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EXTENDED REAL FUNCTIONS IN POINTFREE TOPOLOGY

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Abstract: In pointfree topology, a continuous real function on a frame L is a map $\mathfrak{L}(\mathbb{R}) \to L$ from the frame of reals into L. The discussion of continuous real functions with possibly infinite values can be easily brought to pointfree topology by replacing the frame $\mathfrak{L}(\mathbb{R})$ with the frame of extended reals $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ (i.e. the pointfree counterpart of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$). One can even deal with arbitrary (not necessarily continuous) extended real functions. The main purpose of this paper is to investigate the algebra of extended real functions on a frame. Our results make it possible to study the class $\mathsf{D}(L)$ of almost real valued functions. In particular, we show that for extremally disconnected L, $\mathsf{D}(L)$ becomes an order-complete archimedean f-ring with unit.

Keywords: Frame, locale, sublocale, frame of reals, frame of extended reals, scale, real function, extended real function, lattice ordered ring, ring of continuous functions in pointfree topology.

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Introduction

As in the classical setting ([9]), in the pointfree context of frames and locales each frame L has associated with it the ring of its real functions

$$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$$

(where $\mathcal{S}(L)$ denotes the dual of the co-frame of all sublocales of L) and this in such a way that the correspondence for frames extends that for spaces ([10], [12]). To be precise, if $\mathsf{F}(L)$ is the ring associated with a frame L and $\mathcal{O}X$ the frame of open sets of a space X then the classical function ring $\mathbb{R}^X$ is isomorphic to $\mathsf{F}(\mathcal{O}X)$.

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The important feature of this approach is that, every function having $\mathfrak{L}(\mathbb{R})$ as a common domain and $\mathcal{S}(L)$ as a common codomain, the structure of $\mathcal{S}(L)$ is rich enough to allow to distinguish the different types of continuities. In fact, the classes $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$ of lower and upper semicontinuous functions [11] and the ring $\mathsf{C}(L)$ of continuous functions [2] fit nicely in this framework: $f \in \mathsf{F}(L)$ is *lower semicontinuous* if $f(r, —)$ is a closed sublocale for every r, and f is *upper semicontinuous* if $f(—, r)$ is a closed sublocale for every r; $f \in \mathsf{F}(L)$ is *continuous* if $f(r, s)$ is closed for every r, s, i.e. $\mathsf{C}(L) = \mathsf{LSC}(L) \cap \mathsf{USC}(L)$. In addition, $\mathsf{C}(L)$ is a subring of $\mathsf{F}(L)$ [12].

Now, if we replace the frame of reals $\mathfrak{L}(\mathbb{R})$ with the frame of extended reals $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ we may speak about extended real functions, the pointfree counterpart of functions on a space X with values in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We have then the classes

$$\overline{\mathsf{F}}(L), \overline{\mathsf{LSC}}(L), \overline{\mathsf{USC}}(L) \text{ and } \overline{\mathsf{C}}(L)$$

of respectively extended real functions, extended lower semicontinuous real functions, extended upper semicontinuous real functions and extended continuous real functions on the frame L. The purpose of this paper is to study the algebraic structure of these classes. As an application, we present a study of the sublattice $\mathsf{D}(L)$ of almost real functions.

The paper is organized as follows. Section 1 recalls the fundamental notions and facts about frames of reals and sublocale lattices involved here. In Section 2 we introduce extended (continuous) real functions, show how to use scales to generate them and provide some basic examples. Further, we derive formulas for the lattice operations in the algebras $\overline{\mathsf{C}}(L)$ of extended continuous real functions (Section 3). Next, we derive the conditions under which the addition (Section 4) and the multiplication (Section 5) of two real functions is possible in $\overline{\mathsf{C}}(L)$. Finally, in Section 6 we study the sublattice $\mathsf{D}(L)$ of $\overline{\mathsf{C}}(L)$ of all functions whose domain of reality is dense in L, called *almost real functions*. We show that, in general, $\mathsf{D}(L)$ is not a group or a ring under the operations in $\overline{\mathsf{C}}(L)$ (there are only *partial* addition and multiplication in $\mathsf{D}(L)$) but for extremally disconnected frames L the partial operations are total and, in that case, there is a lattice ordered ring isomorphism between $\mathsf{D}(L)$ and the ring $\mathsf{C}(\mathfrak{B}L)$ of continuous functions on the Booleanization $\mathfrak{B}L$ of L, which makes $\mathsf{D}(L)$ an order-complete archimedean f-ring with unit. We then characterize the frames for which the partial operations are total: they are precisely the quasi-F frames of [1].

For general background regarding frames and locales we refer to [13] or [15]. For details concerning the function rings $\mathsf{C}(L)$ we refer to [2]. The basic facts about general real functions and the corresponding function algebras $\mathsf{F}(L)$ can be found in the recent [10] and [12].

1. Background and preliminaries

We begin by briefly recounting the familiar notions involved here. The *frame* $\mathfrak{L}(\mathbb{R})$ *of reals* (see e.g. [2]) is the frame specified by generators (p, q) for $p, q \in \mathbb{Q}$ and defining relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$,
(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$,
(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

It will be useful here to adopt the equivalent description of $\mathfrak{L}(\mathbb{R})$ introduced in [14] with the elements

$$(r, \text{—}) = \bigvee_{s\in\mathbb{Q}} (r, s) \quad \text{and} \quad (\text{—}, s) = \bigvee_{r\in\mathbb{Q}} (r, s)$$

as primitive notions. Specifically, the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is equivalently given by generators $(r, \text{—})$ and $(\text{—}, r)$ for $r \in \mathbb{Q}$ subject to the defining relations

(r1) $(r, \text{—}) \wedge (\text{—}, s) = 0$ whenever $r \ge s$,
(r2) $(r, \text{—}) \vee (\text{—}, s) = 1$ whenever $r < s$,
(r3) $(r, \text{—}) = \bigvee_{s>r}(s, \text{—})$, for every $r \in \mathbb{Q}$,
(r4) $(\text{—}, r) = \bigvee_{s<r}(\text{—}, s)$, for every $r \in \mathbb{Q}$,
(r5) $\bigvee_{r\in\mathbb{Q}}(r, \text{—}) = 1$,
(r6) $\bigvee_{r\in\mathbb{Q}}(\text{—}, r) = 1$.

With $(p, q) = (p, \text{—}) \wedge (\text{—}, q)$ one goes back to (R1)–(R4).

Besides $\mathfrak{L}(\mathbb{R})$ (as given by the latter description) we also consider its subframes $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ of *upper* and *lower reals* generated by the $(r, \text{—})$ and $(\text{—}, r)$, $r \in \mathbb{Q}$, respectively.

Remark 1. It should be pointed out that $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ can equivalently be defined as the frames specified, respectively, by the generators $(r, \text{—})$, $r \in \mathbb{Q}$, subject to the relations (r3) and (r5), and the generators $(\text{—}, r)$, $r \in \mathbb{Q}$, subject to (r4) and (r6). This can be seen quite easily, say for the frame $\mathfrak{L}_u(\mathbb{R})$, by mapping each generator $(r, \text{—})$ to the corresponding open interval $\langle r, \text{—}\rangle$ in $\mathbb{Q}$ (and analogously for $\mathfrak{L}_l(\mathbb{R})$): For the resulting homomorphism

$h : \mathfrak{L}_u(\mathbb{R}) \to \mathfrak{O}\mathbb{Q}$ and any of its elements $a = \bigvee\{(r,\text{—}) \mid r \in S\}$, $S \subseteq \mathbb{Q}$, we obviously have

$$h(a) = \bigcup\{\langle r,\text{—}\rangle \mid r \in S\} = \{u \in \mathbb{Q} \mid u > r \text{ for some } r \in S\}.$$

Now, if $h(a) = h(b)$ where $b = \bigvee\{(r,\text{—}) \mid r \in T\}$ then, for each $v \in T$, $\langle v,\text{—}\rangle \subseteq h(a)$ so that $p > v$ implies $p > r$ for some $r \in S$, therefore $(p,\text{—}) \leq (r,\text{—})$ by (r3) and hence $(p,\text{—}) \leq a$ which shows, again by (r3), that $(v,\text{—}) \leq a$, therefore $b \leq a$ and finally $a = b$ by symmetry. Thus h is one-one, and by the usual homomorphism $\mathfrak{O}\mathbb{Q} \to \mathfrak{L}(\mathbb{R})$, $\langle r,\text{—}\rangle \mapsto (r,\text{—})$ and $\langle \text{—},r\rangle \mapsto (\text{—},r)$, it then follows that the homomorphism $\mathfrak{L}_u(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$, $(r,\text{—}) \mapsto (r,\text{—})$, is also one-one, as claimed.

For each $p < q$ in $\mathbb{Q}$ we have also the *closed interval frame* $\mathfrak{L}[p,q]$ defined by

$$\uparrow\big((\text{—},p) \vee (q,\text{—})\big) = \{a \in \mathfrak{L}(\mathbb{R}) \mid a \geq (\text{—},p) \vee (q,\text{—})\}.$$

By dropping (r5) and (r6) in the descriptions of $\mathfrak{L}(\mathbb{R})$, $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ above, we have the *extended* variants of the frames introduced, namely:

$$\mathfrak{L}\left(\overline{\mathbb{R}}\right), \quad \mathfrak{L}_u\left(\overline{\mathbb{R}}\right), \quad \text{and} \quad \mathfrak{L}_l\left(\overline{\mathbb{R}}\right).$$

Remark 2. The frame $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ of extended reals is isomorphic to $\mathfrak{L}[p,q]$ for any $p < q$ in $\mathbb{Q}$, as we show next. Let $p < q$ in $\mathbb{Q}$. Consider an order isomorphism ψ from the open rational interval $\langle p,q\rangle$ into $\mathbb{Q}$, as for instance

$$\psi(r) = \begin{cases} \frac{1}{q-r} - \frac{2}{q-p} & \text{if } \frac{p+q}{2} \leq r < q, \\ \frac{2}{q-p} - \frac{1}{r-p} & \text{if } p < r \leq \frac{p+q}{2}. \end{cases}$$

Let $\varphi = \psi^{-1}$ and define $\Phi : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to \mathfrak{L}[p,q]$ on generators by

$$\Phi(r,\text{—}) = (\text{—},p) \vee (\varphi(r),\text{—}) \quad \text{and} \quad \Phi(\text{—},r) = (\text{—},\varphi(r)) \vee (q,\text{—}).$$

Then Φ turns defining relations (r1)–(r4) into equalities in $\mathfrak{L}[p,q]$ (which means that it is a frame homomorphism):

(r1) $\Phi(r,\text{—}) \wedge \Phi(\text{—},s) = (\text{—},p) \vee (\varphi(r),\varphi(s)) \vee (q,\text{—})$ and, consequently, $\Phi(r,\text{—}) \wedge \Phi(\text{—},s) = (\text{—},p) \vee (q,\text{—}) = 0_{\mathfrak{L}[p,q]}$, whenever $r \geq s$.
(r2) $\Phi(r,\text{—}) \vee \Phi(\text{—},s) = (\text{—},p) \vee (\varphi(r),\text{—}) \vee (\text{—},\varphi(s)) \vee (q,\text{—})$. Hence $\Phi(r,\text{—}) \vee \Phi(\text{—},s) = 1$ whenever $r < s$.
(r3) $\bigvee_{s>r} \Phi(s,\text{—}) = \bigvee_{s>r}(\text{—},p) \vee (\varphi(s),\text{—}) = (\text{—},p) \vee \big(\bigvee_{s>r}(\varphi(s),\text{—})\big) = (\text{—},p) \vee (\varphi(r),\text{—}) = \Phi(r,\text{—})$.
(r4) Similar to (r3).

Further, define $\Psi_0 : \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}\left(\overline{\mathbb{R}}\right)$ by

$$\Psi_0(r,s) = \begin{cases} 0 & \text{if } s < p \text{ or } q < r, \\ (-,\psi(s)) & \text{if } r \le p \le s < q, \\ (\psi(r),\psi(s)) & \text{if } p < r < s < q, \\ (\psi(r),-) & \text{if } p < r < q \le s. \end{cases}$$

Since $\Psi_0((-,p)\vee(q,-)) = 0$, it induces a $\Psi : \mathfrak{L}[p,q] \to \mathfrak{L}\left(\overline{\mathbb{R}}\right)$. One can easily check that Ψ is a frame homomorphism in a similar way as before. Moreover, $\Psi\circ\Phi$ is the identity map:

$$\begin{aligned} \Psi\circ\Phi(r,-) &= \Psi\big((-,p)\vee(\varphi(r),-)\big) = (\psi\circ\varphi(r),-) = (r,-), \\ \Psi\circ\Phi(-,r) &= \Psi\big((-,\varphi(r))\vee(q,-)\big) = (-,\psi\circ\varphi(r)) = (-,r), \end{aligned}$$

Finally, Φ is onto since for each $r < s$ in $\mathbb{Q}$ we have

$$(-,p)\vee(r,s)\vee(q,-) = \begin{cases} (-,p)\vee(q,-) = \Phi(0) & \text{if } s < p \text{ or } q < r, \\ (-,s)\vee(q,-) = \Phi(-,\psi(s)) & \text{if } r \le p \le s < q, \\ \Phi(\psi(r),\psi(s)) & \text{if } p < r < s < q, \\ (-,p)\vee(r,-) = \Phi(\psi(r),-) & \text{if } p < r < q \le s. \end{cases}$$

Remark 3. As a consequence of the isomorphism $\mathfrak{L}\left(\overline{\mathbb{R}}\right) \simeq \mathfrak{L}[p,q]$, we have that $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ is compact (besides, of course, of being completely regular): $\mathfrak{L}(\mathbb{R})$ is well known to be complete in its natural uniformity [2], hence any closed quotient of $\mathfrak{L}(\mathbb{R})$ is complete in the image uniformity, but that is totally bounded on $\mathfrak{L}[p,q]$, and any totally bounded complete uniform frame is compact (see [3]).

Another consequence of the isomorphism $\mathfrak{L}\left(\overline{\mathbb{R}}\right) \simeq \mathfrak{L}[p,q]$ is that the spectrum $\Sigma\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ of $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ is homeomorphic to the space $\overline{\mathbb{R}}$ of extended reals.

Remark 4. One might think that, alternatively, $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ could be defined by the generators $(p,q) \in \mathbb{Q}\times\mathbb{Q}$ subject to the relations (R1)–(R3). That, however, is a different frame, as the following shows. Let L be the frame in question, M the frame obtained from $\mathfrak{L}(\mathbb{R})$ by adding a new top, and $h : L \to M$ the homomorphism determined by $(p,q) \mapsto (p,q)$, given by the obvious fact that this assignment preserves the relations (R1)–(R3). Now, since $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ is regular, as noted earlier, L cannot be isomorphic to $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ because M is a homomorphic image of L which is not regular, and taking homomorphic images of frames preserves regularity.

Remark 5. The basic homomorphism $\varrho : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to \mathfrak{L}(\mathbb{R})$ factors as

$$\mathfrak{L}\left(\overline{\mathbb{R}}\right) \xrightarrow{\nu_\omega} {\downarrow}\omega \xrightarrow{k} \mathfrak{L}(\mathbb{R}), \quad \omega = \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\}$$

where $\nu_\omega = (\cdot) \wedge \omega$ and k is an isomorphism (it is obviously onto and has a right inverse by the very definition of $\mathfrak{L}(\mathbb{R})$). One has also analogous situations for $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$.

Regarding the sublocale lattice we adopt the approach of [15]. A subset S of a frame (locale) L is a *sublocale* of L if, whenever $A \subseteq S$, $a \in L$ and $b \in S$, then $\bigwedge A \in S$ and $a \to b \in S$. The set of all sublocales of L forms a co-frame under inclusion, in which arbitrary meets coincide with intersection, $\{1\}$ is the bottom, and L is the top.

For notational reasons, it seems appropriate to make the co-frame of all sublocales of L into a frame $\mathcal{S}(L)$ by considering the dual ordering: $S_1 \leq S_2$ iff $S_2 \subseteq S_1$. Thus, given $\{S_i \in \mathcal{S}(L) : i \in I\}$, we have $\bigvee_{i\in I} S_i = \bigcap_{i\in I} S_i$ and $\bigwedge_{i\in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i\in I} S_i\}$. Also, $\{1\}$ is the top and L is the bottom in $\mathcal{S}(L)$ that we simply denote by 1 and 0, respectively. We recall that $\mathcal{S}(L)$ is isomorphic to the frame $N(L)$ of nuclei on L (as in [13]).

For any $a \in L$, the sets $\mathfrak{c}(a) = {\uparrow}a$ and $\mathfrak{o}(a) = \{a \to b : b \in L\}$ are the *closed* and *open* sublocales of L, respectively. They are complements of each other in $\mathcal{S}(L)$. Furthermore, the map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathcal{S}(L)$ providing an isomorphism between L and the subframe $\mathfrak{c}L$ of $\mathcal{S}(L)$ consisting of all closed sublocales. On the other hand, denoting by $\mathfrak{o}L$ the subframe of $\mathcal{S}(L)$ generated by all $\mathfrak{o}(a)$, the correspondence $a \mapsto \mathfrak{o}(a)$ establishes a dual poset embedding $L \to \mathfrak{o}L$.

2. Extended real functions

Definition 1. An *extended continuous real function* on a frame L is a frame homomorphism $f : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to L$.

We denote by $\overline{\mathsf{C}}(L)$ the collection of all extended continuous real functions on L. Note that the correspondence $L \mapsto \overline{\mathsf{C}}(L)$ is functorial in the obvious way.

Remark 6. By the familiar (dual) adjunction between the contravariant functors $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}$ and $\Sigma : \mathsf{Frm} \to \mathsf{Top}$ there is a natural isomorphism

$\mathsf{Frm}(L, \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \Sigma L)$. Combining this for $L = \mathfrak{L}\left(\overline{\mathbb{R}}\right)$ with the homeomorphism $\Sigma(\mathfrak{L}\left(\overline{\mathbb{R}}\right)) \simeq \overline{\mathbb{R}}$ one obtains

$$\overline{\mathsf{C}}(\mathcal{O}X) = \mathsf{Frm}(\mathfrak{L}\left(\overline{\mathbb{R}}\right), \mathcal{O}X) \simeq \mathsf{Top}(X, \overline{\mathbb{R}}),$$

which justifies the preceding definition.

Let $\mathcal{S}(L)$ be the frame of all sublocales of L. We define

$$\overline{\mathsf{F}}(L) = \overline{\mathsf{C}}(\mathcal{S}(L)).$$

The elements of $\overline{\mathsf{F}}(L)$ will be called the *extended real functions* on L. An extended real function f is *lower semicontinuous* (resp. *upper semicontinuous*) if $f(r,\text{—})$ (resp. $f(\text{—},r)$) is closed for every $r \in \mathbb{Q}$.

By the isomorphism $L \simeq \mathfrak{c}L$ it is immediate that $\overline{\mathsf{C}}(L)$ is equivalent to the set of all $f \in \overline{\mathsf{F}}(L)$ such that $f(p,q)$ is closed for every $p, q \in \mathbb{Q}$ and $\overline{\mathsf{C}}(L) = \overline{\mathsf{LSC}}(L) \cap \overline{\mathsf{USC}}(L)$.

$\overline{\mathsf{C}}(L)$ is partially ordered as $\mathsf{C}(L)$ (see [2]), i.e. given $f, g \in \overline{\mathsf{C}}(L)$ we have

$$\begin{aligned} f \le g \;\equiv\;& f(r,\text{—}) \le g(r,\text{—}) \quad \text{for all } r \in \mathbb{Q} \\ \Leftrightarrow\;& g(\text{—},r) \le f(\text{—},r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

There is a useful way of specifying extended continuous real functions on L with the help of the so called extended scales. An *extended scale* in L is a map

$$\sigma : \mathbb{Q} \to L$$

such that $\sigma(r) \vee \sigma(s)^* = 1$ whenever $r < s$. An extended scale is a *scale* if $\bigvee\{\sigma(r) \mid r \in \mathbb{Q}\} = 1 = \bigvee\{\sigma(r)^* \mid r \in \mathbb{Q}\}$.

Note 1. The terminology *scale* used here differs from its use in [13] where it refers to maps to L from the unit interval of $\mathbb{Q}$ and not all of $\mathbb{Q}$. In [2] the term *descending trail* is used.

Remark 7. An extended scale σ in L is necessarily antitone. If every $\sigma(r)$ is complemented, then σ is an extended scale if and only if it is antitone.

The following two basic lemmas have a straightforward proof.

Lemma 1. *For any extended scale σ in L the formulas*

$$f(r,\text{—}) = \bigvee\{\sigma(s) \mid s > r\} \quad \textit{and} \quad f(\text{—},r) = \bigvee\{\sigma(s)^* \mid s < r\} \quad (r \in \mathbb{Q})$$

determine an $f \in \overline{\mathsf{C}}(L)$. Moreover, $f \in \mathsf{C}(L)$ if and only if σ is a scale. ∎

In particular, any extended scale σ in $\mathcal{S}(L)$ determines an $f \in \overline{\mathsf{F}}(L)$, which is in $\mathsf{F}(L)$ iff σ is a scale.

Lemma 2. *Let $f, g \in \overline{\mathsf{C}}(L)$ be determined by the extended scales σ_1 and σ_2, respectively. Then:*
(a) $f(r,\text{—}) \le \sigma_1(r) \le f(\text{—},r)^*$ for every $r \in \mathbb{Q}$.
(b) $f \le g$ if and only if $\sigma_1(r) \le \sigma_2(s)$ for every $r > s$ in $\mathbb{Q}$. ■

Example 1. (Constant functions) For each $r \in \mathbb{Q}$, consider $\sigma_r : \mathbb{Q} \to L$ such that

$$\sigma_r(s) = 0 \ \ (s \ge r), \quad \sigma_r(s) = 1 \ \ (s < r),$$

clearly a scale in L, and let $\mathbf{r} \in \mathsf{C}(L)$ be the function defined by it, called the *constant* function determined by r. Explicitly, then, for each $s \in \mathbb{Q}$

$$\mathbf{r}(s,\text{—}) = \begin{cases} 0 & \text{if } s \ge r \\ 1 & \text{if } s < r \end{cases} \quad \text{and} \quad \mathbf{r}(\text{—},s) = \begin{cases} 1 & \text{if } s > r \\ 0 & \text{if } s \le r, \end{cases}$$

or alternatively

$$\mathbf{r}(p,q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

One can similarly define two extended constant real functions $+\infty$ and $-\infty$ generated by the extended scales $\sigma_{+\infty} : r \mapsto 1$ $(r \in \mathbb{Q})$ and $\sigma_{-\infty} : r \mapsto 0$ $(r \in \mathbb{Q})$. They are defined for each $r \in \mathbb{Q}$ by

$$+\infty(r,\text{—}) = 1 = -\infty(\text{—},r) \quad \text{and} \quad +\infty(\text{—},r) = 0 = -\infty(r,\text{—})$$

and constitute particular examples of extended continuous real functions which are not continuous real functions. By the preceding lemma, they are precisely the top and bottom elements of the poset $\overline{\mathsf{C}}(L)$.

Remark 8. In particular, defining $+\infty$ and $-\infty$ in $\overline{\mathsf{C}}(\mathcal{S}(L)) = \overline{\mathsf{F}}(L)$, these are the top and bottom elements of $\overline{\mathsf{F}}(L)$. Since $+\infty$ and $-\infty$ are continuous, they are also the top and bottom elements of $\overline{\mathsf{LSC}}(L)$ and $\overline{\mathsf{USC}}(L)$ (this corrects the erroneous statement in [10] that there is no bottom in $\overline{\mathsf{LSC}}(L)$ and no top in $\overline{\mathsf{USC}}(L)$).

Example 2. (Characteristic functions) The classical characteristic functions of clopen subsets of a space have the following pointfree counterpart: for complemented $a \in L$,

$$\sigma(r) = 1 \ \ (r < 0), \quad \sigma(r) = a \ \ (0 \le r < 1), \quad \sigma(r) = 0 \ \ (r \ge 1)$$

is a scale describing a function $\chi_a \in \mathsf{C}(L)$, called the *characteristic function* of a. Specifically, χ_a is defined for each $r \in \mathbb{Q}$ by

$$\chi_c(r,\text{—}) = \begin{cases} 1 & \text{if } r < 0 \\ a & \text{if } 0 \le r < 1 \\ 0 & \text{if } r \ge 1 \end{cases} \quad \text{and} \quad \chi_c(\text{—},r) = \begin{cases} 0 & \text{if } r \le 0 \\ a^* & \text{if } 0 < r \le 1 \\ 1 & \text{if } r > 1. \end{cases}$$

On the other hand, the construction of the constant real functions $\boldsymbol{+\infty}$ and $\boldsymbol{-\infty}$ can also be extended for any arbitrary complemented element a of L by taking the extended scale $\sigma : r \mapsto a$ $(r \in \mathbb{Q})$. We denote by ξ_a the corresponding extended continuous real function and call it the *extended characteristic function* of a. Specifically, ξ_a is defined for each $r \in \mathbb{Q}$ by

$$\xi_a(r,\text{—}) = a \quad \text{and} \quad \xi_a(\text{—},r) = a^*.$$

In particular, $\xi_1 = \boldsymbol{+\infty}$ and $\xi_0 = \boldsymbol{+\infty}$.

These ξ_a correspond, in classical terms, to the extended functions with value $+\infty$ on some clopen set and value $-\infty$ on the complement.

An extended continuous real function $f \in \overline{\mathsf{C}}(L)$ is said to be *bounded* if there exist $p < q$ in $\mathbb{Q}$ such that $\mathbf{p} \le f \le \mathbf{q}$, i.e. $f(q,\text{—}) = f(\text{—},p) = 0$. From $\mathbf{p} \le f$ it follows that $\bigvee_{r\in\mathbb{Q}} f(r,\text{—}) \ge \bigvee_{r\in\mathbb{Q}} \mathbf{p}(r,\text{—}) = 1$. Similarly, from $f \le \mathbf{q}$ it follows that $\bigvee_{r\in\mathbb{Q}} f(\text{—},r) \ge \bigvee_{r\in\mathbb{Q}} \mathbf{q}(\text{—},q) = 1$. Consequently $f \in \mathsf{C}(L)$. In particular, any bounded $f \in \overline{\mathsf{F}}(L)$ is in $\mathsf{F}(L)$.

In connection with Remark 2 we can now prove that

Lemma 3. *The following partially ordered sets are isomorphic for any frame L and any $p < q \in \mathbb{Q}$:*

(i) $\overline{\mathsf{C}}(L)$.
(ii) $\mathsf{Frm}(\mathfrak{L}[p,q], L)$.
(iii) $\{f \in \mathsf{C}(L) \mid \mathbf{p} \le f \le \mathbf{q}\}$.

Proof: The isomorphism between $\overline{\mathsf{C}}(L)$ and $\mathsf{Frm}(\mathfrak{L}[p,q], L)$ follows immediately from Remark 2. Now, given a frame homomorphism $f : \mathfrak{L}[p,q] \to L$ let $\widehat{f} : \mathfrak{L}(\mathbb{R}) \to L$ be defined by $\widehat{f}(r,s) = f\big((r,s) \vee (\text{—},p) \vee (q,\text{—})\big)$ for every $r < s \in \mathbb{Q}$. Clearly $\widehat{f}$ is a frame homomorphism satisfying

$$\widehat{f}(\text{—},p) = \bigvee_{r<p} \widehat{f}(r,p) = f\big((\text{—},p) \vee (q,\text{—})\big) = 0$$

and

$$\widehat{f}(q,\text{—}) = \bigvee_{q<s} \widehat{f}(s,q) = f((\text{—},p) \vee (q,\text{—})) = 0$$

and thus $\mathbf{p} \leq \widehat{f} \leq \mathbf{q}$. Conversely, given a bounded frame homomorphism $\widehat{f} : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{p} \leq \widehat{f} \leq \mathbf{q}$, it follows that the restriction of $\widehat{f}$ to $\mathfrak{L}[p,q]$ is a frame homomorphism (since $\widehat{f}(-,p) = \widehat{f}(q,-) = 0$). ■

Corollary. *For any frame L and any $p < q \in \mathbb{Q}$, the posets*

$$\overline{\mathsf{F}}(L), \mathsf{Frm}(\mathfrak{L}[p,q], \mathcal{S}(L)) \text{ and } \big\{f \in \mathsf{F}(L) \mid \mathbf{p} \leq f \leq \mathbf{q}\big\}$$

are isomorphic. ■

3. Algebra in $\overline{\mathsf{C}}(L)$: Lattice operations

Recall that the operations on the algebra $\mathsf{C}(L)$ are determined by the lattice-ordered ring operations of $\mathbb{Q}$ as follows (see [2] for more details):

(1) For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p,q) = \bigvee\{f(r,s) \wedge g(t,u) \mid \langle r,s\rangle \diamond \langle t,u\rangle \subseteq \langle p,q\rangle\}$$

where $\langle \cdot, \cdot \rangle$ stands for open interval in $\mathbb{Q}$ and the inclusion on the right means that $x \diamond y \in \langle p,q\rangle$ whenever $x \in \langle r,s\rangle$ and $y \in \langle t,u\rangle$.

(2) $(-f)(p,q) = f(-q,-p)$.

(3) For each $r \in \mathbb{Q}$, a nullary operation $\mathbf{r}$ defined by

$$\mathbf{r}(p,q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

(4) For each $0 < \lambda \in \mathbb{Q}$, $(\lambda \cdot f)(p,q) = f(\frac{p}{\lambda}, \frac{q}{\lambda})$.

Indeed, these stipulations define maps from $\mathbb{Q} \times \mathbb{Q}$ to L and turn the defining relations (R1)–(R4) of $\mathfrak{L}(\mathbb{R})$ into identities in L and consequently determine frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$. The result that $\mathsf{C}(L)$ is an f-ring follows from the fact that any identity in these operations which is satisfied by $\mathbb{Q}$ also holds in $\mathsf{C}(L)$.

In particular, each $\mathsf{F}(L)$, coinciding with $\mathsf{C}(\mathcal{S}(L))$, is an f-ring with operations defined by the aforementioned formulas. What about $\overline{\mathsf{C}}(L)$ (and $\overline{\mathsf{F}}(L)$)?

In this section we deal with the algebraic aspects of the extended reals and their extended function algebras. First, we have the following easy description of the operations $\wedge$, $\vee$, $-(\cdot)$ and $\lambda\cdot(\cdot)$ for any $0<\lambda\in\mathbb{Q}$.

Proposition 1. *Let $f,g\in\overline{\mathsf{C}}(L)$ and $0<\lambda\in\mathbb{Q}$. Then:*

(1) *$\sigma_{f\vee g}: r\mapsto f(r,\text{—})\vee g(r,\text{—})$ is an extended scale in L that determines the extended function $f\vee g\in\overline{\mathsf{C}}(L)$ given by $(f\vee g)(r,\text{—})=f(r,\text{—})\vee g(r,\text{—})$ and $(f\vee g)(\text{—},r)=f(\text{—},r)\wedge g(\text{—},r)$. This is precisely the join of f and g in $\overline{\mathsf{C}}(L)$.*

(2) *$\sigma_{f\wedge g}: r\mapsto f(r,\text{—})\wedge g(r,\text{—})$ is an extended scale in L that determines the extended function $f\wedge g\in\overline{\mathsf{C}}(L)$ given by $(f\wedge g)(r,\text{—})=f(r,\text{—})\wedge g(r,\text{—})$ and $(f\wedge g)(\text{—},r)=f(\text{—},r)\vee g(\text{—},r)$. This is precisely the meet of f and g in $\overline{\mathsf{C}}(L)$.*

(3) *$\sigma_{-f}: r\mapsto f(\text{—},-r)$ is an extended scale in L that determines the extended function $-f\in\overline{\mathsf{C}}(L)$ given by $(-f)(r,\text{—})=f(\text{—},-r)$ and $(-f)(\text{—},r)=f(-r,\text{—})$.*

(4) *$\sigma_{\lambda\cdot f}: r\mapsto f(\frac{r}{\lambda},\text{—})$ is an extended scale in L that determines the extended function $\lambda\cdot f\in\overline{\mathsf{C}}(L)$ given by $(\lambda\cdot f)(r,\text{—})=f(\frac{r}{\lambda},\text{—})$ and $(\lambda\cdot f)(\text{—},r)=f(\text{—},\frac{r}{\lambda})$.*

Proof: We only prove assertion (1), the remaining ones can be checked in a similar way.

First, $\sigma_{f\vee g}$ is an extended scale since, for every $s<r$,

$$(f(s,\text{—})\vee g(s,\text{—}))\vee(f(r,\text{—})\vee g(r,\text{—}))^*=$$
$$=(f(s,\text{—})\vee g(s,\text{—})\vee f(r,\text{—})^*)\wedge(f(s,\text{—})\vee g(s,\text{—})\vee g(r,\text{—})^*)\geq 1$$

(because $f(r,\text{—})^*\geq f(\text{—},r)$ and $g(r,\text{—})^*\geq g(\text{—},r)$). Then, using Lemma 1, we get

$$(f\vee g)(r,\text{—})=\bigvee_{s>r}(f(s,\text{—})\vee g(s,\text{—}))=f(r,\text{—})\vee g(r,\text{—})$$

and

$$(f\vee g)(\text{—},r)=\bigvee_{s<r}\big(f(s,\text{—})\vee g(s,\text{—})\big)^*=f(\text{—},r)\wedge g(\text{—},r).$$

(For the latter identity notice that if $s<r$, then $(f(s,\text{—})\vee g(s,\text{—}))^*=f(s,\text{—})^*\wedge g(s,\text{—})^*\leq f(\text{—},r)\wedge g(\text{—},r)$; conversely,

$$f(\text{—},r)\wedge g(\text{—},r)=\bigvee_{s_1,s_2<r}(f(\text{—},s_1)\wedge g(\text{—},s_2))\leq\bigvee_{s<r}(f(s,\text{—})^*\wedge g(s\text{—})^*).)$$

Now, the fact that this is precisely the join of f and g in $\overline{\mathsf{C}}(L)$ is obvious. ■

In conclusion, we have:

Corollary. *The poset* $\overline{\mathsf{F}}(L)$ *has binary joins and meets;* $\overline{\mathsf{USC}}(L)$, $\overline{\mathsf{LSC}}(L)$, $\overline{\mathsf{C}}(L)$, $\mathsf{F}(L)$, $\mathsf{USC}(L)$, $\mathsf{LSC}(L)$ *and* $\mathsf{C}(L)$ *are closed under these joins and meets.* ■

Remark 9. Note that in all these cases the formulas above, when applied to elements of the form (p, q), coincide with those of [2]. In fact, let $f, g \in \overline{\mathsf{C}}(L)$, $r \in \mathbb{Q}$, $0 < \lambda \in \mathbb{Q}$ and $p, q \in \mathbb{Q}$. Then $(f \vee g)(p, q)$ is equal to

$$\begin{aligned}(f \vee g)(p, \text{—}) \wedge (f \vee g)(\text{—}, q) &= \big(f(p, \text{—}) \vee g(p, \text{—})\big) \wedge \big(f(\text{—}, q) \wedge g(\text{—}, q)\big)\\ &= \big(f(p, q) \wedge g(\text{—}, q)\big) \vee \big(g(p, q) \wedge f(\text{—}, q)\big)\\ &= \big(\bigvee_{s<q} f(p, q) \wedge g(s, q)\big) \vee \big(\bigvee_{r<q} f(r, q) \wedge g(p, q)\big).\end{aligned}$$

The latter is equal to $\bigvee\{f(r, s) \wedge g(t, u) \mid \langle r, s\rangle \vee \langle t, u\rangle = \langle r \vee t, s \vee u\rangle \subseteq \langle p, q\rangle\}$: indeed, if $s < q$ then

$$\langle p, q\rangle \vee \langle s, q\rangle = \{x \vee y \mid x \in \langle p, q\rangle, y \in \langle s, q\rangle\} = \langle p \vee s, q\rangle \subseteq \langle p, q\rangle;$$

on the other hand, if $r < q$, then

$$\langle r, q\rangle \vee \langle p, q\rangle = \{x \vee y \mid x \in \langle r, q\rangle, y \in \langle p, q\rangle\} = \langle r \vee p, q\rangle \subseteq \langle p, q\rangle.$$

Hence the inequality $\leq$ follows. Conversely, let r, s, t and u such that $\langle r, s\rangle \vee \langle t, u\rangle \subseteq \langle p, q\rangle$, i.e. such that $p \leq r \vee t$ and $s \vee u \leq q$. We distinguish several cases:

- $p \leq r$ and $t \geq q$: then $f(r, s) \wedge g(t, u) \leq f(p, q) \wedge g(t, q) = 0$.
- $p \leq r$ and $t < q$: then

$$f(r, s) \wedge g(t, u) \leq f(p, q) \wedge g(t, q) \leq \bigvee_{r<q} f(p, q) \wedge g(r, q).$$

- $p \leq t$ and $r \geq q$: then $f(r, s) \wedge g(t, u) \leq f(r, q) \wedge g(p, q) = 0$.
- $p \leq t$ and $r < q$: then

$$f(r, s) \wedge g(t, u) \leq f(r, q) \wedge g(p, q) \leq \bigvee_{s<q} f(s, q) \wedge g(p, q).$$

Concerning meets, we have

$$\begin{aligned}(f\wedge g)(p,q) &= (f\wedge g)(p,\text{—})\wedge(f\wedge g)(\text{—},q)\\ &= \big(f(p,\text{—})\wedge g(p,\text{—})\big)\wedge\big(f(\text{—},q)\vee g(\text{—},q)\big)\\ &= \big(f(p,q)\wedge g(p,\text{—})\big)\vee\big(f(p,\text{—})\wedge g(p,q)\big)\\ &= \big(\bigvee_{p<r} f(p,q)\wedge g(p,r)\big)\vee\big(\bigvee_{p<s} f(p,s)\wedge g(p,q)\big)\end{aligned}$$

and the latter is equal to $\bigvee\{f(r,s)\wedge g(t,u)\mid \langle r,s\rangle\wedge\langle t,u\rangle=\langle r\wedge t,s\wedge u\rangle\subseteq\langle p,q\rangle\}$. In fact, if $p<r$ then

$$\langle p,q\rangle\wedge\langle p,r\rangle=\{x\wedge y\mid x\in\langle p,q\rangle,y\in\langle s,q\rangle\}=\langle p,q\wedge r\rangle\subseteq\langle p,q\rangle,$$

and if $p<s$ then

$$\langle p,s\rangle\wedge\langle p,q\rangle=\{x\wedge y\mid x\in\langle p,s\rangle,y\in\langle p,q\rangle\}=\langle p,s\wedge q\rangle\subseteq\langle p,q\rangle.$$

Hence the inequality $\leq$ follows. Conversely, let r, s, t and u such that $\langle r,s\rangle\wedge\langle t,u\rangle\subseteq\langle p,q\rangle$, i.e. such that $p\leq r\wedge t$ and $s\wedge u\leq q$. Here we also distinguish several cases:

- $s\leq q$ and $p\geq u$: then $f(r,s)\wedge g(t,u)\leq f(p,q)\wedge g(p,u)=0$.
- $s\leq q$ and $u<p$: then

$$f(r,s)\wedge g(t,u)\leq f(p,q)\wedge g(p,u)\leq\bigvee_{p<r} f(p,q)\wedge g(p,r).$$

- $u\leq q$ and $p\geq s$: then $f(r,s)\wedge g(t,u)\leq f(p,s)\wedge g(p,q)=0$.
- $u\leq q$ and $p<s$: then

$$f(r,s)\wedge g(t,u)\leq f(p,s)\wedge g(p,q)\leq\bigvee_{p<sr} f(p,s)\wedge g(p,q).$$

Finally, we have

$$(-f)(p,q)=(-f)(p,\text{—})\wedge(-f)(\text{—},q)=f(\text{—},-p)\wedge f(-q,\text{—})=f(-q,-p)$$

and

$$(\lambda\cdot f)(p,q)=(\lambda\cdot f)(p,\text{—})\wedge(\lambda\cdot f)(\text{—},q)=f(\tfrac{p}{\lambda},\text{—})\wedge f(\text{—},\tfrac{q}{\lambda})=f(\tfrac{p}{\lambda},\tfrac{q}{\lambda}).$$

Remark 10. As a consequence of the above analysis of the operations $\vee$, $\wedge$ and $-(\cdot)$ we note that, by the arguments in [2] for the case of $\mathsf{C}(L)$, they satisfy all identities which hold for the corresponding operations of $\mathbb{Q}$. Hence, $\overline{\mathsf{C}}(L)$ is a *distributive lattice* with join $\vee$, meet $\wedge$ and an *inversion* given by $-(\cdot)$. Moreover, it is, of course, *bounded*, with top $+\infty$ and bottom $-\infty$. Further, again by arguments in [2], the partial order determined by this lattice structure is exactly the one mentioned earlier: $f \vee g = g$ iff $f(r,\text{—}) \leq g(r,\text{—})$ for all $r \in \mathbb{Q}$. Finally, the isomorphism $\mathfrak{L}\left(\overline{\mathbb{R}}\right) \simeq \mathfrak{L}[p,q]$ described in Remark 2 induces a bounded lattice isomorphism

$$\overline{\mathsf{C}}(L) \simeq \{f \in \mathsf{C}(L) \mid \mathbf{p} \leq f \leq \mathbf{q}\}.$$

Notice that the $\overline{\mathsf{C}}h : \overline{\mathsf{C}}(L) \to \overline{\mathsf{C}}(M)$ determined by frame homomorphisms $h : L \to M$ are bounded lattice homomorphisms that preserve inversion.

4. Algebra in $\overline{\mathsf{C}}(L)$: Addition

Things become more complicated in the case of addition and multiplication. This is not a surprise if we think of the typical indeterminacies

$$-\infty + \infty \quad \text{and} \quad \mathbf{0} \cdot \infty.$$

In the classical case, given $f, g : X \to \overline{\mathbb{R}}$, the condition

$$f^{-1}(\{+\infty\}) \cap g^{-1}(\{-\infty\}) = \varnothing = f^{-1}(\{-\infty\}) \cap g^{-1}(\{+\infty\}) \tag{1}$$

ensures that the addition $f+g$ can be defined for all $x \in X$ just by the natural convention $\lambda + (+\infty) = +\infty = (+\infty) + \lambda$ and $\lambda + (-\infty) = -\infty = (-\infty) + \lambda$ for all $\lambda \in \mathbb{R}$ together with the usual $(+\infty) + (+\infty) = +\infty$ and the same for $-\infty$. Clearly enough, condition (1) is equivalent to

$$(f \vee g)^{-1}(\{+\infty\}) \cap (f \wedge g)^{-1}(\{-\infty\}) = \varnothing.$$

This leads naturally to the following:

Notation. For each $f \in \overline{\mathsf{C}}(L)$ let

$$a_f^+ = \bigvee_{r\in\mathbb{Q}} f(\text{—}, r), \quad a_f^- = \bigvee_{r\in\mathbb{Q}} f(r, \text{—}) \quad \text{and} \quad a_f = a_f^+ \wedge a_f^- = \bigvee_{r<s} f(r,s) = f(\omega).$$

Note that a_f is the pointfree counterpart of the *domain of reality* $f^{-1}(\mathbb{R})$ of an $f : X \to \overline{\mathbb{R}}$. Note also that $a_f^+ \vee a_f^- = 1$. Of course, $a_f = 1$ whenever $f \in \mathsf{C}(L)$.

Definition 2. Let $f, g \in \overline{\mathsf{C}}(L)$. We say that f and g are *sum compatible* if

$$a^+_{f\vee g} \vee a^-_{f\wedge g} = 1.$$

Remark 11. Note that $a^+_{f\vee g} \vee a^-_{f\wedge g} = (a^+_f \vee a^-_g) \wedge (a^+_g \vee a^-_f)$ for each $f, g \in \overline{\mathsf{C}}(L)$. Indeed, $a^+_f \vee a^-_f = 1 = a^+_g \vee a^-_g$, $a^+_{f\vee g} = a^+_f \wedge a^+_g$ and $a^-_{f\wedge g} = a^-_f \wedge a^-_g$, hence the equality follows from

$$(a^+_f \wedge a^+_g) \vee (a^-_f \wedge a^-_g) = (a^+_f \vee a^-_f) \wedge (a^+_f \vee a^-_g) \wedge (a^+_g \vee a^-_f) \wedge (a^+_g \vee a^-_g).$$

Consequently f and g are sum compatible if and only if

$$(a^+_f \vee a^-_g) \wedge (a^+_g \vee a^-_f) = 1.$$

Remark 12. Obviously, any $f, g \in \mathsf{C}(L)$ are sum compatible since $a^+_{f\vee g} = a^-_{f\wedge g} = 1$.

Proposition 2. *Let $f, g \in \overline{\mathsf{C}}(L)$ be sum compatible. Then the map $\sigma_{f+g} : \mathbb{Q} \to L$ defined by*

$$\sigma_{f+g}(r) = \bigvee\{f(s,\text{—}) \wedge g(t,\text{—}) \mid s + t = r\},$$

is an extended scale of L.

Proof: Let $f, g \in \overline{\mathsf{C}}(L)$ be sum compatible. We first note that for each $r \in \mathbb{Q}$

$$\begin{aligned}\sigma_{f+g}(r) \wedge \big(\bigvee_{t\in\mathbb{Q}} f(\text{—},t) \wedge g(\text{—},r-t)\big) &= \\ = \bigvee_{s,t\in\mathbb{Q}} f(s,\text{—}) \wedge g(r-s,\text{—}) \wedge f(\text{—},t) \wedge g(\text{—},r-t) &= 0\end{aligned}$$

since $f(s,\text{—}) \wedge f(\text{—},t) = 0$ in case $t \le s$ and $g(r-s,\text{—}) \wedge g(\text{—},r-t)$ in case $t > s$. Hence, $\bigvee_{t\in\mathbb{Q}} f(\text{—},t) \wedge g(\text{—},r-t) \le \sigma_{f+g}(r)^*$.

On the other hand, let $r < s$ and $t \in \mathbb{Q}$ such that $0 < 2t \le s - r$. For each $q \in \mathbb{Q}$ such that $q > \frac{s}{2}$ we have that $r - q < s - q < q$ and so $f(\text{—},q) = f(\text{—},s-q) \vee f(r-q,q)$, $g(\text{—},q) = g(\text{—},s-q) \vee g(r-q,q)$ and

$$\begin{aligned}f(\text{—},q) \wedge g(\text{—},q) &= \big(f(\text{—},s-q) \wedge g(\text{—},q)\big) \vee \big(f(r-q,q) \wedge g(\text{—},q)\big) = \\ &= \big(f(\text{—},s-q) \wedge g(\text{—},q)\big) \vee \big(f(r-q,q) \wedge g(\text{—},s-q)\big) \vee \\ &\quad \vee \big(f(r-q,q) \wedge g(r-q,q)\big).\end{aligned}$$

Now we have that

$$f(\text{—},s-q) \wedge g(\text{—},q) \le \bigvee_{t\in\mathbb{Q}} f(\text{—},t) \wedge g(\text{—},s-t), \tag{2}$$

$$f(r-q,q) \wedge g(\text{—},s-q) \le f(\text{—},q) \wedge g(\text{—},s-q) \le \bigvee_{t\in\mathbb{Q}} f(\text{—},t) \wedge g(\text{—},s-t) \tag{3}$$

and

$$\begin{aligned} f(r-q,q)\wedge g(r-q,q) &= \Big(\bigvee_{r-q<p<q-t} f(p,p+t)\Big)\wedge\Big(\bigvee_{r-q<p'<q-t} g(p',p'+t)\Big) \\ &= \bigvee_{r-q<p,p'<q-t} f(p,p+t)\wedge g(p',p'+t). \end{aligned}$$

If $p+t+p'+t<s$ then

$$f(p,p+t)\wedge g(p',p'+t)\le f(\text{—},p+t)\wedge g(\text{—},s-p-t)\le \bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t)$$

and otherwise if $p+t+p'+t\ge s$ then $p'\ge s-2t-p\ge r-p$ and so

$$f(p,p+t)\wedge g(p',p'+t)\le f(p,\text{—})\wedge g(r-p,\text{—})\le \sigma_{f+g}(r).$$

Hence

$$f(p,p+t)\wedge g(p',p'+t)\le \sigma_{f+g}(r)\vee\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t)$$

and we conclude that

$$f(r-q,q)\wedge g(r-q,q)\le \sigma_{f+g}(r)\vee\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t). \tag{4}$$

It follows immediately from (2), (3) and (4) that

$$f(\text{—},q)\wedge g(\text{—},q)\le \sigma_{f+g}(r)\vee\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t).$$

Hence

$$\begin{aligned} a_f^+\wedge a_g^+ &= \bigvee_{q\in\mathbb{Q}}\big(f(\text{—},q)\wedge g(\text{—},q)\big)=\bigvee_{q>\frac{s}{2}}\big(f(\text{—},q)\wedge g(\text{—},q)\big) \\ &\le \sigma_{f+g}(r)\vee\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t)\le \sigma_{f+g}(r)\vee\sigma_{f+g}(s)^* \end{aligned}$$

Similarly it can be proved that

$$a_f^-\wedge a_g^- = \bigvee_{q\in\mathbb{Q}}\big(f(q,\text{—})\wedge g(q,\text{—})\big)\le \sigma_{f+g}(r)\vee\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t)$$

and we may then conclude that

$$\begin{aligned} 1 &= a^+_{f\vee g}\vee a^+_{f\wedge g}=\big(a_f^+\wedge a_g^+\big)\vee\big(a_f^-\wedge a_g^-\big) \\ &\le \sigma_{f+g}(r)\vee\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t)\le \sigma_{f+g}(r)\vee\sigma_{f+g}(s)^*. \quad \blacksquare \end{aligned}$$

Proposition 3. *Let $f,g\in\overline{\mathsf{C}}(L)$ be sum compatible. Then:*

(1) *The extended real function* $f+g$ *generated by* σ_{f+g} *is defined for each* $r \in \mathbb{Q}$ *by*

$$(f+g)(r,\text{—}) = \bigvee_{s\in\mathbb{Q}} f(s,\text{—})\wedge g(r-s,\text{—}) \text{ and } (f+g)(\text{—},r) = \bigvee_{s\in\mathbb{Q}} f(\text{—},s)\wedge g(\text{—},r-s).$$

(2) $(f+g)(p,q) = \bigvee\{f(r,s)\wedge g(t,u) \mid \langle r,s\rangle + \langle t,u\rangle \subseteq \langle p,q\rangle\}$.

Proof: (1) For each rational r, we have immediately

$$(f+g)(r,\text{—}) = \bigvee_{s>r} \sigma_{f+g}(s) = \bigvee_{s>r}\bigvee_{t\in\mathbb{Q}} f(t,\text{—})\wedge g(s-t,\text{—}) = \bigvee_{s\in\mathbb{Q}} f(s,\text{—})\wedge g(r-s,\text{—}).$$

On the other hand, let $s < r$ in $\mathbb{Q}$. It follows from Proposition 2 that $\sigma_{f+g}(s) \vee \bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},r-t) = 1$ and so $\sigma_{f+g}(s)^* \leq \bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},r-t)$. Hence

$$(f+g)(\text{—},r) = \bigvee_{s<r} \sigma_{f+g}(s)^* \leq \bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},r-t).$$

Moreover

$$\begin{aligned}\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},r-t) &= \bigvee_{t\in\mathbb{Q}}\bigvee_{s<r} f(\text{—},t)\wedge g(\text{—},s-t)\\ &= \bigvee_{s<r}\bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},s-t) \leq \bigvee_{s<r}\sigma_{f+g}(s)^* = (f+g)(\text{—},r)\end{aligned}$$

and hence

$$(f+g)(\text{—},r) = \bigvee_{t\in\mathbb{Q}} f(\text{—},t)\wedge g(\text{—},r-t).$$

(2) Let $p,q,r,s \in \mathbb{Q}$ with $p<q$. Since

$$\langle r,s\rangle + \langle t,u\rangle = \{x+y \mid x\in\langle r,s\rangle, y\in\langle t,u\rangle\} = \langle r+t, s+u\rangle,$$

it readily follows that $\langle r,s\rangle + \langle t,u\rangle \subseteq \langle p,q\rangle$ if and only if $p \leq r+t$ and $q \geq s+u$. Consequently

$$\begin{aligned}\sigma_{f+g}(p) \wedge \bigvee_{s\in\mathbb{Q}} f(\text{—},s)\wedge g(\text{—},q-s) &= \bigvee_{r,s\in\mathbb{Q}} f(r,s)\wedge g(p-r,q-s) \leq\\ &\leq \bigvee\{f(r,s)\wedge g(t,u) \mid \langle r,s\rangle + \langle t,u\rangle \subseteq \langle p,q\rangle\}.\end{aligned}$$

Conversely, if $p \leq r+t$ and $q \geq s+u$, then $p-r\leq t$ and $u \leq q-s$ and so $f(r,s)\wedge g(t,u) \leq \bigvee_{r,s\in\mathbb{Q}} f(r,s)\wedge g(p-r,q-s) = \sigma_{f+g}(p)\wedge\bigvee_{s\in\mathbb{Q}} f(\text{—},s)\wedge g(\text{—},q-s)$. ■

We have finally the following characterization.

Theorem 1. *Let $f, g \in \overline{\mathsf{C}}(L)$. The map $\sigma_{f+g} : \mathbb{Q} \to L$ defined by*

$$\sigma_{f+g}(r) = \bigvee_{s\in\mathbb{Q}} f(s,\text{—}) \wedge g(r-s,\text{—}),$$

is an extended scale of L if and only if f and g are sum compatible.

Proof: Sufficiency follows from Proposition 2. For necessity, it follows from Proposition 3(1) that

$$\begin{aligned} a^+_{f+g} = \bigvee_{r\in\mathbb{Q}} \bigvee_{s\in\mathbb{Q}} f(\text{—},s) \wedge g(\text{—},r-s) \ &= \ \bigvee_{s\in\mathbb{Q}} \bigvee_{r\in\mathbb{Q}} f(\text{—},s) \wedge g(\text{—},r-s) \\ = \bigvee_{s\in\mathbb{Q}} f(\text{—},s) \wedge a^+_g \ &= \ a^+_f \wedge a^+_g \ = \ a^+_{f\vee g} \end{aligned}$$

and similarly $a^-_{f+g} = a^+_g = a^-_f \wedge a^-_g = a^-_{f\wedge g}$. Hence $1 = a^+_{f+g} \vee a^-_{f+g} \leq a^+_{f\vee g} \vee a^-_{f\vee g}$. ■

Corollary. *Let $f, g \in \overline{\mathsf{F}}(L)$ be sum compatible. Then $f+g \in \overline{\mathsf{F}}(L)$. Furthermore, if $f, g \in \overline{\mathsf{C}}(L)$ (resp. $\overline{\mathsf{LSC}}(L)$, resp. $\overline{\mathsf{USC}}(L)$) then $f+g \in \overline{\mathsf{C}}(L)$ (resp. $\overline{\mathsf{LSC}}(L)$, resp. $\overline{\mathsf{USC}}(L)$).* ■

Remark 13. (1) Any $f \in \overline{\mathsf{C}}(L)$ and $\mathbf{r}$ are sum compatible, and (2) For any $f \in \overline{\mathsf{C}}(L)$, f and $-f$ are sum compatible iff $f \in \mathsf{C}(L)$ and then, of course, $f + (-f) = \mathbf{0}$. We omit the details.

5. Algebra in $\overline{\mathsf{C}}(L)$: Multiplication

We turn now to the case of multiplication. In the classical case, given $f, g : X \to \overline{\mathbb{R}}$ the condition

$$f^{-1}\{-\infty,+\infty\} \cap g^{-1}\{0\} = \varnothing = f^{-1}\{0\} \cap g^{-1}\{-\infty,+\infty\} \tag{5}$$

ensures that the multiplication $f \cdot g$ can be defined for all $x \in X$ just by the natural conventions $\lambda \cdot (\pm\infty) = \pm\infty = (\pm\infty) \cdot \lambda$ for all $\lambda > 0$ and $\lambda \cdot (\pm\infty) = \mp\infty = (\pm\infty) \cdot \lambda$ for all $\lambda < 0$ together with the usual $(\pm\infty) \cdot (\pm\infty) = +\infty$ and $(\pm\infty) \cdot (\mp\infty) = -\infty$.

Clearly enough, condition (5) is equivalent to

$$\big(f^{-1}\{-\infty,+\infty\} \cup g^{-1}\{-\infty,+\infty\}\big) \cap \big(f^{-1}\{0\} \cup g^{-1}\{0\}\big) = \varnothing. \tag{6}$$

Now recall that in a frame L, a *cozero element* is an element of the form

$$\operatorname{coz} f = f((\text{—},0) \vee (0,\text{—})) = \bigvee\{f(p,0) \vee f(0,q) \mid p < 0 < q \text{ in } \mathbb{Q}\}$$

for some $f \in \mathsf{C}(L)$. This is the pointfree counterpart to the notion of a cozero set for ordinary continuous real functions. For information on the map $\operatorname{coz} : \mathsf{C}(L) \to L$ we refer to [5]. As usual, $\operatorname{Coz} L$ will denote the *cozero lattice* of all cozero elements of L.

For an extended $f \in \overline{\mathsf{C}}(L)$, we shall continue to write $\operatorname{coz} f = f(-,0) \vee f(0,-)$. Note that $a_f^+ \vee \operatorname{coz} f = 1 = a_f^- \vee \operatorname{coz} f$. Condition (6) leads naturally to the following:

Definition 3. Let $f, g \in \overline{\mathsf{C}}(L)$. We say that f and g are *product compatible* if $(a_f \wedge a_g) \vee (\operatorname{coz} f \wedge \operatorname{coz} g) = 1$.

Remark 14. Note that $(a_f \wedge a_g) \vee (\operatorname{coz} f \wedge \operatorname{coz} g) = (a_f \vee \operatorname{coz} f) \wedge (a_f \vee \operatorname{coz} g) \wedge (a_g \vee \operatorname{coz} f) \wedge (a_g \vee \operatorname{coz} g) = (a_f \vee \operatorname{coz} g) \wedge (a_g \vee \operatorname{coz} f)$. Hence f and g are product compatible if and only if

$$(a_f \vee \operatorname{coz} g) \wedge (a_g \vee \operatorname{coz} f) = 1.$$

Remark 15. Evidently, any $f, g \in \mathsf{C}(L)$ are product compatible since $a_f = a_g = 1$.

Proposition 4. *Let $\mathbf{0} \leq f, g \in \overline{\mathsf{C}}(L)$ be product compatible. Then the map $\sigma_{f\cdot g} : \mathbb{Q} \to L$ defined by*

$$\sigma_{f\cdot g}(r) = \bigvee_{s>0} f(s,-) \wedge g(\tfrac{r}{s},-) \ \ (r \geq 0), \quad \sigma_{f\cdot g}(r) = 1 \ \ (r < 0),$$

is an extended scale of L.

Proof: Let $f, g \in \overline{\mathsf{C}}(L)$ be product compatible. We first note that for each $s > 0$

$$\sigma_{f\cdot g}(s) \wedge \Big(\bigvee_{t>0} f(-,t) \wedge g(-,\tfrac{s}{t})\Big) = \bigvee_{r,t>0} f(r,-) \wedge g(\tfrac{s}{r},-) \wedge f(-,t) \wedge g(-,\tfrac{s}{t}) = 0$$

since $f(r,-) \wedge f(-,t) = 0$ in case $t \leq r$ and $g(\frac{s}{r},-) \wedge g(-,\frac{s}{t}) = 0$ otherwise. Hence, $\bigvee_{t>0} f(-,t) \wedge g(-,\frac{s}{t}) \leq \sigma_{f\cdot g}(s)^*$.

If $r < s$ with $r < 0$ then clearly $\sigma_{f\cdot g}(r) \vee \sigma_{f\cdot g}(s)^* = 1$. On the other hand, if $0 = r < s$ then for each $q \in \mathbb{Q}$ such that $q > \sqrt{s}$ we have $0 < \frac{s}{q} < q$ and thus

$$\begin{aligned}1 &= \big(a_f \wedge a_g\big) \vee \big(\operatorname{coz} f \wedge \operatorname{coz} g\big) = \big(a_f^+ \wedge a_g^+\big) \vee \big(f(0,\text{—}) \wedge g(0,\text{—})\big)\\ &= \Big(\bigvee_{q>\sqrt{s}} \big(f(\text{—},q) \wedge g(\text{—},q)\big)\Big) \vee \sigma_{f\cdot g}(0)\\ &= \Big(\bigvee_{q>\sqrt{s}} \big(f(\text{—},\tfrac{s}{q}) \wedge g(\text{—},q)\big) \vee \big(f(0,q) \wedge g(\text{—},\tfrac{s}{q})\big) \vee \big(f(0,q) \wedge g(0,q)\big)\Big) \vee \sigma_{f\cdot g}(0)\\ &\le \Big(\bigvee_{t>0} f(\text{—},t) \wedge g(\text{—},\tfrac{s}{t})\Big) \vee \sigma_{f\cdot g}(0) \le \sigma_{f\cdot g}(s)^* \vee \sigma_{f\cdot g}(0).\end{aligned}$$

Finally, let $0 < r < s$. For each $0 < q \in \mathbb{Q}$ such that $q^2 > s$ we have $\frac{r}{q} < \frac{s}{q} < q$ and thus $f(\text{—},q) = f(\text{—},\frac{s}{q}) \vee f(\frac{r}{q},q)$, $g(\text{—},q) = g(\text{—},\frac{s}{q}) \vee g(\frac{r}{q},q)$ and

$$\begin{aligned}f(\text{—},q) \wedge g(\text{—},q) &= \big(f(\text{—},\tfrac{s}{q}) \wedge g(\text{—},q)\big) \vee \big(f(\tfrac{r}{q},q) \wedge g(\text{—},q)\big)\\ &= \big(f(\text{—},\tfrac{s}{q}) \wedge g(\text{—},q)\big) \vee \big(f(\tfrac{r}{q},q) \wedge g(\text{—},\tfrac{s}{q})\big) \vee \big(f(\tfrac{r}{q},q) \wedge g(\tfrac{r}{q},q)\big).\end{aligned}$$

Now we have that

$$f(\text{—},\tfrac{s}{q}) \wedge g(\text{—},q) \le \bigvee_{t>0} f(\text{—},t) \wedge g(\text{—},\tfrac{s}{t}), \tag{7}$$

$$f(\tfrac{r}{q},q) \wedge g(\text{—},\tfrac{s}{q}) \le f(\text{—},q) \wedge g(\text{—},\tfrac{s}{q}) \le \bigvee_{t>0} f(\text{—},t) \wedge g(\text{—},\tfrac{s}{t}) \tag{8}$$

and for each $0 < t \in \mathbb{Q}$ such that $1 < t^2 < \frac{s}{r}$,

$$\begin{aligned}f(\tfrac{r}{q},q) \wedge g(\tfrac{r}{q},q) &= \Big(\bigvee_{\frac{r}{q}<p<\frac{q}{t}} f(p,pt)\Big) \wedge \Big(\bigvee_{\frac{r}{q}<p'<\frac{q}{t}} g(p',p't)\Big)\\ &= \bigvee_{\frac{r}{q}<p,p'<\frac{q}{t}} f(p,pt) \wedge g(p',p't).\end{aligned}$$

If $pp't^2 < s$ then

$$f(p,pt) \wedge g(p',p't) \le f(\text{—},pt) \wedge g(\text{—},\tfrac{s}{pt}) \le \bigvee_{t>0} f(\text{—},t) \wedge g(\text{—},\tfrac{s}{t})$$

and if $pp't^2 \ge s$ then $p' \ge \frac{s}{t^2 p} > \frac{r}{p}$ and so

$$f(p,pt) \wedge g(p',p't) \le f(p,\text{—}) \wedge g(\tfrac{r}{p},\text{—}) \le \sigma_{f\cdot g}(r).$$

Hence $f(p,pt) \wedge g(p',p't) \le \sigma_{f\cdot g}(r) \vee \bigvee_{t>0} f(\text{—},t) \wedge g(\text{—},\frac{s}{t})$ and we conclude that

$$f(\tfrac{r}{q},q) \wedge g(\tfrac{r}{q},q) \le \sigma_{f\cdot g}(r) \vee \bigvee_{t>0} f(\text{—},t) \wedge g(\text{—},\tfrac{s}{t}). \tag{9}$$

It follows immediately from (7), (8) and (9) that

$$f(—,q)\wedge g(—,q)\le \sigma_{f\cdot g}(r)\vee \bigvee_{t>0} f(—,t)\wedge g(—,\tfrac{s}{t}).$$

Hence

$$\begin{aligned} a_f\wedge a_g = a_f^+\wedge a_g^+ &= \bigvee_{q\in\mathbb{Q}}\big(f(—,q)\wedge g(—,q)\big) = \bigvee_{q>\sqrt{s}}\big(f(—,q)\wedge g(—,q)\big)\\ &\le \sigma_{f\cdot g}(r)\vee \bigvee_{t>0} f(—,t)\wedge g(—,\tfrac{s}{t}) \le \sigma_{f\cdot g}(r)\vee\sigma_{f\cdot g}(s)^*\end{aligned}$$

Similarly it can be proved that

$$\operatorname{coz} f\wedge \operatorname{coz} g = \bigvee_{q>0}\big(f(q,—)\wedge g(q,—)\big)\le \sigma_{f\cdot g}(r)\vee \bigvee_{t>0} f(—,t)\wedge g(—,\tfrac{s}{t})$$

and we may finally conclude that

$$\begin{aligned} 1 &= \big(a_f\wedge a_g\big)\vee\big(\operatorname{coz} f\wedge \operatorname{coz} g\big)\\ &\le \sigma_{f\cdot g}(r)\vee \bigvee_{t\in\mathbb{Q}} f(—,t)\wedge g(—,\tfrac{s}{t}) \le \sigma_{f\cdot g}(r)\vee\sigma_{f\cdot g}(s)^*. \qquad \blacksquare\end{aligned}$$

Proposition 5. *Let $\mathbf{0}\le f,g\in \overline{\mathsf{C}}(L)$ be product compatible. Then:*

(1) *The extended real function $f\cdot g$ generated by $\sigma_{f\cdot g}$ is defined for each $r\in\mathbb{Q}$ by*

$$(f\cdot g)(r,—)=\begin{cases}\bigvee_{s>0} f(s,—)\wedge g(\tfrac{r}{s},—) & \text{if } r\ge 0\\ 1 & \text{if } r<0\end{cases}$$

and

$$(f\cdot g)(—,r)=\begin{cases}\bigvee_{s>0} f(—,s)\wedge g(—,\tfrac{r}{s}) & \text{if } r> 0\\ 0 & \text{if } r\le 0.\end{cases}$$

(2) $(f\cdot g)(p,q)=\bigvee\{f(r,s)\wedge g(t,u)\mid \langle r,s\rangle\cdot\langle t,u\rangle\subseteq\langle p,q\rangle\}$.

Proof: (1) For each rational r, we have $(f\cdot g)(r,—)=\bigvee_{s>r}\sigma_{f\cdot g}(s)$ and so

$$(f\cdot g)(r,—)=\begin{cases}\bigvee_{s>r}\bigvee_{t\in\mathbb{Q}} f(t,—)\wedge g(\tfrac{s}{t},—)=\bigvee_{s>0} f(s,—)\wedge g(\tfrac{r}{s},—) & \text{if } r\ge 0,\\ 1 & \text{if } r<0.\end{cases}$$

On the other hand, $(f\cdot g)(—,r)=\bigvee_{s<r}\sigma_{f\cdot g}(s)^*$ for each r. Hence $(f\cdot g)(—,r)=0$ if $r\le 0$. In case $r>0$, then for each $0<s<r$, it follows from Proposition 4 that $\sigma_{f\cdot g}(s)\vee\bigvee_{t>0} f(—,t)\wedge g(—,\tfrac{r}{t})=1$ and so $\sigma_{f\cdot g}(s)^*\le\bigvee_{t>0} f(—,t)\wedge g(—,\tfrac{r}{t})$. Hence

$$(f\cdot g)(—,r)=\bigvee_{0<s<r}\sigma_{f\cdot g}(s)^*\le\bigvee_{t>0} f(—,t)\wedge g(—,\tfrac{r}{t}).$$

Moreover

$$\begin{aligned}\bigvee_{t>0} f(-,t)\wedge g(-,\tfrac{r}{t}) &= \bigvee_{t>0}\bigvee_{0<s<r} f(-,t)\wedge g(-,\tfrac{s}{t})\\ &= \bigvee_{0<s<r}\bigvee_{t>0} f(-,t)\wedge g(-,\tfrac{s}{t}) \le \bigvee_{0<s<r}\sigma_{f\cdot g}(s)^* = (f\cdot g)(-,r)\end{aligned}$$

and hence

$$(f\cdot g)(-,r) = \bigvee_{t>0} f(-,t)\wedge g(-,\tfrac{r}{t}).$$

(2) Let $p,q\in\mathbb{Q}$ with $0\le p<q$ (the case $p<0$ is similar). Then

$$\begin{aligned}\sigma_{f\cdot g}(p)\wedge\sigma_{f\cdot g}(q) &= \bigvee_{r,s>0}\big(f(r,-)\wedge g(\tfrac{p}{r},-)\wedge f(-,s)\wedge g(-,\tfrac{q}{s})\big)\\ &= \bigvee\{(f(r,s)\wedge g(\tfrac{p}{r},\tfrac{q}{s})\mid 0<r<s, 0\le\tfrac{p}{r}<\tfrac{q}{s}\}\\ &\le \bigvee\{f(r,s)\wedge g(t,u)\mid \langle r,s\rangle\cdot\langle t,u\rangle\subseteq\langle p,q\rangle\}\end{aligned}$$

since $\langle r,s\rangle\cdot\langle\frac{p}{r},\frac{q}{s}\rangle=\langle p,q\rangle$ for $0<r<s$ and $0\le\frac{p}{r}<\frac{q}{s}$. Conversely, if $\langle r,s\rangle\cdot\langle t,u\rangle\subseteq\langle p,q\rangle$ then either $s,u<0$ or $r,t>0$. If $s,u<0$, then $f(r,s)\wedge g(t,u)=0$; on the other hand, if $r,t>0$ we have that $\langle r,s\rangle\cdot\langle t,u\rangle=\langle rt,su\rangle\subseteq\langle p,q\rangle$ and so $p\le rt$ and $q\ge su$. Hence

$$f(r,s)\wedge g(t,u)\le f(r,s)\wedge g(\tfrac{p}{r},\tfrac{q}{s})\le\bigvee_{0<r,s}\big(f(r,s)\wedge g(\tfrac{p}{r},\tfrac{q}{s})\big)=\sigma_{f\cdot g}(p)\wedge\sigma_{f\cdot g}(q).\quad\blacksquare$$

Finally, we have

Theorem 2. *Let* $\mathbf{0}\le f,g\in\overline{\mathsf{C}}(L)$. *The map* $\sigma_{f\cdot g}:\mathbb{Q}\to L$ *defined by*

$$\sigma_{f\cdot g}(r)=\bigvee_{s>0} f(s,-)\wedge g(\tfrac{r}{s},-)\ \ (r\ge 0),\qquad \sigma_{f\cdot g}(r)=1\ \ (r<0),$$

is an extended scale of L *if and only if* f *and* g *are product compatible.*

Proof: Sufficiency follows from Proposition 4. For necessity, it follows from Proposition 5 (1) that

$$\begin{aligned}a^+_{f\cdot g} &= \bigvee_{r\in\mathbb{Q}}\bigvee_{s>0} f(-,s)\wedge g(-,\tfrac{r}{s}) \;=\; \bigvee_{s>0}\bigvee_{r\in\mathbb{Q}} f(-,s)\wedge g(-,\tfrac{r}{s})\\ &= \bigvee_{s>0} f(-,s)\wedge a^+_g \;=\; a^+_f\wedge a^+_g \;=\; a_f\wedge a_g\end{aligned}$$

and

$$\operatorname{coz}(f\cdot g) \;=\; (f\cdot g)(0,-) \;=\; f(0,-)\wedge g(0,-) \;=\; \operatorname{coz} f\wedge\operatorname{coz} g.$$

Hence $1=a^+_{f\cdot g}\vee\operatorname{coz}(f\cdot g)=\big(a_f\wedge a_g\big)\vee\big(\operatorname{coz} f\wedge\operatorname{coz} g\big)$. $\blacksquare$

Corollary. *Let $\mathbf{0} \le f, g \in \overline{\mathsf{F}}(L)$ be product compatible. Then $f \cdot g \in \overline{\mathsf{F}}(L)$. Furthermore, if $f, g \in \overline{\mathsf{C}}(L)$ (resp. $\overline{\mathsf{LSC}}(L)$, resp. $\overline{\mathsf{USC}}(L)$) then $f \cdot g \in \overline{\mathsf{C}}(L)$ (resp. $\overline{\mathsf{LSC}}(L)$, resp. $\overline{\mathsf{USC}}(L)$).* ■

Remark 16. For any $f \in \overline{\mathsf{C}}(L)$ and $r \neq 0$ in $\mathbb{Q}$, f and $\mathbf{r}$ are product compatible and $\mathbf{r} \cdot f = r \cdot f$ for $r > 0$ as defined in Proposition 1.

On the other hand, f and $\mathbf{0}$ are product compatible iff $f \in \mathsf{C}(L)$ and then, of course, $\mathbf{0} \cdot f = \mathbf{0}$. We omit the details.

6. Almost real functions

To begin with, recall that for any frame L,

(1) $a \in L$ is called *dense* if $a^* = 0$ or, equivalently, $a^{**} = 1$, and
(2) L is called *extremally disconnected* if it satisfies the Stone identity $a^* \vee a^{**} = 1$ for each $a \in L$.

Obviously, the latter means that the sublattice $BL = \{a \in L \mid a \vee a^* = 1\}$ of complemented elements of L coincides with the Boolean frame

$$\mathfrak{B}L = \{a \in L \mid a = a^{**}\}$$

of L, called the Booleanization of L [7]. Regarding the latter, the map $\beta_L : L \to \mathfrak{B}L$, $a \mapsto a^{**}$, is a dense homomorphism (that is, $\beta_L(a) = 0$ implies $a = 0$), and up to isomorphism the unique such homomorphism with Boolean image.

Now, for any frame L, let

$$\mathsf{D}(L) = \left\{ f \in \overline{\mathsf{C}}(L) \mid a_f \text{ is dense} \right\}.$$

Note that this definition extends a familiar classical notion to the pointfree setting. For any space X, recall that $\mathsf{D}(X)$ is the set of all extended real-valued continuous functions $u : X \to \overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with the usual topology, for which $u^{-1}[\mathbb{R}]$ is dense in X. Now, as remarked earlier, $\mathsf{Top}(X, \overline{\mathbb{R}}) \simeq \overline{\mathsf{C}}(\mathfrak{O}X)$ by the map

$$u \mapsto \widetilde{u}, \quad \widetilde{u}(r, \text{—}) = u^{-1}[\uparrow r] \quad \text{and} \quad \widetilde{u}(\text{—}, r) = u^{-1}[\downarrow r]$$

where

$$\uparrow r = \{x \in \overline{\mathbb{R}} \mid r < x\} \quad \text{and} \quad \downarrow r = \{x \in \overline{\mathbb{R}} \mid x < r\}.$$

Moreover, this map makes $\mathsf{D}(X)$ correspond exactly to the present $\mathsf{D}(\mathfrak{O}X)$: for $f = \widetilde{u}$,

$$\begin{aligned} a_f &= \bigvee\{f(r,\text{—}) \wedge f(\text{—},s) \mid r,s \in \mathbb{Q}\} = \bigvee\{u^{-1}[\uparrow\! r] \cap u^{-1}[\downarrow\! s] \mid r,s \in \mathbb{Q}\} \\ &= \bigvee\{u^{-1}[\,]\!]r,s[\![\,] \mid r,s \in \mathbb{Q}\} = u^{-1}[\mathbb{R}], \end{aligned}$$

where $]\!]\cdot,\cdot[\![$ stands for open interval in $\mathbb{R}$, showing that $u \in \mathsf{D}(X)$ iff $\widetilde{u} \in \mathsf{D}(\mathfrak{O}X)$.

Remark 17. $\mathsf{D}(L)$ is a (not bounded) sublattice with inversion of $\overline{\mathsf{C}}(L)$: all (non-extended) constant functions in $\overline{\mathsf{C}}(L)$ belong to $\mathsf{D}(L)$; $f \vee g$, $f \wedge g \in \mathsf{D}(L)$ for any $f, g \in \mathsf{D}(L)$ because

$$a_{f\vee g} = (a_f \wedge a_g^+) \vee (a_f^+ \wedge a_g) \quad \text{ and } \quad a_{f\wedge g} = (a_f \wedge a_g^-) \vee (a_f^- \wedge a_g);$$

further, $-f \in \mathsf{D}(L)$ for any $f \in \mathsf{D}(L)$ since $a_{-f} = a_f$.

Remark 18. Any $f \in \mathsf{D}(L)$ such that $a_f = f(\omega) = 1$ factors through the basic homomorphism $\varrho : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to \mathfrak{L}(\mathbb{R})$. In particular, for any Boolean L, each $f \in \mathsf{D}(L)$ factors through ϱ because, in that case, a_f is dense just means $a_f = 1$. Hence, for any Boolean L, the map $f \mapsto f\varrho$ from $\mathsf{C}(L)$ to $\overline{\mathsf{C}}(L)$ induces an isomorphism $\mathsf{C}(L) \to \mathsf{D}(L)$.

Remark 19. The correspondence $L \mapsto \mathsf{D}(L)$ is functorial for skeletal homomorphisms, that is, the $h : L \to M$ which take dense elements to dense elements: for any skeletal $h : L \to M$ and $f \in \mathsf{D}(L)$, $a_{hf} = hf(\omega) = h(a_f)$ is dense so that $hf \in \mathsf{D}(M)$.

Remark 20. Concerning the addition and multiplication in $\overline{\mathsf{C}}(L)$ of sum compatible, resp. product compatible, pairs, the result is not necessarily in $\mathsf{D}(L)$ for $f, g \in \mathsf{D}(L)$. But on the other hand, $\mathsf{D}(L)$ has its own sum and product for certain $f, g \in \mathsf{D}(L)$ which we describe next.

For any dense $a \in L$, the homomorphism $\nu_a = (\cdot) \wedge a : L \to \downarrow\! a$ is skeletal and hence determines the map $\mathsf{D}(L) \to \mathsf{D}(\downarrow\! a)$, $f \mapsto \nu_a f$, which is one-one because ν_a is also dense and $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ is regular. Further, for any $f \in \mathsf{D}(L)$ such that $a_f \geq a$, $a_{\nu_a f} = \nu_a(a_f) = a_f \wedge a = a$ (the unit of $\downarrow\! a$) so that we have a factorization

$$\begin{tikzcd} \mathfrak{L}\left(\overline{\mathbb{R}}\right) \arrow[rr, "\nu_a f"] \arrow[dr, "\varrho"'] & & \downarrow\! a \\ & \mathfrak{L}(\mathbb{R}) \arrow[ur, "f_a"'] & \end{tikzcd}$$

as noted earlier (Remark 18), where $f_a(r,\text{—}) = f(r,\text{—}) \wedge a$ and $f_a(\text{—},r) = f(\text{—},r) \wedge a$. In particular, for any $f,g \in \mathsf{D}(L)$, $a = a_f \wedge a_g$ is dense and then $f_a, g_a \in \mathsf{C}(\downarrow a)$. Now, if there exists $h \in \mathsf{D}(L)$ such that $a_h \geq a$ and $h_a = f_a + g_a$ (resp. $h_a = f_a \cdot g_a$) in the usual ring structure of $\mathsf{C}(\downarrow a)$ then this will be unique and we put $h = f + g$ (resp. $h = f \cdot g$), referring to the operations given this way as the *partial addition* (resp. *partial multiplication*) of $\mathsf{D}(L)$.

In conclusion, for $\diamond = +, \cdot$, the partial operation $\diamond$ on $\mathsf{D}(L)$ is defined for all pairs $f,g \in \mathsf{D}(L)$ for which

> *there exists $h \in \mathsf{D}(L)$ such that $a_h \geq a_f \wedge a_g$ and $h_{a_f\wedge a_g} = f_{a_f\wedge a_g} \diamond g_{a_f\wedge a_g}$ in $\mathsf{C}(\downarrow(a_f \wedge a_g))$.*

Note that these $f+g$ or $f\cdot g$ may well be defined for some $f,g \in \mathsf{D}(L)$ which are not sum (resp. product) compatible in $\overline{\mathsf{C}}(L)$. Thus, for any $f \in \mathsf{D}(L)$, $a_f = a_{-f}$ and since $f_a + (-f)_a = \mathbf{0}_a$ for $a = a_f$ it follows that $f + (-f) = \mathbf{0}$ in the partial addition of $\mathsf{D}(L)$, in contrast with the earlier observation (Remark 13) that f and $-f$ are sum compatible for $f \in \overline{\mathsf{C}}(L)$ iff $f \in \mathsf{C}(L)$. Similarly, $\mathbf{0} \cdot f = \mathbf{0}$ in the partial multiplication of $\mathsf{D}(L)$ whereas f and $\mathbf{0}$ are product compatible in $\overline{\mathsf{C}}(L)$ again iff $f \in \mathsf{C}(L)$, as noted earlier (Remark 16).

Theorem 3. *For any L, there exists an inversion lattice embedding $\delta_L : \mathsf{D}(L) \to \mathsf{C}(\mathfrak{B}L)$ such that*

$$\delta_L(f)(r,\text{—}) = f(r,\text{—})^{**} \qquad \text{and} \qquad \delta_L(f)(\text{—},r) = f(\text{—},r)^{**}$$

which preserves the partial addition and multiplication of $\mathsf{D}(L)$.

Moreover, δ_L is onto if and only if L is extremally disconnected and then the partial operations are total so that δ_L is a lattice-ordered ring isomorphism.

Proof: By what was noted earlier $\beta_L : L \to \mathfrak{B}L$, being skeletal induces a map

$$\mathsf{D}(L) \to \mathsf{D}(\mathfrak{B}L), \quad f \mapsto \beta_L f,$$

and because $\mathfrak{B}L$ is Boolean there is an isomorphism

$$\mathsf{D}(\mathfrak{B}L) \to \mathsf{C}(\mathfrak{B}L), \quad h \mapsto h^{\#},$$

such that $h = h^{\#}\varrho$. Next, since β_L is also dense, $f \mapsto \beta_L f$ is one-one by regularity, and hence the composite

$$\delta_L : \mathsf{D}(L) \to \mathsf{C}(\mathfrak{B}L), \quad f \mapsto (\beta_L f)^{\#},$$

is one-one. Further, given the nature of $\varrho : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to \mathfrak{L}(\mathbb{R})$,

$$\delta_L(f)(r,\text{—}) = (\beta_L f)^{\#}(r,\text{—}) = (\beta_L f)^{\#}\varrho(r,\text{—}) = \beta_L f(r,\text{—}) = f(r,\text{—})^{**}$$

and analogously for $(\text{—}, r)$, as claimed. Finally, since either of the two factors of δ_L is an inversion lattice homomorphism the same holds for δ_L.

Now, for any $f, g \in \mathsf{D}(L)$ such that $h = f + g$ is defined, if $a = a_f \wedge a_g$ then $h_a = f_a + g_a$ in $\mathsf{C}(\downarrow a)$ as described above. Further, let $\beta_L^{(a)} : \downarrow a \to \mathfrak{B}L$ be the map induced by β_L and hence such that $\beta_L = \beta_L^{(a)}\nu_a$, given by the fact that $\beta_L(a) = a^{**} = 1$. Then, for any $k = f, g$ or h,

$$\beta_L^{(a)} k_a \varrho = \beta_L^{(a)} \nu_a k = \beta_L k = \delta_L(k)\varrho$$

so that $\beta_L^{(a)} k_a = \delta_L(k)$ since ϱ is onto. Finally, given that $\beta_L^{(a)} h_a = \beta_L^{(a)} f_a + \beta_L^{(a)} g_a$ because $h_a = f_a + g_a$, this shows $\delta_L(h) = \delta_L(f) + \delta_L(g)$, and the same argument obviously applies to the multiplication.

Concerning the second part of the proposition, let δ_L be onto. Now, as $\mathfrak{B}L$ is Boolean, any $a \in \mathfrak{B}L$ determines its characteristic function $\chi_a \in \mathsf{C}(\mathfrak{B}L)$, given by the scale σ such that

$$\sigma(r) = 1 \;\; (r < 0), \quad \sigma(r) = a \;\; (0 \le r < 1), \quad \sigma(r) = 0 \;\; (r \ge 1).$$

Then, immediately, $\mathbf{0} \le \chi_a \le \mathbf{1}$, and, as is familiar, $(\chi_a)^2 = \chi_a$. Next, if $h \in \mathsf{D}(L)$ such that $\delta_L(h) = \chi_a$ by hypothesis then also $\mathbf{0} \le h \le \mathbf{1}$ (by the obvious fact that $\delta_L(\mathbf{r}) = \mathbf{r}$ for any $r \in \mathbb{Q}$) so that h is bounded. Hence h^2 is defined and then $\delta_L(h^2) = \delta_L(h)^2$ readily implies that $h^2 = h$, again by the nature of δ_L. Now, given that h is bounded it factors through $\mathfrak{L}(\mathbb{R})$ so that $h = k\varrho$ for some $k \in \mathsf{C}(L)$ and consequently $\beta_L k = \chi_a$ by canceling ϱ. Further, $k^2 = k$ because β_L is dense and hence $\operatorname{coz}(k)$ is complemented, with complement $\operatorname{coz}(\mathbf{1} - k)$ by the familiar rules concerning the coz map. Further $a = \operatorname{coz}(\chi_a)$ since χ_a is the characteristic function of a on $\mathsf{C}(\mathfrak{B}L)$ and therefore

$$a = \operatorname{coz}(\chi_a) = \operatorname{coz}(\beta_L k) = \beta_L(\operatorname{coz}(k)) = \operatorname{coz}(k)^{**} = \operatorname{coz}(k),$$

showing that any $a \in \mathfrak{B}L$ is complemented in L, that is, L is extremally disconnected.

Conversely, if L is extremally disconnected then, for any $h \in \mathsf{C}(\mathfrak{B}L)$, $\sigma : \mathbb{Q} \to L$, $r \mapsto h(r,\text{—})$, is an extended scale in L, being obviously antitone with each $\sigma(r)$ complemented in L by extremal disconnectedness. Hence by Lemma 1 we have $f \in \overline{\mathsf{C}}(L)$ such that

$$f(r,\text{—}) = \bigvee\{\sigma(s) \mid s > r\} = \bigvee\{h(s,\text{—}) \mid s > r\} = h(r,\text{—})$$

and

$$f(\text{—},r) = \bigvee\{\sigma(s)^* \mid s < r\} = \bigvee\{h(s,\text{—})^* \mid s < r\}$$

for which $a_f^+ = \bigvee\{h(s,\text{—})^* \mid s \in \mathbb{Q}\}$ and $a_f^- = \bigvee\{h(s,\text{—}) \mid s \in \mathbb{Q}\}$. Now, given that $h \in \mathsf{C}(\mathfrak{B}L)$ and join in $\mathfrak{B}L$ is $(\bigvee -)^{**}$ in L,

$$(\bigvee\{h(\text{—},s)^* \mid s \in \mathbb{Q}\})^{**} = 1 = (\bigvee\{h(s,\text{—})^* \mid s \in \mathbb{Q}\})^{**}$$

where $(a_f^+)^{**}$ is above the first element, $(a_f^-)^{**}$ equal to the last, showing that in fact $f \in \mathsf{D}(L)$. Further,

$$f(r,\text{—})^{**} = h(r,\text{—}) \quad \text{and} \quad f(\text{—},r)^{**} = h(\text{—},r)$$

where the first part is obvious and the second results from

$$(\bigvee\{h(\text{—},s) \mid s < r\})^{**} = h(\text{—},r)$$

and

$$h(\text{—},r) \geq h(s,\text{—})^* \geq h(\text{—},s)$$

for $r > s$. In all, then, $f \in \mathsf{D}(L)$ and $\delta_L(f) = h$, showing that δ_L is onto.

Next, the latter fact has the immediate consequence that, for any dense $a \in L$ and $h \in \mathsf{C}(\downarrow a)$, there exists $k \in \mathsf{D}(L)$ for which $\nu_a k = h\varrho$: since $\beta_L^{(a)} h \in \mathsf{C}(\mathfrak{B}L)$ there exists $k \in \mathsf{D}(L)$ such that $\delta_L(k) = \beta_L^{(a)} h$ and then

$$\big(\beta_L^{(a)} h\big)\varrho = \delta_L(k)\varrho = \beta_L k = \beta_L^{(a)} \nu_a k$$

showing that $h\varrho = \nu_a k$ because $\beta_L^{(a)}$ is dense. Now, this in turn can be used to see that the partial addition and multiplication of $\mathsf{D}(L)$ are in fact total. For any $f, g \in \mathsf{D}(L)$, take $a = a_f \wedge a_g$ and the corresponding $f_a, g_a \in \mathsf{C}(\downarrow a)$ as described earlier. Then, taking the case of the addition, there exists $k \in \mathsf{D}(L)$ such that $\nu_a k = (f_a + g_a)\varrho$ by what has just been shown; further, since $f_a + g_a \in \mathsf{C}(\downarrow a)$ we also have $a = a_{\nu_a k} = \nu_a(a_k) = a \wedge a_k$ so that $a \leq a_k$ and hence $k = f + g$ by the definition of $+$. Of course, the argument for the product $f \cdot g$ is exactly the same, and in all this proves the final part of the theorem. ∎

In particular, for extremally disconnected L, the isomorphism $\mathsf{D}(L) \cong \mathsf{C}(\mathfrak{B}L)$ shows, by familiar facts concerning the functor $\mathsf{C}(\cdot)$ ([4], [6]), that $\mathsf{D}(L)$ becomes an order-complete archimedean f-ring with unit.

The above $\delta_L : \mathsf{D}(L) \to \mathsf{C}(\mathfrak{B}L)$ is actually the composite of two separate maps, each with a certain interest of its own, namely

$$\varphi_L : \mathsf{D}(L) \to \varinjlim_{a\in \Delta L} \mathsf{C}(\downarrow a) \quad \text{and} \quad \tau_L : \varinjlim_{a\in \Delta L} \mathsf{C}(\downarrow a) \to \mathsf{C}(\mathfrak{B}L),$$

where ΔL is the filter of all dense $a \in L$ and τ_L is the obvious map determined by the embeddings

$$\mathsf{C}(\downarrow a) \to \mathsf{C}(\mathfrak{B}L), \quad h \mapsto \beta_L^{(a)} h \quad (a \in \Delta L)$$

and the connecting maps

$$\mathsf{C}(\downarrow a) \to \mathsf{C}(\downarrow b), \quad h \mapsto h(\cdot) \wedge b \quad (a \geq b \text{ in } \Delta L)$$

while φ_L, more elaborately, results as follows: If $D_a(L) = \{f \in \mathsf{D}(L) \mid a_f \geq a\}$ for each $a \in \Delta L$ then $D_a(L) \subseteq D_b(L)$ whenever $a \geq b$ and $\mathsf{D}(L) = \bigcup\{D_a(L) \mid a \in \Delta L\}$, saying that $\mathsf{D}(L) = \varinjlim_{a\in \Delta L} D_a(L)$, given that ΔL is a filter. On the other hand, as noted earlier, any $f \in D_a(L)$ determines $f_a \in \mathsf{C}(\downarrow a)$ such that $\nu_a f = f_a \varrho$ for the familiar $\varrho : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to \mathfrak{L}(\mathbb{R})$ and $f \mapsto f_a$ then provides an embedding $D_a(L) \to \mathsf{C}(\downarrow a)$ for each $a \in \Delta L$, evidently compatible with the identical embeddings $D_a(L) \to D_b(L)$ and the connecting maps $\mathsf{C}(\downarrow a) \to \mathsf{C}(\downarrow b)$ for $a \geq b$. As a result, these $f \mapsto f_a$ induce a map $\varphi_L : \mathsf{D}(L) \to \varinjlim_{a\in \Delta L} \mathsf{C}(\downarrow a)$ such that $\tau_L \varphi_L$ takes any $f \in D_a(L)$ to $\beta_L^{(a)} f_a$ and since

$$\beta_L^{(a)} f_a \varrho = \beta_L^{(a)} \nu_a f = \beta_L f = \delta_L(f)\varrho$$

it follows that $\tau_L \varphi_L = \delta_L$.

Now we have, as a consequence of the present theorem:

Corollary. *For any extremally disconnected L, φ_L and τ_L are isomorphisms.*

Proof: Since $\delta_L = \tau_L \varphi_L$ is an isomorphism here it is enough to show the same for one of these factors, and we do that for φ_L. Now, this is evidently one-one since δ_L is and hence it only has to be verified that it is onto, and by the properties of updirected colimits this is saying that, for each $a \in \Delta L$ and $h \in \mathsf{C}(\downarrow a)$ there exists $f \in D_a(L)$ for which $f_a = h$. Now, by the proof of the theorem, there exists $f \in \mathsf{D}(L)$ such that $\nu_a f = h\varrho$ and hence

$a \wedge a_f = \nu_a f(\omega) = h\varrho(\omega) = a$, the top of $\downarrow a$. Thus $a \le a_f$ so that $f \in D_a(L)$, and since $\nu_a f = f_a \varrho$ this shows $f_a = h$. ■

We end with a characterization of the frames L where the partial operations on $\mathsf{D}(L)$ are indeed total. For that we need a couple of lemmas.

Lemma 4. *For each* $f \in \overline{\mathsf{C}}(L)$, $a_f \in \operatorname{Coz} L$.

Proof: As described in Remark 2, using any order isomorphism $\varphi : \mathbb{Q} \to \{r \in \mathbb{Q} \mid 0 < r < 1\}$ one obtains an isomorphism

$$\Phi : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to \mathfrak{L}[0,1] = \uparrow((-,0) \vee (1,-)) \subseteq \mathfrak{L}(\mathbb{R})$$

such that

$$\Phi(r,-) = \nu(\varphi(r),-), \quad \Phi(-,r) = \nu(-,\varphi(r))$$

where $\nu : \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}[0,1]$ is the usual quotient map. In particular, then, for $\omega = \bigvee\{(r,-) \wedge (-,s) \mid r < s \text{ in } \mathbb{Q}\}$,

$$\begin{aligned}\Phi(\omega) &= \bigvee\{\nu(\varphi(r),\varphi(s)) \mid r < s \text{ in } \mathbb{Q}\} = \\ &= \nu(\bigvee\{(p,q) \mid 0 < p < q < 1\} = \nu(0,1),\end{aligned}$$

the second step by the nature of φ. Consequently, for any $f \in \overline{\mathsf{C}}(L)$, $a_f = f(\omega) = \widetilde{f}(0,1)$ where $\widetilde{f} = f\left(\Phi^{-1}\right)\nu \in \mathsf{C}(L)$ and hence

$$a_f = \operatorname{coz}\left(\left(\widetilde{f}\right)^+ \wedge \left(\mathbf{1} - \widetilde{f}\right)^+\right)$$

by the properties of coz. ■

Recall from [1] that an onto frame homomorphism $\kappa : L \to M$ is called a C^*-*quotient map* if for each $f \in \mathsf{C}^*(M)$ (that is, each bounded $f \in \mathsf{C}(M)$) there exists $\widetilde{f} \in \mathsf{C}(L)$ such that $\kappa \widetilde{f} = f$. Similarly, we say that an onto frame homomorphism $\kappa : L \to M$ is a $\overline{\mathsf{C}}$-*quotient map* if for each $f \in \overline{\mathsf{C}}(M)$ there exists a frame homomorphism $\overline{f} : \mathfrak{L}[0,1] \to L$ such that the diagram

$$\begin{tikzcd}
L \arrow[r, "\kappa"] & M \\
\mathfrak{L}[0,1] \arrow[u, "\overline{f}"] & \mathfrak{L}\left(\overline{\mathbb{R}}\right) \arrow[l, "\Phi"] \arrow[u, "f"']
\end{tikzcd}$$

commutes. We have:

Lemma 5. *Any dense* C^*-*quotient map is a* $\overline{\mathsf{C}}$-*quotient map.*

Proof: Consider the diagram

$$\begin{tikzcd} L \arrow[rr, "\kappa"] & & M \\ \mathfrak{L}(\mathbb{R}) \arrow[u, "\widetilde{f}"] \arrow[r, "\nu"'] & \mathfrak{L}[0,1] \arrow[ul, "\overline{f}"'] & \mathfrak{L}\left(\overline{\mathbb{R}}\right) \arrow[u, "f"'] \arrow[l, "\Phi"] \end{tikzcd}$$

where κ is the quotient map involved, f arbitrary, Φ and ν as before, and $\kappa\widetilde{f} = f\left(\Phi^{-1}\right)\nu$ by hypothesis as the latter is bounded. Then

$$\begin{aligned} \kappa\widetilde{f}((\text{—},0)\vee(1,\text{—})) &= f(\Phi^{-1})\nu((\text{—},0)\vee(1,\text{—})) = \\ &= f(\Phi^{-1})((\text{—},0)\vee(1,\text{—})) = f(0) = 0 \end{aligned}$$

so that $\widetilde{f}((\text{—},0)\vee(1,\text{—})) = 0$ because κ is dense, and therefore $\widetilde{f} = \overline{f}\nu$. Further, $\kappa\overline{f}\nu = f\left(\Phi^{-1}\right)\nu$, hence $\kappa\overline{f} = f\left(\Phi^{-1}\right)$ and finally $f = \kappa\overline{f}\Phi$. ■

Recall also from [1] that a completely regular frame L is coined *quasi-F* if for every dense $a \in \operatorname{Coz} L$, the open quotient map $\nu_a : L \to {\downarrow}a$ is a C^*-quotient map. Each extremally disconnected frame is quasi-F [1] (for more information on quasi-F frames see [1] or [8]). Finally, we conclude:

Proposition 6. *The following are equivalent for a completely regular frame L:*

(i) *L is quasi-F.*
(ii) *The partial addition in $\mathsf{D}(L)$ is total.*
(iii) *The partial multiplication in $\mathsf{D}(L)$ is total.*

Proof: (i) ⇒ (iii): Let L be a quasi-F frame and consider arbitrary $f, g \in \mathsf{D}(L)$. By Lemma 4, $a_f = \operatorname{coz}\widetilde{f}$ and $a_g = \operatorname{coz}\widetilde{g}$ for some $\widetilde{f}, \widetilde{g} \in \mathsf{C}(L)$ and therefore, by the well-known properties of cozero elements, the dense element $a = a_f \wedge a_g = \operatorname{coz}\widetilde{f} \wedge \operatorname{coz}\widetilde{g} = \operatorname{coz}\left(\widetilde{f}\cdot\widetilde{g}\right)$ is also a cozero element. Hence, by the hypothesis, $\nu_a : L \to {\downarrow}a$ is a C^*-quotient map. Take the $f_a, g_a \in \mathsf{C}({\downarrow}a)$ as described earlier. Then we have $f_a \cdot g_a \in \mathsf{C}({\downarrow}a)$ and $(f_a \cdot g_a)\varrho \in \overline{\mathsf{C}}({\downarrow}a)$. Now, since $\nu_a : L \to {\downarrow}a$ is a $\overline{\mathsf{C}}$-quotient map by Lemma 5, $(f_a \cdot g_a)\varrho = \nu_a h$ for some $h : \mathfrak{L}\left(\overline{\mathbb{R}}\right) \to L$. Then $a_h \wedge a = \nu_a h(\omega) = (f_a \cdot g_a)\varrho(\omega) = (f_a \cdot g_a)(1) = a$ so that $a \leq a_h$ and hence $\nu_a h = h_a\varrho$ for $h_a \in \mathsf{C}({\downarrow}a)$. Finally, $h_a = f_a \cdot g_a$ since $h_a\varrho = (f_a \cdot g_a)\varrho$, and given that $a_f \wedge a_g = a \leq a_h$ it follows that $h = f \cdot g$.

(iii) $\Rightarrow$ (i): Let $a \in \operatorname{Coz} L$ be dense and $g \in \mathsf{C}^*(\downarrow a)$ (of course, we may assume without loss of generality that $\mathbf{0} \leq g \leq \mathbf{1}$). Then there exists $f \in \mathsf{C}(L)$ (here again we may assume that $\mathbf{0} \leq f \leq \mathbf{1}$) such that $\operatorname{coz} f = f(0, \text{—}) = a$. Set

$$\sigma_1(r) = 1 \ \ (r < 1), \quad \sigma_1(r) = f\left(\text{—}, \tfrac{1}{r}\right) \ \ (r \geq 1) \quad \text{ and}$$

$$\sigma_2(r) = 1\,(r<0),\ \sigma_2(r) = \bigvee_{r<s<1} f(s,\text{—}) \wedge g(\tfrac{r}{s},\text{—})\,(0 \leq r < 1),\ \sigma_2(r) = 0\,(r \geq 1).$$

σ_1 is clearly an extended scale in L since

$$\sigma_1(r) \vee \sigma_1(s)^* = f\left(\text{—}, \tfrac{1}{r}\right) \vee f\left(\text{—}, \tfrac{1}{s}\right)^* \geq f\left(\text{—}, \tfrac{1}{r}\right) \vee f\left(\tfrac{1}{s}, \text{—}\right) = 1$$

for any $1 \leq r < s$. Applying Lemma 1, it generates $h_1 \in \overline{\mathsf{C}}(L)$ given by

$$\begin{aligned} h_1(r,\text{—}) &= \bigvee_{s>r} \sigma_1(s) = \begin{cases} 1 & \text{if } r < 1 \\ \bigvee_{r<s} f\left(\text{—}, \tfrac{1}{s}\right) & \text{if } r \geq 1 \end{cases} = \begin{cases} 1 & \text{if } r < 1 \\ f\left(\text{—}, \tfrac{1}{r}\right) & \text{if } r \geq 1 \end{cases} \\ h_1(\text{—}, r) &= \bigvee_{s<r} \sigma_1(s)^* = \begin{cases} 0 & \text{if } r \leq 1 \\ \bigvee_{1<s<r} f\left(\text{—}, \tfrac{1}{s}\right)^* = f\left(\tfrac{1}{r}, \text{—}\right) & \text{if } r > 1. \end{cases} \end{aligned}$$

Moreover, $a_{h_1} = \bigvee_{r>1} h_1(\text{—}, r) = \bigvee_{r>1} f\left(\tfrac{1}{r}, \text{—}\right) = f(0, \text{—}) = a$, hence $h_1 \in \mathsf{D}(L)$.

On the other hand, σ_2 is also an extended scale in L. Indeed, it can be checked in a way similar to the proof in Proposition 4 (the proof now becomes slightly simpler because both f and g are bounded) that $\sigma_2(r) \vee \sigma_2(s)^* = 1$ for each $0 \leq r < s < 1$. Therefore, σ_2 generates an $h_2 \in \overline{\mathsf{C}}(L)$, given by

$$\begin{aligned} h_2(r,\text{—}) &= \bigvee_{s>r} \sigma_2(s) = \begin{cases} 1 & \text{if } r < 0 \\ \bigvee_{r<s<1} f(s,\text{—}) \wedge g(\tfrac{r}{s},\text{—}) & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \\ h_2(\text{—}, r) &= \bigvee_{s<r} \sigma_2(s)^* = \begin{cases} 0 & \text{if } r \leq 0 \\ \bigvee_{s>0} f(\text{—}, s) \wedge g\left(\text{—}, \tfrac{r}{s}\right) & \text{if } 0 < r \leq 1 \\ 1 & \text{if } r > 1. \end{cases} \end{aligned}$$

Since $\mathbf{0} \leq h_2 \leq \mathbf{1}$, then $a_{h_2} = 1$ and hence $h_1 \in \mathsf{D}(L)$.

Now we know, by the hypothesis that the product of h_1 and h_2 exists in $\mathsf{D}(L)$, that there is an $h \in \mathsf{D}(L)$ such that $a_h \geq a$ and $h_a = (h_1)_a \cdot (h_2)_a$ in $\mathsf{C}(\downarrow a)$. Since $a_{(h\wedge\mathbf{1})\vee\mathbf{0}} = ((h \wedge \mathbf{1}) \vee \mathbf{0})(\omega) = 1$, there exists $\widetilde{g} \in \mathsf{C}(L)$ (recall Remark 18) such that $\widetilde{g}\,\varrho = (h \wedge \mathbf{1}) \vee \mathbf{0}$. Then $\nu_a\,\widetilde{g}(r,\text{—}) = g(r,\text{—})$ for every $r \in \mathbb{Q}$, as can be easily checked, and thus ν_a is a C^*-quotient map.

The equivalence (i) $\Leftrightarrow$ (ii) can be proved in a similar way. ■

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