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CHARACTERISTIC-FREE RESOLUTIONS FOR WEYL AND SPECHT MODULES

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ABSTRACT: We construct explicit resolutions of Weyl modules by divided powers and of co-Specht modules by permutational modules. We also prove Conjecture 3.4 of Boltje-Hartmann [BH10] on resolutions of co-Specht modules.

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Introduction

Schur algebras are fundamental tools in the representation theory of the general linear group $\mathrm{GL}_n(R)$ and of the symmetric group. In fact, over infinite fields, the category of homogeneous polynomial representations of degree r of $\mathrm{GL}_n(R)$ is equivalent to the category of finite dimensional modules for the Schur algebra $S_R(n,r)$. If $r \leq n$, the use of the Schur functor (see [Gre07, §6]) allows us to relate these categories with the category of finite dimensional representations of the symmetric group Σ_r.

Introduced by I. Schur in his doctoral dissertation [Sch01] in 1901, for the field of complex numbers, Schur algebras were generalized for arbitrary infinite fields by J.A. Green in [Gre07]. The Schur algebra $S_R(n,r)$, for a commutative ring R with identity, was introduced by K. Akin and D. Buchsbaum in [AB88] and Green in [Gre93].

If R is a noetherian commutative ring, $S_R(n,r)$ is quasi-hereditary. So it is natural to ask for the construction of projective resolutions of Weyl modules, which are the standard modules in this case.

In their work on characteristic-free representation theory of the general linear group [ABW82], [AB85], [AB88] Akin, Buchsbaum, and Weyman study the problem of constructing resolutions of Weyl modules in terms of direct sums of tensor products of divided powers of R^n. Moreover, they ask for these resolutions to be finite and universal (defined over the integers).

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This task was accomplished for Weyl modules associated with two-part partitions in [AB85]. Although the existence of these resolutions was proved in [AB88], no explicit construction of such complex is known.

In this paper we use the theory of Schur algebras to give an answer to this problem.

Denote by $\Lambda(n;r)$ (respectively $\Lambda^+(n;r)$) the set of all compositions (respectively partitions) of r into at most n parts. For each $\lambda \in \Lambda^+(n;r)$ write W_λ^R for the Weyl module over $S_R(n,r)$ associated to λ. The Schur algebra $S_R(n,r)$ has a decomposition of the identity as a sum over $\mu \in \Lambda(n;r)$ of orthogonal idempotents ξ_μ. The projective module $S_R(n,r)\,\xi_\mu$ is isomorphic as $\mathrm{GL}_n(R)$-module to the tensor product of divided powers $D_{\mu_1}(R^n) \otimes_R \cdots \otimes_R D_{\mu_n}(R^n)$. Hence, the construction of a universal projective resolution of the Weyl module W_λ^R in term of direct sums of the projective modules $S_R(n,r)\,\xi_\mu$ gives an answer to the problem posed by Akin and Buchsbaum.

The Borel-Schur algebra $S_R^+(n,r)$ is a subalgebra of $S_R(n,r)$ introduced by Green in [Gre93] (see also [Gre90]). It is an algebra with interesting properties. All the idempotents ξ_μ are elements of $S_R^+(n,r)$ and

$$S_R(n,r) \otimes_{S_R^+(n,r)} S_R^+(n,r)\,\xi_\mu \cong S_R(n,r)\,\xi_\mu.$$

Moreover, for every $\lambda \in \Lambda^+(n,r)$ there exists a rank one $S_R^+(n,r)$-module R_λ such that $S_R(n,r) \otimes_{S_R^+(n,r)} R_\lambda \cong W_\lambda^R$.

Woodcock [Woo94] proved that if R is an infinite field then the modules R_λ, $\lambda \in \Lambda^+(n,r)$ are acyclic with respect to the induction functor $S_R(n,r) \otimes_{S_R^+(n,r)} -$. The first author proved in [San93] that $S_R^+(n,r)\,\xi_\mu$ are principal projective modules in this case. So applying the induction functor to a projective resolution of R_λ we obtain a resolution of W_λ^R by direct sums of modules $S_R(n,r)\,\xi_\mu$.

In Theorem 5.2 we show that the modules R_λ are acyclic with respect to the induction functor $S_R(n,r) \otimes_{S_R^+(n,r)} -$ in the case of an arbitrary commutative ring R. Then we construct a universal resolution of R_λ by direct sums of modules $S_R^+(n,r)\,\xi_\mu$, $\mu \in \Lambda(n;r)$. Applying the induction functor we obtain a universal resolution of the Weyl modules by direct sums of modules $S_R(n,r)\,\xi_\mu$, $\mu \in \Lambda(n;r)$.

At this point, we only need to apply the Schur functor to the resolution of W_λ^R just obtained to get a resolution by permutational modules of the co-Specht Σ_r-module associated with λ. With this result we prove Conjecture 3.4 of [BH10].

The paper is organized as follows. In Section 1 we introduce the combinatorial notation we use. In Section 2 we recall the definition of Schur algebra and give a new version of the formula for the product of two basis elements of $S_R(n,r)$, which seems to be well suited to our work with divided powers.

Section 3 is technical and is written for the convenience of the reader. It explains how the reduced bar construction looks like for augmented algebras in the monoidal category of S-bimodules, where S is an arbitrary commutative ring.

Section 4 is dedicated to the Borel-Schur subalgebra $S_R^+(n,r)$. We apply the reduced bar construction to obtain a projective resolution for every rank one module R_μ, $\mu \in \Lambda(n;r)$.

Section 5 contains the main result (Theorem 5.2) of the paper. In Sections 6 and 7 we explain how our results prove Conjecture 3.4 in [BH10].

In appendix we show that the definition of Schur algebra used in this paper is equivalent to the one given in [Gre93]. Then we construct an explicit isomorphism between $D_\lambda(R^n)$ and $S_R(n,r)\,\xi_\lambda$, as it seems to be absent in published form. We also recall the theory of divided powers.

1. Combinatorics

In this section we collect the combinatorial notation used in the paper. We will give general definitions that include as partial cases the usual tools in the subject such as multi-indices, composition, etc.

We denote by R a commutative ring with identity e, and n and r are arbitrary fixed positive integers. For any natural number s we denote by $\mathbf{s}$ the set $\{1,\dots,s\}$ and by Σ_s the symmetric group on $\mathbf{s}$. Given a finite set X, we write:

- $\mu = (\mu_x)_{x\in X}$ and $|\mu| = \sum_{x\in X}\mu_x$, for each map $\mu\colon X \to \mathbb{N}_0$ given by $x \mapsto \mu_x$.
- $\Lambda(X,r) := \{\,\mu\colon X\to\mathbb{N}_0 \mid |\mu| = r\}$.
- $\mathrm{wt}(u) \in \Lambda(X,r)$, for the map defined by

$$\mathrm{wt}(u)_x = \#\{\,s \mid u_s = x,\ s = 1,\dots,r\},$$

for each $u \in X^r$.

The symmetric group Σ_r acts on the right of X^r by

$$(x_1, \ldots, x_r)\,\sigma = \left(x_{\sigma(1)}, \ldots, x_{\sigma(r)}\right).$$

Identifying $\mathrm{wt}\,(u)$ with the Σ_r-orbit of $u \in X^r$ we can think of $\Lambda\,(X, r)$ as the set of Σ_r-orbits on X^r. We will write $u \in \omega$ if $\mathrm{wt}\,(u) = \omega$.

Next we consider several partial cases of the definitions given above. We write $I\,(n, r)$ for $\mathbf{n}^r$. The elements of $I\,(n, r)$ are called *multi-indices* and will be usually denoted by the letters i, j. We identify the sets $(\mathbf{n} \times \mathbf{n})^r$ and $I\,(n, r) \times I\,(n, r)$ via the map

$$((i_1, j_1)\,, \ldots, (i_r, j_r)) \mapsto ((i_1, \ldots, i_r)\,, (j_1, \ldots, j_r))\,.$$

Similarly $(\mathbf{n} \times \mathbf{n} \times \mathbf{n})^r$ will be identified with $I\,(n, r) \times I\,(n, r) \times I\,(n, r)$.

The sets $\Lambda\,(\mathbf{n}, r)$, $\Lambda\,(\mathbf{n} \times \mathbf{n}; r)$, and $\Lambda\,(\mathbf{n} \times \mathbf{n} \times \mathbf{n}; r)$ will be denoted by $\Lambda\,(n; r)$, $\Lambda\,(n, n; r)$, and $\Lambda\,(n, n, n; r)$, respectively. We can think of the elements of $\Lambda\,(n; r)$ as *the compositions* of r into at most n parts, and we will write $\Lambda^+\,(n; r)$ for those $(\lambda_1, \ldots, \lambda_n) \in \Lambda\,(n; r)$ that verify $\lambda_1 \geq \cdots \geq \lambda_n$ (the *partitions* of r into at most n parts). The elements of $\Lambda\,(n, n; r)$ are functions from $\mathbf{n} \times \mathbf{n}$ to $\mathbb{N}_0$ and can be considered as $n \times n$ matrices of non-negative integers $(\omega_{s,t})_{s,t=1}^n$ such that $\sum_{s,t=1}^n \omega_{st} = r$. Similarly, the elements of $\Lambda\,(n, n, n; r)$ can be interpreted in terms of 3-dimensional $n \times n \times n$ tensors.

Note that the weight function on $I\,(n, r)$ coincides with the one defined in [Gre07]. Now, given $\omega = (\omega_{st})_{s,t=1}^n \in \Lambda\,(n, n; r)$ and $\theta = (\theta_{s,t,q})_{s,t,q=1}^n \in \Lambda\,(n, n, n; r)$ we will define ω^1, $\omega^2 \in \Lambda\,(n; r)$ and θ^1, θ^2, $\theta^3 \in \Lambda\,(n, n, n; r)$ by

$$\begin{aligned}
\left(\omega^1\right)_t, &= \sum_{s=1}^n \omega_{st}, & \left(\omega^2\right)_s &= \sum_{t=1}^n \omega_{st}; & & \\
\left(\theta^1\right)_{tq} &= \sum_{s=1}^n \theta_{stq}, & \left(\theta^2\right)_{sq} &= \sum_{t=1}^n \theta_{stq}, & \left(\theta^3\right)_{st} &= \sum_{q=1}^n \theta_{stq}.
\end{aligned}$$

Remark 1.1. *If i, j, $k \in I\,(n, r)$ then it can be seen from the definition of the weight function that*

$$\begin{aligned}
\mathrm{wt}\,(i, j)^1 &= \mathrm{wt}\,(j)\,, & \mathrm{wt}\,(i, j)^2\,, &= \mathrm{wt}\,(i)\,; & & \\
\mathrm{wt}\,(i, j, k)^1 &= \mathrm{wt}\,(j, k)\,, & \mathrm{wt}\,(i, j, k)^2 &= \mathrm{wt}\,(i, k)\,, & \mathrm{wt}\,(i, j, k)^3 &= \mathrm{wt}\,(i, j)\,.
\end{aligned}$$

2. The Schur algebra $\mathbf{S_R}(\mathbf{n}, \mathbf{r})$

In this section we recall the definition of the Schur algebra $S_R(n,r)$ over a commutative ring and give a new version of the formula for the product of two basis elements of this algebra (cf. [Gre07, (2.3.b)] and [Gre90, (2.6)]).

Let $\{ e_s \mid 1 \leq s \leq n\}$ be the standard basis of R^n. For every $i \in I(n,r)$ define $e_i = e_{i_1} \otimes \cdots \otimes e_{i_r}$. Then $\{ e_i \mid i \in I(n,r)\}$ is a basis for $(R^n)^{\otimes r}$ and $\mathrm{End}_R\left((R^n)^{\otimes r}\right)$ has basis $\{ e_{i,j} \mid i, j \in I(n,r)\}$, where the map $e_{i,j}$ is defined by

$$e_{i,j} e_k := \delta_{jk} e_i, \;\; i, j, k \in I(n,r).$$

The action of Σ_r on $I(n,r)$ extends on $(R^n)^{\otimes r}$ to $e_i \sigma := e_{i\sigma}$ and turns $(R^n)^{\otimes r}$ into a right $R\Sigma_r$-module.

Definition 2.1. The Schur algebra $S_R(n,r)$ is the endomorphism algebra of $(R^n)^{\otimes r}$ in the category of $R\Sigma_r$-modules.

The action of Σ_r on $(R^n)^{\otimes r}$ induces a Σ_r-action on $\mathrm{End}_R\left((R^n)^{\otimes r}\right)$ by $(f\sigma)(v) := f\left(v\sigma^{-1}\right)\sigma$. On the basis elements this action is $e_{i,j}\sigma = e_{i\sigma, j\sigma}$. We have, by Lemma 2.4 in [Fei82], $S_R(n,r) \cong \left(\mathrm{End}_R\left(R^n\right)^{\otimes r}\right)^{\Sigma_r}$.

For $\omega \in \Lambda(n,n;r)$ define $\xi_\omega \in S_R(n,r)$ by

$$\xi_\omega := \sum_{(i,j)\in\omega} e_{i,j}.$$

Since $\Lambda(n,n;r)$ is identified with the set of Σ_r-orbits on $I(n,r) \times I(n,r)$ via wt, the set $\{ \xi_\omega \mid \omega \in \Lambda(n,n;r)\}$ is a basis for $S_R(n,r)$.

Remark 2.2. *The definition of Schur algebra we use is equivalent to the one given by J.A. Green in* [Gre93] *(cf. Theorem 8.1). Note that Green writes $\xi_{i,j}$ where we have $\xi_{\mathrm{wt}(i,j)}$ for i, $j \in I(n,r)$. In the case R is an infinite field, this definition of $S_R(n,r)$ is also equivalent to the one given in* [Gre07].

For $\lambda \in \Lambda(n;r)$ we write ξ_λ for $\xi_{\mathrm{diag}(\lambda)}$. It is immediate from the definition, that $\sum_{\lambda\in\Lambda(n;r)} \xi_\lambda$ is an orthogonal idempotent decomposition of the identity of $S_R(n,r)$.

Next we will deduce a product formula for any two basis elements of $S_R(n,r)$. Let $\theta \in \Lambda(n,n,n;r)$. We write $[\theta] \in \mathbb{N}_0$ for the product of binomial coefficients:

$$\prod_{s,t=1}^{n} \binom{\left(\theta^2\right)_{st}}{\theta_{s1t}, \theta_{s2t}, \ldots, \theta_{snt}}.$$

Proposition 2.3. *For any ω, $\pi \in \Lambda(n,n;r)$ we have*

$$\xi_\omega \xi_\pi = \sum_{\theta \in \Lambda(n,n,n;r)\colon\ \theta^3=\omega,\quad \theta^1=\pi} [\theta]\, \xi_{\theta^2}.$$

Proof: We have

$$\xi_\omega \xi_\pi = \left(\sum_{(i,j)\in\omega} e_{i,j} \right) \left(\sum_{(l,k)\in\pi} e_{l,k} \right) = \sum_{(i,j,k)\colon\ (i,j)\in\omega,\ (j,k)\in\pi} e_{i,k}.$$

Note that the right hand side of the above formula is Σ_r-invariant.

From Remark 1.1 it follows that if $\mathrm{wt}\,(i,j,k) = \theta \in \Lambda(n,n,n;r)$, then $(i,j) \in \omega$ if and only if $\omega = \theta^3$, and $(j,k) \in \pi$ if and only if $\pi = \theta^1$. Thus

$$\xi_\omega \xi_\pi = \sum_{\theta\colon\ \theta^3=\omega,\ \theta^1=\pi} \ \sum_{(i,j,k)\in\theta} e_{i,k}.$$

Let us fix $\theta \in \Lambda(n,n,n;r)$ such that $\theta^1 = \pi$ and $\theta^3 = \omega$. Then $(i,j,k) \in \theta$ implies that $(i,k) \in \theta^2$. Thus $\sum_{(i,j,k)\in\theta} e_{i,k}$ is a multiple of ξ_{θ^2}, where the multiplicity is given by the number

$$\#\left\{ j \,|\, (i,j,k) \in \theta \right\}$$

for each pair $(i,k) \in \theta^2$. Now we fix $(i,k) \in \theta^2$ and define the sets X_{sq} by

$$X_{sq} := \left\{ 1 \le t \le r \,|\, i_t = s,\ k_t = q \right\}.$$

For every j such that $(i,j,k) \in \theta$ and $1 \le s, q \le n$ we define the function

$$\begin{aligned} J_{sq} \colon X_{sq} &\to \mathbf{n} \\ t &\mapsto j_t. \end{aligned}$$

Then defining $\mathrm{wt}\,(J_{sq})$ by

$$\mathrm{wt}\,(J_{sq})_v = \#\left\{ t \in X_{sq} \,|\, J_{sq} = v \right\},$$

we get $\mathrm{wt}\,(J_{sq}) = (\theta_{s1q}, \dots, \theta_{snq})$. On the other hand if we have a collection of functions $(J_{sq})_{s,q=1}^n$ such that $\mathrm{wt}\,(J_{sq}) = (\theta_{s1q}, \dots, \theta_{snq})$, then we can define $j \in I(n,r)$ by $j_t = J_{sq}(t)$ for $t \in X_{sq}$. Since $\mathbf{r}$ is the disjoint union of the sets X_{sq} the multi-index j is well defined. Moreover,

$$\mathrm{wt}\,(i,j,k)_{stq} = \#\left\{ 1 \le v \le r \,|\, i_v = s,\ j_v = t,\ k_v = q \right\} = \mathrm{wt}\,(J_{sq})_t = \theta_{stq}.$$

Thus $(i,j,k) \in \theta$. This gives a one-to-one correspondence between the set of those $j \in I(n,r)$ such that $(i,j,k) \in \theta$ and the set of collections of functions $(J_{sq})_{s,q=1}^{n}$, $J_{sq}\colon X_{sq} \to \mathbf{n}$ such that $\mathrm{wt}\,(J_{sq}) = (\theta_{s1q}, \dots, \theta_{snq})$ for all $1 \le s, q \le n$. The number of possible choices for each J_{sq} is

$$\binom{\theta_{s1q} + \cdots + \theta_{snq}}{\theta_{s1q}, \dots, \theta_{snq}} = \binom{\left(\theta^2\right)_{sq}}{\theta_{s1q}, \dots, \theta_{snq}}.$$

Since the choices of J_{sq} for different pairs (s, q) can be done independently we get that the number of elements in $\{\, j \mid (i, j, k) \in \theta\}$ is given by $[\theta]$. ■

Remark 2.4. *Given ω, $\pi \in \Lambda\,(n, n; r)$ if there is $\theta \in \Lambda\,(n, n, n; r)$ such that $\theta^3 = \omega$ and $\theta^1 = \pi$ then*

$$\omega^1 = \left(\theta^3\right)^1 = \left(\theta^1\right)^2 = \pi^2.$$

Thus if $\omega^1 \ne \pi^2$ then $\xi_\omega \xi_\pi = 0$. From this and the definition of ξ_λ it follows that

$$\xi_\lambda \xi_\omega = \begin{cases} \xi_\omega, & \lambda = \omega^2 \\ 0, & \textit{otherwise} \end{cases} \qquad\qquad \xi_\omega \xi_\lambda = \begin{cases} \xi_\omega, & \lambda = \omega^1 \\ 0, & \textit{otherwise.} \end{cases}$$

3. The reduced bar construction

In this section we recall the construction of the reduced bar resolution. This is a partial case of the construction described in Chapter IX, §7 of [ML63]. We will use in this section a slightly different notation, namely we write i, j, k for natural numbers.

Let A be a ring with identity e and S a subring of A. We assume that in the category of rings there is a splitting $p\colon A \to S$ of the inclusion map $S \to A$. We denote the kernel of p by I. Then I is an S-bimodule. Denote by $\widetilde{p}$ the map from A to I given by $a \mapsto a - p\,(a)$. Obviously $\widetilde{p}$ is a homomorphism of S-bimodules and the restriction of $\widetilde{p}$ to I is the identity map.

For every left A-module M we define the complex $B_k\,(A, S, M)$, $k \ge -1$, as follows. We set $B_{-1}\,(A, S, M) = M$, $B_0\,(A, S, M) = A \otimes M$ and for $k \ge 1$, $B_k\,(A, S, M) = A \otimes I^{\otimes k} \otimes M$, where all the tensor products are taken over S. Next we define A-module homomorphisms $d_{kj}\colon B_k\,(A, S, M) \to B_{k-1}\,(A, S, M)$, $0 \le j \le k$, and S-module homomorphisms $s_k\colon B_k\,(A, S, M) \to$

$B_{k+1}(A, S, M)$ by

$$\begin{aligned} d_{0,0}(a \otimes m) &:= am \\ d_{k,0}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= aa_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m \\ d_{k,j}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= a \otimes a_1 \otimes \dots a_j a_{j+1} \otimes \dots a_k \otimes m \quad 1 \le j \le k-1 \\ d_{k,k}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= a \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k m \\ s_{-1}(m) &:= e \otimes m \\ s_k(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &:= e \otimes \widetilde{p}(a) \otimes a_1 \otimes \cdots \otimes a_k \otimes m \qquad 0 \le k. \end{aligned}$$

It is simple to see that

$$\begin{aligned} d_{k,i} d_{k+1,j} &= d_{k,j-1} d_{k+1,i} & 0 \le i < j \le k \\ s_{k-1} d_{k,j} &= d_{k+1,j+1} s_k & 1 \le j \le k. \end{aligned}$$

Moreover for $k \ge 0$

$$\begin{aligned} d_{0,0} s_{-1}(m) &= d_{0,0}(e \otimes m) = m \\ d_{k+1,0} s_k(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= \widetilde{p}(a) \otimes a_1 \otimes \cdots \otimes a_k \otimes m \\ d_{k+1,1} s_k(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= e \otimes \widetilde{p}(a) a_1 \otimes \cdots \otimes a_k \otimes m \\ s_{-1} d_{0,0}(a \otimes m) &= s_{-1}(am) = e \otimes am \\ s_{k-1} d_{k,0}(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= e \otimes aa_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m. \end{aligned}$$

Note that in the last identity we used the fact that $aa_1 \in I$ which implies $\widetilde{p}(aa_1) = aa_1$. From the formulas above we have $d_{0,0} s_{-1} = \mathrm{id}_M$ and, taking into account that $\widetilde{p}(a) = a - p(a)$ and $p(a) \in S$,

$$\begin{aligned} &(d_{k+1,0} s_k - d_{k+1,1} s_k + s_{k-1} d_{k,0})(a \otimes a_1 \otimes \cdots \otimes a_k \otimes m) = \\ &\quad = a \otimes a_1 \otimes \cdots \otimes a_k \otimes m - e \otimes p(a) a_1 \otimes \cdots \otimes a_k \otimes m \\ &\qquad - e \otimes aa_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m + e \otimes p(a) a_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m \\ &\qquad + e \otimes aa_1 \otimes a_2 \cdots \otimes a_k \otimes m \\ &\quad = a \otimes a_1 \otimes \cdots \otimes a_k \otimes m. \end{aligned}$$

Thus $d_{k+1,0} s_k - d_{k+1,1} s_k + s_{k-1} d_{k,0} = \mathrm{id}_{B_k(A,S,M)}$, $k \ge 0$.

Define $d_k \colon B_k(A, S, M) \to B_{k-1}(A, S, M)$ by

$$d_k := \sum_{t=0}^{k} (-1)^t d_{k,t}.$$

The above computations show that

Proposition 3.1. *The sequence $(B_k(A,S,M), d_k)_{k\geq -1}$ is a complex of left A-modules. Moreover*

$$\begin{aligned} d_0 s_{-1} &= \mathrm{id}_{B_{-1}(A,S,M)} \\ d_{k+1}s_k + s_{k-1}d_k &= \mathrm{id}_{B_k(A,S,M)} \qquad\qquad 0 \leq k. \end{aligned}$$

Thus s_k, $k \geq -1$, give a splitting of $B_*(A,S,M)$ in the category of S-modules. In particular, $(B_k(A,S,M), d_k)_{k\geq -1}$ is exact.

Definition 3.2. The complex $(B_k(A,S,M), d_k)_{k\geq -1}$ is called the *reduced bar resolution*.

4. Borel-Schur algebras

In this section we introduce the Borel-Schur algebra $S_R^+(n,r)$ and apply the results of the previous section to the construction of the reduced bar resolution for the irreducible $S_R^+(n,r)$-modules. We should remark that $S_R^+(n,r)$ can be identified with the Borel subalgebra $S(B^+)$ defined in [Gre93, §8]. In case R is an infinite field the algebra $S_R^+(n,r)$ can be also identified with the algebra $S(B^+)$ used in [Gre90], [San93], and [Woo94].

On the set $I(n,r)$ we define the ordering $\leq$ by

$$i \leq j \Leftrightarrow i_1 \leq j_1,\ i_2 \leq j_2,\ \dots,\ i_n \leq j_n.$$

We write $i < j$ if $i \leq j$ and $i \neq j$. Note that $\leq$ is Σ_r invariant. Denote by $\Lambda^s(n,n;r)$ the set

$$\Lambda^s(n,n;r) = \left\{ \omega \in \Lambda(n,n;r) \,\middle|\, \begin{array}{c} \omega \text{ is upper triangular and} \\ \sum_{1\leq k\leq l\leq n} (l-k)\,\omega_{kl} \geq s \end{array} \right\}.$$

Remark 4.1. *Clearly $i \leq j$ and $i < j$ are equivalent to* $\mathrm{wt}(i,j) \in \Lambda^0(n,n;r)$ *and* $\mathrm{wt}(i,j) \in \Lambda^1(n,n;r)$*, respectively.*

For $s \geq 0$ denote by $J_s^R(n,r)$ the R-submodule of $S_R(n,r)$ spanned by the set $\{\xi_\omega \,|\, \omega \in \Lambda^s(n,n;r)\}$. Note that for $s > r(n-1)$ the set $\Lambda^s(n,n;r)$ is empty, therefore $J_s^R(n,r)$ is zero for $s \gg 0$.

Proposition 4.2. *If $\omega \in \Lambda^s(n,n;r)$ and $\pi \in \Lambda^t(n,n;r)$, then $\xi_\omega \xi_\pi \in J_{s+t}^R(n,r)$.*

Proof: By Proposition 2.3 $\xi_\omega\xi_\pi$ is a linear combination of the elements ξ_{θ^2} for $\theta \in \Lambda(n,n,n;r)$ satisfying $\theta^3 = \omega$ and $\theta^1 = \pi$. For any such θ the condition $\omega_{kq} = 0$ for $k > q$ implies that $\theta_{kql} = 0$ for $k > q$, and the condition $\pi_{ql} = 0$

for $q > l$ implies $\theta_{kql} =$ for $q > l$. Therefore for any q and for $k > l$ we have $\theta_{kql} = 0$. Hence θ^2 is upper triangular. Moreover,

$$\begin{aligned}
\sum_{k\le l} (l-k)\left(\theta^2\right)_{kl} &= \sum_{k\le l}\sum_{q=1}^{n} (l-k)\,\theta_{kql} = \sum_{k\le q\le l} (l-k)\,\theta_{kql} \\
&= \sum_{k\le q\le l} (l-q)\,\theta_{kql} + \sum_{k\le q\le l} (q-k)\,\theta_{kql} \\
&= \sum_{q\le l}\left(\sum_{k=1}^{n} (l-q)\,\theta_{kql}\right) + \sum_{k\le q}\left(\sum_{l=1}^{n} (q-k)\,\theta_{kql}\right) \\
&= \sum_{q\le l} (l-q)\left(\theta^1\right)_{ql} + \sum_{k\le q} (q-k)\left(\theta^3\right)_{kq} \\
&= \sum_{q\le l} (l-q)\,\pi_{ql} + \sum_{k\le q} (q-k)\,\omega_{kq} \ge t+s. \qquad \blacksquare
\end{aligned}$$

Define $S_R^+(n,r) := J_0^R(n,r)$ and $J_R = J_R(n,r) := J_1^R(n,r)$. Then for every $\lambda \in \Lambda(n;r)$ we have $\xi_\lambda \in S_R^+(n,r)$. In particular the identity of $S_R(n,r)$ is in $S_R^+(n,r)$. Now it follows from Proposition 4.2 that $S_R^+(n,r)$ is a subalgebra of $S_R(n,r)$ and $J_R(n,r)$ is a nilpotent ideal of $S_R^+(n,r)$.

Definition 4.3. The algebra $S_R^+(n,r)$ is called the *Borel-Schur algebra.*

Let $L_{n,r} = \bigoplus_{\lambda\in\Lambda(n,r)} R\xi_\lambda$. Then $L_{n,r}$ is a commutative R-subalgebra of $S_R^+(n,r)$, and $S^+(n,r) = L_{n,r} \oplus J_R(n,r)$. Since $J_R(n,r)$ is an ideal in $S_R^+(n,r)$, this direct sum decomposition implies that the natural inclusion of $L_{n,r}$ has a splitting in the category of R-algebras. So we can apply to $S_R^+(n,r)$ and $L_{n,r}$ the reduced bar construction described in Section 3.

For every $\lambda \in \Lambda(n;r)$ we have a rank one module $R_\lambda := R\xi_\lambda$ over $L_{n,r}$. Note that ξ_λ acts on $R_\lambda = R\xi_\lambda$ by identity, and ξ_μ, $\mu \neq \lambda$, acts by zero. We will denote in the same way the module over $S_R^+(n,r)$ obtained from R_λ by inflating along the natural projection of $S_R^+(n,r)$ on $L_{n,r}$.

Note that if R is a field the algebra $L_{n,r}$ is semi-simple, and so $J_R(n,r)$ is the radical of $S_R^+(n,r)$. In this case $\{\, R_\lambda \mid \lambda \in \Lambda(n,r)\}$ is a complete set of pairwise non isomorphic simple modules over $S_R^+(n,r)$. For more details the reader is referred to [San93].

For $\lambda \in \Lambda(n;r)$ we denote the resolution $B_*(S^+(n,r), L_{n,r}, R_\lambda)$ defined in Section 3 by $B_*^+(R_\lambda)$. Then

$$B_k^+(R_\lambda) := S_R^+(n,r) \otimes J_R(n,r) \otimes \cdots \otimes J_R(n,r) \otimes R_\lambda, \tag{1}$$

where all tensor products are over $L_{n,r}$ and there are k factors $J_R(n,r)$.

Let M be a right $L_{n,r}$-module and N a left $L_{n,r}$-module. It follows from Corollary 9.3 in [ML63] that $M \otimes_{L_{n,r}} N = \bigoplus_{\lambda\in\Lambda(n;r)} M\xi_\lambda \otimes_{R\xi_\lambda} \xi_\lambda N$. Hence $M \otimes_{L_{n,r}} N \cong \bigoplus_{\lambda\in\Lambda(n;r)} M\xi_\lambda \otimes_R \xi_\lambda N$.

Therefore $B_k^+(R_\lambda)$ is the direct sum over all sequences $\mu^{(1)}, \dots, \mu^{(k+1)} \in \Lambda(n;r)$ of the $S^+(n,r)$-modules

$$S_R^+(n,r)\,\xi_{\mu^{(1)}} \otimes \xi_{\mu^{(1)}} J_R(n,r)\,\xi_{\mu^{(2)}} \otimes \cdots \otimes \xi_{\mu^{(k)}} J_R(n,r)\,\xi_{\mu^{(k+1)}} \otimes \xi_{\mu^{(k+1)}} R_\lambda, \tag{2}$$

where all tensor products are over R. Since $\xi_{\mu^{(k+1)}} R_\lambda = 0$ unless $\mu^{(k+1)} = \lambda$, the summation is in fact over the sequences $\mu^{(1)}, \dots, \mu^{(k)} \in \Lambda(n,r)$.

Recall that it is said that $\nu \in \Lambda(n;r)$ *dominates* $\mu \in \Lambda(n;r)$ if

$$\sum_{s=1}^{t} \nu_s \geq \sum_{s=1}^{t} \mu_s$$

for all t. If ν dominates μ we write $\nu \trianglerighteq \mu$. If ν strictly dominates μ we write $\nu \triangleright \mu$. Note that if $i, j \in I(n,r)$ and $i \leq j$ or $i < j$, then $\mathrm{wt}(i) \trianglerighteq \mathrm{wt}(j)$ or $\mathrm{wt}(i) \triangleright \mathrm{wt}(j)$, respectively.

Proposition 4.4. *Let $\nu, \mu \in \Lambda(n;r)$. Then $\xi_\nu J_R(n,r)\,\xi_\mu = 0$ unless $\nu \triangleright \mu$. If $\nu \triangleright \mu$ then*

$$\left\{ \xi_\omega \,\middle|\, \omega \in \Lambda^1(n,n;r),\ \omega^1 = \mu,\ \omega^2 = \nu \right\}$$

is an R-basis of the free R-module $\xi_\nu J_R(n,r)\,\xi_\mu$.

Proof: From Remark 2.4 it follows that the set

$$\left\{ \xi_\omega \,\middle|\, \omega \in \Lambda^1(n,n;r),\ \omega^1 = \mu,\ \omega^2 = \nu \right\}$$

is an R-basis for $\xi_\nu J_R(n,r)\,\xi_\mu$. Suppose this is not empty. Then there is $\omega \in \Lambda^1(n,n;r)$ with $\omega^1 = \mu$ and $\omega^2 = \nu$. Let $(i,j) \in \omega$. Then by Remark 4.1 we know that $i < j$. Therefore, we have $\mathrm{wt}(i) \triangleright \mathrm{wt}(j)$. By Remark 1.1 we get

$$\nu = \omega^2 = \mathrm{wt}(i,j)^2 = \mathrm{wt}(i) \triangleright \mathrm{wt}(j) = \mathrm{wt}(i,j)^1 = \omega^1 = \mu. \quad \blacksquare$$

Corollary 4.5. *Let N be the length of the maximal strictly decreasing sequence in $(\Lambda(n;r), \triangleright)$. Then $B_k^+(R_\lambda) = 0$ for $k > N$.*

We conclude that the resolutions $B_*^+(R_\lambda)$ are finite, for all $\lambda \in \Lambda(n;r)$. We also have $B_0^+(R_\lambda) \cong S_R^+(n,r)\xi_\lambda$, and for $k \geq 1$

$$B_k^+(R_\lambda) = \bigoplus_{\mu^{(1)} \rhd \ldots \rhd \mu^{(k)} \rhd \lambda} S_R^+(n,r)\,\xi_{\mu^{(1)}} \otimes_R \xi_{\mu^{(1)}} J_R \xi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \xi_{\mu^{(k)}} J_R \xi_\lambda,$$

where all $\mu^{(s)}$, $1 \leq s \leq k$, are elements of $\Lambda(n;r)$. Given $\mu \rhd \lambda$ we denote by $\Omega_k^+(\lambda,\mu)$ the set

$$\left\{ (\omega_1,\ldots,\omega_k) \in \Lambda^1(n,n;r)^k \,\middle|\, \substack{(\omega_1)^2=\mu,(\omega_k)^1=\lambda \\ (\omega_1)^1=(\omega_2)^2,\ldots,(\omega_{k-1})^1=(\omega_k)^2} \right\}.$$

Then

$$\begin{aligned} B_0^+(R_\lambda) &\cong S_R^+(n,r)\,\xi_\lambda \\ B_k^+(R_\lambda) &\cong \bigoplus_{\mu \rhd \lambda} \left(S_R^+(n,r)\,\xi_\mu\right)^{\#\Omega_k^+(\lambda,\mu)}, \qquad k \geq 1 \end{aligned} \tag{3}$$

as $S_R^+(n,r)$-modules. In particular, $B_k^+(R_\lambda)$ is a projective $S_R^+(n,r)$-module for any $k \geq 0$. So we have the following result

Proposition 4.6. *Let $\lambda \in \Lambda(n;r)$. Then $B^+(R_\lambda)$ is a projective resolution of the module R_λ over $S_R^+(n,r)$.*

Define

$$b_k(\lambda) := \left\{ (\omega_0,\omega_1,\ldots,\omega_k) \,\middle|\, \omega_0 \in \Lambda^0(n,n;r),\ (\omega_1,\ldots,\omega_k) \in \Omega_k^+\left(\lambda,(\omega_0)^1\right) \right\}.$$

Then the set

$$\left\{ \xi_{\omega_0} \otimes \cdots \otimes \xi_{\omega_k} \,\middle|\, (\omega_j)_{j=0}^k \in b_k(\lambda) \right\}$$

is an R-basis of $B_k^+(R_\lambda)$. The differential ∂ of $B_*^+(R_\lambda)$ in terms of this basis looks like

$$\partial_k(\xi_{\omega_0} \otimes \cdots \otimes \xi_{\omega_k}) = \sum_{t=0}^{k-1} (-1)^t\, \xi_{\omega_0} \otimes \cdots \otimes \xi_{\omega_t}\xi_{\omega_{t+1}} \otimes \cdots \otimes \xi_{\omega_k}. \tag{4}$$

Let R' be another commutative ring with identity and $\phi\colon R \to R'$ a homomorphism of rings. It is now clear that

$$B_*^+(R_\lambda) \otimes_R R' \cong B_*^+(R'_\lambda)$$

as chain complexes of R'-modules.

5. Induction and Woodcock's theorem

In this section we explain how the results of Woodcock [Woo94] can be applied to prove that the $S_R^+(n,r)$-module R_λ, $\lambda \in \Lambda^+(n,r)$, is acyclic for the induction functor $S_R^+(n,r)\text{-mod} \to S_R(n,r)\text{-mod}$ We start with a definition.

Definition 5.1. For $\lambda \in \Lambda^+(n,r)$ the $S_R(n,r)$-module

$$W_\lambda^R := S_R(n,r) \otimes_{S_R^+(n,r)} R_\lambda$$

is called the *Weyl module* for $S_R(n,r)$ associated with λ.

It follows from Theorem 8.1 in [Gre93] that this definition is equivalent to the definition of Weyl module given in that work. It is proved in Theorem 7.1(ii) of [Gre93] that W_λ^R is a free R-module. In fact in this theorem an explicit description of an R-basis for W_λ^R is given. It turns out that this basis does not depend on the ring R but only on the partition λ. Moreover the coefficients in the formulas for the action of $S_R(n,r)$ on W_λ^R are in $\mathbb{Z}$ and do not depend on R. This implies that if $\phi\colon R \to R'$ is a ring homomorphism then there is an isomorphism of $S_{R'}(n,r)$-modules

$$W_\lambda^{R'} \cong W_\lambda^R \otimes_R R'. \tag{5}$$

For each $\lambda \in \Lambda^+(n;r)$ define the complex

$$B_*\left(W_\lambda^R\right) := S_R(n,r) \otimes_{S_R^+(n,r)} B_*^+(R_\lambda).$$

Note that $B_{-1}\left(W_\lambda^R\right) = W_\lambda^R$. Now, for any $\mu \in \Lambda(n;r)$

$$S_R(n,r) \otimes_{S_R^+(n,r)} S_R^+(n,r)\,\xi_\mu \cong S_R(n,r)\,\xi_\mu$$

is a projective $S_R(n,r)$-module. From (3) it follows that for $k \geq 1$ the $S_R(n,r)$-module $B_k\left(W_\lambda^R\right)$ is isomorphic to a direct sum of modules $S_R(n,r)\,\xi_\mu$, $\mu \rhd \lambda$. Also $B_0\left(W_\lambda^R\right) \cong S_R(n,r)\,\xi_\lambda$. Thus $B_k\left(W_\lambda^R\right)$ is a projective $S_R(n,r)$-module for $k \geq 0$.

Denote by $\widetilde{b^k}(\lambda)$ the set

$$\left\{ (\omega_0, \omega_1, \dots, \omega_k) \,\middle|\, \omega_0 \in \Lambda(n,n;r)\,,\ (\omega_1, \dots, \omega_k) \in \Omega_k^+\left(\lambda, (\omega_0)^1\right) \right\}.$$

Then $B_0\left(W_\lambda^R\right)$ has an R-basis $\left\{ \xi_\omega \,\middle|\, \omega \in \Lambda(n;r)\,,\ \omega^1 = \lambda \right\}$. For $k \geq 1$

$$B_k\left(W_\lambda^R\right) = \bigoplus_{\substack{\mu^{(1)} \rhd \dots \rhd \mu^{(k)} \rhd \lambda \\ \mu^{(s)} \in \Lambda(n;r),\ 1 \leq s \leq k}} S_R(n,r)\,\xi_{\mu^{(1)}} \otimes_R \xi_{\mu^{(1)}} J_R \xi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \xi_{\mu^{(k)}} J_R \xi_\lambda$$

has an R-basis

$$\left\{ \xi_{\omega_0} \otimes \xi_{\omega_1} \otimes \cdots \otimes \xi_{\omega_k} \,\middle|\, (\omega_0, \ldots, \omega_k) \in \widetilde{b_k}(\lambda) \right\}.$$

The differential ∂ of $B_*\left(W_\lambda^R\right)$ in the terms of these bases is given again by (4). From these facts it follows that, if $\phi\colon R' \to R$ is a homomorphism of rings, then

$$B_*\left(W_\lambda^R\right) \otimes_R R' \cong B_*\left(W_\lambda^{R'}\right). \tag{6}$$

Theorem 5.2. *Let $\lambda \in \Lambda^+(n;r)$. The complex $B_*\left(W_\lambda^R\right)$ is a projective resolution of W_λ^R over $S_R(n,r)$.*

Proof: Fix $\lambda \in \Lambda^+(n;r)$ and denote the complex $B_*\left(W_\lambda^R\right)$ by $\mathbb{X}(R)$. Then all R-modules in $\mathbb{X}(R)$ are free R-modules. Moreover we know from (6) that $\mathbb{X}(Z) \otimes_{\mathbb{Z}} R \cong \mathbb{X}(R)$. Now by the Universal Coefficient Theorem (cf. for example Theorem 8.22 in [Rot79]) we have a short exact sequence

$$0 \to H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} R \to H_k(\mathbb{X}(R)) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{k-1}(\mathbb{X}(Z)), R) \to 0. \tag{7}$$

Thus to show that the complexes $\mathbb{X}(R)$ are acyclic it is enough to check that the complex $\mathbb{X}(\mathbb{Z})$ is acyclic. Now $H_k(\mathbb{X}(\mathbb{Z}))$ is a finitely generated abelian group. Therefore

$$H_k(\mathbb{X}(\mathbb{Z})) \cong \mathbb{Z}^t \oplus \bigoplus_{p \text{ prime}} \bigoplus_{s \geq 1} \left(\mathbb{Z}/p^s\mathbb{Z} \right)^{t_{ps}},$$

where only finitely many of the integers t, t_{ps} are different from zero. For every prime p denote by $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$. Then we get

$$H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p \cong \overline{\mathbb{F}}_p^{\sum_{s \geq 1} t_{ps}}$$

and also

$$H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^t.$$

Hence if we show that $H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p = 0$ for all prime numbers p and $H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, this will imply that $H_k(\mathbb{X}(\mathbb{Z})) = 0$.

Let $\mathbb{K}$ be one of the fields $\overline{\mathbb{F}}_p$, p prime, or $\mathbb{Q}$. Then, by the Universal Coefficient Theorem, $H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{K}$ is a submodule of $H_k(\mathbb{X}(\mathbb{K}))$. Therefore it is enough to show that $H_k(\mathbb{X}(\mathbb{K})) = 0$. For this we use Theorem 5.1 in [Woo94]. That result can be applied because $\mathbb{K}$ is an infinite field.

Note that the algebra $S_R(n,r)$ has an anti-involution $\mathscr{J}: S_R(n,r) \to S_R(n,r)$ defined on the basis elements by $\mathscr{J}: \xi_\omega \mapsto \xi_{\omega^t}$. The image of

$S_R^+(n,r)$ under $\mathscr{J}$ is the subalgebra $S_R^-(n,r)$ of $S_R(n,r)$. Now for each $S_{\mathbb{K}}^+(n,r)$-module M we define a structure of $\mathscr{J}\left(S_{\mathbb{K}}^+(n,r)\right)$-module on $M^* := \mathrm{Hom}_{\mathbb{K}}(M,\mathbb{K})$ by $(\xi\theta)(m) := \theta(\mathscr{J}(\xi)m)$, for $\theta \in M^*$, $\xi \in \mathscr{J}\left(S_{\mathbb{K}}^+(n,r)\right)$. This induces a contravariant equivalence of categories $\mathscr{J}_*\colon S_{\mathbb{K}}^+(n,r)\text{-mod} \to S_{\mathbb{K}}^-(n,r)\text{-mod}$. There is also a similarly defined contravariant auto-equivalence functor $\mathscr{J}_*\colon S_{\mathbb{K}}(n,r)\text{-mod} \to S_{\mathbb{K}}(n,r)\text{-mod}$. By Theorem 7.1 in [San93] the functors

$$\begin{aligned} \mathscr{J}_* \circ \mathrm{Hom}_{S_{\mathbb{K}}^-(n,r)}(S_{\mathbb{K}}(n,r),-) \circ \mathscr{J}_*\colon S_{\mathbb{K}}^+(n,r)\text{-mod} &\to S_{\mathbb{K}}(n,r)\text{-mod} \\ S_{\mathbb{K}}(n,r) \otimes_{S_{\mathbb{K}}^+(n,r)} -\colon S_{\mathbb{K}}^+(n,r)\text{-mod} &\to S_{\mathbb{K}}(n,r)\text{-mod} \end{aligned}$$

are isomorphic. Now since $\mathscr{J}_*$ is a contravariant equivalence of abelian categories, $\mathscr{J}_*(B(\mathbb{K}_\lambda))$ is an injective resolution of $\mathscr{J}_*(\mathbb{K}_\lambda)$. By Theorem 5.1 of [Woo94] the complex

$$\mathrm{Hom}_{S_{\mathbb{K}}^-(n,r)}(S_{\mathbb{K}}(n,r), J_*(B_*(\mathbb{K}_\lambda)))$$

is exact. Applying $\mathscr{J}_*$ we get that also the complex $\mathbb{X}(\mathbb{K})$ is exact. ■

Corollary 5.3. *Let $\lambda \in \Lambda^+(n;r)$ and $P \to R_\lambda$ be a projective resolution of R_λ over $S_R^+(n,r)$. Then $S_R(n,r) \otimes_{S^+(n,r)} P \to W_\lambda^R$ is a projective resolution of W_λ^R over $S_R(n,r)$.*

Proof: The homology groups of the complex $S_R(n,r) \otimes_{S^+(n,r)} P \to 0$ are the tor-groups $Tor_k^{S^+(n,r)}(S_R(n,r), R_\lambda)$. But these groups are zero for $k \geq 1$ by Theorem 5.2. And in degree zero we get W_λ^R. ■

From Theorem 8.4 it follows that the resolution $B_*\left(W_\lambda^R\right)$ is a resolution of W_λ^R by divided power modules over $\mathrm{GL}_n(R)$. Moreover, this resolution is universal in the sense of Akin and Buchsbaum. Namely, we have

$$B_*\left(W_\lambda^R\right) = B_*\left(W_\lambda^{\mathbb{Z}}\right) \otimes_{\mathbb{Z}} R.$$

So this resolution answers the question posed by Akin and Buchsbaum referred in the introduction of the paper.

6. The Schur algebra and the symmetric group

In this section we recall the connection which exists between the categories of representations of the symmetric group Σ_r and of the Schur algebra $S_R(n,r)$. Good references on this subject are [Gre07] and [Jam98].

Suppose that $n \geq r$. Then there is $\delta = (1, \dots, 1, 0, \dots, 0) \in \Lambda(n; r)$. For every $\sigma \in \Sigma_r$ we denote by $\omega(\sigma)$ the element of $\Lambda(n, n; r)$ defined by

$$\omega(\sigma)_{st} := \begin{cases} 1, & 1 \leq t \leq r,\ s = \sigma t \\ 0, & \text{otherwise.} \end{cases}$$

This gives a one-to-one correspondence between Σ_r and the elements ω of $\Lambda(n, n; r)$ such that $\omega^1 = \omega^2 = \delta$. It follows from the Proposition 2.3 that the map

$$\begin{aligned} \Sigma_r &\to \xi_\delta S_R(n, r) \xi_\delta \\ \sigma &\mapsto \xi_{\omega(\sigma)} \end{aligned}$$

is multiplicative. In fact, if $\theta \in \Lambda(n, n, n; r)$ is such that $\theta^3 = \omega(\sigma_1)$ and $\theta^1 = \omega(\sigma_2)$, then

$$\theta_{stq} = \begin{cases} 1, & 1 \leq q \leq r,\ s = \sigma_1 t,\ t = \sigma_2 q \\ 0, & \text{otherwise} \end{cases}.$$

Such θ is unique and for this θ we have $\theta^2 = \omega(\sigma_1\sigma_2)$ and $[\theta] = 1$. Therefore $\xi_{\omega(\sigma_1)}\xi_{\omega(\sigma_2)} = \xi_{\omega(\sigma_1\sigma_2)}$. We can consider $\xi_\delta S_R(n, r) \xi_\delta$ as an algebra with identity ξ_δ. Then $\sigma \mapsto \omega(\sigma)$ induces an isomorphism of algebras $\phi\colon R\Sigma_r \overset{\cong}{\to} \xi_\delta S_R(n, r) \xi_\delta$. It is obvious that if M is an $S_R(n, r)$-module then $\xi_\delta M$ is an $\xi_\delta S_R(n, r) \xi_\delta$-module. In fact the map $M \mapsto \xi_\delta M$ is functorial. Now we can consider $\xi_\delta M$ as $R\Sigma_r$-module via ϕ. The resulting functor $\mathfrak{S}\colon S_R(n, r)$-mod $\to R\Sigma_r$-mod was named *Schur functor* in [Gre07]. This terminology should not be mixed up with Schur functors in [ABW82]. It is obvious that $\mathfrak{S}$ is exact.

Given $\lambda \in \Lambda^+(n; r)$, we can apply the functor $\mathfrak{S}$ to the complex $B_*\left(W_\lambda^R\right)$. We obtain an exact sequence $\mathfrak{S}\left(B_*\left(W_\lambda^R\right)\right)$, which is a resolution by permutation modules of the co-Specht module corresponding to λ. This will be explained in more detail in the next section.

7. The Boltje-Hartmann complex

In this section we show that for $\lambda \in \Lambda^+(n; r)$ the complex $\mathfrak{S}\left(B_*\left(W_\lambda^R\right)\right)$ is isomorphic to the complex constructed in [BH10]. Thus we prove Conjecture 3.4 of [BH10].

We start by summarizing notation and conventions of [BH10].

Let $\lambda \in \Lambda(n;r)$. The *diagram* of shape λ is the subset of $\mathbb{N}^2$

$$[\lambda] = \{ (s,t) \mid 1 \le t \le \lambda_s,\ 1 \le s \le n \},$$

and a *tableau* of shape λ or λ-*tableau* is a map $T\colon [\lambda] \to \mathbb{N}$. The content $c(T)$ of T is defined by

$$c(T)_t = \# \{ (s,q) \in [\lambda] \mid T(s,q) = t \}.$$

If all values of T are no greater than n we consider $c(T)$ as an element of $\Lambda(n;r)$. For every pair $\lambda,\ \mu \in \Lambda(n;r)$, denote by $\mathscr{T}(\lambda,\mu)$ the set of tableaux of shape λ and content μ. We say that a tableau $T \in \mathscr{T}(\lambda,\mu)$ is row semistandard if for every $1 \le s \le n$

$$T(s,1) \le T(s,2) \le \cdots \le T(s,\lambda_s).$$

The set of row semistandard tableaux of shape λ and content μ will be denoted by $\mathscr{T}^{rs}(\lambda,\mu)$. Following [BH10] we denote by $\mathscr{T}(\lambda)$ (respectively $\mathscr{T}^{rs}(\lambda)$) the set $\mathscr{T}(\lambda,\delta)$ (respectively $\mathscr{T}^{rs}(\lambda,\delta)$), where $\delta = (1^r, 0^{n-r})$ like in the previous section. Note that every element $\mathfrak{t}$ of $\mathscr{T}(\lambda)$ is a bijection from $[\lambda]$ to $\mathbf{r}$. Therefore we can define a left action of Σ_r on $\mathscr{T}(\lambda)$ by

$$\begin{gathered} \Sigma_r \times \mathscr{T}(\lambda) \to \mathscr{T}(\lambda) \\ (\sigma,\mathfrak{t}) \mapsto \sigma\mathfrak{t}. \end{gathered}$$

Define the projection $\mathfrak{r}\colon \mathscr{T}(\lambda) \to \mathscr{T}^{rs}(\lambda)$ that turns $\mathfrak{t} \in \mathscr{T}(\lambda)$ into the row semistandard tableau obtained from $\mathfrak{t}$ by rearranging the elements of each row of $\mathfrak{t}$ in increasing order. Then

$$\begin{gathered} \Sigma_r \times \mathscr{T}^{rs}(\lambda) \to \mathscr{T}^{rs}(\lambda) \\ (\sigma,\mathfrak{t}) \mapsto \mathfrak{r}(\sigma\mathfrak{t}) \end{gathered}$$

is a left transitive action of Σ_r on $\mathscr{T}^{rs}(\lambda)$, and the stabilizer of

$$\mathfrak{t}_\lambda := \begin{array}{ccccc} 1 & \dots & \dots & \dots & \lambda_1 \\ \lambda_1+1 & \dots & \dots & \lambda_1+\lambda_2 & \\ \vdots & \vdots & \vdots & & \\ \lambda_1+\cdots+\lambda_{n-1}+1 & \dots & \lambda_1+\cdots+\lambda_n & & \end{array}$$

is $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_n}$. Following [BH10], we define the left permutational Σ_r- module M^λ as the linear span over R of the elements in $\mathscr{T}^{rs}(\lambda)$, with the action induced on M^λ by the formula above.

Let $V_\lambda = \xi_\lambda\left((R^n)^{\otimes r}\right) = \bigoplus_{i\in\lambda} Re_i$. Then V_λ is a right Σ_r-submodule of $(R^n)^{\otimes r}$. We will consider V_λ as a left Σ_r-module via the transitive Σ_r-action

$\sigma e_i = e_{i\sigma^{-1}}$ on the basis $\{ e_i \mid i \in \lambda\}$. Obviously, V_λ is a permutational module. Let $l(\lambda) := (1^{\lambda_1}, \dots, n^{\lambda_n})$. Then the stabilizer of $e_{l(\lambda)}$ is Σ_λ, and so the Σ_r-modules M^λ and V_λ are isomorphic. Let us describe the isomorphism induced by $e_{l(\lambda)} \to \mathfrak{t}_\lambda$ explicitly.

For every $i \in \lambda$, we define $\mathfrak{t}(i) \in \mathscr{T}^{rs}(\lambda)$ as the row semistandard λ-tableau whose row s contains the λ_s integers q satisfying $i_q = s$, $s \in \mathbf{n}$.

We have $\mathfrak{t}(l(\lambda)) = \mathfrak{t}_\lambda$. Next we check that the correspondence $e_i \mapsto \mathfrak{t}(i)$ is Σ_r-invariant.

Let $i \in \lambda$, $\sigma \in \Sigma_r$ and $1 \le s \le n$. Suppose s occurs at positions $\nu_1 < \cdots < \nu_{\lambda_s}$ in i. Then

$$s = i_{\nu_q} = i_{\sigma^{-1}(\sigma(\nu_q))}, \qquad\qquad 1 \le q \le \lambda_s.$$

Hence s occurs at the positions $\sigma(\nu_1), \dots, \sigma(\nu_{\lambda_s})$ in $i\sigma^{-1}$. Since s is arbitrary we get $\mathfrak{t}\left(i\sigma^{-1}\right) = \mathfrak{r}(\sigma\mathfrak{t}(i))$. Therefore the map $e_i \mapsto \mathfrak{t}(i)$ is Σ_r-invariant, and defines the Σ_r-isomorphism between V_λ and M^λ referred above.

As a consequence, for λ, $\mu \in \Lambda(n; r)$ we get an isomorphism

$$\mathrm{Hom}_{R\Sigma_r}\left(M^\mu, M^\lambda\right) \to \mathrm{Hom}_{R\Sigma_r}\left(V^\mu, V^\lambda\right).$$

Since $(R^n)^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n;r)} V_\lambda$ these isomorphisms can be assembled into an isomorphism of algebras

$$\bigoplus_{\lambda,\mu \in \Lambda(n;r)} \mathrm{Hom}_{R\Sigma_r}\left(M^\mu, M^\lambda\right) \xrightarrow{\cong} S_R(n, r), \tag{8}$$

where we consider the direct sum on the left hand side as an algebra, with the product of two composable maps given by their composition, and the product of two non-composable maps is defined to be zero.

We will describe (8) explicitly below. Let

$$\Omega(\lambda, \mu) = \left\{ \omega \in \Lambda(n, n; r) \,\middle|\, \omega^1 = \mu,\ \omega^2 = \lambda \right\}.$$

Then there is a bijection between the sets $\mathscr{T}^{rs}(\lambda, \mu)$ and $\Omega(\lambda, \mu)$. To prove this, we define for each $T \in \mathscr{T}^{rs}(\lambda, \mu)$ the matrix $\omega(T) \in \Lambda(n, n; r)$ by

$$\omega_{st} := \#\{ (s, q) \in [\lambda] \mid T(s, q) = t\} = (\text{number of } t\text{'s in row } s \text{ of } T).$$

Clearly $\omega(T)^2 = \lambda$ and $\omega(T)^1 = \mu$. Thus $\omega(T) \in \Omega(\lambda, \mu)$. Conversely, if $\omega \in \Omega(\lambda, \mu)$ we define $T(\omega)$ as the tableau of shape λ whose s-th row is the sequence $(1^{\omega_{s1}}, \dots, n^{\omega_{sn}})$. Then $T(\omega)$ is row semistandard. Moreover, t occurs in T exactly $\left(\omega^1\right)_t = \mu_t$ times, and so $T(\omega)$ has content μ. It is easy to see that these constructions are mutually inverse.

For every $T \in \mathscr{T}^{rs}(\lambda,\mu)$ Boltje and Hartmann define a map $\theta_T\colon M^\mu \to M^\lambda$ by the rule: for $\mathfrak{t} \in \mathscr{T}^{rs}(\mu)$ the element $\theta_T\mathfrak{t} \in M^\lambda$ is equal to the sum of all λ-tableaux $\mathfrak{s} \in \mathscr{T}^{rs}(\lambda)$ with the following property, for each $1 \le s \le n$: if the s-th row of T contains precisely q entries equal to t then the s-th row of $\mathfrak{s}$ contains precisely q entries from the t-th row of $\mathfrak{t}$.

We can reformulate this rule in terms of $\omega(T)$ as follows: the element $\theta_T\mathfrak{t}$ is the sum of those $\mathfrak{s} \in \mathscr{T}^{rs}(\lambda)$ that, for each $1 \le s,t \le n$, the s-th row of $\mathfrak{s}$ contains precisely $\omega(T)_{st}$ entries from the t-th row of $\mathfrak{t}$.

The set $\{\,\theta_T \mid T \in \mathscr{T}^{rs}(\lambda,\mu)\}$ is an R-basis of $\mathrm{Hom}_{R\Sigma_r}\left(M^\mu, M^\lambda\right)$. On the other hand $\{\,\xi_\omega \mid \omega \in \Omega(\lambda,\mu)\}$ is an R-basis of $\xi_\lambda S_R(n,r)\,\xi_\mu$. Now we have the following result.

Proposition 7.1. *Under* (8) θ_T, $T \in \mathscr{T}^{rs}(\lambda,\mu)$, *corresponds to* $\xi_{\omega(T)}$.

Proof: Let $T \in \mathscr{T}^{rs}(\lambda,\mu)$ and $j \in \mu$. Then

$$\xi_{\omega(T)} e_j = \sum_{(i,j)\in\omega(T)} e_i.$$

Now for all $1 \le s,t \le n$

$$\#\{\,1 \le q \le r \mid i_q = s,\ j_q = t\} = \mathrm{wt}\,(i,j)_{st} = \omega(T)_{st}\,.$$

Thus $\xi_{\omega(T)}e_j$ is the sum of e_i, for those $i \in \lambda$ such that i is obtained from j by replacing $\omega(T)_{st}$ occurences of t by s. Therefore $\mathfrak{t}(i)$ is obtained from $\mathfrak{t}(j)$ by moving exactly $\omega(T)_{st}$ elements from the t-th row of $\mathfrak{t}(j)$ to the s-th row of $\mathfrak{t}(i)$.

Moreover, if $\mathfrak{s} \in \mathscr{T}^{rs}(\lambda)$ is obtained from $\mathfrak{t}(j)$ by moving exactly $\omega(T)_{st}$ elements from the t-th row of $\mathfrak{t}(j)$ to s-th row of $\mathfrak{s}$, then for the corresponding $i = i(\mathfrak{s}) \in \lambda$ we have $(i,j) \in \omega(T)_{st}$. This proves the proposition. ∎

Boltje and Hartmann define $\mathrm{Hom}^{\wedge}_{R\Sigma_r}\left(M^\mu, M^\lambda\right)$ as the R-span of those elements θ_T with $T \in \mathscr{T}^{rs}(\lambda,\mu)$ such that for every $1 \le s \le n$ the s-th row of T does not contain any entry smaller than s. This is equivalent to the requirement that $\omega(T)$ is an upper triangular matrix. Therefore, under (8) the subspace $\mathrm{Hom}^{\wedge}_{R\Sigma_r}\left(M^\mu, M^\lambda\right)$ is mapped to $\xi_\lambda S^+(n,r)\,\xi_\mu$, since we know that $\xi_\lambda S^+_R(n,r)\,\xi_\mu$ is the R-linear span of

$$\{\,\xi_\omega \mid \omega \text{ is upper triangular and } \omega \in \Omega(\lambda,\mu)\}\,.$$

As $S^+_R(n,r) \cong J_R(n,r) \oplus L_{n,r}$, we have $\mathrm{Hom}^{\wedge}_{R\Sigma_r}\left(M^\mu, M^\lambda\right) \cong \xi_\lambda J_R(n,r)\,\xi_\mu$ if $\mu \rhd \lambda$.

In Section 3.2 of [BH10] there is defined the complex $\widetilde{C}_*^\lambda$ as follows. $\widetilde{C}_{-1}^\lambda$ is the co-Specht module that corresponds to the partition $\lambda \in \Lambda^+(n;r)$, $\widetilde{C}_0^\lambda = \mathrm{Hom}_R\left(M^\lambda, R\right)$. For $k \geq 1$ the $R\Sigma_r$-module $\widetilde{C}_k^\lambda$ is defined as the direct sum over all sequences $\mu^{(1)} \rhd \ldots \rhd \mu^{(k)} \rhd \lambda$

$$\mathrm{Hom}_R\left(M^{\mu^{(1)}}, R\right) \otimes_R \mathrm{Hom}^\wedge_{R\Sigma_r}\left(M^{\mu^{(2)}}, M^{\mu^{(1)}}\right) \otimes_R \otimes_R \cdots \otimes_R \mathrm{Hom}^\wedge_{R\Sigma_r}\left(M^\lambda, M^{\mu^{(k)}}\right). \quad (9)$$

The differential d_k, $k \geq 1$, in $\widetilde{C}_*^\lambda$ is given by the formula

$$d_k\left(f_0 \otimes f_1 \otimes \cdots \otimes f_{k-1}\right) = \sum_{t=0}^{k} (-1)^t f_0 \otimes \cdots \otimes f_t f_{t+1} \otimes \ldots f_{k-1}. \quad (10)$$

Note that we have arranged the factors in the definition of $\widetilde{C}_k^\lambda$ in a different order from the one used in [BH10].

Theorem 7.2. *For $\lambda \in \Lambda^+(n;r)$, the complex $\widetilde{C}_*^\lambda$ is isomorphic to $\mathfrak{S}B_*\left(W_\lambda^R\right)$.*

Proof: We will establish the isomorphism only in non-negative degrees. The isomorphism in the degree -1 will follow, since the complex $\mathfrak{S}B_*\left(W_\lambda^R\right)$ is exact, and the complex $\widetilde{C}_*^\lambda$ is exact in degrees 0 and -1 by Theorems 4.2 and 4.3 in [BH10].

We define the complex $\widehat{C}_*^\lambda$ in the same way as the complex $\widetilde{C}_*^\lambda$ with the only difference that the summands (9) are replaced by

$$\mathrm{Hom}_{R\Sigma_r}\left(M^{\mu^{(1)}}, M^\delta\right) \otimes_R \mathrm{Hom}^\wedge_{R\Sigma_r}\left(M^{\mu^{(2)}}, M^{\mu^{(1)}}\right) \otimes_R \otimes_R \cdots \otimes_R \mathrm{Hom}^\wedge_{R\Sigma_r}\left(M^\lambda, M^{\mu^{(k)}}\right).$$

Then it is straightforward that the isomorphism (8) induces an isomorphism between the complexes $\mathfrak{S}B_*\left(W_\lambda^R\right)$ and $\widehat{C}_*^\lambda$ in non-negative degrees.

To show that the complexes $\widehat{C}_*^\lambda$ and $\widetilde{C}_*^\lambda$ are isomorphic in the non-negative degrees it is enough to find for every $\nu \in \Lambda(n;r)$ an isomorphism of Σ_r-modules $\phi_\nu\colon \mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right) \to \mathrm{Hom}_R\left(M^\nu, R\right)$, such that for all $\mu \in \Lambda(n;r)$, $f \in \mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right)$, and $h \in \mathrm{Hom}_{R\Sigma_r}\left(M^\mu, M^\nu\right)$ we have $\phi_\mu(fh) = \phi_\nu(f)\, h$.

Note that the action of Σ_r on $\mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right)$ is given by composition with $\theta_{T(\omega(\sigma))}$, $\sigma \in \Sigma_r$. The action of Σ_r on $\mathrm{Hom}_R\left(M^\nu, R\right)$ is given by the formula $\left(\sigma f\right)(m) = f\left(\sigma^{-1}m\right)$, for $f \in \mathrm{Hom}_R\left(M^\nu, R\right)$, $m \in M^\nu$, and $\sigma \in \Sigma_r$.

Let $f \in \mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right)$, $m \in M^\nu$. We define $\phi_\nu\left(f\right)(m)$ to be the coefficient of $\mathfrak{t}_\delta$ in $f(m)$. Note that f can be recovered from $\phi_\nu\left(f\right)$ in a unique way. In fact, $\{\, \sigma\mathfrak{t}_\delta \,|\, \sigma \in \Sigma_r \}$ is an R-basis of M^δ. Now the coefficient of $\sigma\mathfrak{t}_\delta$ in $f(m)$ is the same as the coefficient of $\mathfrak{t}_\delta$ in $\sigma^{-1}f(m) = f\left(\sigma^{-1}m\right)$, and so equals $\phi_\nu\left(f\right)\left(\sigma^{-1}m\right)$. This shows that ϕ_ν is injective.

Now let $g \in \mathrm{Hom}_R\left(M^\nu, R\right)$. We define $\psi_\nu(g) \in \mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right)$ by

$$m \mapsto \sum_{\sigma \in \Sigma_r} g\left(\sigma^{-1}m\right)\left(\sigma\mathfrak{t}_\delta\right).$$

We have to check that $\psi_\nu(g)$ is Σ_r-invariant. Let $\sigma' \in \Sigma$. Then

$$\begin{aligned}\psi_\nu(g)\left(\sigma' m\right) &= \sum_{\sigma \in \Sigma_r} g\left(\sigma^{-1}\sigma' m\right)\left(\sigma\mathfrak{t}_\delta\right) \xlongequal{\left(\sigma''\right)^{-1} = \sigma^{-1}\sigma'} \sum_{\sigma'' \in \Sigma_r} g\left(\left(\sigma''\right)^{-1} m\right)\left(\sigma'\sigma''\mathfrak{t}_\delta\right) \\ &= \sigma'\left(\psi_\nu(g)(m)\right).\end{aligned}$$

Since all the elements $\sigma\mathfrak{t}_\delta$, $\sigma \in \Sigma_r$, are linearly independent, we get that $\phi_\nu\left(\psi_\nu(g)\right)(m) = g(m)$ for all $g \in \mathrm{Hom}_R\left(M^\nu, R\right)$ and $m \in M^\nu$. Therefore ϕ_ν is surjective.

Now we check that ϕ_ν is a homomorphism of Σ_r-modules. For this we have to see that for all $\sigma \in \Sigma_r$, $f \in \mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right)$, and $m \in M^\nu$ there holds

$$\phi_\nu\left(\theta_{T(\omega(\sigma))}f\right)(m) = \phi_\nu(f)\left(\sigma^{-1}m\right). \tag{11}$$

We will show that $\theta_{T(\omega(\sigma))}$ acts by permutation on the basis $\{\, \sigma'\mathfrak{t}_\delta \,|\, \sigma' \in \Sigma_r \}$ of M^δ. Then $\phi_\nu\left(\theta_{T(\omega(\sigma))}f\right)(m)$ will be the coefficient of some $\sigma'\mathfrak{t}_\delta$ in $f(m)$.

Let $\sigma' \in \Sigma_r$. Then $\sigma'\mathfrak{t}_\delta$ is obtained from $\mathfrak{t}_\delta$ by applying σ' to every entry of $\mathfrak{t}_\delta$. Thus $\sigma'\mathfrak{t}_\delta$ is a δ-tableau with the entry $\sigma'(s)$ in the row s.

We know that $\omega(\sigma)_{st}$ is non-zero only if $t = \sigma^{-1}s$. Now by our reformulation of Boltje-Hartmann rule, $\theta_{T(\omega(\sigma))}\left(\sigma'\mathfrak{t}_\delta\right)$ is the sum of those $\mathfrak{s}$ for which the s-th row of $\mathfrak{s}$ contains precisely one entry from the $\sigma^{-1}s$ row of $\sigma'\mathfrak{t}_\delta$. Of course such $\mathfrak{s}$ is unique and the entry in the row s is $\sigma'\left(\sigma^{-1}s\right)$. Thus $\theta_{T(\omega(\sigma))}\left(\sigma'\mathfrak{t}_\delta\right) = \sigma'\sigma^{-1}\mathfrak{t}_\delta$.

Therefore the coefficient of $\mathfrak{t}_\delta$ in $\theta_{T(\omega(\sigma))}f(m)$ is the same as coefficient of $\sigma\mathfrak{t}_\delta$ in $f(m)$, which is also the coefficient of $\mathfrak{t}_\delta$ in $f\left(\sigma^{-1}m\right)$. So (11) holds.

It is left to check that for every ν, $\mu \in \Lambda(n;r)$, and $h \in \mathrm{Hom}_{R\Sigma_r}(M^\mu, M^\nu)$, $f \in \mathrm{Hom}_{R\Sigma_r}\left(M^\nu, M^\delta\right)$, we have $\phi_\mu(fh) = \phi_\nu(f)\,h$. But this is immediate since both $\phi_\mu(fh)(m)$ and $\phi_\nu(f)(h(m))$ are the coefficients of $\mathbf{t}_\delta$ in $fh(m)$, for every $m \in M^\mu$. ■

8. Appendix

In this appendix we discuss the connection between the general linear group and Schur algebras as they are defined in this paper. We start with the proof that our definition of Schur algebra is equivalent to the one given in Green [Gre93].

We define $A_R(n)$ to be the commutative ring $R[c_{s,t}\colon s,t \in \mathbf{n}]$ in the indeterminates c_{st}.

For every i, $j \in I(n,r)$ we denote by $c_{i,j}$ the product $c_{i_1,j_1}\dots c_{i_r,j_r}$. Then (i,j) and $(i',j') \in I(n,r) \times I(n,r)$ are on the same Σ_r-orbit if and only if $c_{i,j} = c_{i',j'}$. We will write c_ω for $c_{i,j}$ if $(i,j) \in \omega$.

Denote by $A_R(n,r)$ the R-submodule of $A_R(n)$ of all homogeneous polynomials of degree r in the c_{st}. Then $\{\, c_\omega \,|\, \omega \in \Lambda(n,n;r)\}$ is an R-basis of $A_R(n,r)$ and $A_n = \bigoplus_{r\ge 0} A_R(n,r)$. It is well known (see [Gre93]) that $A_R(n,r)$ has the structure of a coassociative coalgebra with the structure maps given by

$$\triangle(c_{i,j}) := \sum_{k\in I(n,r)} c_{i,k}\otimes c_{k,j}, \qquad\qquad \varepsilon(c_{i,j}) := \delta_{i,j}.$$

The Schur algebra $S_R^{Gr}(r,n)$ in the sense of Green is the R-algebra dual to the coalgebra $A_R(n,r)$. Let $\left\{ \widehat{\xi}_\omega \,\middle|\, \omega \in \Lambda(n,n;r)\right\}$ be the R-basis of $S_R^{Gr}(n,r)$ dual to $\{\, c_\omega \,|\, \omega \in \Lambda(n,n;r)\}$. Define a map ϕ from $S_R^{Gr}(n,r)$ to $\mathrm{End}_R\left((R^n)^{\otimes r}\right)$ by

$$\phi(f)\,e_j := \sum_{i\in I(n,r)} f(c_{i,j})\,e_i.$$

Theorem 8.1. *The map ϕ provides an R-algebra isomorphism between $S_R^{Gr}(n,r)$ and $S_R(n,r)$.*

Proof: Clearly ϕ maps the identity of $S_R^{Gr}(n,r)$ into the identity map of $\mathrm{End}_R\left((R^n)^{\otimes r}\right)$. Let f_1, $f_2 \in S_R^{Gr}(n,r)$. Then the product (see [Gre07] or [Gre93]) of f_1 and f_2 is defined by

$$(f_1 f_2)(c_{i,j}) = \sum_{k\in I(n,r)} f_1(c_{i,k}) f_2(c_{k,j}).$$

Therefore

$$\phi(f_1 f_2) e_j = \sum_{i\in I(n,r)} \sum_{k\in I(n,r)} f_1(c_{i,k}) f_2(c_{k,j}) e_i.$$

On the other hand

$$\phi(f_1)\phi(f_2) e_j = \phi(f_1) \sum_{k\in I(n,r)} f_2(c_{k,j}) e_k = \sum_{k\in I(n,r)} \sum_{i\in I(n,r)} f_2(c_{k,j}) f_1(c_{i,k}) e_i.$$

Therefore $\phi(f_1 f_2) = \phi(f_1)\phi(f_2)$. We have also $\phi\left(\widehat{\xi}_\omega\right) = \xi_\omega$. In fact

$$\phi\left(\widehat{\xi}_\omega\right) e_j = \sum_{i\in I(n,r)} \widehat{\xi}_\omega(c_{i,j}) e_i = \sum_{i\in I(n,r)\colon (i,j)\in\omega} e_i = \xi_\omega e_j.$$

Thus the result follows. ■

To each $c_{s,t} \in A_R(n)$ we can associate a function $\widetilde{c}_{s,t}\colon \mathrm{GL}_n(R) \to R$ defined by $\widetilde{c}_{s,t}(g) = g_{st}$, $g = (g_{s,t})_{s,t=1}^n \in \mathrm{GL}_n(R)$. The correspondence $c_{s,t} \mapsto \widetilde{c}_{s,t}$ induces an algebra homomorphism ψ from $A_R(n)$ to the algebra of maps from $\mathrm{GL}_n(R)$ to R. The homomorphism ψ is injective if R is an infinite field, but this is not true for arbitrary commutative rings or even finite fields. We write $\widetilde{c}_\omega$ for $\psi(c_\omega)$, $\omega \in \Lambda(n,n;r)$. Then for all $g \in \mathrm{GL}_n(R)$ and $(i,j) \in \omega$ we have

$$\widetilde{c}_\omega(g) = \prod_{s,t=1}^n g_{st}^{\omega_{st}}, \qquad \widetilde{c}_{i,j}(g) = \prod_{q=1}^r g_{i_q,j_q}. \tag{12}$$

Next we give an overview of the functor of divided powers and explain the relation between divided powers and principal projective modules over the Schur algebra $S_R(n,r)$.

We start by recalling the definition of the algebra of divided powers $D(M)$ associated to an R-module M. A good reference on this subject is [Rob63].

We denote by $\mathscr{D}(M)$ the free algebra over R generated by variables $X_{(m,k)}$, $m \in M$ and $k \in \mathbb{N}$. Now $D(M)$ is defined as the quotient of $\mathscr{D}(M)$ by the ideal generated by

$$\begin{aligned} & X_{(m,0)} - 1 && (m \in M) \\ & X_{(am,k)} - a^k X_{(m,k)} && (a \in R,\ m \in M,\ k \geq 0) \\ & X_{(m,k)} X_{(m,l)} - \binom{k+l}{k} X_{(m,k+l)} && (m \in M,\ k,\ l \geq 0) \\ & X_{(m_1+m_2,l)} - \sum_{k=0}^{l} X_{(m_1,k)} X_{(m_2,l-k)} && (m_1,\ m_2 \in M,\ l \geq 0)\,. \end{aligned} \tag{13}$$

We will denote the image of $X_{(m,k)}$ in $D(M)$ by $m^{(k)}$. Then, besides the relations determined by (13), it also holds

$$\begin{aligned} 0^{(k)} &= 0, && k \geq 1 \\ m^{(k_1)} \dots m^{(k_s)} &= \binom{k_1 + \dots + k_s}{k_1, \dots, k_s} m^{(k_1+\dots+k_s)}, && m \in M;\ k_1, \dots, k_s \geq 0 \\ (m_1 + \dots + m_s)^{(k)} &= \sum_{k_1+\dots+k_s=k} m_1^{(k_1)} \dots m_s^{(k_s)}, && m_1, \dots, m_s \in M; k \geq 0. \end{aligned}$$

If M and N are two R-modules and $f\colon M \to N$ is a homomorphism of R-modules, we can define an R-algebra homomorphim $D(f) : D(M) \to D(N)$ by $D(f)\left(m^{(k)}\right) = (f(m))^{(k)}$, $m \in M$, $k \geq 0$. Note that the rings $\mathscr{D}(M)$ are graded, with $deg\left(X_{(m,k)}\right) = k$, $m \in M$, $k \geq 0$. As all the relations (13) are homogeneous, the ring $D(M)$ is graded as well. Now the map $D(f)$ preserves the grading for any map of R-modules f. Therefore D is a functor from the category of R-modules to the category of graded R-algebras. We denote by $D_k(M)$ the k-th homogeneous component of $D(M)$. Given a homomorphism of R-modules $f\colon M \to N$ we define $D_k(f)$ to be the restriction of $D(f)$ to $D_k(M)$. In this way we obtain the endofunctor D_k on the category of R-modules.

Define the map $\tau_k\colon \mathrm{GL}(M) \to \mathrm{End}_R(D_k(M))$ by $\tau_k(f) := D_k(f)$. This map is obviously multiplicative and thus extends to a representation

$$\tau_k\colon R\mathrm{GL}(M) \to \mathrm{End}_R(D_k(M))$$

of the group $\mathrm{GL}(M)$. Thus we get a structure of $\mathrm{GL}(M)$-module on $D_k(M)$. As the category of $\mathrm{GL}(M)$-modules is monoidal we can define for every

$\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n;r)$ the $\mathrm{GL}(M)$-module $D_\lambda(M)$ by

$$D_\lambda(M) = D_{\lambda_1}(M) \otimes_R \cdots \otimes_R D_{\lambda_n}(M).$$

Now we shall give a more explicit description of $D_\lambda(M)$ in the case $M = R^n$.

Recall that $\{e_1, \ldots, e_n\}$ denotes the standard basis of R^n. By Theorem IV.2 in [Rob63]

$$\left\{ e_1^{(k_1)} \ldots e_n^{(k_n)} \,\middle|\, k_t \in \mathbb{N}, \sum_{t=1}^{n} k_t = k \right\}$$

is a basis of $D_k(R^n)$ for $k \geq 0$. Thus the set

$$\left\{ e^{(\pi)} \,\middle|\, \pi \in \Lambda(n,n;r), \pi^1 = \lambda \right\},$$

where $e^{(\pi)} = \bigotimes_{t=1}^{n} \prod_{s=1}^{n} e_s^{(\pi_{st})}$, is an R-basis of the R-module $D_\lambda(R^n)$.

Proposition 8.2. *Let $\lambda \in \Lambda(n;r)$. Then the action of $\mathrm{GL}_n(R)$ on $D_\lambda(R^n)$ is given by*

$$ge^{(\pi)} = \sum_{\theta \in \Lambda(n,n,n;r)\colon\, \theta^1 = \pi} [\theta]\, \widetilde{c}_{\theta^3}(g)\, e^{\left(\theta^2\right)}, \;\; \pi \in \Lambda(n,n;r), \;\; \pi^1 = \lambda, \;\; g \in \mathrm{GL}_n(R).$$

Proof: Let $g = (g_{st})_{s,t=1}^{n} \in \mathrm{GL}_n(R)$ and $\pi \in \Lambda(n,n;r)$ such that $\pi^1 = \lambda$. Then

$$\begin{aligned} ge^{(\pi)} &= \bigotimes_{t=1}^{n} \prod_{s=1}^{n} (ge_s)^{(\pi_{st})} = \bigotimes_{t=1}^{n} \prod_{s=1}^{n} \left(\sum_{q=1}^{n} g_{qs} e_q \right)^{(\pi_{st})} \\ &= \bigotimes_{t=1}^{n} \prod_{s=1}^{n} \sum_{\nu \in \Lambda(n;\pi_{st})} \prod_{q=1}^{n} g_{q,s}^{\nu_q} e_q^{(\nu_q)} \\ &= \sum_{\nu\colon \mathbf{n}\times\mathbf{n} \to \Lambda(n)} \bigotimes_{t=1}^{n} \prod_{s,q=1}^{n} g_{q,s}^{\nu(s,t)_q} e_q^{\left(\nu(s,t)_q\right)}, \end{aligned}$$

where the summation is over the set $X(\pi)$ of functions

$$\nu\colon \mathbf{n} \times \mathbf{n} \to \Lambda(n) = \bigcup_{r \geq 0} \Lambda(n;r)$$

such that $\nu(s,t) \in \Lambda(n;\pi_{st})$. There is a one-to-one correspondence between $X(\pi)$ and the set $Y(\pi) := \left\{ \theta \in \Lambda(n,n,n;r) \,\middle|\, \theta^1 = \pi \right\}$. In fact, let $\theta \in Y(\pi)$.

Then we can define $\nu\colon \mathbf{n}\times\mathbf{n}\to\Lambda(n)$ by

$$(s,t)\mapsto(\theta_{1st},\dots,\theta_{nst}).$$

Since $\left(\theta^1\right)_{st}=\pi_{st}$ we get $\nu(s,t)\in\Lambda(n;\pi_{st})$ for all $1\le s,t\le n$. Now let $\nu\in X(\pi)$. Then we can define $\theta\in\Lambda(n,n,n;r)$ by $\theta_{qst}=\nu(s,t)_q$. Since $\nu(s,t)\in\Lambda(n;\pi_{st})$ we get that $\theta^1=\pi$. Thus $\theta\in Y(\pi)$. It is easy to see that these constructions are mutually inverse. Therefore

$$ge^{(\pi)}=\sum_{\theta\in\Lambda(n,n,n;r)\colon\,\theta^1=\pi}\ \bigotimes_{t=1}^{n}\prod_{s,q=1}^{n} g_{q,s}^{\theta_{qst}}e_q^{(\theta_{qst})}.$$

Since R is a commutative ring it is left to show that for every $\theta\in Y(\pi)$ we have

$$\prod_{t,s,q=1}^{n} g_{qs}^{\theta_{qst}}=\widetilde{c}_{\theta^3}(g),\qquad\qquad \bigotimes_{t=1}^{n}\prod_{s,q=1}^{n} e_q^{(\theta_{qst})}=[\theta]\,e^{\left(\theta^2\right)}.$$

For the first formula we have

$$\prod_{t,s,q=1}^{n} g_{qs}^{\theta_{qst}}=\prod_{s,q=1}^{n} g_{qs}^{\sum_{t=1}^{n}\theta_{qst}}=\prod_{s,q=1}^{n} g_{qs}^{\left(\theta^3\right)_{qs}}\overset{(12)}{=}\widetilde{c}_{\theta^3}(g).$$

For the second identity we fix $t\in\mathbf{n}$ and compute the product

$$\prod_{s,q=1}^{n} e_q^{(\theta_{qst})}=\prod_{q=1}^{n}\left(\prod_{s=1}^{n} e_q^{(\theta_{qst})}\right)=\prod_{q=1}^{n}\binom{\left(\theta^2\right)_{qt}}{\theta_{q1t},\dots,\theta_{qnt}}e_q^{\left(\left(\theta^2\right)_{qt}\right)}.$$

Now the result follows. ∎

The group $\mathrm{GL}_n(R)$ acts naturally on R^n, by multiplication. Thus $\mathrm{GL}_n(R)$ acts on $(R^n)^{\otimes r}$ diagonally. Denote by $\rho_{n,r}$ the corresponding representation of $\mathrm{GL}_n(R)$. Since for every $g\in\mathrm{GL}_n(R)$ the endomorphism $\rho_{n,r}(g)$ commutes with the action of Σ_r on $(R^n)^{\otimes r}$, we get that $\mathrm{Im}(\rho_{n,r})\subset S_R(n,r)$.

Proposition 8.3. *Let $g\in\mathrm{GL}_n(R)$. Then $\rho_{n,r}(g)=\sum_{\omega\in\Lambda(n,n;r)}\widetilde{c}_\omega(g)\,\xi_\omega$.*

Proof: Let $i \in I(n,r)$. Then we have

$$\begin{aligned} ge_i &= ge_{i_1} \otimes \cdots \otimes ge_{i_r} \\ &= \sum_{j \in I(n,r)} \left(g_{j_1,i_1} \ldots g_{j_r,i_r}\right) e_{j_1} \otimes \cdots \otimes e_{j_r} \\ &\overset{(12)}{=} \sum_{j \in I(n,r)} \widetilde{c}_{\mathrm{wt}(j,i)}(g)\, e_j = \sum_{\omega \in \Lambda(n,n;r)} \widetilde{c}_\omega(g) \sum_{j\colon (j,i)\in\omega} e_j = \sum_{\omega \in \Lambda(n,n;r)} \widetilde{c}_\omega(g)\, \xi_\omega e_i. \end{aligned}$$

As the set $\{ e_i \,|\, i \in I(n,r)\}$ is a basis of $(R^n)^{\otimes r}$, we see that $\rho_{n,r}(g) = \sum_{\omega \in \Lambda(n,n;r)} \widetilde{c}_\omega(g)\, \xi_\omega$. ■

Theorem 8.4. *Let $\lambda \in \Lambda(n,r)$ and consider $S_R(n,r)\,\xi_\lambda$ as a $\mathrm{GL}_n(R)$-module via the homomorphism $\rho_{n,r}$. Then the map*

$$\begin{aligned} \psi\colon D_\lambda(R^n) &\to S_R(n,r)\,\xi_\lambda \\ e^{(\pi)} &\mapsto \xi_\pi \end{aligned}$$

is an isomorphism of $\mathrm{GL}_n(R)$-modules.

Proof: It is clear that ψ is an isomorphism of R-modules. Thus it is enough to show that ψ commutes with the action of $\mathrm{GL}_n(R)$. Let $g \in \mathrm{GL}_n(R)$ and $\pi \in \Lambda(n,n;r)$, $\pi^1 = \lambda$. Then by Proposition 8.2 we have

$$\psi\left(ge^{(\pi)}\right) = \sum_{\theta \in \Lambda(n,n,n;r)\colon \theta^1=\pi} [\theta]\, \widetilde{c}_{\theta^3}(g)\, \psi\left(e^{(\theta^2)}\right) = \sum_{\theta \in \Lambda(n,n,n;r)\colon \theta^1=\pi} [\theta]\, \widetilde{c}_{\theta^3}(g)\, \xi_{\theta^2}.$$

On the other hand by Proposition 8.3 and Proposition 2.3

$$\begin{aligned} g\psi\left(e^{(\pi)}\right) &= \left(\sum_{\omega \in \Lambda(n,n;r)} \widetilde{c}_\omega(g)\, \xi_\omega\right) \xi_\pi \\ &= \sum_{\omega \in \Lambda(n,n;r)} \widetilde{c}_\omega(g) \sum_{\theta \in \Lambda(n,n,n;r)\colon \theta^3=\omega,\ \theta^1=\pi} [\theta]\, \xi_{\theta^2} \\ &= \sum_{\theta \in \Lambda(n,n,n;r)\colon \theta^1=\pi} [\theta]\, \widetilde{c}_{\theta^3}(g)\, \xi_{\theta^2}. \end{aligned}$$

■

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