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A NOTE ON EFFECTIVE DESCENT MORPHISMS OF TOPOLOGICAL SPACES AND RELATIONAL ALGEBRAS

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Dedicated to Eraldo Giuli on the occasion of his seventieth birthday

ABSTRACT: We formulate two open problems related to and, in a sense, suggested by the Reiterman-Tholen characterization of effective descent morphisms of topological spaces.

KEYWORDS: monad, effective descent morphism, relational algebra, ultrafilter monad.
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Introduction

Apart from Grothendieck's original presentations, Grothendieck descent theory has been described in several old and recent survey articles that also mention various results obtained in the last twenty years (see e.g. [13] and [12]). Applying this theory to a concrete category $\mathbf{C}$, one needs to find a sufficiently large class of effective descent morphisms in $\mathbf{C}$, or, better, to give a complete characterization of effective descent morphisms in $\mathbf{C}$. Restricting ourselves to what was called *global descent* in [13], and using a result of J. Bénabou and J. Roubaud [2] (also mentioned in [13] and [12]), the effective descent morphisms can be defined as follows:

Definition 0.1. A morphism $f : X \to Y$ in a category $\mathbf{C}$ with pullbacks is an effective descent morphism if and only the pullback functor $f^* : (\mathbf{C} \downarrow Y) \to (\mathbf{C} \downarrow X)$ is monadic.

When $\mathbf{C} = \mathbf{Top}$ is the category of topological spaces, the characterization problem is very hard. And although it was solved by J. Reiterman and W. Tholen [15] almost twenty years ago, the solution suggested, in a sense, further open problems to be solved in order to understand it fully. In particular, it is important to consider topological spaces as "generalized preorders" and as relational algebras over the ultrafilter monad in the sense of M. Barr [1],

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and then to see what can be done for an arbitrary monad (on the category of sets). The aim of the present paper is to introduce two of these open problems, formulated below as Problem 2.2 and Problem 2.4, and to explain that in the case of topological spaces some partial solutions of Problem 2.2 easily follow from the known results.

Apart from Introduction the paper contains three sections: Section 1 briefly recalls the relationship between preorders, relational algebras, and topological spaces. In particular it recalls M. Barr's result saying that topological spaces defined via ultrafilter convergence are the same as relational algebras over the ultrafilter monad. (A more complete overview would include some material from [3] and related papers.) Our open problems are formulated in Section 2 in the context of relational algebras, while the context of topological spaces is considered in Section 3.

Throughout the paper $T = (T, \eta, \mu)$ will always denote a non-trivial monad on the category **Sets** (of sets). Non-triviality means that there exists at least one T-algebra with at least two elements in its underlying set. Recall that non-triviality is equivalent to the injectivity of all $\eta_X : X \to T(X)$ ($X \in \mathbf{Sets}$).

1. T-Preorders are the same as relational T-algebras

A preorder, that is, a set equipped with a reflexive and transitive relation, can equivalently be defined as a small category whose domain map and codomain map are jointly monic. In the same way, but using the notion of T-*category* in the sense of A. Burroni [4] instead, one can define a T-*preorder*, which will then become nothing but a *relational T-algebra* in the sense of M. Barr [1]. These relational algebras are also special cases of reflexive and transitive lax algebras in the sense of [7] and of more special *reflexive and transitive (T, V)-algebras*, also called (T, V)-*categories*, in the sense of [10]. The formal definition is:

Definition 1.1. (a) A relational T-algebra (or a T-preorder) is a pair (X, R), in which X is a set and $R : T(X) \to X$ is a relation with

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & T(X) \arrow[d, "R"] \arrow[l, phantom, "\subseteq", shift right=6] \arrow[dr, phantom, "\subseteq"] & T^2(X) \arrow[l, "T(R)"'] \arrow[d, "\mu_X"] \\
 & X & T(X); \arrow[l, "R"]
\end{tikzcd}$$

that is, R is reflexive in the sense that $(\eta_X(x), x) \in R$ for each $x \in X$, and R is transitive in the sense that

$$((u,t) \in T(R) \ \& \ (t,x) \in R) \ \Rightarrow \ (\mu_X(u), x) \in R$$

for $u \in T^2(X)$, $t \in T(X)$, and $x \in X$.

(b) A homomorphism $f : (X,R) \to (Y,S)$ of relational T-algebras is a map $f : X \to Y$ with

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "R"'] \arrow[dr, phantom, "\subseteq"] & T(Y) \arrow[d, "S"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

The category of T-preorders and their homomorphisms will be denoted by $\mathbf{RelAlg}(T)$.

We will usually write $t \to x$ instead of $(t,x) \in R$, and even $u \to t$ instead of $(u,t) \in T(R)$. In this notation the conditions on R required in 1.1(a) become

$$(\forall x \in X)\ (\eta_X(x) \to x) \ \ \text{(reflectivity)}, \tag{1.1}$$

$$\begin{aligned} &(\forall u \in T^2(X))\ (\forall t \in T(X))\ (\forall x \in X) \\ &\qquad (u \to t \to x \ \Rightarrow \ \mu_X(u) \to x) \ \ \text{(transitivity)}, \end{aligned} \tag{1.2}$$

and the condition required in 1.1(b) becomes

$$(\forall t \in T(X))\ (\forall x \in X)\ (t \to x \ \Rightarrow \ T(f)(t) \to f(x)). \tag{1.3}$$

According to Definition 1.1(a), a relational T-algebra is a "generalized T-algebra", while homomorphisms between "ordinary" T-algebras in the sense of 1.1(b) are the same as "ordinary" homomorphisms. That is, the category $\mathbf{RelAlg}(T)$ of relational T-algebras contains the category $\mathbf{Alg}(T)$ of T-algebras as a full subcategory. On the other hand, $\mathbf{RelAlg}(T)$ itself is a full subcategory in the category $\mathbf{Rel}(T)$ of *T-relations*, that is, pairs (X,R), in which X is a set and $R : T(X) \to X$ is an arbitrary relation. Such pairs (X,R) are called *relational T-prealgebras* in [1], and they are special cases of lax algebras in the sense of [7] and of *(T,V)-algebras* in the sense of [10].

Example 1.2. Let T be the identity monad. Then $\mathbf{RelAlg}(T)$ becomes the category of ordinary preorders, while $\mathbf{Alg}(T)$, which can be identified with the category of sets, is embedded in $\mathbf{RelAlg}(T)$ as the full subcategory of discrete (pre)orders.

Example 1.3. Again, the relational T-algebras in the sense of Definition 1.1 are the same as the relational T-algebras in the sense of [1], and our $\mathbf{Alg}(T) \subset \mathbf{RelAlg}(T) \subset \mathbf{Rel}(T)$ becomes the same as $\mathbf{S}^{\mathbb{T}} \subset \mathbf{S}^{R(\mathbb{T})} \subset \mathbf{S}^{P(\mathbb{T})}$ in [1]. Let T be the ultrafilter monad; that is, T is the monad on **Sets** determined by the adjunction

$$(\mathbf{Bool})^{\mathrm{op}} \underset{P}{\overset{U}{\rightleftarrows}} \mathbf{Sets}. \tag{1.4}$$

where: **Bool** denotes the category of Boolean algebras; $U(B)$, for a Boolean algebra B, is the set of ultrafilters in B; and $P(S)$, for a set S, is the Boolean algebra of subsets in B. Then $\mathbf{Alg}(T)$ becomes the category of compact Hausdorff spaces (E. Manes [14]), while $\mathbf{RelAlg}(T)$ becomes the category **Top** of all topological spaces (M. Barr [1]), defined via the ultrafilter convergence. That is, a T-preorder (X, R) is a set X equipped with the convergence relation

$$R = \{(t, x) \,|\, t \to x\} = \{(t, x) \,|\, \text{ the ultrafilter } t \text{ converges to the point } x\}.$$

making (X, r) a topological space.

Example 1.4. Since every variety $\mathbf{C}$ of universal algebras has the free-forgetful adjunction

$$\mathbf{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Sets}\ ,\ \eta : 1_{\mathbf{Sets}} \to UF,\ \varepsilon : FU \to 1_{\mathbf{C}}, \tag{1.5}$$

and $\mathbf{C} \cong \mathbf{Alg}(T)$ for the corresponding monad $T = (UF, \eta, U\mu F)$, we conclude that every variety of algebras has the corresponding category of relational algebras. The non-triviality of T is then equivalent to the non-triviality of $\mathbf{C}$ (as it is defined in universal algebra).

2. Effective descent morphisms of relational algebras

As explained in [11], effective descent morphisms of (ordinary) preorders have a simple description:

Theorem 2.1. *A morphism $f : (X, R) \to (Y, S)$ of preorders is an effective descent morphism if and only if the induced map*

$$\{(x, x', x'') \mid (x, x'), (x', x'') \in R\} \longrightarrow \{(y, y', y'') \mid (y, y'), (y', y'') \in S\} \tag{2.1}$$

is surjective.

If one tries to extend Theorem 2.1 to T-preorders (=relational T-algebras), the first question to ask would be: What is the "T-version" of the map (2.1)? To answer this question, given an object (X, R) in $\mathbf{Rel}(T)$, consider the diagram

$$\begin{tikzcd}
 & & T(R)\times_{T(X)} R \arrow[ddl] \arrow[ddr] \arrow[d, dotted] & & \\
 & & \check{R}\times_{T(X)} R \arrow[ddl, dotted] \arrow[dr, dotted] & & \\
 & T(R) \arrow[d, dotted] \arrow[ddl] \arrow[ddr] & & R \arrow[ddl] \arrow[ddr] & \\
 & \check{R} \arrow[dl, dotted] \arrow[dr, dotted] & & & \\
T^2(X) & & T(X) & & X
\end{tikzcd} \tag{2.2}$$

in which:

- the solid arrows represent R as a span $T(X) \to X$ and $T(R)$ as a span $T^2(R) \to T(X)$, and then represent the composite of these spans as a span $T^2(X) \to X$;
- $\check{R}$ is the relation $T^2(X) \to T(X)$ associated with the span $T(R) : T^2(X) \to T(X)$, that is, $\check{R}$ is simply the image of $T(R)$ in $T^2(X) \times T(X)$;
- the dotted arrows are the canonical maps defined accordingly.

We observe:

- The maps

$$T(R) \to \check{R} \ \text{ and } \ T(R) \times_{T(X)} R \to \check{R} \times_{T(X)} R \tag{2.3}$$

are bijections whenever the canonical map

$$T(T(X) \times X) \to T^2(X) \times T(X) \tag{2.4}$$

is injective.

- Given a morphism $f : (X, R) \to (Y, S)$ in $\mathbf{RelAlg}(T)$, or, more generally, in $\mathbf{Rel}(T)$, consider the induced maps

$$T(R) \times_{T(X)} R \to T(S) \times_{T(Y)} S \tag{2.5}$$

and

$$\check{R} \times_{T(X)} R \to \check{S} \times_{T(Y)} S; \tag{2.6}$$

each of them can be considered as a T-version of the map (2.1) – since in the case of the identity monad they can be identified with each other and each of them becomes nothing but the map (2.1).

Therefore extending Theorem 2.1 to T-preorders, it would be natural to replace the map (2.1) either with the map (2.5), or with the map (2.6). Furthermore, since the class of effective descent morphisms is always pullback stable, this suggests:

Problem 2.2. *What is the relationship of the following conditions on a morphism $f : (X, R) \to (Y, S)$ in* $\mathbf{RelAlg}(T)$*:*

(a) *$f : (X, R) \to (Y, S)$ is an effective descent morphism in* $\mathbf{RelAlg}(T)$*;*
(b) *the map (2.5) is surjective;*
(c) *every pullback of f in* $\mathbf{RelAlg}(T)$ *satisfies (b);*
(d) *every pullback of f in* $\mathbf{Rel}(T)$ *satisfies (b);*
(e) *the map (2.6) is surjective;*
(f) *every pullback of f in* $\mathbf{RelAlg}(T)$ *satisfies (e);*
(g) *every pullback of f in* $\mathbf{Rel}(T)$ *satisfies (e).*

In fact only the trivial implications between conditions 2.2(b)-(g), namely

$$\begin{array}{ccccc} (d) & \Longrightarrow & (c) & \Longrightarrow & (b) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (g) & \Longrightarrow & (f) & \Longrightarrow & (e) \end{array},$$

are known for a general T, while, as follows from simple observations in [11], all these conditions are equivalent to each other when T is the identity monad. However, we have:

Theorem 2.3. (a) *If, for every two sets A and B, the canonical map*

$$T(A \times B) \to T(A) \times T(B) \tag{2.7}$$

is injective, then 2.2(b) is equivalent to 2.2(e), and therefore also 2.2(c) is equivalent to 2.2(f), and 2.2(d) is equivalent to 2.2(g).

(b) *If for every two maps $A \to C$ and $B \to C$ with the same codomain, the canonical map*

$$T(A \times_C B) \to T(A) \times_{T(C)} T(B) \tag{2.8}$$

is surjective, then 2.2(g) implies 2.2(a).

Proof: (a) is obvious since the map (2.4) is a special case of (2.7).

(b) is a special case of Theorem 3.3 in [8], since to say that the map (2.6) is surjective is the same as to say that $f : (X, R) \to (Y, S)$ is a $*$-quotient morphism in $\mathbf{Rel}(T)$ in the sense of [8] (not just for f in $\mathbf{RelAlg}(T)$, but also for any f in $\mathbf{Rel}(T)$). Let us, however, point out that the requirement on T in [8], namely "preservation of Beck-Chevalley squares" is formally different from our requirement; the fact that this difference is irrelevant follows from the results of [9]. ■

The map (2.1) in Theorem 2.1 can be described, using the ordinal $\mathbf{3}$, as

$$\hom(\mathbf{3}, f) : \hom(\mathbf{3}, (X, R)) \to \hom(\mathbf{3}, (X, R)),$$

and so Theorem 2.1 can be reformulated as: *a morphism of preorders is an effective descent morphism if and only if the ordinal* $\mathbf{3}$ *is projective with respect to it.* This suggests:

Problem 2.4. *Given a monad T, describe a class P of objects in* $\mathbf{RelAlg}(T)$ *such that a morphism f in* $\mathbf{RelAlg}(T)$ *is an effective descent morphism if and only if every object in P is projective with respect to it. Under what conditions on T is it possible?*

3. Effective descent morphisms of topological spaces

There are two known characterizations of effective descent morphisms of topological spaces, due to J. Reiterman and W. Tholen [15] and its reformulation due to M. M. Clementino and D. Hofmann [5].

Reiterman-Tholen characterization:

Theorem 3.1 ([15]). *A surjective map $f : X \to Y$ is an effective descent map if, and only if, for every family of ultrafilters $\mathfrak{f}_i$ on Y converging to $y_i \in Y$, $i \in I$, such that the y_i's converge to $y \in Y$ with respect to an ultrafilter $\mathfrak{u}$ on I, there is an ultrafilter $\mathfrak{v}$ on X converging to a point $x \in f^{-1}y$ such that $\bigcup_{i\in U} A_i \in \mathfrak{v}$ for all $U \in \mathfrak{u}$, where A_i is the set of adherence points of the filterbase $f^{-1}\mathfrak{f}_i$ which belong to $f^{-1}y_i$.*

Clementino-Hofmann characterization:

Let T be the ultrafilter monad defined as in Example 1.3. The endofunctor Ult of $\mathbf{RelAlg}(T) = \mathbf{Top}$ was defined in [5] by

$$\mathrm{Ult}(X, R) = (R, T(R) \times_{T(X)} R),$$

where $T(R) \times_{T(X)} R$ is constructed as in (2.2).

Theorem 3.2 ([5]). *A continuous map $f : X \to Y$ between topological spaces is of effective descent if and only if* $\mathrm{Ult}(\mathrm{Ult}(f))$ *is surjective.*

Using these two characterizations we obtain the following partial solution of Problem 2.2 in the case of ultrafilter monad:

Theorem 3.3. *If T is the ultrafilter monad, then the conditions 2.2(a)-(d) and 2.2(g) are equivalent to each other.*

Proof: 2.2(a) $\Leftrightarrow$ 2.2(b) follows from Theorem 3.2 (that is, from the Clementino-Hofmann characterization of the effective descent maps of topological spaces) since the map (2.5) is the same as the map $\mathrm{Ult}(\mathrm{Ult}(f))$.

2.2(b) $\Leftrightarrow$ 2.2(c) follows from 2.2(a) $\Leftrightarrow$ 2.2(b) and the fact that the class of effective descent morphisms in any category with pullbacks is pullback stable (see e.g. [12, Corollary 4.3]).

2.2(b) $\Leftrightarrow$ 2.2(d) is a consequence of the following three facts:

(i) the map (2.5) is the same as the map $\mathrm{Ult}(\mathrm{Ult}(f))$;

(ii) the map $\mathrm{Ult}(\mathrm{Ult}(f))$ is surjective if and only if f is a $*$-quotient map in the sense of [15], as shown in Section 5 of [5] (although we do not know if it is the same as a $*$-quotient map in the sense of [8]);

(iii) the class of $*$-quotient maps in the sense of [15] is pullback stable in $\mathbf{Rel}(T)$ by [15, Lemma 4.2] (noting that $\mathbf{Rel}(T)$ is the same as $\mathbf{PsTop}$ in [15]).

Equivalently, instead of (ii) and (iii) one can refer to [6, Theorem 6.2].

As already observed above, the implication 2.2(d) $\Rightarrow$ 2.2(g) is trivial (since so is 2.2(b) $\Rightarrow$ 2.2(e)), and 2.2(g) $\Rightarrow$ 2.2(a) holds by Theorem 2.3(b) (that is, in fact by [8, Theorem 3.3]). ■

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