_i'|,$$

with p_i and p_i', for any $i = 1, ..., N$; we obtain

$$\begin{aligned} \sum_{i=1}^N \int_\Omega |\partial_i (u-k)_-|^{p_i} \xi \ &\le \ C_\varepsilon \sum_{i=1}^N \int_\Omega |\partial_i (u-k)_-|^{p_i} \xi_i^{p_i} \overline{\xi}_i \\ &+ \ C_\varepsilon' \sum_{i=1}^N \int_\Omega |(u-k)_-|^{p_i} |\xi_i'|^{p_i} \overline{\xi}_i + \int_\Omega |f| |(u-k)_-| \xi. \end{aligned}$$

Noting that $\xi_i^{p_i} \overline{\xi}_i = \xi$ and choosing ε such that $1 - C_\varepsilon > 0$, we arrive at

$$\sum_{i=1}^N \int_\Omega |\partial_i (u-k)_-|^{p_i} \xi \ \le \ C \sum_{i=1}^N \int_\Omega |(u-k)_-|^{p_i} |\xi_i'|^{p_i} \overline{\xi}_i + C \int_\Omega |f| |(u-k)_-| \xi.$$

We estimate the second term on the right hand side of the previous inequality using Poincaré type inequality (5), with $r = p_i$, the ε-version of Young's inequality and the assumptions on f

$$\begin{aligned}\int_\Omega |f||(u-k)_-|\xi &\le C'_\varepsilon \sum_{i=1}^N \int_{B_{k,Q_\rho^N}} |f|^{p'_i} + C''_\varepsilon \sum_{i=1}^N \int_\Omega |\partial_i((u-k)_-\xi)|^{p_i} \\ &\le C'_\varepsilon \sum_{i=1}^N \|f\|^{p'_i}_{L^\infty(\Omega)} |B_{k,Q_\rho^N}| + C''_\varepsilon \sum_{i=1}^N \int_\Omega |\partial_i(u-k)_-|^{p_i}\xi \\ &+ C''_\varepsilon \sum_{i=1}^N \int_\Omega |(u-k)_-|^{p_i}|\xi'_i|^{p_i}\overline{\xi}_i\end{aligned}$$

where B_{k,Q_ρ^N} is defined in (7). In conclusion, choosing ε conveniently, that is such that $1 - C''_\varepsilon > 0$, we obtain (13); obviously also (14) follows, for any $i = 1, ..., N$. ■

In a similar way it is also possible to prove the proposition below.

Proposition 3.2. *Let u be the weak solution of problem* (1) *and $k \in \mathbb{R}$. There exists a constant $C > 0$ depending only on the data such that*

$$\sum_{i=1}^N \int_\Omega |\partial_i(u-k)_+|^{p_i}\xi \le C\left[\sum_{j=1}^N \int_\Omega |(u-k)_+|^{p_j}|\xi'_j|^{p_j}\overline{\xi}_j + |A_{k,Q_\rho^N}|\right], \tag{15}$$

Moreover for any $i = 1, ..., N$

$$\int_\Omega |\partial_i(u-k)_+|^{p_i}\xi \le C\left[\sum_{j=1}^N \int_\Omega |(u-k)_+|^{p_j}|\xi'_j|^{p_j}\overline{\xi}_j + |A_{k,Q_\rho^N}|\right], \tag{16}$$

where A_{k,Q_ρ^N} is defined in (6), *Q_ρ^N in* (9) *and ξ as before.*

Proof: We proceed as in the previous proposition but we use as a test function in the weak formulation of (1), $\phi_+ = (u-k)_+\xi$ instead of $\phi_- = -(u-k)_-\xi$. ■

4. Auxiliary lemmas

We consider $0 < \rho < 1$, sufficiently small so that $Q_\rho^N \subset \Omega$, and we define the essential oscillation of the weak solution u in Q_ρ^N

$$\omega = \operatorname*{ess\,osc}_{Q_\rho^N} u = \mu_+ - \mu_-,$$

where

$$\mu_+ = \operatorname*{ess\,sup}_{Q_\rho^N} u \quad \text{and} \quad \mu_- = \operatorname*{ess\,inf}_{Q_\rho^N} u.$$

Then, we construct the rescaled parallelepiped $Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}$, defined in (10), with $a = \omega/2^\lambda$, where $\lambda > 1$ is to be fixed later, depending only on the data (see (29)). We assume, without loss of generality, that

$$\rho < \frac{\omega}{2^\lambda}. \tag{17}$$

Instead, if this is not true, we have $\omega \leq 2^\lambda \rho$ and there is nothing to prove because the oscillation is comparable to the "radius". Now (17) implies that

$$Q_{\rho,\frac{\omega}{2^\lambda},(p_i)} \subset Q_\rho^N,$$

see Remark 2.4, and the relation

$$\operatorname*{ess\,osc}_{Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u \leq \omega,$$

which will be the starting point of an iteration process that leads to the main result. We consider a subparallelepiped of $Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}$, namely

$$Q_{\rho,\frac{\omega}{2},(p_i)} \subset Q_{\rho,\frac{\omega}{2^\lambda},(p_i)} \subset Q_\rho^N \subset \Omega.$$

The proof of the Hölder continuity of the weak solution u of (1) now follows from the analysis of two complementary cases. For $Q_{\rho,\frac{\omega}{2},(p_i)}$ either

The First Alternative: there exists $\nu_0 \in (0,1)$ such that

$$\left| \left\{ x \in Q_{\rho,\frac{\omega}{2},(p_i)} : u(x) < \mu_- + \frac{\omega}{2} \right\} \right| \leq \nu_0 |Q_{\rho,\frac{\omega}{2},(p_i)}|, \tag{18}$$

or this does not hold. Then for any $\nu \in (0,1)$,

The Second Alternative holds:

$$\left| \left\{ x \in Q_{\rho,\frac{\omega}{2},(p_i)} : u(x) < \mu_- + \frac{\omega}{2} \right\} \right| > \nu |Q_{\rho,\frac{\omega}{2},(p_i)}|. \tag{19}$$

Since $\mu_+ - \frac{\omega}{2} = \mu_- + \frac{\omega}{2}$, (19) is equivalent to

$$\left| \left\{ x \in Q_{\rho,\frac{\omega}{2},(p_i)} : u(x) \geq \mu_+ - \frac{\omega}{2} \right\} \right| \leq (1-\nu)|Q_{\rho,\frac{\omega}{2},(p_i)}| = \nu_1 |Q_{\rho,\frac{\omega}{2},(p_i)}|, \tag{20}$$

for any $\nu_1 \in (0,1)$.

Now we start the analysis assuming that (18) holds in $Q_{\rho,\frac{\omega}{2},(p_i)}$ for some $\nu_0 \in (0,1)$, that will be determined depending only on the data, that is u is essentially away from its infimum. We show that going down to a smaller parallelepiped the oscillation decreases by a small factor that we can exhibit.

Lemma 4.1. *Assume* (17) *is in force. There exists a* $\nu_0 \in (0,1)$ *such that if* (18) *holds, then*

$$u(x) > \mu_- + \frac{\omega}{4}, \quad \text{a.e. in } Q_{\frac{\rho}{2},\frac{\omega}{2},(p_i)}.$$

Proof: We consider a sequence of parallelepipeds $Q_h = Q_{\rho_h,\frac{\omega}{2},(p_i)}$ where

$$\rho_h = \frac{\rho}{2} + \frac{\rho}{2^{h+1}}, \quad h = 0, 1, ... \tag{21}$$

We note that $Q_h \subset \Omega$, for any h, since the sequence $\{\rho_h\}$ is decreasing and

$$\lim_{h\to\infty} \rho_h = \frac{\rho}{2} < \rho_h \leq \rho_0 = \rho.$$

Let us also consider a sequence

$$k_h = \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{h+2}}, \quad h = 0, 1, ..., \tag{22}$$

and cutoff functions, ξ_h, defined as followed

$$\xi_h = \prod_{i=1}^{N} \xi_{h,i}^{p_i}, \quad \text{with} \quad \xi_{h,i} = \xi_h(x_i), \quad \forall\ i = 1, ..., N.$$

$\xi_h \in C_0^1(Q_h)$ is a nonnegative function, $0 \leq \xi_{h,i} \leq 1$, for any $i = 1, ..., N$, that vanishes outside of the set Q_h, is equal to unity in Q_{h+1} and

$$|\xi'_{h,i}| \leq \frac{2^{(h+1)\frac{p_N}{p_i}}}{\rho^{\frac{p_N}{p_i}} \left(\frac{\omega}{2}\right)^{\frac{p_i - p_N}{p_i}}}, \qquad \forall\ i = 1, ..., N. \tag{23}$$

By the definitions of k_h, the sets B_{k_h,Q_h} and ξ_h, we have

$$\begin{aligned}
\left(\frac{\omega}{2^{h+3}}\right)^{\overline{p}} |B_{k_{h+1},Q_{h+1}}| &= (k_h - k_{h+1})^{\overline{p}} |B_{k_{h+1},Q_{h+1}}| \\
&= \int_{B_{k_{h+1},Q_{h+1}}} (k_h - k_{h+1})^{\overline{p}} \leq \int_{B_{k_{h+1},Q_{h+1}}} (k_h - u)^{\overline{p}} \\
&\leq \int_{B_{k_h,Q_{h+1}}} (k_h - u)^{\overline{p}}\ \xi_h^{\overline{p}} = \int_\Omega (u - k_h)_-^{\overline{p}}\ \xi_h^{\overline{p}}.
\end{aligned}$$

Now we use Hölder's inequality with exponents $N/(N - \overline{p}) > 1$ and $N/\overline{p}$ to obtain

$$\left(\frac{\omega}{2^{h+3}}\right)^{\overline{p}} |B_{k_{h+1},Q_{h+1}}| \leq \left(\int_\Omega (u - k_h)_-^{\overline{p}^*}\ \xi_h^{\overline{p}^*}\right)^{\frac{N-\overline{p}}{N}} |B_{k_h,Q_h}|^{\frac{\overline{p}}{N}},$$

where $\overline{p}^*$ is defined in (3). So, by the anisotropic Sobolev inequality (4), we have

$$\left(\frac{\omega}{2^{h+3}}\right)^{\overline{p}}|B_{k_{h+1},Q_{h+1}}| \le C\left\{\prod_{i=1}^{N}\left(\int_\Omega |\partial_i[(u-k_h)_-\xi_h]|^{p_i}\right)^{\frac{\overline{p}}{p_iN}}\right\}|B_{k_h,Q_h}|^{\frac{\overline{p}}{N}}$$

$$\le C\left\{\prod_{i=1}^{N}\left(\int_\Omega |\partial_i(u-k_h)_-|^{p_i}\xi_h+\int_\Omega (u-k_h)_-^{p_i}|\xi'_{h,i}|^{p_i}\overline{\xi_h}_i\right)^{\frac{\overline{p}}{p_iN}}\right\}|B_{k_h,Q_h}|^{\frac{\overline{p}}{N}},$$

recalling that $\xi_{h,i}\le 1$, $p_i\ge 2$ for any i and

$$\partial_i\xi_{h,i}=\xi_{h,i}^{p_i-1}\xi'_{h,i}\,\overline{\xi_h}_i.$$

Now we use (14) with $k=k_h$ and $\xi=\xi_h$ to estimate the first terms of the product in the right hand side of the previous inequality. We obtain

$$\left(\frac{\omega}{2^{h+3}}\right)^{\overline{p}}|B_{k_{h+1},Q_{h+1}}| \le$$

$$\le C\left\{\prod_{i=1}^{N}\left[\sum_{j=1}^{N}\int_\Omega (u-k_h)_-^{p_j}|\xi'_{h,j}|^{p_j}\overline{\xi_h}_j+|B_{k_h,Q_h}|\right]^{\frac{\overline{p}}{p_iN}}\right\}|B_{k_h,Q_h}|^{\frac{\overline{p}}{N}}$$

$$= C\left[\sum_{j=1}^{N}\int_\Omega (u-k_h)_-^{p_j}|\xi'_{h,j}|^{p_j}\overline{\xi_h}_j+|B_{k_h,Q_h}|\right]|B_{k_h,Q_h}|^{\frac{\overline{p}}{N}}.$$

By (23), the definition of $(u-k_h)_-$ and ξ_h we have

$$\begin{aligned}\sum_{j=1}^{N}\int_\Omega (u-k_h)_-^{p_j}|\xi'_{h,j}|^{p_j}\overline{\xi_h}_j &\le \sum_{j=1}^{N}\int_{B_{k_h,Q_h}}(k_h-u)^{p_j}|\xi'_{h,j}|^{p_j}\\ &\le \sum_{j=1}^{N}\left(\frac{\omega}{2}\right)^{p_j}\frac{C\,2^{(h+1)p_N}}{\rho^{p_N}\left(\frac{\omega}{2}\right)^{p_j-p_N}}|B_{k_h,Q_h}|.\end{aligned}$$

We note that

$$k_h-u=\mu_-+\frac{\omega}{4}+\frac{\omega}{2^{h+2}}-u\le\frac{\omega}{2}.$$

We arrive at

$$\begin{aligned}\left(\frac{\omega}{2^{h+3}}\right)^{\overline{p}}|B_{k_{h+1},Q_{h+1}}| &\le C\left[\left(\frac{\omega}{2}\right)^{p_N}\frac{2^{(h+1)p_N}}{\rho^{p_N}}+1\right]|B_{k_h,Q_h}|^{1+\frac{\overline{p}}{N}}\\ &\le C\frac{2^{(h+2)p_N}}{\rho^{p_N}}\left(\frac{\omega}{2}\right)^{p_N}|B_{k_h,Q_h}|^{\frac{\overline{p}}{N}+1},\end{aligned}$$

by the assumption (17). Simplifying

$$|B_{k_{h+1},Q_{h+1}}| \leq C\frac{2^{(h+3)(p_N+\overline{p})}}{\rho^{p_N}}\left(\frac{\omega}{2}\right)^{p_N-\overline{p}}|B_{k_h,Q_h}|^{\frac{\overline{p}}{N}+1}.$$

Now we divide both terms of the previous inequality by

$$|Q_{h+1}| = 2^N\left(\frac{\omega}{2}\right)^{N-\frac{Np_N}{\overline{p}}}\rho_{h+1}^{\frac{Np_N}{\overline{p}}},$$

defined in (11), to get

$$\frac{|B_{k_{h+1},Q_{h+1}}|}{|Q_{h+1}|} \leq C2^{(p_N+\overline{p})h}\,\frac{1}{\rho^{p_N}}\,\frac{\rho_h^{\frac{Np_N}{\overline{p}}\left(1+\frac{\overline{p}}{N}\right)}}{\rho_{h+1}^{\frac{Np_N}{\overline{p}}}}\left(\frac{|B_{k_h,Q_h}|}{|Q_h|}\right)^{1+\frac{\overline{p}}{N}}.$$

We note that, by (21),

$$\frac{1}{\rho^{p_N}}\,\frac{\rho_h^{\frac{Np_N}{\overline{p}}\left(1+\frac{\overline{p}}{N}\right)}}{\rho_{h+1}^{\frac{Np_N}{\overline{p}}}} = \frac{1}{\rho^{p_N}}\,\frac{\rho^{\frac{Np_N}{\overline{p}}+p_N}}{\rho^{\frac{Np_N}{\overline{p}}}}\,\frac{\left(\frac{1}{2}+\frac{1}{2^{h+1}}\right)^{\frac{Np_N}{\overline{p}}+p_N}}{\left(\frac{1}{2}+\frac{1}{2^{h+2}}\right)^{\frac{Np_N}{\overline{p}}}} \leq 2^{\frac{Np_N}{\overline{p}}}.$$

In conclusion, we arrive at the following inequality

$$\frac{|B_{k_{h+1},Q_{h+1}}|}{|Q_{h+1}|} \leq C\,2^{h(p_N+\overline{p})}\left(\frac{|B_{k_h,Q_h}|}{|Q_h|}\right)^{1+\frac{\overline{p}}{N}},$$

where the constant C depends only upon the data. So we can use Lemma 2.3 if we define

$$y_h = \frac{|B_{k_h,Q_h}|}{|Q_h|}, \quad b = 2^{p_N+\overline{p}} > 1 \quad \text{and} \quad \varepsilon = \frac{\overline{p}}{N}$$

and we have that if

$$y_0 \leq C^{-\frac{N}{\overline{p}}}\,2^{-(p_N+\overline{p})\frac{N^2}{\overline{p}^2}} \tag{24}$$

then $y_h \to 0$ as $h \to \infty$. We observe, by (21) and (22), that

$$y_0 = \frac{|B_{k_0,Q_0}|}{|Q_0|} = \frac{|B_{k,Q_{\rho,\frac{\omega}{2},(p_i)}}|}{|Q_{\rho,\frac{\omega}{2},(p_i)}|}$$

Therefore, we can take

$$\nu_0 \leq C^{-\frac{N}{\overline{p}}}\,2^{-(p_N+\overline{p})\frac{N^2}{\overline{p}^2}}$$

and so (24) is equivalent to the assumption (18). This complete the proof of the Lemma. ■

Now we prove another lemma, useful to prove the Hölder continuity of the weak solution of problem (1). This lemma states that, if (18) does not hold, then u is strictly below its supremum μ_+ in a smaller parallelepiped.

Lemma 4.2. *Assume* (17) *is in force. If* (19)*, or equivalently* (20) *holds, then there exists* $\lambda > 1$*, depending only on the data such that*

$$u \le \mu_+ - \frac{\omega}{2^{\lambda+1}}, \quad \text{a.e. in } \frac{1}{2} Q_{\frac{\rho}{2}, \frac{\omega}{2^\lambda}, (p_i)}$$

where $\frac{1}{2} Q_{\frac{\rho}{2}, \frac{\omega}{2^\lambda}, (p_i)}$ *is defined in* (12).

Proof: We proceed as before. We consider a sequence

$$\rho_h = \frac{\rho}{2} + \frac{\rho}{2^{h+1}}, \quad h = 0, 1, 2, ...$$

and a sequence of parallelepipeds $Q_h = \frac{1}{2} Q_{\rho_h, \frac{\omega}{2^\lambda}, (p_i)}$. Let us consider

$$k_h = \mu_+ - \frac{\omega}{2^{\lambda+1}} - \frac{\omega}{2^{\lambda+1+h}}, \quad h = 0, 1, ...$$

an increasing sequence, that is

$$k_0 = \mu_+ - \frac{\omega}{2^\lambda} \le k_h < \lim_{h \to +\infty} k_h = \mu_+ - \frac{\omega}{2^{\lambda+1}}$$

and cutoff functions, ξ_h, defined as follows

$$\xi_h = \prod_{i=1}^N \xi_{h,i}^{p_i}, \quad \text{with} \quad \xi_{h,i} = \xi_h(x_i) \quad \forall\ i = 1, ..., N,$$

$\xi_h \in C_0^1(Q_h)$ is a nonnegative function, $0 \le \xi_{h,i} \le 1$, for any $i = 1, ..., N$, that vanishes outside of the set Q_h, is equal to unity in Q_{h+1} and

$$|\xi'_{h,i}| \le \frac{2^{(h+3)\frac{p_N}{p_i}}}{\rho^{\frac{p_N}{p_i}} \frac{1}{2} \left(\frac{\omega}{2^\lambda}\right)^{\frac{p_i - p_N}{p_i}}}, \qquad \forall\ i = 1, ..., N.$$

Using the same tools of the previous lemma, and (16) instead of (14), we arrive at the following inequality

$$\left(\frac{\omega}{2^{\lambda+h+2}}\right)^{\overline{p}} |A_{k_{h+1}, Q_{h+1}}| \le C \left[\sum_{j=1}^N \int_\Omega (u - k_h)_+^{p_j} |\xi'_{h,j}|^{p_j} \overline{\xi_h}_j + |A_{k_h, Q_h}| \right] |A_{k_h, Q_h}|^{\frac{\overline{p}}{N}}.$$

We note that

$$u - k_h = u - \mu_+ + \frac{\omega}{2^{\lambda+1}} + \frac{\omega}{2^{\lambda+1+h}} \le \frac{\omega}{2^\lambda}$$

and by the choice of ξ_h and (17) we obtain

$$\left(\frac{\omega}{2^{\lambda+h+2}}\right)^{\overline{p}} |A_{k_{h+1},Q_{h+1}}| \leq C\left(\frac{\omega}{2^{\lambda}}\right)^{p_N} \frac{2^{p_N(h+4)}}{\rho^{p_N}} |A_{k_h,Q_h}|^{1+\frac{\overline{p}}{N}},$$

and so

$$\frac{|A_{k_{h+1},Q_{h+1}}|}{|Q_{h+1}|} \leq C 2^{h(p_N+\overline{p})} \left(\frac{|A_{k_h,Q_h}|}{|Q_h|}\right)^{1+\frac{\overline{p}}{N}}.$$

So we can use Lemma 2.3 if we define

$$y_h = \frac{|A_{k_h,Q_h}|}{|Q_h|}, \quad b = 2^{p_N+\overline{p}} > 1 \quad \text{and} \quad \varepsilon = \frac{\overline{p}}{N}$$

and we have that if

$$y_0 \leq C^{-\frac{N}{\overline{p}}}\, 2^{-(p_N+\overline{p})\frac{N^2}{\overline{p}^2}} := \nu_* \tag{25}$$

then $y_h \to 0$ as $h \to \infty$. We observe, by the definition of $\{\rho_h\}$ and $\{k_h\}$, that

$$y_0 = \frac{|A_{k_0,Q_0}|}{|Q_0|} = \frac{\left|A_{k,\frac{1}{2}\,Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}\right|}{\left|\frac{1}{2}\,Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|}.$$

So if we show that

$$\left|\left\{x \in \frac{1}{2}\,Q_{\rho,\frac{\omega}{2^\lambda},(p_i)} : u > \mu_+ - \frac{\omega}{2^\lambda}\right\}\right| \leq \nu_* \left|\frac{1}{2} Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|, \tag{26}$$

for some $\lambda > 1$, depending only on the data, the lemma is proved.

We use (16) with

$$\xi = \prod_{i=1}^{N} \xi_i^{p_i}, \quad \xi_i = \xi(x_i), \quad 0 \leq \xi_i \leq 1, \quad \forall\ i = 1, ..., N,$$

$\xi \in C_0^1(Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)})$ vanishes outside of the set $Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}$ and is equal to unity in $\frac{1}{2}\,Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}$,

$$|\xi_i'| \leq \frac{2}{\left(\frac{\omega}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}} \rho^{\frac{p_N}{p_i}}}, \quad \forall\ i = 1, ..., N,$$

and

$$k = \mu_+ - \frac{\omega}{2^s}.$$

We have

$$\begin{aligned}\int_{A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}}|\partial_i u|^{p_i} &\leq C\left\{\sum_{i=1}^N\int_\Omega (u-k)_+^{p_i}|\xi_i'|^{p_i}\overline{\xi}_i+|A_{k,Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}}|\right\}\\ &\leq C\left\{\sum_{i=1}^N\left(\frac{\omega}{2^s}\right)^{p_i}\left(\frac{\omega}{2^\lambda}\right)^{p_N-p_i}\frac{2^{p_N}}{\rho^{p_N}}+1\right\}|A_{k,Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}}|\\ &\leq C\left(\frac{\omega}{2^s}\right)^{p_N}\frac{2^{p_N}}{\rho^{p_N}}|A_{k,Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}}|, \qquad (27)\end{aligned}$$

for all $i=1,...,N$ and for any $s\leq\lambda$, by the fact that

$$u-k=u-\mu^++\frac{\omega}{2^s}<\frac{\omega}{2^s}$$

$$\frac{1}{2^s}\geq\frac{1}{2^\lambda}\quad\text{and}\quad\frac{\omega}{2^s\rho}>1,$$

since $s\leq\lambda$ and (17) holds. Now we apply Lemma 2.1 to the one variable function $u(x_1,...,x_{N-1},\cdot)$ for the levels

$$k=\mu_+-\frac{\omega}{2^s}\quad\text{and}\quad l=\mu_+-\frac{\omega}{2^{s+1}},$$

and $A_{l,Q^1_{\frac{\rho}{2}}}$, $B_{k,Q^1_{\frac{\rho}{2}}}$, $A_{k,Q^1_{\frac{\rho}{2}}}\cap B_{l,Q^1_{\frac{\rho}{2}}}$, subsets of $\mathbb{R}$, defined respectively in (6), (7) and (8), with

$$Q^1_{\frac{\rho}{2}}=\left(-\frac{\rho}{2},\frac{\rho}{2}\right).$$

and

$$-\frac{1}{2}\rho^{\frac{p_N}{p_i}}\left(\frac{\omega}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}}\leq x_i\leq\frac{1}{2}\rho^{\frac{p_N}{p_i}}\left(\frac{\omega}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}},\quad\forall\ i=1,...,N-1.$$

So we obtain

$$\frac{\omega}{2^{s+1}}|A_{l,Q^1_{\frac{\rho}{2}}}|\leq C\frac{\rho^2}{|B_{k,Q^1_{\frac{\rho}{2}}}|}\int_{A_{k,Q^1_{\frac{\rho}{2}}}\cap B_{l,Q^1_{\frac{\rho}{2}}}}|\partial_N u|dx_N.$$

We can suppose that

$$|B_{k,Q^1_{\frac{\rho}{2}}}|\geq\frac{1}{2}|Q^1_{\frac{\rho}{2}}|=\frac{\rho}{2}.$$

In fact, if this is not true,

$$|B_{k,Q^1_{\frac{\rho}{2}}}|<\frac{\rho}{2}\quad\Rightarrow\quad|\overline{A_{k,Q^1_{\frac{\rho}{2}}}}|>\frac{\rho}{2}, \qquad (28)$$

where $\overline{A} = A \cup \partial A$, for some set A. Let $a_i = \rho^{p_N/p_i}(\omega/2)^{(p_i-p_N)/p_i}$ for any $i = 1, ..., N$, then

$$\left|\left\{x \in Q_{\rho,\frac{\omega}{2},(p_i)} : u(x) \geq \mu_+ - \frac{\omega}{2}\right\}\right| = \int_{-a_1}^{a_1} dx_1 \int_{-a_2}^{a_2} dx_2 ...$$

$$\begin{aligned}
... &\int_{-a_{N-1}}^{a_{N-1}} \left|\left\{x_N \in (-\rho, \rho) : u(x_1, ..., x_N) \geq \mu_+ - \frac{\omega}{2}\right\}\right| dx_{N-1} \\
\geq &\int_{-a_1}^{a_1} dx_1 ... \int_{-a_{N-1}}^{a_{N-1}} \left|\left\{x_N \in (-\rho, \rho) : u(x) \geq \mu_+ - \frac{\omega}{2^s}\right\}\right| dx_{N-1} \\
\geq &\int_{-a_1}^{a_1} dx_1 ... \int_{-a_{N_1}}^{a_{N-1}} \left|\left\{x_N \in \left(-\frac{\rho}{2}, \frac{\rho}{2}\right) : u(x) \geq \mu_+ - \frac{\omega}{2^s}\right\}\right| dx_{N-1}
\end{aligned}$$

since

$$\mu_+ - \frac{\omega}{2} \leq \mu_+ - \frac{\omega}{2^s} \quad \text{if } s \geq 1 \quad \text{and} \quad \frac{\rho}{2} < \rho.$$

So, using (28), we arrive at

$$\left|\left\{x \in Q_{\rho,\frac{\omega}{2},(p_i)} : u(x) \geq \mu_+ - \frac{\omega}{2}\right\}\right| \geq \frac{1}{4}|Q_{\rho,\frac{\omega}{2},(p_i)}|$$

and this inequality contradicts the second alternative (20). Hence we obtain

$$\frac{\omega}{2^{s+1}}|A_{l,Q^1_{\frac{\rho}{2}}}| \leq C\rho \int_{A_{k,Q^1_{\frac{\rho}{2}}} \cap B_{l,Q^1_{\frac{\rho}{2}}}} |\partial_N u| dx_N.$$

Integrating over $x_1, ..., x_{N-1}$ the previous inequality, we arrive at

$$\frac{\omega}{2^{s+1}}|A_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda}(p_i)}}| \leq C\rho \int_{A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} \cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}} |\partial_N u| dx.$$

By Hölder's inequality with exponents p_N and p'_N, we get

$$\frac{\omega}{2^{s+1}}|A_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}| \leq$$

$$\leq C\rho \left(\int_{A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} \cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}} |\partial_N u|^{p_N} \right)^{\frac{1}{p_N}} |A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} \cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{1-\frac{1}{p_N}}$$

$$\leq C\rho\left(\int_{A_{k,\frac{1}{2}Q_{\rho\frac{\omega}{2^\lambda},(p_i)}}}|\partial_N u|^{p_N}\right)^{\frac{1}{p_N}}|A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}\cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{1-\frac{1}{p_N}}.$$

Now we use inequality (27), with $i=N$, and arrive at

$$\frac{\omega}{2^{s+1}}|A_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|\leq C\rho\frac{2\,\omega}{2^s\rho}|A_{k,Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{\frac{1}{p_N}}|A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}\cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{1-\frac{1}{p_N}}$$

and so

$$|A_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|\leq C|A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}\cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{1-\frac{1}{p_N}}\left|\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|^{\frac{1}{p_N}},$$

using the facts that

$$|A_{k,Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}}|\leq|Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}|\quad\text{and}\quad|Q_{2\rho,\frac{\omega}{2^\lambda},(p_i)}|^{\frac{1}{p_N}}=2^{N\left(\frac{1}{p_N}+\frac{1}{\overline{p}}\right)}\left|\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|^{\frac{1}{p_N}}.$$

On the left hand side we replace $|A_{\mu_+-\frac{\omega}{2^{s+1}},\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|$ by the smaller quantity $|A_{\mu_+-\frac{\omega}{2^\lambda},\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|$, taking $1\leq s\leq\lambda-1$ and so $\lambda\geq 2$, to obtain

$$|A_{\mu_+-\frac{\omega}{2^\lambda},\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{\frac{p_N}{p_N-1}}\leq C\left|\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|^{\frac{1}{p_N-1}}|A_{k,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}\cap B_{l,\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|,$$

for $s=1,...,\lambda-1$. Let us sum with respect to s and replace the right side of the resulting inequality by the larger quantity $|\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}|$, to get

$$(\lambda-1)|A_{\mu_+-\frac{\omega}{2^\lambda},\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|^{\frac{p_N}{p_N-1}}\leq C\left|\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|^{\frac{p_N}{p_N-1}}$$

$$\Downarrow$$

$$|A_{\mu_+-\frac{\omega}{2^\lambda},\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}}|\leq\left(\frac{C}{\lambda-1}\right)^{\frac{p_N-1}{p_N}}\left|\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}\right|.$$

We obtain (26) if λ is chosen so large that

$$\left(\frac{C}{\lambda-1}\right)^{\frac{p_N-1}{p_N}}\leq\nu_*\quad\text{and}\quad\lambda\geq 2,\tag{29}$$

where ν_* defined in (25). This conclude the proof of (26) and so that of the lemma. ■

5. Recursive argument, main theorem and final remarks

Before proving the Hölder continuity of the weak solution of the problem (1), we present two corollaries of the previous lemmas.

Corollary 5.1. *If the First Alternative* (18) *is true and* (17) *is in force, then there exists a constant* $\sigma_0 \in (0,1)$ *such that*

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u \leq \sigma_0 \omega.$$

Proof: By Lemma 4.1, we have

$$\operatorname*{ess\,inf}_{Q_{\frac{\rho}{2},\frac{\omega}{2},(p_i)}} u \geq \mu_- + \frac{\omega}{4}$$

and so

$$\operatorname*{ess\,inf}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u \geq \mu_- + \frac{\omega}{4},$$

since

$$Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)} \subset Q_{\frac{\rho}{2},\frac{\omega}{2},(p_i)}.$$

Hence

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u = \operatorname*{ess\,sup}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u - \operatorname*{ess\,inf}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u \leq \mu_+ - \mu_- - \frac{\omega}{4} = \frac{3}{4}\omega.$$

■

Corollary 5.2. *If the Second Alternative* (19) *holds and* (17) *is in force, there exists a constant* $\sigma_1 \in (0,1)$*, depending only on the data, such that*

$$\operatorname*{ess\,osc}_{\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u \leq \sigma_1 \omega.$$

Proof: By Lemma 4.2, there exists $\lambda > 1$ such that

$$u \leq \mu_+ - \frac{\omega}{2^{\lambda+1}}, \quad \text{a.e. in } \frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}.$$

Then

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u \leq \mu_+ - \frac{\omega}{2^{\lambda+1}},$$

and so

$$\operatorname*{ess\,osc}_{\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u = \operatorname*{ess\,sup}_{\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u - \operatorname*{ess\,inf}_{\frac{1}{2}Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u \leq \mu_+ - \mu_- - \frac{\omega}{2^{\lambda+1}} = \left(1 - \frac{1}{2^{\lambda+1}}\right)\omega.$$

■

We finally prove the Hölder continuity of weak solutions of problem (1) through an iterative scheme. An immediate consequence of Corollaries 5.1 and 5.2 is the following

Proposition 5.3. *There exists a constant $\sigma \in (0,1)$, that depends only on the data, such that*

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u \le \sigma\omega.$$

Proof: We note that

$$Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)} \subseteq \frac{1}{2} Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}$$

and then

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} u \le \sigma\omega,$$

where $\sigma = \max\{\sigma_0, \sigma_1\}$. ■

Proposition 5.4. *There exists a positive constant C, depending only on the data, such that, defining the sequences*

$$\rho_h = C^{-h}\rho \quad \text{and} \quad \omega_h = \sigma^h \omega, \quad h = 0,1,2,...,$$

where $\sigma \in (0,1)$ is given by the previous proposition, and constructing the family of parallelepipedes $Q_h = Q_{\rho_h,\frac{\omega_h}{2^\lambda},(p_i)}$, where $\lambda > 1$ is given in (29), we have

$$Q_{h+1} \subset Q_h \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_h} u \le \omega_h, \quad \text{for all } h = 0,1,2,...$$

Proof: The starting relation

$$\operatorname*{ess\,osc}_{Q_0} u \le \omega \tag{30}$$

holds, since we are assuming (17). We find, for any $i = 1, ..., N$,

$$\begin{aligned}
\left(\frac{\omega}{2}\right)^{\frac{p_i-p_N}{p_i}} \left(\frac{\rho}{4}\right)^{\frac{p_N}{p_i}} &= \left(\frac{\omega}{2}\right)^{\frac{p_i-p_N}{p_i}} \left(\frac{2^\lambda}{\omega_1}\right)^{\frac{p_i-p_N}{p_i}} \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}} \frac{\rho^{\frac{p_N}{p_i}}}{4^{\frac{p_N}{p_i}}} \\
&= \left(\frac{\omega}{\omega_1}\right)^{\frac{p_i-p_N}{p_i}} \left(\frac{2^\lambda}{2}\right)^{\frac{p_i-p_N}{p_i}} \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}} \frac{\rho^{\frac{p_N}{p_i}}}{4^{\frac{p_N}{p_i}}} \\
&= \sigma^{\frac{p_N-p_i}{p_i}} 2^{(\lambda-1)\frac{p_i-p_N}{p_i} - \frac{2p_N}{p_i}} \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}} \rho^{\frac{p_N}{p_i}} \\
&= \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p_i-p_N}{p_i}} \rho_1^{\frac{p_N}{p_i}},
\end{aligned}$$

where

$$\rho_1 = \sigma^{\frac{p_N - p_i}{p_N}} 2^{(\lambda-1)\frac{p_i - p_N}{p_N} - 2} \rho = C^{-1}\rho,$$

$C > 4$. So $Q_1 \subset Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}$. From Proposition 5.3, we conclude that

$$\operatorname*{ess\,osc}_{Q_1} u \leq \operatorname*{ess\,osc}_{Q_{\frac{\rho}{4},\frac{\omega}{2},(p_i)}} \leq \sigma\omega = \omega_1,$$

which puts us back to the setting of (30). The entire process can now be repeated inductively starting from Q_1. ■

Lemma 5.5. *There exist constants $\gamma > 1$ and $\alpha \in (0,1)$, that can be determined a priori in terms of the data, such that, for all parallelepipeds*

$$Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}, \quad \text{with} \quad 0 < \rho \leq \rho_0,$$

we have

$$\operatorname*{ess\,osc}_{Q_{\rho,\frac{\omega}{2^\lambda},(p_i)}} u \leq \gamma\,\omega \left(\frac{\rho}{\rho_0}\right)^{\alpha}.$$

Proof: See, for example, Lemma 4.9 of [29] (p. 45). ■

Now we have all the tools to prove the main result of this paper, namely *Theorem 1.1*; for more details, see Theorem 4.10 of [29] (p. 46).

Remark 5.6. If we consider the isotropic case, that is all the p_i's are equal to 2 (or, more generally all are equal to p), we essentially recover the now classical proof of Hölder continuity of weak solutions for elliptic equations, presented in [15].

Remark 5.7. We want also to underline that the result presented in this work also holds for more general datum f. As we expect by the isotropic case, Theorem 1.1 is also true if we suppose that $f \in L^m(\Omega)$, with $m > N/\overline{p}$. As a matter of fact if f belongs to $L^m(\Omega)$ with $m > N/\overline{p}$ it is known that weak solutions of (1) are bounded (see [8] and [27]). We present the result in the case of f bounded only for simplicity. To be complete, we want to note that, in this general case, to prove the same result we have to slightly change the proofs. For f in $L^m(\Omega)$, it is possible to prove the following energy estimates, where the hypothesis $m > N/\overline{p}$ is necessary:

$$\int_\Omega |\partial_i (u-k)_-|^{p_i} \xi \leq C \left[\sum_{j=1}^{N} \int_\Omega |(u-k)_-|^{p_j} |\xi_j'|^{p_j} \overline{\xi}_j + |B_{k,Q_\rho^N}|^{1-\frac{1}{m}} \right],$$

$$\int_\Omega |\partial_i (u-k)_+|^{p_i} \xi \le C \left[\sum_{j=1}^N \int_\Omega |(u-k)_+|^{p_j} |\xi_j'|^{p_j} \overline{\xi}_j + |A_{k,Q_\rho^N}|^{1-\frac{1}{m}} \right],$$

for any $i = 1, ..., N$, instead of (14) and (16). Moreover we have to substitute the assumption (17) with the following

$$\frac{\omega}{2^\lambda} > \rho^{1-\frac{N}{m\overline{p}}}.$$

But this fact does not substantially change the proofs. In fact, if it is not true we always have that the oscillation of u is comparable to the "radius" of the set that we are considering and so there is nothing to prove. Moreover, by the assumption on m, we also have

$$\frac{\omega}{2^\lambda} > \rho$$

and it ensures that

$$Q_{\rho,\frac{\omega}{2^\lambda},(p_i)} \subset Q_\rho^N,$$

and the starting point of the iteration process that leads to the main result is satisfied.

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