Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 11–10

DIFFERENCE AND DIFFERENTIAL EQUATIONS FOR DEFORMED LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

A. BRANQUINHO AND M.N. REBOCHO

ABSTRACT: Reflection parameters of Laguerre-Hahn orthogonal polynomials on the unit circle are studied.

KEYWORDS: Carathéodory function, measures on the unit circle, matrix Sylvester differential equations, semi-classical class.
AMS SUBJECT CLASSIFICATION (2000): 33C45, 39B42.

1. Introduction and notation

Let μ be a positive Borel measure with infinite support on the unit circle $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\,\}$ with moments

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} z^{-k}\, d\mu\,,\ k \in \mathbb{Z},\ z = e^{i\theta},$$

satisfying $c_k \in \ell^1(\mathbb{T})$, for all $k \in \mathbb{Z}$, and $\bar{c}_k = c_{-k}$, for all $k \geq 0$, where the bar denotes the complex conjugate. With μ one associates the *Carathéodory function*, defined by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\,,$$

and the sequence of orthogonal polynomials with respect to μ, called *orthogonal polynomials on the unit circle* (OPUC),

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_n(z)\overline{\phi_m(z)}\, d\mu = h_n \delta_{n,m}\,,\ z = e^{i\theta}\,,\quad h_n \geq 0\,,\ n, m \geq 0\,. \tag{1}$$

If μ is absolutely continuous associated with a weight w, then we say that $\{\phi_n\}$ is orthogonal with respect to w.

Received April 6, 2011.
This work was supported by CMUC, Department of Mathematics, University of Coimbra. The second author was supported by FCT, Fundação para a Ciência e Tecnologia, with grant ref. SFRH/BPD/45321/2008.

Throughout this paper we will consider that ϕ_n is monic, for all $n \in \mathbb{N}$, thus $\{\phi_n\}$ will be called a *monic orthogonal polynomial sequence* and it will be denoted by MOPS.

Note that

$$\phi_0(z) = 1\,,\ \phi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & \overline{c}_1 & \cdots & \overline{c}_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \cdots & \overline{c}_1 \\ 1 & z & \cdots & z^n \end{vmatrix}\,,\ n \in \mathbb{N}\,, \tag{2}$$

where Δ_n is the minor of the Toeplitz matrix associated with μ, defined as

$$\Delta_{-1} = 1\,,\ \Delta_0 = c_0\,,\ \Delta_n = \begin{vmatrix} c_0 & \cdots & c_n \\ \vdots & \ddots & \vdots \\ \overline{c}_n & \cdots & c_0 \end{vmatrix}\,,\ n \in \mathbb{N}\,.$$

The numbers $a_n = \phi_n(0),\ n \in \mathbb{N}$, known as *reflection parameters*, satisfy

$$1 - |a_n|^2 = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}\,,\ n \in \mathbb{N}\,.$$

The *associated polynomials of the second kind* are defined by

$$\Omega_0(z) = 1,\ \Omega_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\,d\mu(\theta)\,,\ n \geq 1\,,$$

and the *functions of the second kind* are given by

$$Q_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\,d\mu(\theta)\,,\ n = 0, 1, \ldots$$

Throughout the text we will use the matrices Y_n defined as

$$Y_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\,,\ Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix},\ n \in \mathbb{N}\,, \tag{3}$$

where ϕ_n^* and Ω_n^* denote the reciprocal polynomial of ϕ_n and Ω_n, respectively. We recall that the reciprocal polynomial p^* of a polynomial p of exact degree n is defined by $p^*(z) = z^n \overline{p}(1/z)$. We write the Szegő recurrence relations in the matrix form (see [14]),

$$Y_n = \mathcal{A}_n Y_{n-1},\ \ \mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n\, z & 1 \end{bmatrix}\,,\ n \in \mathbb{N}\,. \tag{4}$$

where $\mathcal{A}_n$ are the so-called *transfer matrices*. It is well known that the reflection parameters describe completely the sequence of orthogonal polynomials, $\{\phi_n\}$, in (4) they satisfy $|a_n| < 1$, for all $n \in \mathbb{N}$ and, conversely, given an arbitrary sequence of complex numbers (a_n) with the only restriction $|a_n| < 1$, for all $n \in \mathbb{N}$, the polynomials ϕ_n defined by (4) are orthogonal with respect to a unique probability measure with infinite support on $\mathbb{T}$ such that $a_n = \phi_n(0)$, for all $n \in \mathbb{N}$ (see [9]).

The present paper deals with measures supported on the unit circle whose corresponding Carathéodory function satisfies a Riccati type differential equation

$$zAF' = BF^2 + CF + D\,,\; A \not\equiv 0\,. \tag{5}$$

where A, B, C, D are co-prime polynomials. The class of Carathéodory functions satisfying (5) and the corresponding sequences of orthogonal polynomials is known as the *Laguerre-Hahn class on the unit circle* (see [3, 5]). Some well-known families of polynomials on the unit circle orthogonal with respect to Carathéodory functions of this type are the *Laguerre-Hahn affine orthogonal polynomials*, which correspond to the case $B \equiv 0$ in (5), as well as the *semi-classical orthogonal polynomials*, which correspond to the case $B \equiv 0$ and D a specific polynomial in (5). Also, fractional linear transformations of the above mentioned Carathéodory functions are members of the Laguerre-Hahn class, whenever the resulting transformation is a Carathéodory function.

The aim of this work is twofold: on the one hand, to study difference equations for the reflection parameters of Laguerre-Hahn OPUC, and on the other hand, to study continuous equations that follow when deformations of Laguerre-Hahn Carathéodory functions occur through a dependence on a certain parameter, t.

So far, the analysis of orthogonal polynomials with respect to deformed measures via a $t-$dependence parameter has been carried out for the semi-classical class. This has been done in a vast list of papers, and we refer the reader to [2, 10, 11, 12, 15, 16]. It is well-known the connection of semi-classical orthogonal polynomials with integrable systems, Painlevé equations (discrete and continuous) Toda lattices, monodromy theory (see the introduction of [11, 12, 15, 16] and its list of references). For example, in [11, 12] continuous Painlevé equations were found from the reflection parameters of semi-classical orthogonal polynomials on the unit circle.

This paper is organized as follows: in Section 2 we begin by presenting differential systems for the matrices Y_n corresponding to Carathéodory functions satisfying (5) (cf. Theorem 1). Then we shown that compatibility between the above mentioned differential system and the recurrence relation written in the matrix form (4) leads to semi-discrete Lax equations for the corresponding matrices $\mathcal{A}_n$ in (4) (cf. Corollary 2). In Section 3 we study the t-dependent case. In Section 4 we present an example: we study the reflection parameters of the sequence of OPUC with respect to a non semi-classical measure, the sum of a Jacobi measure with the Lebesgue measure on the unit circle.

Throughout the paper we will use f' to denote the derivative of f with respect to z and $\dot{f}$ to denote the derivative of f with respect to t.

2. Differential linear system for Laguerre-Hahn OPUC and discrete Lax equations

We begin by presenting a differential linear system for Laguerre-Hahn orthogonal polynomials. The differential relations (5) that follow were deduced in [4, Theorem 3], thus we omit here the proof of it.

Theorem 1 ([4]). *Let F be a non-rational Carathéodory function, let $\{Y_n\}$ be the corresponding sequence defined by (3) and let $\mathcal{Q}_n = \begin{bmatrix} -Q_n \\ Q_n^* \end{bmatrix}$, $n \geq 1$. The following statements are equivalent:*
(a) F satisfies the differential equation with polynomial coefficients

$$zAF' = BF^2 + CF + D\,; \tag{6}$$

(b) $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ satisfy

$$zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \tag{7}$$

$$zA\mathcal{Q}_n' = \left(\mathcal{B}_n + (BF + C/2)\, I\right) \mathcal{Q}_n\,,\ n \in \mathbb{N}\,, \tag{8}$$

where $\mathcal{B}_n$ are matrices of bounded degree polynomials,

$$\mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}, \tag{9}$$

and

$$\mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}. \tag{10}$$

Remark . Eqs. (7) are particular cases of matrix Riccati equations, known as matrix Sylvester differential equations (see [1, 17]).

As a consequence of the previous Theorem the following result holds (see [4, Theorem 4] and [11, Corollary 2.4]).

Theorem 2. *Let $\{\phi_n\}$ be a MOPS with respect to a weight w, let $\{Q_n\}$ be the sequence of functions of the second kind, and* $\widehat{Y}_n = \begin{bmatrix} \phi_n & -Q_n/w \\ \phi_n^* & Q_n^*/w \end{bmatrix}$, $n \geq 1$. *Then,*

$$\frac{w'}{w} = \frac{C}{zA} \tag{11}$$

if, and only if, $\widehat{Y}_n$ *satisfies*

$$zA\widehat{Y}_n' = (\mathcal{B}_n - C/2\ I)\widehat{Y}_n\,,\ n \in \mathbb{N}\,, \tag{12}$$

where $\mathcal{B}_n$ is the matrix associated with the equation $zAF' = CF + D$ satisfied by the Carathéodory function of w.

Corollary 1 ([4]). *Under the conditions of the preceding theorem, the matrix $\mathcal{B}_n$ given by (9) satisfies*

$$\operatorname{tr}(\mathcal{B}_n) = nA\,,\ n \in \mathbb{N}\,, \tag{13}$$

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_1) - A\sum_{k=1}^{n-1} l_{k,2}\,,\ n \geq 2\,, \tag{14}$$

where $\operatorname{tr}(\mathcal{B}_n)$ *and* $\det(\mathcal{B}_n)$ *denote the trace and the determinant of* $\mathcal{B}_n$, *respectively, and*

$$\det(\mathcal{B}_1) = A\left(2zA\overline{a}_1 - h_1(D+B) + C(|a_1|^2+1)\right)/(2\,h_1) + BD - C^2/4\,,$$

$a_1 = \phi_1(0),\ h_1 = 1 - |a_1|^2$.

Corollary 2 ([4]). *Eqs. (7) are equivalent to the discrete Lax equations*

$$zA\mathcal{A}_n' = \mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1}\,,\ n \geq 2\,, \tag{15}$$

Proof: To prove that (7) implies (15) we use the matrix form of the recurrence relation (4) to get a compatibility relation that yields (15) (this was done in [4, Corollary 3]).

To prove the converse, let us multiply (15) by Y_{n-1}. Then we obtain

$$zA(\mathcal{A}_nY_{n-1})' - zA\mathcal{A}_nY_{n-1}' = \mathcal{B}_n\mathcal{A}_nY_{n-1} - \mathcal{A}_n\mathcal{B}_{n-1}Y_{n-1}.$$

Taking into account the recurrence relations (4) we get

$$zAY_n' - \mathcal{B}_nY_n = \mathcal{A}_n(zAY_{n-1}' - \mathcal{B}_{n-1}Y_{n-1}).$$

Iterating we get

$$zAY_n' - \mathcal{B}_nY_n = \mathcal{A}_n \cdots \mathcal{A}_2(zAY_1' - \mathcal{B}_1Y_1).$$

Using $\mathcal{A}_n \cdots \mathcal{A}_2 = Y_nY_1^{-1}$ in the preceding equation yields $zAY_n' = \mathcal{B}_nY_n - Y_n\widetilde{C}$ with $\widetilde{C} = -Y_1^{-1}(zAY_1' - \mathcal{B}_1Y_1)\,,\ n \geq 2\,.$ ∎

3. Differential equations for deformed Laguerre-Hahn

Henceforth we consider Carathéodory functions satisfying differential equations $zAF' = BF^2 + CF + D$, where A, B, C, D are co-prime polynomials now depending on a parameter t (we consider a general t-dependence). Notice that the corresponding orthogonal polynomials, as well as its reflection parameters, will then also depend on t. Thus, one of the topics to be studied is the t-differential equations satisfied by the reflection parameters.

Theorem 3. *Let F satisfy $zAF' = BF^2 + CF + D$, where A, B, C, D are polynomials that depend on a parameter, t, and let the corresponding Y_n satisfy the matrix Sylvester equations $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$, $n \geq 1$, and let $\mathcal{L}$ be a nonsingular matrix such that $zA\mathcal{L}' = \mathcal{C}\mathcal{L}$. Then, the matrices H_n defined by*

$$\mathcal{H}_n = (\dot{Y}_n + Y_n\dot{\mathcal{L}}\mathcal{L}^{-1})Y_n^{-1}\,,\ n \geq 1\,, \tag{16}$$

satisfy

$$\frac{\partial}{\partial t}\left(\frac{\mathcal{B}_n}{zA}\right) = \mathcal{H}_n' + \mathcal{H}_n\frac{\mathcal{B}_n}{zA} - \frac{\mathcal{B}_n}{zA}\mathcal{H}_n\,,\ n \geq 1\,. \tag{17}$$

Furthermore,

$$\dot{\mathcal{A}}_n = \mathcal{H}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{H}_{n-1}\,,\ n \geq 1\,, \tag{18}$$

where $\mathcal{A}_n$ are the transfer matrices corresponding to $\{Y_n\}$.

Proof: Firstly we deduce (17).

On the one hand, taking derivatives with respect to z in (16), we have

$$\frac{\partial(\dot{Y}_n)}{\partial z} = \mathcal{H}_n'Y_n + \mathcal{H}_nY_n' - Y_n'(\dot{\mathcal{L}}\mathcal{L}^{-1}) - Y_n(\dot{\mathcal{L}}\mathcal{L}^{-1})'\,.$$

If we use $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$ and

$$(\dot{\mathcal{L}}\mathcal{L}^{-1})' = \frac{\partial}{\partial t}\left(\frac{\mathcal{C}}{zA}\right) + \frac{\mathcal{C}}{zA}\dot{\mathcal{L}}\mathcal{L}^{-1} - \dot{\mathcal{L}}\mathcal{L}^{-1}\frac{\mathcal{C}}{zA}$$

in the previous equation there follows

$$\frac{\partial(\dot{Y}_n)}{\partial z} = \mathcal{H}_n' Y_n + \mathcal{H}_n \left(\frac{\mathcal{B}_n}{zA} Y_n - Y_n \frac{\mathcal{C}}{zA} \right) - \left(\frac{\mathcal{B}_n}{zA} Y_n - Y_n \frac{\mathcal{C}}{zA} \right) (\dot{\mathcal{L}}\mathcal{L}^{-1}) \\ - Y_n \frac{\partial}{\partial t} \left(\frac{\mathcal{C}}{zA} \right) - Y_n \frac{\mathcal{C}}{zA} \dot{\mathcal{L}}\mathcal{L}^{-1} + Y_n \dot{\mathcal{L}}\mathcal{L}^{-1} \frac{\mathcal{C}}{zA} . \quad (19)$$

On the other hand, taking derivatives with respect to t in $zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, we get

$$\frac{\partial(Y_n')}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\mathcal{B}_n}{zA} \right) Y_n + \frac{\mathcal{B}_n}{zA} \dot{Y}_n - \dot{Y}_n \frac{\mathcal{C}}{zA} - Y_n \frac{\partial}{\partial t} (\mathcal{C}/zA) \ ,$$

and the use of (16) in the preceding equation yields

$$\frac{\partial(Y_n')}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\mathcal{B}_n}{zA} \right) Y_n + \frac{\mathcal{B}_n}{zA} (\mathcal{H}_n Y_n - Y_n \dot{\mathcal{L}}\mathcal{L}^{-1}) \\ - (\mathcal{H}_n Y_n - Y_n \dot{\mathcal{L}}\mathcal{L}^{-1}) \frac{\mathcal{C}}{zA} - Y_n \frac{\partial}{\partial t} \left(\frac{\mathcal{C}}{zA} \right) . \quad (20)$$

The compatibility of (19) and (20) gives

$$\mathcal{H}_n' Y_n + \mathcal{H}_n \frac{\mathcal{B}_n}{zA} Y_n = \frac{\partial}{\partial t} \left(\frac{\mathcal{B}_n}{zA} \right) Y_n + \frac{\mathcal{B}_n}{zA} \mathcal{H}_n Y_n \, .$$

Since Y_n is nonsingular, for $z \neq 0$, there follows (17).

To deduce (18) we use the recurrence relation (4),

$$\frac{\partial}{\partial t} (\mathcal{A}_n Y_{n-1}) = \mathcal{H}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n Y_{n-1} \mathcal{L}\mathcal{L}^{-1} \, ,$$

thus obtaining

$$\dot{\mathcal{A}}_n Y_{n-1} + \mathcal{A}_n \dot{Y}_{n-1} = \mathcal{H}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n Y_{n-1} \mathcal{L}\mathcal{L}^{-1} \, .$$

The use of (16) to $n-1$ in the above equation yields

$$\dot{\mathcal{A}}_n Y_{n-1} = \mathcal{H}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n \mathcal{H}_{n-1} Y_{n-1} \, .$$

Since Y_n is nonsingular, for $z \neq 0$, there follows (18). ∎

Remark . We have proceeded in similarity with [16], that is, we started by defining a matrix $\mathcal{H}_n$, in (16), and then Eq. (17) was deduced. Note the similarity between our Eq. (17) and [16, Eq. (24)] that holds for the semi-classical class (see also [11]).

Remark . Equations (17) and (18) enclose differential equations in t for the corresponding sequences of reflection parameters.

Further properties of $\mathcal{H}_n$ follow. We denote by $\mathcal{H}_n^{(i,j)}$ the element in position (i,j) of the matrix $\mathcal{H}_n$.

Lemma 1. *Let $\mathcal{H}_n$ be given by (16). The following assertions hold:*
(i) For all $n \in \mathbb{N}$, $\operatorname{tr}(\mathcal{H}_n)$ does not depend on z. Furthermore,

$$\operatorname{tr}(\mathcal{H}_n) = \frac{\dot{h}_n}{h_n} + \operatorname{tr}(\dot{\mathcal{L}}\mathcal{L}^{-1}), \tag{21}$$

where h_n is the same constant appearing in (1).
(ii) For all $n \in \mathbb{N}$,

$$\det(\mathcal{H}_n) = \det(\mathcal{H}_1) + \sum_{k=2}^{n} \xi_k, \tag{22}$$

with $\xi_k = -\frac{1}{1-|a_k|^2}\left(|\dot{a}_k|^2(1+\operatorname{tr}(\mathcal{H}_1) + \frac{\partial}{\partial t}\log(a_k/a_1)) + \dot{a}_k\frac{\mathcal{H}_{k-1}^{(2,1)}}{z} + \dot{\overline{a_k}} z\mathcal{H}_{k-1}^{(1,2)}\right)$.

Proof: From (17) we have

$$\frac{\partial}{\partial t}\left(\frac{l_{n,1}}{zA}\right) = (\mathcal{H}_n^{(1,1)})' + \frac{1}{zA}\left(\Theta_{n,1}\mathcal{H}_n^{2,1} - \Theta_{n,2}\mathcal{H}_n^{1,2}\right), \tag{23}$$

$$\frac{\partial}{\partial t}\left(\frac{l_{n,2}}{zA}\right) = (\mathcal{H}_n^{(2,2)})' + \frac{1}{zA}\left(-\Theta_{n,1}\mathcal{H}_n^{(2,1)} + \Theta_{n,2}\mathcal{H}_n^{(1,2)}\right). \tag{24}$$

Summing (23) with (24) gives us

$$\frac{\partial}{\partial t}\left(\frac{l_{n,1}+l_{n,2}}{zA}\right) = (\mathcal{H}_n^{(1,1)})' + (\mathcal{H}_n^{(2,2)})'.$$

Taking into account (13) we have $l_{n,1}+l_{n,2} = nA$, thus we get

$$\frac{\partial}{\partial t}\left(\frac{n}{z}\right) = (\mathcal{H}_n^{(1,1)})' + (\mathcal{H}_n^{(2,2)})',$$

thus,

$$(\mathcal{H}_n^{(1,1)} + \mathcal{H}_n^{(2,2)})' = 0,$$

and we conclude that $\mathcal{H}_n^{(1,1)} + \mathcal{H}_n^{(2,2)}$ does not depend on z.

Now, let us write $\dot{\mathcal{L}}\mathcal{L}^{-1} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_4 \end{bmatrix}$. From (16) we have

$$\mathcal{H}_n^{(1,1)} + \mathcal{H}_n^{(2,2)} = \frac{1}{2h_n\, z^n}\left(\Omega_n^*\dot{\phi}_n + \phi_n^*\dot{\Omega}_n + \Omega_n\dot{\phi}_n^* + \phi_n\dot{\Omega}_n^*\right) + \frac{1}{2h_n\, z^n}\left(\varepsilon_1(\phi_n\Omega_n^* + \phi_n^*\Omega_n) + \varepsilon_4(\phi_n\Omega_n^* + \phi_n^*\Omega_n)\right),$$

that is,

$$\mathcal{H}_n^{(1,1)} + \mathcal{H}_n^{(2,2)} = \frac{1}{2h_n\, z^n}\frac{\partial}{\partial t}\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right) + \frac{(\varepsilon_1+\varepsilon_4)\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)}{2h_n\, z^n}.$$

Taking into account that $\phi_n\Omega_n^* + \phi_n^*\Omega_n = 2h_n\, z^n$, we obtain (21).

Eq. (22) follows from the use of (18). ∎

4. A non semi-classical example: the sum of a Jacobi measure and the Lebesgue measure on the unit circle

We begin by considering the Jacobi weight given by modified by a parameter t,

$$w_{\alpha,\beta}(z) = t^{\alpha} z^{-\alpha-\beta}(z+1)^{2\beta_1}(z+1/t)^{2\alpha} \times \begin{cases} 1 & , \ \theta \notin]\pi-\phi, \pi[\\ 1-\varsigma & , \ \theta \in]\pi-\phi, \pi[\end{cases}, \tag{25}$$

where $\phi \in [0, 2\pi[$, $\beta_1, \beta_2, \alpha, \varsigma \in \mathbb{R}$, $\beta = \beta_1 + i\beta_2$, $2\beta_1 > -1, 2\alpha > -1, \varsigma < 1, t = e^{i\phi}$. $w_{\alpha,\beta}$ is real and positive on $\mathbb{T}$ and it is semi-classical, $\frac{w'_{\alpha,\beta}}{w_{\alpha,\beta}} = \frac{C}{zA}$, with

$$zA(z) = z(z+1)(z+1/t), \tag{26}$$

$$C(z) = (\alpha+\overline{\beta})z^2 + (\alpha-\beta+\frac{\overline{\beta}-\alpha}{t})z - \frac{\alpha+\beta}{t}. \tag{27}$$

The corresponding Carathéodory function, henceforth denoted by $F_{\alpha,\beta}$, satisfies

$$zAF'_{\alpha,\beta} = CF_{\alpha,\beta} + D, \tag{28}$$

where D is a polynomial.

We define the measure μ as

$$d\mu = w_{\alpha,\beta}\, d\theta + \frac{d\theta}{2\pi} \tag{29}$$

and we denote its Carathéodory function by F. It is well-known that F is Laguerre-Hahn affine, non semi-classical (see [3, 6]).

Remark . The sequence of MOPS with respect to (29) are related to perturbation of diagonals of Toeplitz matrices (see[7]). If we denote the Toeplitz matrix of $w_{\alpha,\beta}$ by $\mathbf{T}_{w_{\alpha,\beta}}$, then the Toeplitz matrix associated with μ, $\mathbf{T}$, satisfies

$$\mathbf{T} = I + \mathbf{T}_{w_{\alpha,\beta}}, \tag{30}$$

where I is the identity matrix. The above identity follows taking into account that the moments of μ are defined in terms of the moments $w_n^{\alpha,\beta}$ of $w_{\alpha,\beta}$ by $c_n = w_n^{\alpha,\beta} + \delta_{n,0}\,,\ n \in \mathbb{Z}\,.$

Difference equations.

Lemma 2. *Let μ be the measure given in (29), let F be its Carathéodory function, and let $\phi_n(z) = z^n + \tau_n z^{n-1} + \ldots + \gamma_n z + a_n$ be the corresponding MOPS, $n \geq 1$. Then,*

$$zAF' = CF + D_1\,, \tag{31}$$

with A, C given in (26), (27) and $D_1 = D - C$, with D given in (28). The corresponding $\{Y_n\}$ defined in (3) satisfies $zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, $n \geq 1$, with

$$\mathcal{C} = \begin{bmatrix} C/2 & -D_1 \\ 0 & -C/2 \end{bmatrix}, \tag{32}$$

and $\mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}$, *where*

$$l_{n,1}(z) = (n + \frac{\alpha+\overline{\beta}}{2})z^2 + \xi_{n,1}\, z + \frac{2n+\alpha+\beta}{2t}\,, \tag{33}$$

$$l_{n,2}(z) = -\frac{\alpha+\overline{\beta}}{2}z^2 + \xi_{n,2}\, z - \frac{\alpha+\beta}{2t}\,, \tag{34}$$

$$\Theta_{n,1}(z) = -a_{n+1}(n+1+\alpha+\overline{\beta})z + \frac{a_n}{t}(n+\alpha+\beta)\,, \tag{35}$$

$$\Theta_{n,2}(z) = -\overline{a}_n(n+\alpha+\overline{\beta})z^2 + \frac{\overline{a}_{n+1}}{t}(n+1+\alpha+\beta)z\,, \tag{36}$$

and

$$\xi_{n,1} = -\tau_n + \frac{2n+\alpha-\beta}{2} + \frac{2n+\overline{\beta}-\alpha}{2t} - \overline{a}_n a_{n+1}(n+1+\alpha+\overline{\beta})\,,$$

$$\xi_{n,2} = \frac{\overline{\gamma_n}}{\overline{a}_n}(n-1+\alpha+\overline{\beta}) + \frac{2n+\alpha-\beta}{2} + \frac{2n+\overline{\beta}-\alpha}{2t} - (n+\alpha+\overline{\beta})\tau_n + \frac{\overline{a}_{n+1}}{\overline{a}_n\, t}(n+1+\alpha+\beta)\,.$$

Proof: Eq. (31) follows taking into account (28) and $F = F_{\alpha,\beta}+1$.

The polynomials $\Theta_{n,i}$ are defined as

$$\Theta_{n,1} = \tilde{\Theta}_n/(2h_n z^n)\,,\ \Theta_{n,2} = \hat{\Theta}_n/(2h_n z^n)$$

where

$$\tilde{\Theta}_n = \left\{ zA\left(\frac{Q_n}{\phi_n}\right)' - C\frac{Q_n}{\phi_n}\right\}\phi_n^2\,,\ \hat{\Theta}_n = \left\{ zA\left(\frac{Q_n^*}{\phi_n^*}\right)' - C\frac{Q_n^*}{\phi_n^*}\right\}(\phi_n^*)^2$$

(see [4]). The use of the asymptotic expansions

$$\begin{aligned} Q_n(z) &= 2h_n z^n + \mathcal{O}(z^{n+1}),\ |z|<1\,,\\ Q_n(z) &= 2a_{n+1}h_n z^{-1} + \mathcal{O}(z^{-2}),\ |z|>1\,,\\ Q_n^*(z) &= 2\overline{a}_{n+1}h_n z^{n+1} + \mathcal{O}(z^{n+2}),\ |z|<1\\ Q_n^*(z) &= 2h_n + \mathcal{O}(z^{-1}),\ |z|>1 \end{aligned}$$

give us (35) and (36). Once $\Theta_{n,i}$ are determined, the polynomials $l_{n,i}$ are determined through

$$\begin{cases} zA\phi_n' = (l_{n,1}-C/2)\phi_n - \Theta_{n,1}\phi_n^* \\ zA(\phi_n^*)' = (l_{n,2}-C/2)\phi_n^* - \Theta_{n,2}\phi_n \end{cases}$$

and we get (33) and (34). ■

Theorem 4. *Let $\{\phi_n\}$ be the MOPS with respect to μ defined in (29), and let $\phi_n(z) = z^n + \tau_n z^{n-1} + \ldots + \gamma_n z + a_n$, $n\geq 1$, be the corresponding MOPS. The following holds:*

(a) the reflection parameters a_n satisfy the second order difference equation

$$a_{n+1}\overline{a}_n(n+1+\alpha+\overline{\beta})t = a_n\overline{a}_{n+1}(n+1+\alpha+\beta) + a_n\overline{a}_{n-1}(n-1+\alpha+\overline{\beta})t - a_{n-1}\overline{a}_n(n-1+\alpha+\beta)\,, \quad (37)$$

(b) The sub-leading coefficient of ϕ_n is given by

$$\tau_n = \frac{n+\alpha-\beta}{2} + \frac{n+\overline{\beta}-\alpha}{2t} - \overline{a}_n a_{n+1}(n+1+\alpha+\overline{\beta}) + a_n\overline{a}_{n+1}\frac{n+1+\alpha+\beta}{2t} - a_{n-1}\overline{a}_n\frac{n-1+\alpha+\beta}{2t} + \frac{a_{n+1}(n+1+\alpha+\overline{\beta})}{2a_n} + \frac{a_{n-1}(n-1+\alpha+\beta)}{2ta_n}\,, \quad (38)$$

(c) γ_n is given by

$$\gamma_n = a_n\overline{\tau}_n\frac{n+1+\alpha+\beta}{n-1+\alpha+\beta} - a_n\frac{n+\alpha-\overline{\beta}}{n-1+\alpha+\beta} - a_n t\frac{n-\alpha+\beta}{n-1+\alpha+\beta} + a_n^2\overline{a}_{n+1}\frac{n+1+\alpha+\beta}{n-1+\alpha+\beta} - a_{n+1}t\frac{n+1+\alpha+\overline{\beta}}{n-1+\alpha+\beta}\,. \quad (39)$$

Proof: The discrete Lax equations (15) give

$$z l_{n,1} - \overline{a}_n z\Theta_{n,1} = z l_{n-1,1} - a_n\Theta_{n-1,2} + zA \quad (40)$$
$$a_n l_{n,1} - \Theta_{n,1} = -z\Theta_{n-1,1} + a_n l_{n-1,2} \quad (41)$$
$$-z\Theta_{n,2} + \overline{a}_n z l_{n,2} = \overline{a}_n z l_{n-1,1} - \Theta_{n-1,2} + \overline{a}_n zA \quad (42)$$
$$-a_n\Theta_{n,2} + l_{n,2} = -\overline{a}_n z\Theta_{n-1,1} + l_{n-1,2}\,. \quad (43)$$

Let us evaluate the previous equations at $z=-1$, a zero of A. Summing (40) with (43) and taking into account that $l_{n,1}(-1)+l_{n,2}(-1)=0,\ n\geq 1$, (cf. (13)) there follows

$$a_n\Theta_{n-1,2}(-1) + \overline{a}_n\Theta_{n,1}(-1) + a_n\Theta_{n,2}(-1) + \overline{a}_n\Theta_{n-1,1}(-1) = 0\,, \quad (44)$$

which yields (37).

To get (38) we sum (41) with (43) and use $l_{n,1}(-1)+l_{n,2}(-1)=0,\ n\geq 1$, thus getting

$$l_{n,1}(-1) = -\frac{a_n}{2}\Theta_{n,2}(-1) - \frac{\overline{a}_n}{2}\Theta_{n-1,1}(-1) + \frac{\Theta_{n,1}(-1)}{2a_n} + \frac{\Theta_{n-1,1}(-1)}{2a_n}$$

which yields (38).

Eq. (39) follows from $\xi_{n,1}+\xi_{n,2} = n(1+\frac{1}{t})$, which in turn follows from $l_{n,1}+l_{n,2} = nA$ (cf. (13)). ∎

Differential equations. To study the differential equations for the reflection parameters $a_n(t)$ we will begin with the analysis of the matrix $\mathcal{H}_n$ defined in (16), $\dot{Y}_n = \mathcal{H}_n Y_n - Y_n \dot{\mathcal{L}}\mathcal{L}^{-1}$. Firstly, let us compute the matrix $\dot{\mathcal{L}}\mathcal{L}^{-1}$ in (16).

Lemma 3. *Let $\mathcal{C}$ be the matrix given in (32), let G be a domain in the complex plane not containing the zeroes of zA, and let $z_0 \in G$. Let $\mathcal{L}$ be a nonsingular matrix such that*

$$\begin{cases} zA\mathcal{L}' = \mathcal{C}\mathcal{L} \\ \mathcal{L}(z_0, \cdot) = I\,, \end{cases} \tag{45}$$

where I is the identity matrix. Then,

$$\dot{\mathcal{L}}\mathcal{L}^{-1} = \begin{bmatrix} \dot{k}_1/k_1 + \dfrac{1}{2}\dfrac{E}{zA} & \eta\, w_{\alpha,\beta} \\ 0 & \dot{k}_3/k_3 - \dfrac{1}{2}\dfrac{E}{zA} \end{bmatrix} \tag{46}$$

where the k_i's are functions of t, $\eta = \dfrac{k_1\dot{k}_2 - k_2\dot{k}_1}{k_1k_3}$, $\dfrac{\dot{w}_{\alpha,\beta}}{w_{\alpha,\beta}} = \dfrac{E}{zA}$, with E a polynomial.

Proof: The solution of (45) is given by

$$\mathcal{L}(z,t) = \begin{bmatrix} k_1(t)\sqrt{w_{\alpha,\beta}} & k_2(t)\sqrt{w_{\alpha,\beta}} \\ 0 & \dfrac{k_3(t)}{\sqrt{w_{\alpha,\beta}}} \end{bmatrix}, \quad z \in G, \tag{47}$$

where the k_i's are functions of t and $w_{\alpha,\beta}$ is the Jacobi weight (25). Thus, (46) follows. ■

From the previous Lemma there follows that the singularities of the matrix $\mathcal{H}_n$ defined in (16) occur at the zeroes of zA. Our next step is to analyze the asymptotic expansion of $\mathcal{H}_n$. We will see that it is closely related with the expansion of the matrix that is associated with the MOPS corresponding to the semi-classical weight $w_{\alpha,\beta}$.

Firstly we give a representation of $\{Y_n\}$ associated with μ (such a representation follows from a result on matrix Riccati equations known as Radon's Lemma (see [1, 17], as well as [4, Theorem 5])).

Theorem 5. *Let μ be the measure given in (29) and let $\{Y_n\}$ be the corresponding sequence defined in (3). Let $\{\phi_n^{(\alpha,\beta)}\}$ be the MOPS with respect to* *the Jacobi weight* $w_{\alpha,\beta}$ *given in (25), let* $\{Q_n^{(\alpha,\beta)}\}$ *be the sequence of functions of the second kind, and let*

$$\widehat{Y}_n^{(\alpha,\beta)} = \begin{bmatrix} \phi_n^{(\alpha,\beta)} & -Q_n^{(\alpha,\beta)}/w_{\alpha,\beta} \\ (\phi_n^{(\alpha,\beta)})^* & (Q_n^{(\alpha,\beta)})^*/w_{\alpha,\beta} \end{bmatrix}, \; n \in \mathbb{N}\,. \tag{48}$$

Let G *be a domain in the complex plane not containing the zeroes of* zA*, with* zA *the polynomial given in (26). Then, for* $z \in G$*,*

$$Y_n = \sqrt{w_{\alpha,\beta}}\, \widehat{Y}_n^{(\alpha,\beta)}\, \mathcal{L}^{-1}, \; n \in \mathbb{N}\,, \tag{49}$$

where $\mathcal{L}$ *is a fundamental matrix of (45).*

Therefore, Y_n *satisfies* $\dot{Y}_n = \mathcal{H}_n Y_n - Y_n \dot{\mathcal{L}} \mathcal{L}^{-1}$*,* $n \in \mathbb{N}$*, with*

$$\mathcal{H}_n = \frac{\partial}{\partial t}\left(\widehat{Y}_n^{(\alpha,\beta)}\right)\left(\widehat{Y}_n^{(\alpha,\beta)}\right)^{-1} + \frac{E}{2zA} I\,, \; n \in \mathbb{N}\,, \tag{50}$$

where I *is the identity matrix and the polynomial* E *is such that* $\dfrac{\dot{w}_{\alpha,\beta}}{w_{\alpha,\beta}} = \dfrac{E}{zA}$.

Proof: The representation (49) follows from [4, Theorem 7] (note that in the present situation the polynomial $\tilde{C}$ in [4, Theorem 7] is C and the weight $\tilde{w}$ is now $w_{\alpha,\beta}$).

To get (50) we take derivatives with respect to t in (49), and use $\dot{w}_{\alpha,\beta}/w_{\alpha,\beta} = E/zA$. ■

Theorem 6. *Let us consider the same conditions of the previous Theorem. Let us denote the reflection parameters of* $w_{\alpha,\beta}$ *given in (25) by* $\hat{a}_n$*,* $n \geq 1$*. The reflection parameters of the MOPS corresponding to* μ *defined in (29) satisfy*

$$\dot{a}_n = \frac{1}{t^3}\left(\hat{a}_n(n+\alpha+\overline{\beta}) + \hat{a}_{n-1}(n-1+\alpha+\beta)\right)\,, \; n \geq 1\,. \tag{51}$$

Proof: Let us denote by $\widehat{\mathcal{H}}_n$ the matrix $\widehat{\mathcal{H}}_n = \dfrac{\partial}{\partial t}\left(\widehat{Y}_n^{(\alpha,\beta)}\right)\left(\widehat{Y}_n^{(\alpha,\beta)}\right)^{-1}$ in (50). If we use (50) in eq. (18), $\dot{\mathcal{A}}_n = \mathcal{H}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{H}_{n-1}$, there follows

$$\dot{\mathcal{A}}_n = \widehat{\mathcal{H}}_n \mathcal{A}_n - \mathcal{A}_n \widehat{\mathcal{H}}_{n-1}\,. \tag{52}$$

Notice that, taking into account Theorem 2, $\widehat{Y}_n^{(\alpha,\beta)}$ satisfies

$$\left(\widehat{Y}_n^{(\alpha,\beta)}\right)' = \frac{1}{zA}(\widehat{\mathcal{B}}_n - C/2\, I)\widehat{Y}_n^{(\alpha,\beta)}\,,$$

where $\widehat{\mathcal{B}}_n$ is the matrix associated with the equation (28) satisfied by $F_{\alpha,\beta}$ (note that we are using the notation of the present paper). From [11, Corollary 3.3] we have the expansion of $\widehat{\mathcal{H}}_n$ in terms of the residues of $\frac{1}{zA}(\widehat{\mathcal{B}}_n - C/2\,I)$ at $z = z_j$, where $zA = z(z+1)(z+\frac{1}{t})$,

$$\widehat{\mathcal{H}}_n = \widehat{\mathcal{H}}_{n,\infty} - \sum_{j=1}^{3} \frac{\dot{z}_j}{z - z_j} \Re es \left(\frac{\widehat{\mathcal{B}}_n - C/2\,I}{zA} \right) \Bigg|_{z=z_j} , \tag{53}$$

where $\widehat{\mathcal{H}}_{n,\infty} = \begin{bmatrix} 0 & 0 \\ \dot{\overline{a}}_n & 0 \end{bmatrix}$.

The use of (53) in (52) gives us, from position $(1,2)$,

$$\dot{a}_n = \frac{1}{t^2} \hat{\Theta}_{n-1,1}(-1/t)\,.$$

Hence, (51) follows taking into account the definition of $\hat{\Theta}_{n,1}$, $\hat{\Theta}_{n,1}(z) = -\hat{a}_{n+1}(n+1+\alpha+\overline{\beta})z + \hat{a}_n(n+\alpha+\beta)/t$ (see (35)). ■

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*, Birhäuser, 2003.

[2] A.I. Aptekarev, A. Branquinho, and F. Marcellán, *Toda-Type Differential Equations for the Recurrence Coefficients of Orthogonal Polynomials and Freud Transformation*, J. Comput. Appl. Math. **78** (1997), 139-160.

[3] A. Branquinho and M.N. Rebocho, *Distributional equation for Laguerre-Hahn functionals on the unit circle*, J. Comput. Appl. Math. **233** (2009), 634-642.

[4] A. Branquinho and M.N. Rebocho, *Matrix Sylvester equations in the theory of orthogonal polynomials on the unit circle*, Bull. Belg. Math. Soc. Simon Stevin 17 (2010), 355-376.

[5] A. Cachafeiro and C. Pérez, *A study of the Laguerre-Hahn affine functionals on the unit circle*, J. Comput. Anal. Appl. **6** (2) (2004), 107-123.

[6] A. Cachafeiro, F. Marcellán, and C. Pérez, *Lebesgue perturbation of a quasi-definite Hermitian functional. The positive-definite case*, Linear Algebra and its Applications 369 (2003), 235-250.

[7] K. Castillo, L. Garza, and F. Marcellán, *Perturbations on the subdiagonals of Toeplitz matrices*, Linear Algebra and its Applications 434 (2011), 1563-1579.

[8] Y. Chen and M.E.H. Ismail, *Ladder operators and differential equations for orthogonal polynomials*, J. Phys. A 30 (1997), 7818-7829.

[9] T. Erdélyi, J.S. Geronimo, P. Nevai, and J. Zhang, *A simple proof of "Favards Theore" on the unit circle*, Atti. Sem. Mat. Fis. Univ. Modena 29, 41-46 (1991).

[10] P.J. Forrester and C.M. Ormerod, *Differential equations for deformed Laguerre polynomials*, J. Approx. Theory 162 (2010), no. 4, 653-677.

[11] P.J. Forrester and N.S. Witte, *Bi-orthogonal Polynomials on the Unit Circle, Regular Semi-Classical Weights and Integrable Systems*, Constr. Approx. **24** (2) (2006), 201-237.

[12] P.J. Forrester and N.S. Witte, *Discrete Painlevé equations for a class of PVI τ-functions given as $U(N)$ averages* (2004), arXiv:math.PH/0412065.

[13] Ya.L. Geronimus, *Polynomials orthogonal on a circle and interval*, vol 18, International Series on Applied Mathematics, Consultants Bureau, New York, 1961.
[14] L. Golinskii and P. Nevai, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Commun. Math. Phys. **223** (2001), 223-259.
[15] M.E.H. Ismail and N.S. Witte, *Discriminants and functional equations for polynomials orthogonal on the unit circle*, J. Approx. Theory, **110** (2) (2001), 200-228.
[16] A.P. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comput. Appl. Math. **57** (1995), 215-237.
[17] J. Radon, *Zum Problem von Lagrange*, Hamburger Math. Einzelschr., **6**, 1928.
[18] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory*, Amer. Math. Soc. Colloq. Publ. Volume 54, Part 1. Amer. Math. Soc. Providence, Rhode Island, 2005.

A. Branquinho
CMUC and Department of Mathematics, University of Coimbra, Largo D. Dinis, 3001-454 Coimbra, Portugal.

E-mail address: ajplb@mat.uc.pt

M.N. Rebocho
CMUC and Department of Mathematics, University of Beira Interior, 6201-001 Covilhã, Portugal.

E-mail address: mneves@mat.ubi.pt