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SUPRACONVERGENCE AND SUPERCLOSENESS IN VOLTERRA EQUATIONS

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Abstract: Integro-differential equations of Volterra type arise, naturally, in many applications such as for instance heat conduction in materials with memory, diffusion in polymers, diffusion in porous media. The aim of this paper is to study a finite difference discretization of the mentioned integro-differential equations. Second convergence order with respect to the H^1 norm is established which means that the discretization proposed is supraconvergent in finite difference methods language. As the finite difference method can be seen as a piecewise linear finite element method combined with special quadrature formulas, our result establishes the supercloseness of the gradient in the finite element language. Numerical results illustrating the discussed theoretical results are included.

Key words: Volterra equation, finite difference methods, piecewise linear finite element method, supraconvergence, supercloseness.

Mathematics Subject Classification (2000): 65M06, 65M20, 65M15

1. Introduction

We consider discretizations of the integro-differential equation

$$\frac{\partial u}{\partial t}(t) + Au(t) = \int_0^t B(s,t)u(s)\,ds + f(t), t \in (0,T], \tag{1}$$

subject to Dirichlet boundary conditions

$$u(t) = \psi(t) \ \text{ on } \partial\Omega \times (0,T], \tag{2}$$

and with the initial condition

$$u(0) = u_0. \tag{3}$$

In (1) $u(t)$ denotes a function defined on $\overline{\Omega}\times[0,T]$ when t is fixed, Ω is a simple polygonal domain of $\mathbb{R}^2$, A and $B(s,t)$ represent the following differential operators

$$Au(t) = -\nabla.\Big(A_1\nabla u(t)\Big) + \nabla.(A_0 u(t)) + a_0 u(t),$$

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$$B(s,t)u(t) = -\nabla.\Big(B_1(s,t)\nabla u(t)\Big) + \nabla.(B_0(s,t)u(t)) + b_0(s,t)u(t),$$

where A_1, A_0, a_0 dependent on (x,y), $A_0 = [a_i]$, $A_1 = [a_{ij}], i,j = 1,2$, and $a_{12} = a_{21} = a_m$. B_1, B_0, b_0 dependent on (x,y), s and t, $B_0 = [b_i]$, $B_1 = [b_{ij}], i,j = 1,2$, and $b_{12} = b_{21} = b_m$.

The fully discrete scheme is obtained using the so called MOL approach: a spatial discretization combined with a time integration. The semi-discretization is obtained by a standard finite difference method (FDM) on a nonuniform rectangular grid $\overline{\Omega}_H$ subdividing Ω considering a sequence of grids $\overline{\Omega}_H$, $H \in \Lambda$, with maximal mesh-size H_{max} converging to zero without any restriction on the nonuniformity of $\overline{\Omega}_H$. The resulting semi-discretization is equivalent to a lumped mass semi-discretization obtained combining the piecewise linear finite element method with quadrature rules defined on a triangulation of Ω generated by $\overline{\Omega}_H$. The time integration is defined by an implicit-explicit method. It is shown that the error of the semi-discrete approximation and its gradient are second order convergent being however the truncation error induced by the spatial discretization only of first order. The stability and convergence of the fully discrete scheme is proved.

The finite volume approximation of initial boundary value problem (IBVP) considered in this paper was studied in [31] when a quasi-uniform family of triangulations are used and the authors prove that the semi-discretization error is of second order convergent with respect to L^2-norm. We point out that the authors use the approach introduced in [36] for Galekin methods: the semi-discrete error is splitted into two terms introducing the Ritz-Galerkin projection of the semi-discrete approximation. The same approach was followed in [26], [33] to study the accuracy of semi-discrete finite element approximations for the solutions of the same class of integro-differential IBVP. Second convergence order for the semi-discretization error with respect to H^1-norm was established in [5] for the one-dimensional version of (1) following the approach introduced by Wheeler in [36].

In the present paper we prove error estimates for the semi-discrete finite difference approximation for the solution of (1) and for its gradient avoiding the approach mentioned above. Considering a convenient representation of the semi-discretization error we avoid the split of this error and we reduce the smoothness requirements for the solution of (1), (2), (3), usually needed when such splitting approach is considered. We show, when the domain Ω is a rectangle, that the error and its gradient have second convergence

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supreconvergence result was established. The stability and the convergence of the fully discrete method were also studied. In the convergence analysis we introduced a different approach from the one that is usually followed in the literature (see for instance [36], [32], [33], [37]). Such new approach enable us to assume lower smoothness of the solution of the IBVP (1), (2), (3), than those that we need to assume if the approach introduced in [36] was followed.

The methods studied can be seen into different class of methods : the class of Galerkin methods and the class of finite difference methods. In fact, with respect to the spatial discretization, the methods were constructed considering the variational formulation of the differential problem and replacing the space $H^1_0(\Omega)$ by the space of the piecewise linear functions and using convenient quadrature rules.

We point out that the analysis presented here can be followed if we use in the time integration methods of higher order than Euler's method like Crank-Nicolson method. This remark holds if we replace the rectangular rule considered in the approximation of the time integral by higher approximation methods.

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order while the truncation error is only of first order. This convergence order is lower when the domain presents an oblique side. The convergence of a fully discrete scheme established combining this spatial discretization with an implicit-explicit time integration with an integration rule for the time integral is studied using the new approach mentioned above. It should be pointed out that the results introduced in [14] for elliptic problems with smooth solutions and in [15] for problems with solutions with lower smoothness have a central role in the proof of the main results of the present paper.

As in [15], our finite difference solution can be seen as a lumped mass approximation constructed associating a triangulation T_H to the rectangular grid $\overline{\Omega}_H$ and applying convenient quadrature formulas to each term of the variational form that characterizes the semi-discrete piecewise linear finite element approximation. This means that our finite difference solution can be seen as a fully discrete piecewise linear finite element solution where the triangulations T_H does not satisfy any smoothness requirement, and so our results can be seen as supercloseness results. We do not assume any smoothness requirement to the triangulations T_H. For FDM for elliptic equations and for parabolic equations, this property is usually called supraconvergence ([2], [9], [10], [14], [15], [18], [19], [23], [21], [24],[28]).

The paper is organized as follows. In Section 2 the semi-discrete approximation is introduced and its stability behavior is studied in Section 3. In Section 4 it is established an estimate for the semi-discretization error. The study of the fully discrete scheme obtained combining the semi-discretization analysed in Section 3, an implicit-explicit time integration methods for ordinary differential equations with a quadrature rules for the integral term is presented in Section 5. Finally some numerical experiments illustrating the results of this paper are presented in Section 6.

2. A fully discrete Galerkin approximation

In this section we introduce the Galerkin formulation of our IBVP and its discretization by linear finite elements with quadrature. In order to do that we need to introduce some functional spaces. For $m \in \mathbb{N}_0, p \in [2,+\infty[$, $W^{m,p}(\Omega)$ denotes the Sobolev space with the semi-norm and norm, respectively, given by

$$|v|_{m,p} = \Big(\sum_{|\alpha|=m} \| \frac{\partial^m v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \|^p_{L^p} \Big)^{1/p}, \; \|v\|_{m,p} = \Big(\sum_{|\alpha|\le m} \| \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \|^p_{L^p} \Big)^{1/p},$$

where $\alpha = (\alpha_1, \alpha_2), \alpha_i \in \mathbb{N}_0, i = 1, 2, |\alpha| = \alpha_1 + \alpha_2$. For $p = \infty$ we consider the norm

$$\|v\|_{m,\infty} = \sum_{|\alpha|\le m} \operatorname{ess\,sup}_{\Omega} |\frac{\partial^m v}{\partial x^{\alpha_1}\partial y^{\alpha_2}}|.$$

By $H^m(\Omega)$ we represent the Sobolev space $W^{m,2}(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$. The norm $\|.\|_{m,2}$ is represented by $\|.\|_m$ and in $L^2(\Omega)$ we consider the usual inner product $(.,.)_0$. The subspace of $H^m(\Omega)$ of functions null on the boundary is denoted by $H_0^m(\Omega)$.

By $L^p(0,T;H^m(\Omega))$, $p \in [2,+\infty[$, we denote the space of functions $v : (0,T) \to H^m(\Omega)$ such that

$$\|v\|_{L^p(0,T;H^m(\Omega))} = \Big(\int_0^T \|v(t)\|_m^p\,dt\Big)^{1/p} \tag{4}$$

is finite. We also consider, for $m, r \in \mathbb{N}_0$, the space $W^{r,p}(0,T;H^m(\Omega))$ of functions $v : (0,T) \to H^m(\Omega)$ such that $\dfrac{d^jv}{dt^j} \in L^p(0,T;H^m(\Omega))$ for $j = 0,\ldots,r$, and

$$\|v\|_{W^{r,p}(0,T;H^m(\Omega))} := \Big(\sum_{j=1}^r \int_0^T \|\frac{d^jv}{dt^j}(t)\|_m^p\,dt\Big)^{1/p}, \tag{5}$$

is finite. When $p = 2$ this space is represented by by $H^r(0,T;H^m(\Omega))$ where we consider the inner product

$$(v,w)_{H^r(0,T;H^m(\Omega))} := \sum_{j=0}^r \int_0^T (\frac{d^jv}{dt^j}(t), \frac{d^jw}{dt^j}(t))_{H^m(\Omega)}\,dt\,. \tag{6}$$

In (6), $(.,.)_{H^m(\Omega)}$ denotes the usual inner product in $H^m(\Omega)$. We take $H^0(0,T;H^m(\Omega)) = L^2(0,T;H^m(\Omega))$. By $L^\infty(0,T;H^m(\Omega))$ we represent the space of functions $v : (0,T) \to H^m(\Omega)$ such that

$$\|v\|_{L^\infty(0,T;H^m(\Omega))} := \operatorname{ess\,sup}_{[0,T]} \|v(t)\|_m < \infty. \tag{7}$$

The space of functions $v : (0,T) \to H^m(\Omega)$ such that $\dfrac{d^jv}{dt^j} \in L^\infty(0,T;H^m(\Omega))$ for $j = 0,\ldots,r$, and

$$\|v\|_{W^{r,\infty}(0,T;H^m(\Omega))} := \sum_{j=1}^r \operatorname{ess\,sup}_{[0,T]} \|\frac{d^jv}{dt^j}(t)\|_r < \infty \tag{8}$$

Example 3. *In in what follows we consider the polygonal domain presented in Figure 2 and the grids $\overline{\Omega}_H$ satisfying the condition* **Geom**, *the operators A and B defined by*

$$Au = -\frac{\partial^2u}{\partial x^2} - \frac{\partial^2u}{\partial y^2} - \frac{\partial}{\partial x}\Big(\frac{1}{10}(x+y-\frac{7}{5})\frac{\partial u}{\partial y}\Big) - \frac{\partial}{\partial y}\Big(\frac{1}{10}(x+y-\frac{7}{5})\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial y}(yu)$$

and

$$Bu = -\frac{\partial^2u}{\partial x^2} - \frac{\partial^2u}{\partial y^2}.$$

The boundary conditions and the term f are such that the initial boundary value problem (1), (2), (3) has the solution

$$u(x,y,t) = e^t xy(x-1)(y-1)(x+y-\frac{3}{2}), (x,y) \in \overline{\Omega}.$$

In the time interval $[0, 0.002]$ we consider a grid with step-size $\Delta t = 10^{-6}$. In Table 3 we present the grids used and the numerical results obtained. The notations used were introduced in Example 1.

H_{max}	N_x	N_y	E_H^M	R_{H_1,H_2}
1.15110×10^{-1}	10	11	1.8705×10^{-3}	1.3627
5.7553×10^{-2}	20	22	7.2734×10^{-4}	1.5405
2.8776×10^{-2}	40	44	2.5003×10^{-4}	1.9386
1.0660×10^{-2}	80	88	6.5227×10^{-5}	1.9757
5.3300×10^{-3}	160	176	1.6583×10^{-5}	1.9858
2.6650×10^{-3}	320	352	4.1869×10^{-6}	1.9872
1.3325×10^{-3}	640	704	1.0561×10^{-6}	–

Table 3

We observe that the convergence rate is 2 which is greater than the $\frac{3}{2}$ given in the error estimate (80). Nevertheless, such estimate it is according with the error estimate (81).

7. Conclusions

In this paper numerical methods for the IBVP (1), (2), (3) were proposed. The methods were defined using MOL approach, that is, they were defined combining a spatial discretization, which converts the integro-differential problem in a ordinary differential problem, with a time integration method of the implicit-explicit type. The semi-discrete solution was studied and a

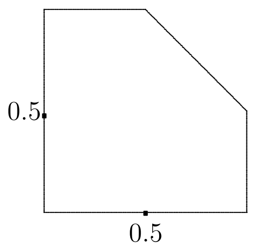

Figure 2: *Polygonal domain.*

In the numerical simulation we consider the grids $\overline{\Omega}_H$ satisfying the condition **Geom**, *the operators A and B defined by*

$$Au = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}((x-y)\frac{\partial u}{\partial y}) - \frac{\partial}{\partial y}((x-y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial y}(yu)$$

and

$$Bu = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}.$$

The boundary conditions are such that the initial boundary value problem (1), (2), (3) has the solution

$$u(x,y,t) = e^t xy(x-1)(y-1)(x+y-\frac{3}{2})\sin(10xy), (x,y) \in \overline{\Omega}.$$

In the time interval $[0, 0.001]$ we consider a grid with step-size $\Delta t = 10^{-6}$. In Table 2 we present the grids used and the numerical results obtained. We use the notations introduced in Example 1. We observe that the convergence rate is $\frac{3}{2}$ which agrees with the error estimate (80).

H_{max}	N_x	N_y	E_H^M	R_{H_1,H_2}
8.5280×10^{-2}	14	15	2.5363×10^{-4}	1.0969
4.2640×10^{-2}	28	30	1.1858×10^{-4}	1.3988
2.1320×10^{-2}	56	60	4.4970×10^{-5}	1.4518
1.0660×10^{-2}	112	120	1.6440×10^{-5}	1.4602
5.3300×10^{-3}	224	240	5.9752×10^{-6}	1.4696
2.6650×10^{-3}	448	480	2.1575×10^{-6}	1.4790
1.3325×10^{-3}	896	960	7.7396×10^{-7}	–

Table 2

is denoted by $W^{r,\infty}(0,T;H^m(\Omega))$.

Let $L^2(0,T;H^{-1}(\Omega))$ be the dual space of $L^2(0,T;H^1(\Omega))$ where $H^{-1}(\Omega)$ denotes the dual space of $H^1(\Omega)$. We define

$$\mathrm{W}(0,T) = \left\{ g \in L^2(0,T;H^1(\Omega)) \text{ such that } \frac{dg}{dt} \in L^2(0,T;H^{-1}(\Omega)) \right\},$$

which is a Hilbert space (see Theorem 25.4 of [35]).

For $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, we consider the following variational formulation of problem (1)-(3): find $u \in \mathrm{W}(0,T)$ such that $u(t) = \psi(t)$ on $\partial\Omega$ and

$$\begin{cases} < \dfrac{du}{dt}(t), v > +a(u(t),v) = \displaystyle\int_0^t b(s,t,u(s),v)\, ds + (f(t),v)_0 \text{ a.e in } (0,T), \\ \qquad\qquad\qquad\qquad\qquad \text{for all } v \in H_0^1(\Omega), \\ u(0) = u_0, \end{cases} \tag{9}$$

where $< .,. >$ denotes dual pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $a(.,.)$ and $b(s,t,.,.)$ are the bilinear forms defined by

$$a(v,w) = (A_1\nabla v, \nabla w)_0 - (A_0 v, \nabla w)_0 + (a_0 v, w)_0, \tag{10}$$

for $v, w \in H^1(\Omega)$, and

$$b(s,t,v,w) = (B_1(s,t)\nabla v, \nabla w)_0 - (B_0(s,t)v, \nabla w)_0 + (b_0(s,t)v, w)_0, \tag{11}$$

for $v, w \in H^1(\Omega)$. In (10) and (11) we use the notation

$$((p_1,p_2),(q_1,q_2))_0 = (p_1,q_1)_0 + (p_2,q_2)_0, p_i, q_i \in L^2(\Omega), i = 1,2.$$

The coefficient functions of the integro-differential equation (1) are assumed to be smooth enough with respect to the space variables x and y, e.g. they are in $W^{m,\infty}(\Omega), m \in \{1,2\}$.

In what follows we introduce the semi-discretization of (9) (see [15]). The spacial grid $\overline{\Omega}_H$ is defined by $\mathbb{R}_H \cap \overline{\Omega}$ where $H = (\mathbf{h},\mathbf{k})$, $\mathbf{h} = (h_j)_{\mathbb{Z}}$, $\mathbf{k} = (k_\ell)_{\mathbb{Z}}$ are two sequences of mesh-sizes and $\mathbb{R}_H = \mathbb{R}_{\mathbf{h}} \times \mathbb{R}_{\mathbf{k}}$ is a non-equidistant grid introduced in $\mathbb{R}^2$ with

$$\mathbb{R}_{\mathbf{h}} = \{x_j \in \mathbb{R} : \, x_{j+1} = x_j + h_j, \, j \in \mathbb{Z}\},$$

where $x_0 \in \mathbb{R}$ given and $\mathbb{R}_{\mathbf{k}}$ is defined analogously with the mesh-size vector $\mathbf{k}$ in place of $\mathbf{h}$ and y_0 in place of x_0. We also introduce

$$\Omega_H := \Omega \cap \mathbb{R}_H, \qquad \partial\Omega_H := \partial\Omega \cap \mathbb{R}_H.$$

Since we are considering polygonal domains, the following compatibility condition between the grid $\overline{\Omega}_H$ and the domain Ω is assumed:

(Geom) The intersection of $\partial\Omega$ with the rectangles $\square := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ spanned by points $(x_j, y_\ell), (x_{j+1}, y_{\ell+1})$ of $\mathbb{R}_H$ is either empty or it is a diagonal of $\square$.

By W_H we denote the space of grid functions on $\overline{\Omega}_H$ and by $W_{H,0}$ the subspace of $\overline{\Omega}_H$ of grid functions vanishing on $\partial\Omega_H$. For convenience we assume that functions in W_H are also defined outside of $\overline{\Omega}_H$ with value equal to zero. For $(x_j, y_\ell) \in \overline{\Omega}_H$, we represent by $\square_{j,\ell}$ the box $(x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega$ and we denote its measure by $\omega_{j,\ell}$. Then

$$(v_H, w_H)_H := \sum_{(x_j,y_\ell)\in\overline{\Omega}_H} \omega_{j,\ell}\, v_{j,\ell}\, \overline{w}_{j,\ell}, \quad \text{for} \quad v_H, w_H \in W_H, \tag{12}$$

defines an inner product on W_H. Let R_H denote the operator of pointwise restriction to the grid $\overline{\Omega}_H$ and let T_H be a triangulation of Ω using the set $\overline{\Omega}_H$ as vertices. By $P_H v_H$ we denote the continuous piecewise linear interpolation of v_H with respect to T_H.

The discrete version of $L^2(0,T;H^1(\Omega))$ is denoted by $L^2(0,T;W_H)$ and it is the space of functions $w_H : [0,T] \to W_H$ such that

$$\int_0^T \|w_H(t)\|_1^2\, dt \tag{13}$$

is finite, where $\|w_H\|_1^2 = \|w_H\|_H^2 + |P_H w_H|_1^2$ being $\|.\|_H$ the norm induced by the inner product (12) and $|.|_1$ the usual semi-norm in $H^1(\Omega)$.

Let W_H^* be the dual space of W_H, and

$$\mathrm{W}_H(0,T) = \left\{ g \in L^2(0,T;W_H) \text{ such that } \frac{dg}{dt} \in L^2(0,T;W_H^*) \right\}.$$

In the time interval $[0, 0.01]$ *we consider a grid with step-size* $\Delta t = 2 \times 10^{-5}$. *In Table 1 we include the* H_{max} *for each partition* $\overline{\Omega}_H$, *the number of points in* x *and* y *axis, respectively,* N_x *and* N_y, *the error* E_H^M

$$Error_H^m = \left(\left(\|e_H(t_M)\|_H^2 + \Delta t \sum_{j=0}^{M} \|P_H e_H(t_j)\|_1^2 \right) \right)^{1/2}$$

where $t_M = 0.01$, *the numerical solution* u_H^j *was computed using the method (55), (56). We also include in Table 1 the rate* R_{H_1,H_2}

$$R_{H_1,H_2} = \frac{\ln\left(\frac{Error_{H_{1,max}}}{Error_{H_{2,max}}}\right)}{\ln\left(\frac{H_{1,max}}{H_{2,max}}\right)}$$

where $H_{1,max}$ *and* $H_{2,max}$ *are the maximal mesh-size of two consecutive grids.*

H_{max}	N_x	N_y	E_H^M	R_{H_1,H_2}
9.706×10^{-2}	22	19	1.729×10^{-3}	—
4.799×10^{-2}	45	35	1.599×10^{-4}	3.380
2.470×10^{-2}	77	84	1.189×10^{-4}	1.956
1.249×10^{-2}	62	171	9.270×10^{-5}	2.550
8.326×10^{-3}	246	234	1.203×10^{-5}	2.023
7.120×10^{-3}	266	289	1.045×10^{-5}	1.955
6.244×10^{-3}	336	317	3.667×10^{-6}	2.244
5.553×10^{-3}	356	378	3.140×10^{-6}	2.206
4.997×10^{-3}	405	407	2.670×10^{-6}	2.182
4.157×10^{-3}	453	487	4.613×10^{-6}	2.249
3.844×10^{-3}	538	509	6.767×10^{-7}	2.430
3.569×10^{-3}	567	577	1.247×10^{-7}	2.191

Table 1

The results presented in Table 1 show that for the error E_H^M *is of second order in* H_{max}.

Example 2. *Let* Ω *be the polygonal domain presented in Figure 2.*

$$
\begin{aligned}
&\|e_H^n\|_H^2 + \Delta t \sum_{m=1}^{n} \|P_H e_H^m\|_1^2 \\
&\quad \le C\Big(H_{max}^4 \Big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^{\infty}(0,T;H^3(\Omega))} + \|u\|^2_{L^2(0,T;H^3(\Omega))} \Big) \\
&\quad + \sigma_{mix} H^3_{max} \sum_{\Delta \in \mathrm{T}_H^{obl}} \mathrm{diam}\Delta (\Big(\|u\|^2_{L^{\infty}(0,T;C^2(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,T;C^2(\Omega_H^{obl}))} \Big) \\
&\quad + \Delta t^2 \Big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))} \Big) \Big).
\end{aligned} \tag{80}
$$

If $\sum_{\Delta \in \mathrm{T}_H^{obl}} \mathrm{diam}\Delta \le C \max_{\Delta \in \mathrm{T}_H^{obl}} \mathrm{diam}\Delta$, *then*

$$
\begin{aligned}
&\|e_H^n\|_H^2 + \Delta t \sum_{m=1}^{n} \|P_H e_H^m\|_1^2 \le C\Big(H_{max}^4 \Big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^{\infty}(0,T;H^3(\Omega))} \\
&\quad + \|u\|^2_{L^2(0,T;H^3(\Omega))} + \sigma_{mix} \Big(\|u\|^2_{L^{\infty}(0,T;C^2(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,T;C^2(\Omega_H^{obl}))} \Big) \Big) \\
&\quad + \Delta t^2 \Big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))} \Big) \Big).
\end{aligned} \tag{81}
$$

6. Numerical simulation

We illustrate in what follows the theoretical results obtained for the integro-differential initial boundary value problem (1), (2), (3).

Example 1. *We start by* $\Omega = (0,1) \times (0,1)$ *and*

$$
Au = -\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 10^{-2}\frac{\partial}{\partial x}\big(xy\frac{\partial u}{\partial y}\big) - 10^{-2}\frac{\partial}{\partial y}\big(xy\frac{\partial u}{\partial x}\big) + \frac{\partial}{\partial x}\big(xu\big) + \frac{\partial}{\partial y}\big(yu\big) - 2,
$$

$$
Bu = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}.
$$

The boundary conditions and the term f *are such that the initial boundary value problem (1), (2), (3) has the solution*

$$
u(x,y,t) = e^t y(x-1)(y-1), (x,y) \in \overline{\Omega}.
$$

The discrete problem has the form: find $u_H \in \mathrm{W}_H(0,T)$ such that $u_H(t) = R_H\psi(t)$ on $\partial\Omega_H$ and

$$
\begin{cases}
< \dfrac{du_H}{dt}(t), v_H >_H + a_H(u_H(t), v_H) = \displaystyle\int_0^t b_H(s,t,u_H(s),v_H)\, ds \\
\qquad\qquad + (f_H(t), v_H)_H \ \text{ a.e. in } (0,T), \text{for all } \ v_H \in W_{H,0}, \\
u_H(0) = u_{0,H},
\end{cases} \tag{14}
$$

where $< .,. >_H$ denotes the duality pairing between W_H and W_H^*, and $u_{0,H} \in W_H$ is an approximation of u_0. In (14) $a_H(\cdot,\cdot)$ and $b_H(s,t,.,.)$ are sesquilinear forms that we define in what follows. We take

$$
a_H(.,.) = \sum_{i=1}^{2} a_{ii,H}(.,.) + \sum_{i=0}^{2} a_{i,H}(.,.) + a_{m,H}(.,.), \tag{15}
$$

where $a_{ii,H}(.,.), a_{i,H}(.,.)$ are sesquilinear forms corresponding to different terms in the continuous sesquilinear form $a(.,.)$ and $a_{m,H}(.,.)$ corresponds to the mixed terms ($a_{12} = a_{21} = a_m$). The sesquilinear form $a_{11,H}(.,.)$ is defined by

$$
a_{11,H}(v_H, w_H) := \sum_{\Delta \in \mathrm{T}_H} a_{11}(\Delta_x) \int_\Delta (P_H v_H)_x (P_H \overline{w}_H)_x \, dxdy, \tag{16}
$$

where Δ_x is the midpoint of the side of $\Delta \in \mathrm{T}_H$ parallel to the x-axis. Similarly, if Δ_y represents the midpoint of the side of Δ parallel to the y-axis, then we define $a_{22,H}(.,.)$ by

$$
a_{22,H}(v_H, w_H) := \sum_{\Delta \in \mathrm{T}_H} a_{22}(\Delta_y) \int_\Delta (P_H v_H)_y (P_H \overline{w}_H)_y \, dxdy. \tag{17}
$$

The approximation of the first order terms is achieved by

$$
a_{1,H}(v_H, w_H) := -\sum_{\Delta \in \mathrm{T}_H} [P_H(a_1 v_H)](\Delta_x) \int_\Delta (P_H \overline{w}_H)_x \, dxdy, \tag{18}
$$

$$
a_{2,H}(v_H, w_H) := -\sum_{\Delta \in \mathrm{T}_H} [P_H(a_2 v_H)](\Delta_y) \int_\Delta (P_H \overline{w}_H)_y \, dxdy. \tag{19}
$$

Finally, we set

$$
a_{0,H}(v_H, w_H) := \big((R_H a_0) v_H, w_H \big)_H. \tag{20}
$$

The function f in the right-hand side of (1) is discretized by the grid function

$$f_H(x_j, y_\ell, t) := \frac{1}{\omega_{j,\ell}} \int_{\square_{j,\ell}} f(x,y,t)dxdy, \quad (x_j, y_\ell) \in \Omega_H. \tag{21}$$

To define the sesquilinear form associated with the mixed derivatives, we consider two special triangulations of Ω that we call $\mathrm{T}_H^{(1)}$ and $\mathrm{T}_H^{(2)}$. They are obtained from the disjoint decomposition

$$\mathbb{R}_H = \mathbb{R}_H^{(1)} \,\dot\cup\, \mathbb{R}_H^{(2)},$$

where the sum $j+\ell$ of the indices of the points (x_j, y_ℓ) in $\mathbb{R}_H^{(1)}$ and in $\mathbb{R}_H^{(2)}$ is even and odd, respectively. In order to simplify the following definitions we introduce $\mathbb{R}_H^{(3)} := \mathbb{R}_H^{(1)}$. To each point $(x_j, y_\ell) \in \mathbb{R}_H$ we associate the four (open) triangles $\Delta_{j,\ell}^{(i)}$, $i = 1,2,3,4$, that have an angle $\pi/2$ at (x_j, y_ℓ) and two of the four horizontal/vertical neighbor grid points of (x_j, y_ℓ) as further vertices. We then define for $\nu \in \{1,2\}$ the triangulations

$$\begin{aligned}
&\mathrm{T}_{H,1}^{(\nu)} := \Big\{\Delta_{j,\ell}^{(i)} \subset \Omega : (x_j, y_\ell) \in \mathbb{R}_H^{(\nu)}, i \in \{1,2,3,4\}\,, \Big\}\\
&\mathrm{T}_{H,2}^{(\nu)} := \Big\{\Delta_{j,\ell}^{(i)} \subset (\Omega \setminus \cup\{\Delta | \Delta \in \mathrm{T}_{H,1}^{(\nu)}\}) : (x_j, y_\ell) \in \mathbb{R}_H^{(\nu+1)}, i \in \{1,2,3,4\}\Big\},\\
&\mathrm{T}_H^{(\nu)} := \mathrm{T}_{H,1}^{(\nu)} \cup \mathrm{T}_{H,2}^{(\nu)}\,.
\end{aligned} \tag{22}$$

By T_H^{obl} we denote the set of triangles which have one side on the oblique part of $\partial\Omega$. T_H^{obl} is empty for a domain Ω that is the union of rectangles. Figure 1 shows an example of a triangulation.

For $\nu = 1,2$ the continuous piecewise linear interpolation $P_H^{(\nu)} v_H$ of a grid function $v_H \in W_H$ with respect to the triangulations $\mathrm{T}_H^{(\nu)}$ is well-defined.

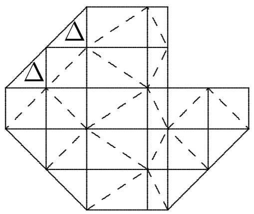

Figure 1: Triangulation $\mathrm{T}_H^{(\nu)}$. Δ indicates triangles of $\mathrm{T}_{H,2}^{(\nu)}$.

by (56), holds the following

$$\begin{aligned}
&\|e_H^n\|_H^2 + \Delta t \sum_{m=1}^n \|P_H e_H^m\|_1^2 \le C\Big(H_{max}^2\Big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^2(0,T;H^2(\Omega))}\Big)\\
&\quad + \Delta t^2\Big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\Big)\Big),
\end{aligned} \tag{78}$$

provided that $u \in W^{1,\infty}(0,T;H^2(\Omega)) \cap H^2(0,T;C(\Omega))$ *and*

$$\begin{aligned}
&\|e_H^n\|_H^2 + \Delta t \sum_{m=1}^n \|P_H e_H^m\|_1^2\\
&\quad \le C\Big(H_{max}^4\Big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \|u\|^2_{L^2(0,T;H^3(\Omega))}\Big)\\
&\quad + \sigma_{mix} H_{max}^{3-2/p}\Big(\|u\|^2_{L^\infty(0,T;W^{1,p}(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,T;W^{1,p}(\Omega_H^{obl}))}\Big)\\
&\quad + \Delta t^2\Big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\Big)\Big),
\end{aligned} \tag{79}$$

for $p \in [2,\infty)$, *provided that* $u \in W^{1,\infty}(0,T;H^3(\Omega)) \cap H^2(0,T;C(\Omega))$.

In (79) $\sigma_{mix} = 1$ *if* Ω *has an oblique side and* $a_m \neq 0$ *or* $b_m \neq 0$, $\sigma_{mix} = 0$ *if* Ω *is a rectangle or* $a_m = b_m = 0$.

□

Remark 2. *Considering Corollary 6.2 of* [15], *under the assumptions of Corollary 1, if the coefficients functions are in* $W^{2,\infty}(\Omega)$, $u \in L^\infty(0,T;C^2(\overline{\Omega} \cup \Omega_0))$, *where* Ω_0 *is a neighborhood of the oblique part of* $\partial\Omega$, *then we can state the following estimate*

is obtained using Bramble-Hilbert Lemma. As $b_H(s,t,.,.)$ and $\dfrac{\partial b_H}{\partial s}(s,t,.,.)$ are uniformly continuous, from (73), we obtain

$$\begin{aligned} |\tau_{n,2}(e_H^{m+1})| &\leq C\Delta t b_c \sum_{j=0}^{m}\int_{t_j}^{t_{j+1}} \big(\|P_H u(s)\|_1 + \|P_H R_H \frac{du}{dt}(s)\|_1\big)\, ds \|P_H e_H^{m+1}\|_1 \\ &\leq \frac{1}{4\gamma_2^2} C\Delta t^2 b_c^2 T \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))} + \gamma_2^2 \|P_H e_H^{m+1}\|_1^2, \end{aligned} \tag{75}$$

where $\gamma_2 \neq 0$ is an arbitrary constant.

Combining the estimations (72), (75) with the estimates for $\tau(e_H^{m+1})$ obtained considering Proposition 1 we get

$$\begin{aligned} \tau(e_H^{m+1}) \leq \frac{1}{4\gamma_3^2}\tau_c^{(\mu)}(t_{m+1})^2 + (\gamma_3^2+\gamma_2^2)\|P_H e_H^{m+1}\|_1^2 + \gamma_1^2 \|e_H^{m+1}\|_H^2 \\ +C\Big(\frac{1}{4\gamma_1^2}\Delta t \|R_H u\|^2_{H^2(t_m,t_{m+1};W_H)} + \frac{1}{4\gamma_2^2} b_c^2 T \Delta t^2 \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\Big) \end{aligned} \tag{76}$$

where $\mu \in \{1,2\}$, $\tau_c^{(1)}(t_{m+1})^2$ and $\tau_c^{(2)}(t_{m+1})^2$ are given by (52) and (53), respectively, with $t = t_{m+1}$.

From (71) and (76), it can be deduced, following the proof of Theorem 4, that the errors $e_H^j, j = 0,\ldots,m+1$, satisfy

$$\begin{aligned} &\|e_H^{m+1}\|_H^2 - \|e_H^m\|_H^2 + 2\Delta t(a_e - \epsilon^2 - \gamma_2^2 - \gamma_3^2)\|P_H e_H^{m+1}\|_1^2 \\ &\leq \Delta t^2 \frac{b_c^2 T}{2\epsilon^2}\sum_{j=0}^{m}\|P_H e_H^j\|_1^2 + 2\Delta t(\lambda+\gamma_1^2)\|e_H^{m+1}\|_H^2 + \Delta t \frac{1}{2\gamma_3^2}\tau_c^{(\mu)}(t_{m+1})^2 \\ &\quad +C\Delta t\Big(\frac{1}{2\gamma_1^2}\Delta t\|R_H u\|^2_{H^2(t_m,t_{m+1};W_H)} + \frac{b_c^2 T}{2\gamma_2^2}\Delta t^2 \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\Big), \end{aligned} \tag{77}$$

which leads, following again the proof of Theorem 4, to (68).

□

Corollary 1. *Under the assumptions of Theorem 1, there exists a positive constant C which does not depend on H and Δt such that, for $\Delta t \in (0, \Delta t_0)$ with Δt_0 satisfying (69), for the error $e_H^n = R_H u(t_n) - u_H^n$, with u_H^n defined*

For each triangle $\Delta \in \mathrm{T}_H^{(\nu)}$, (x_Δ, y_Δ) denotes the vertex of Δ associated with its angle $\pi/2$, $(\tilde{x}_\Delta, y_\Delta)$ denotes the vertex that has the y-coordinate of (x_Δ, y_Δ) and $(x_\Delta, \tilde{y}_\Delta)$ denotes the other vertex of Δ. Then, for $\nu \in \{1,2\}$,

$$a_m(\Delta_x) := \begin{cases} a_m(x_\Delta, y_\Delta) & \text{if } \Delta \in \mathrm{T}_{H,1}^{(\nu)} \\ a_m(\tilde{x}_\Delta, y_\Delta) & \text{if } \Delta \in \mathrm{T}_{H,2}^{(\nu)} \end{cases},$$

$$a_m(\Delta_y) := \begin{cases} a_m(x_\Delta, y_\Delta) & \text{if } \Delta \in \mathrm{T}_{H,1}^{(\nu)} \\ a_m(x_\Delta, \tilde{y}_\Delta) & \text{if } \Delta \in \mathrm{T}_{H,2}^{(\nu)} \end{cases},$$

and

$$a_{m,H}(v_H, w_H) := \frac{1}{2}\big(a^{(1)}_{m,H}(v_H,w_H) + a^{(2)}_{m,H}(v_H,w_H)\big), \tag{23}$$

for $v_H \in W_H, w_H \in W_{H,0}$. In (23) we use the notation

$$\begin{aligned} a^{(\nu)}_{m,H}(v_H,w_H) \;:=& \sum_{\Delta\in \mathrm{T}_H^{(\nu)}} \int_\Delta \Big(a_m(\Delta_x)(P_H^{(\nu)}v_H)_x (P_H^{(\nu)}\overline{w}_H)_y \\ & + a_m(\Delta_y)(P_H^{(\nu)}v_H)_y (P_H^{(\nu)}\overline{w}_H)_x\Big) dxdy. \end{aligned} \tag{24}$$

The definition of the sesquilinear form

$$b_H(s,t,.,.) = \sum_{i=1}^{2} b_{ii,H}(s,t,.,.) + \sum_{i=0}^{2} b_{i,H}(s,t,.,.) + b_{m,H}(s,t,.,.) \tag{25}$$

is analogous to the definition of $a_H(.,.)$ with the convenient replacements.

The semi-discrete approximation defined by the semi-discrete variational problem (14) is obtained solving an ordinary differential system. To define such system we introduce the following finite difference operators

$$\begin{aligned} A_H v_H \;=& -\delta_x^{(1/2)}(a_{11}\delta_x^{(1/2)}v_H) - \delta_x(a_{12}\delta_y v_H) - \delta_y(a_{21}\delta_x v_H) \\ & -\delta_y^{(1/2)}(a_{22}\delta_y^{(1/2)}v_H) + \delta_x(a_1 v_H) + \delta_y(a_2 v_H) + a_0 v_H, \end{aligned} \tag{26}$$

where

$$\delta_x^{(1/2)} v_H(x_i,y_j) = \frac{v_H(x_{i+1/2},y_j) - v_H(x_{i-1/2},y_j)}{h_{i+1/2}},$$

$$\delta_x^{(1/2)} v_H(x_{i+1/2},y_j) = \frac{v_H(x_{i+1},y_j) - v_H(x_i,y_j)}{h_{i+1}},$$

$$\delta_x v_H(x_i,y_j) = \frac{v_H(x_{i+1},y_j) - v_H(x_{i-1},y_j)}{h_{i+1}+h_i},$$

and $h_{i+1/2} = \dfrac{h_i + h_{i+1}}{2}$. The corresponding operators in y-direction are defined analogously.

The finite difference operator B_H is defined as A_H with the coefficient of A replaced by the correspondent coefficients of B.

If the operator A (or B) contains mixed derivatives then A_H (or B_H) acts, next to oblique parts of the boundary, on grid points outside $\overline{\Omega}_H$. As in [15], in this case the missing quantities in forming $A_H u_H$ (or $B_H u_H$) are determined by auxiliary variables which are obtained by a kind of anti-symmetric extension. For example, let $(x_j, y_\ell) \in \Omega_H$ be a grid point such that $(x_{j-1}, y_{\ell+1}) \notin \overline{\Omega}_H$. In the approximation of $(a_m u_x)_y$ the auxiliary value $u_{j-1,\ell+1}$ is then determined by

$$u_{j-1,\ell+1} - \psi_{j-1,\ell} = -(u_{j,\ell} - \psi_{j,\ell+1}). \tag{27}$$

Considering the procedure adopted in [2], [5] and in [15], it can be shown that the solution $u_H \in \mathrm{W}_H(0,T)$ of (14) solves the finite difference problem

$$\begin{cases} \dfrac{du_H}{dt}(t) + A_H u_H(t) = \displaystyle\int_0^t B_H(s,t) u_H(s)\, ds + f_H(t) \text{ in } \Omega_H, \\ u_H(t) = R_H \psi(t) \text{ on } \partial\Omega_H, \\ u_H(0) = u_{0,H}. \end{cases} \tag{28}$$

We assume in what follows that $a_H(.,.)$ is continuous, that is, there exists a positive constant a_c such that

$$|a_H(v_H, w_H)| \le a_c \|P_H v_H\|_1 \|P_H w_H\|_1, \text{ for all } v_H, w_H \in W_{H,0}, \tag{29}$$

and $a_H(.,.)$ is coercive, that is, there exists a positive constant a_e and $\lambda \in \mathbb{R}$ such that

$$a_H(v_H, v_H) \ge a_e \|P_H v_H\|_1^2 - \lambda \|v_H\|_H^2, \text{ for all } v_H \in W_{H,0}. \tag{30}$$

We also suppose that $b_H(s,t,.,.)$ is bounded uniformly with respect to s, t, that is, there exists a positive constant b_c such that

$$|b_H(s,t,v_H,w_H)| \le b_c \|P_H v_H\|_1 \|P_H w_H\|_1, \text{for all } v_H, w_H \in W_{H,0},\ s,t \in [0,T]. \tag{31}$$

Considering that (41) holds with $t = t_{m+1}$, from (70), we deduce

$$(D_{-t} e_H^{m+1}, e_H^{m+1})_H + a_H(e_H^{m+1}, e_H^{m+1}) = \Delta t \sum_{j=0}^{m} b_H(t_j, t_{m+1}, e_H^j, e_H^{m+1}) + \tau_{cd}(e_H^{m+1}) \tag{71}$$

with

$$\tau_{cd}(e_H^{n+1}) = \tau(e_H^{n+1}) + \tau_n(e_H^{n+1}),$$

where $\tau(e_H^{m+1})$ is defined by (43) with $e_H(t)$ replaced by e_H^{m+1} and

$$\tau_n(e_H^{m+1}) = \tau_{n,1}(e_H^{m+1}) + \tau_{n,2}(e_H^{m+1}),$$

$$\begin{aligned} \tau_{n,1}(e_H^{m+1}) = {} & \Big(D_{-t} R_H u(t_{m+1}) - R_H \frac{du}{dt}(t_{m+1}), e_H^{m+1}\Big)_H, \\ \tau_{n,2}(e_H^{m+1}) = {} & \int_0^{t_{m+1}} b_H(s, t_{m+1}, R_H u(s), e_H^{m+1}) ds \\ & - \Delta t \sum_{j=0}^{m} b_H(t_j, t_{m+1}, R_H u(t_j), e_H^{m+1}). \end{aligned} \tag{72}$$

We remark that an estimate for $\tau(e_H^{m+1})$ is obtained considering Proposition 1. For $\tau_{n,1}(e_H^{m+1})$ we have

$$\begin{aligned} |\tau_{n,1}(e_H^{m+1})| & \le C \int_{t_m}^{t_{m+1}} \|R_H \frac{d^2 u}{dt^2}(s)\|_H \|e_H^{m+1}\|_H \\ & \le C \Delta t \frac{1}{4\gamma_1^2} \|R_H u\|^2_{H^2(t_m, t_{m+1}; W_H)} + \gamma_1^2 \|e_H^{m+1}\|_H^2, \end{aligned} \tag{73}$$

where $\gamma_1 \neq 0$ is an arbitrary constant.

The estimate for $\tau_{n,2}(e_H^{m+1})$

$$\begin{aligned} |\tau_{n,2}(e_H^{m+1})| \le C \Delta t \sum_{j=0}^{m} \int_{t_j}^{t_{m+1}} & \Big(|\frac{\partial b_H}{\partial s}(s, t_{m+1}, R_H u(s), e_H^{m+1})| \\ & + |b_H(s, t_{m+1}, R_H \frac{du}{dt}(s), e_H^{n+1})| \Big) ds, \end{aligned} \tag{74}$$

Theorem 6. *Under the assumptions of Theorem 1, if* $\dfrac{\partial b_H}{\partial s}(s,t,.,.)$ *is uniformly continuous*

$$|\frac{\partial b_H}{\partial s}(s,t,u_H,v_H)| \leq b_c \|P_H u_H\|_1 \|P_H v_H\|_1, \forall u_H, v_H \in W_{H,0}, s,t \in [0,T], \tag{67}$$

then there exists a positive constant which does not depend on H, Δt *and* u *such that the error* $e_H^n = R_H u(t_n) - u_H^n$, *with* u_H^n *defined by (56), satisfies the inequality*

$$\begin{aligned} \|e_H^n\|_H^2 + \Delta t \sum_{m=0}^{n} \|P_H e_H^m\|_1^2 \quad & \leq \tilde{C}\Big(2\Delta t(a_e - \epsilon^2 - \gamma_2^2 - \gamma_3^2)\|P_H e_H^0\|_1^2 + \|e_H^0\|_H^2 \\ & + \tau_c^{(\mu)}(t_m)^2 + \Delta t \sum_{m=1}^{n} \frac{1}{2\gamma_3^2} \\ & + C\Delta t^2 \Big(\frac{1}{4\gamma_1^2}\|R_H u\|^2_{H^2(0,T;W_H)} + \frac{b_c^2 T^2}{4\gamma_2^2}\|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\Big)\Big) \end{aligned} \tag{68}$$

with

$$\tilde{C} = \frac{exp\Big(\frac{T\max\{2(\lambda+\eta_1^2), \frac{b_c^2T^2}{2\epsilon^2}\}}{\min\{1-2\Delta t_0(\lambda+\lambda_1^2), 2(a_e-\epsilon^2-\gamma_2^2-\gamma_3^2)\}}\Big)}{\min\{1 - 2(\lambda+\gamma_1^2)\Delta t_0, 2(a_e - \epsilon^2 - \gamma_2^2 - \gamma_3^2)\}},$$

for $\epsilon, \gamma_i \neq 0, i = 1,2,3,$ *such that*

$$a_e - \epsilon^2 - \gamma_2^2 - \gamma_3^2 > 0,$$

and for $\Delta t \in (0, \Delta t_0)$, *where* Δt_0 *is fixed by*

$$1 - 2(\lambda + \gamma_1^2)\Delta t_0 > 0. \tag{69}$$

In (68), for $\mu \in \{1,2\}$, $\tau_c^{(\mu)}(t_m)$ *is defined by (52) and (53), respectively, for* $\mu = 1$ *and* $\mu = 2$ *when* $t = t_m$.

Proof: It is easy to show that

$$\begin{aligned} (D_{-t} e_H^{m+1}, e_H^{m+1})_H \; & = (D_{-t} R_H u(t_{m+1}), e_H^{m+1})_H + a_H(u_H^{m+1}, e_H^{m+1}) \\ & -\Delta t \sum_{j=0}^{m} b_H(t_j, t_{m+1}, u_H^j, e_H^{m+1}) - (f_H^{m+1}, e_H^{m+1})_H. \end{aligned} \tag{70}$$

3. Stability analysis

In the stability analysis we consider homogeneous boundary conditions ($\psi = 0$) and we require some smoothness on the solution of the variational problem (14), namely, we assume that u_H is in $C^1([0,T]; W_{H,0})$, that is, $u_H : [0,T] \to W_{H,0}$ such that $\dfrac{du_H}{dt} : [0,T] \to W_{H,0}$ is continuous when in $W_{H,0}$ we consider the norm $\|.\|_H$.

The proofs of Theorems 1 and 2 differ in small details from the proofs of Theorem 1 and 2 in [5] and for this reason they are omitted.

Theorem 1. *Let us suppose that* $a_H(.,.)$ *and* $b_H(s,t,.,.)$ *satisfy (30) and (31), respectively. If the solution* u_H *of (14) is in* $C^1([0,T]; W_{H,0})$ *then*

$$\begin{aligned} & \|u_H(t)\|_H^2 + \int_0^t \|P_H u_H(s)\|_1^2 \, ds \\ & \qquad \leq \frac{1}{\min\{1, 2(a_e - \epsilon^2)\}} e^{Ct}\Big(\|u_H(0)\|_H^2 + \frac{1}{2\eta_2}\int_0^t e^{-Cs}\|f_H(s)\|_H^2 ds\Big), \end{aligned} \tag{32}$$

for $t \in [0,T]$, *where*

$$C = \frac{\max\{2(\lambda+\eta^2), \frac{b_c^2 T}{2\epsilon^2}\}}{\min\{1, 2(a_e - \epsilon^2)\}} \tag{33}$$

and $\epsilon \neq 0$ *is such that*

$$a_e - \epsilon^2 > 0. \tag{34}$$

□

Theorem 2. *Let us suppose that* $a_H(.,.)$ *satisfies (30) with* $\lambda = 0$, $b_H(s,t,.,.)$ *satisfies (31),*

$$\exists b_e > 0 \text{ such that } b_H(t,t,v_H,v_H) \geq b_e \|P_H v_H\|_1^2, \tag{35}$$

for all $v_H \in W_{H,0}, t \in [0,T]$, *and*

$$\exists b_d > 0 \text{ such that } |\frac{\partial b_H}{\partial t}(s,t,v_H,w_H)| \leq b_d \|P_H v_H\|_1 \|P_H w_H\|_1, \tag{36}$$

for all $v_H, w_H \in W_{H,0},\ s,t \in [0,T]$

If the solution u_H of (14) is in $C^1([0,T];W_{H,0})$, then

$$\int_0^t \|\frac{du_H}{ds}(s)\|_H^2 ds + \|P_H u_H(t)\|_1^2 + \int_0^t \|P_H u_H(s)\|_1^2\, ds$$
$$\leq \frac{2\max\{1,a_c\}}{\min\{1, a_e-\eta^2, 2b_e-\epsilon^2\}} e^{Ct}\Big(\|P_H u_H(0)\|_1^2 + \int_0^t e^{-Cs} g_H(s)ds\Big),\ t\in[0,T], \tag{37}$$

where

$$g_H(s) = a_c\|P_H u_H(0)\|_1^2 + \int_0^s \|f_H(\varsigma)\|_H^2\, d\varsigma,$$

and ϵ, η are such that

$$a_e - \eta^2 > 0,\ 2b_e - \epsilon^2 > 0, \tag{38}$$

with

$$C = \frac{\max\{\frac{b_c^2 T}{\eta^2}, \frac{b_d^2 T}{\epsilon^2}\}}{\min\{1, a_e-\eta^2, 2b_e-\epsilon^2\}}\,. \tag{39}$$

□

4. Convergence analysis

Let $e_H(t) = R_H u(t) - u_H(t)$ be the error induced by the spatial discretization introduced in Section 2. In what follows we establish a supraconvergent-superconvergent upper bound for $e_H(t)$ avoiding the split of this error introduced in [36] and largely followed in the literature. In fact, following [36], an estimate for $e_H(t)$ is obtained estimating $\rho_H(t) = R_H u(t) - \tilde{u}_H(t)$ and $\theta_H(t) = \tilde{u}_H(t) - u_H(t)$ with $\tilde{u}_H(t)$ defined by

$$a_H(\tilde{u}_H(t), w_H) = (g_H(t), w_H)_H, w_H \in W_{H,0},$$

where

$$g_H(t) = \int_0^t (B(s,t)u(s))_H\, ds + f_H(t)_H - (\frac{du}{dt}(t))_H,$$

being $(B(s,t)u(s))_H$ and $(\frac{du}{dt}(t))_H$ defined by (21) with f replaced by $B(s,t)u(s)$ and $\frac{du}{dt}(t)$ respectively.

An estimate for $\rho_H(t)$, depending on certain norm of $u(t)$, can be obtained considering the convergence analysis for finite difference scheme in the stationary case as for instance in [15]. In this particular case, assuming that

Choosing in (63) Δt, ϵ and η satisfying (60) and (59) we get

$$\|u_H^n\|_H^2 + \Delta t\sum_{m=0}^{n}\|P_H u_H^m\|_1^2 \leq \sum_{m=0}^{n-1} C\Big(\|u_H^m\|_H^2 + \Delta t\sum_{j=0}^{m}\|P_H u_H^j\|_1^2\Big)$$
$$+\frac{1}{\min\{1-2(\lambda+\eta^2)\Delta t, 2(a_e-\epsilon^2)\}}\Big(\|u_H^0\|_H^2 + 2\Delta t(a_e-\epsilon^2)\|P_H u_H^0\|_1^2$$
$$+\frac{\Delta t}{2\eta^2}\sum_{m=1}^{n}\|f_H^m\|_H^2\Big). \tag{64}$$

with

$$C = \frac{\Delta t\max\{2(\lambda+\eta^2), \frac{b_c^2 T}{2\epsilon^2}\}}{\min\{1-2(\lambda+\eta^2)\Delta t, 2(a_e-\epsilon^2)\}}.$$

Finally the application of the discrete Gronwall lemma to (64) leads to (58).

□

The stability of (56) is now established:

Theorem 5. *Under the assumptions of Theorem 1, the solution u_H^n of (56) with $f_H(t^{n+1}) = 0$ satisfies the estimation*

$$\|u_H^n\|_H^2 + \Delta t\sum_{m=0}^{n}\|P_H u_H^n\|_1^2 \leq \tilde{C}\Big(\|u_H^0\|_H^2 + 2(a_e-\epsilon^2)\Delta t\|P_H u_H^0\|_1^2\Big) \tag{65}$$

with

$$\tilde{C} = \frac{exp\Big(\frac{T\max\{2\lambda, \frac{b_c^2 T^2}{2\epsilon^2}\}}{\min\{1-2\lambda\Delta t_0, 2(a_e-\epsilon^2)\}}\Big)}{\min\{1-2\lambda\Delta t_0, 2(a_e-\epsilon^2)\}},$$

for $\epsilon \neq 0$ satisfying (59) and $\Delta t \in (0, \Delta t_0)$ where Δt_0 is such that

$$1 - 2\lambda\Delta t_0 > 0. \tag{66}$$

□

For λ positive we conclude the stability of (56) without any condition on the time step-size Δt. In this case the method (56) is unconditionally stable. Otherwise, it is conditionally stable.

Let $e_H^n = R_H u(t_n) - u_H^n$ be the error for the solution u_H^n defined by (56). An estimation for this error is established in the next result.

As we have

$$b_c\Delta t\sum_{j=0}^{m}\|P_Hu_H^j\|_1\|P_Hu_H^{m+1}\|_1 \le \frac{b_c^2T\Delta t}{4\epsilon^2}\sum_{j=0}^{m}\|P_Hu_H^j\|_1^2+\epsilon^2\|P_Hu_H^{m+1}\|_1^2,$$

and

$$(f_H^{m+1},u_H^{m+1})_H \le \frac{1}{4\eta^2}\|f_H^{m+1}\|_H^2+\eta^2\|u_H^{m+1}\|_H^2,$$

for all $\epsilon\neq 0,\eta\neq 0$, from (61) we deduce

$$\begin{aligned}\|u_H^{m+1}\|_H^2-\|u_H^m\|_H^2 \quad &+2\Delta t(a_e-\epsilon^2)\|P_Hu_H^{m+1}\|_1^2 \le \frac{b_c^2T\Delta t^2}{2\epsilon^2}\sum_{j=0}^{m}\|P_Hu_H^j\|_1^2\\ &+\Delta t\frac{1}{2\eta^2}\|f_H^{m+1}\|_H^2+2(\lambda+\eta^2)\Delta t\|u_H^{m+1}\|_H^2.\end{aligned} \tag{62}$$

Summing (62) over $m=0,\ldots,n-1$, we get

$$\begin{aligned}\|u_H^n\|_H^2-\|u_H^0\|_H^2+2\Delta t(a_e-\epsilon^2)\sum_{m=0}^{n-1}\|P_Hu_H^{m+1}\|_1^2 &\le \frac{b_c^2T\Delta t^2}{2\epsilon^2}\sum_{m=0}^{n-1}\sum_{j=0}^{m}\|P_Hu_H^j\|_1^2\\ &+\frac{\Delta t}{2\eta^2}\sum_{m=0}^{n-1}\|f_H^{m+1}\|_H^2+2(\lambda+\eta^2)\Delta t\sum_{m=0}^{n-1}\|u_H^{m+1}\|_H^2,\end{aligned}$$

and consequently

$$\begin{aligned}(1-2(\lambda+\eta^2)\Delta t)\|u_H^n\|_H^2+2\Delta t(a_e-\epsilon^2)\sum_{m=0}^{n}\|P_Hu_H^m\|_1^2&\\ \le \|u_H^0\|_H^2+2\Delta t(a_e-\epsilon^2)\|P_Hu_H^0\|_1^2+\frac{\Delta t}{2\eta^2}\sum_{m=1}^{n}\|f_H^m\|_H^2&\\ +\sum_{m=0}^{n-1}\Big(\frac{b_c^2T\Delta t}{2\epsilon^2}\Delta t\sum_{j=0}^{m}\|P_Hu_H^j\|_1^2+2(\lambda+\eta^2)\Delta t\|u_H^m\|_H^2\Big).&\end{aligned} \tag{63}$$

$a_H(.,.)$ is elliptic, and when Ω is a rectangular domain we have

$$\|P_H\rho_H(t)\|_1^2 \le H_{max}^{2\mu}\Big(\|u(t)\|_{\mu+1}^2+\int_0^t\|u(s)\|_{\mu+1}^2\,ds\Big),$$

for $\mu\in\{1,2\}$.

An estimate for $\Theta_H(t):=\|\theta_H(t)\|_H^2+\int_0^t\|P_H\theta_H(s)\|_1^2ds$ is obtained constructing an ordinary differential problem for $\Theta_H(t)$ which depends on $\|\frac{d\rho_H}{dt}(t)\|_H$. Consequently following the proof of Theorem 1, it can be shown that the upper bound for $\Theta_H(t)$ depends on

$$H_{max}^{2\mu}\Big(\|\frac{du}{dt}(t)\|_{\mu+1}^2+\int_0^t\|\frac{du}{dt}(s)\|_{\mu+1}^2\,ds\Big)$$

and on $H_{max}^4\|\frac{du}{dt}(t)\|_2^2$. So, in order to get an upper bound with fourth order we need to assume that $\frac{du}{dt}\in L^\infty(0,T;H^3(\Omega))$.

The approach that we follow enable us to reduce the smoothness required above for $u(t)$. We start by noting that $e_H(t)$ satisfies the equality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_H(t)\|_H^2 \;&= \big(R_H\frac{du}{dt}(t),e_H(t)\big)_H+a_H(u_H(t),e_H(t))\\ &-\int_0^t b_H(s,t,u_H(s),e_H(t))\,ds-(f_H(t),e_H(t))_H.\end{aligned} \tag{40}$$

As

$$(f_H(t),e_H(t))_H=\big((\frac{du}{dt}(t))_H,e_H(t)\big)_H+\Big(\big(Au(t)-\int_0^tB(s,t)u(s)\,ds\big)_H,e_H(t)\Big)_H, \tag{41}$$

where $\big(\frac{du}{dt}(t)\big)_H$, $\big(Au(t)-\int_0^tB(s,t)u(s)\,ds\big)_H$ are defined by (21) with f replaced by $\frac{du}{dt}(t)$ and $Au(t)-\int_0^tB(s,t)u(s)\,ds$, respectively, from (40) we obtain

$$\frac{1}{2}\frac{d}{dt}\|e_H(t)\|_H^2+a_H(e_H(t),e_H(t))=\int_0^tb_H(s,t,e_H(s),e_H(t))\,ds+\tau(e_H(t)), \tag{42}$$

where

$$\tau(e_H(t)) = \tau_d(e_H(t)) + \tau_A(e_H(t)) + \tau_{int}(e_H(t)), \tag{43}$$

$$\tau_d(e_H(t)) = (R_H\frac{du}{dt}(t), e_H(t))_H - ((\frac{du}{dt}(t))_H, e_H(t))_H, \tag{44}$$

$$\tau_A(e_H(t)) = a_H(R_H u(t), e_H(t)) - ((Au(t))_H, e_H(t))_H \tag{45}$$

and

$$\tau_{int}(e_H(t)) = \int_0^t \Big(((B(s,t)u(s))_H, e_H(t))_H - b_H(s,t,R_H u(s), e_H(t)) \Big)\, ds. \tag{46}$$

The estimations for $\tau_d(e_H(t))$, $\tau_A(e_H(t))$ and $\tau_{int}(e_H(t))$ are obtained in what follows considering the results presented in [15] for elliptic operators. Let $\tau(v_H)$ be defined by (43) with $e_H(t)$ replaced by $v_H \in W_{H,0}$.

Considering now Lemmas 5.1, 5.4, 5.5 and 5.7 of [15] we state the following proposition where we denote by C a positive constant that does not depend on u and H and which is not necessarily the same in all expressions.

Proposition 1. *Let the grids $\overline{\Omega}_H$, $H \in \Lambda$, satisfy condition (Geom) and let $\mu \in \{1,2\}$ and assume that the coefficients of A are in $W^{\mu,\infty}(\Omega)$ and the coefficients of B are in $W^{\mu,\infty}(\Omega)$ for $t, s \in [0,T]$. Then, for $v_H \in W_{H,0}$, $\tau(v_H)$ satisfies*

$$|\tau(v_H)| \le \tau^{(\mu)}(u(t))\|P_H v_H\|_1,$$

where

$$\begin{aligned}\tau^{(1)}(u(t)) \le C\Big(&\Big(\sum_{\Delta\in \mathrm{T}_H}(\mathrm{diam}\Delta)^2\|u(t)\|^2_{H^2(\Delta)}\Big)^{1/2}\\ &+\Big(\sum_{\Delta\in \mathrm{T}_H}(\mathrm{diam}\Delta)^4\|\frac{du}{dt}(t)\|^2_{H^2(\Delta)}\Big)^{1/2}\\ &+\int_0^t\Big(\sum_{\Delta\in \mathrm{T}_H}(\mathrm{diam}\Delta)^2\|u(s)\|^2_{H^2(\Delta)}\Big)^{1/2}ds\Big)\\ \le CH_{max}\Big(&\|u(t)\|_2 + \|\frac{du}{dt}(t)\|_2 + \int_0^t \|u(s)\|_2\, ds\Big),\end{aligned} \tag{47}$$

discretize the time integral term. We study in what follows the qualitative behavior of the solution of (56) having the following result an important role.

Lemma 1. *(Discrete Gronwall inequality) Let $\{\eta_n\}$ be a sequence of nonnegative real numbers satisfying*

$$\eta_n \le \sum_{j=0}^{n-1}\omega_j\eta_j + \beta_n \quad \text{for } n \ge 0,$$

where $\omega_j \ge 0$ and $\{\beta_n\}$ is a nondecreasing sequence of nonnegative numbers. Then

$$\eta_n \le \beta_n \exp\big(\sum_{j=0}^{n-1}\omega_j\big) \quad \text{for } n \ge 1. \tag{57}$$

Theorem 4. *Under the assumptions of Theorem 1, the solution of (55) satisfies*

$$\begin{aligned}&\|u_H^n\|_H^2 + \Delta t\sum_{m=0}^{n}\|P_H u_H^n\|_1^2\\ &\quad\le \tilde{C}\Big(\|u_H^0\|_H^2 + \Delta t 2(a_e-\epsilon^2)\|P_H u_H^0\|_1^2 + \frac{\Delta t}{2\eta^2}\sum_{m=1}^{n}\|f_H^m\|_H^2\Big),\end{aligned} \tag{58}$$

where $\eta \neq 0, \epsilon \neq 0$, ϵ is such that

$$a_e - \epsilon^2 > 0, \tag{59}$$

the time step-size Δt satisfies

$$1 - 2(\lambda+\eta^2)\Delta t > 0, \tag{60}$$

and

$$\tilde{C} = \frac{exp\Big(\frac{T\max\{2(\lambda+\eta^2), \frac{b_c^2 T}{2\epsilon^2}\}}{\min\{1-2(\eta^2+\eta^2)\Delta t, 2(a_e-\epsilon^2)\}}\Big)}{\min\{1-2(\lambda+\eta^2)\Delta t, 2(a_e-\epsilon^2)\}}.$$

Proof: Taking in (55) $v_H = u_H^{m+1}$ and considering the coercivity of $a_H(.,.)$ ((30)) and the uniform continuity of $b_H(s,t,.,.)$ ((31)) we establish

$$\begin{aligned}(D_{-t}u_H^{m+1}, u_H^{m+1})_H &+ a_e\|P_H u_H^{m+1}\|_1^2 - \lambda\|u_H^{m+1}\|_H^2\\ &\le b_c\Delta t\sum_{j=0}^{m}\|P_H u_H^j\|_1\|P_H u_H^{m+1}\|_1 + (f_H^{m+1}, u_H^{m+1})_H.\end{aligned} \tag{61}$$

thai is

$$\tau_c^{(2)}(t)^2 \leq CH_{max}^4\Big(\|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \|\frac{du}{dt}\|^2_{L^\infty(0,T;H^2(\Omega)} + \|u\|^2_{L^2(0,t;H^3(\Omega))}\Big)$$

$$+C\sigma_{mix}H^3_{max}\Big(\sum_{\Delta\in T_H^{obl}}\mathrm{diam}\Delta\Big)\Big(|u|^2_{L^\infty(0,T;C^2(\Omega_H^{obl}))} + |u|^2_{L^2(0,t;C^2(\Omega_H^{obl}))}\Big).$$

We point out that, in this case, if $\sum_{\Delta\in T_H^{obl}}\mathrm{diam}\Delta \leq C\max_{\Delta\in T_H^{obl}}\mathrm{diam}\Delta$, *then* $\tau_c^{(2)}(t) \leq CH^2_{max}$.

5. Fully discrete approximations

We introduce in $[0,T]$ a uniform grid $\{t_n, n=0,\ldots,N\}$ with $t_0 = 0, t_N = T$, $t_n - t_{n-1} = \Delta t$. By D_{-t} we denote the backward finite difference operator with respect to time variable. Let u_H^n be the fully discrete approximation in W_H such that $u_H^n = R_H\psi(t_n)$ on $\partial\Omega_H$ and

$$\begin{cases} (D_{-t}u_H^{n+1}, v_H)_H + a_H(u_H^{n+1}, v_H) = \Delta t\sum_{j=0}^{n} b(t_j, t_{n+1}, u_H^j, v_H) + (f_H^{n+1}, v_H)_H, \\ \qquad n = 0,\ldots,N-1, \quad \forall v_H \in W_{H,0}, \\ u_H^0 = u_{0,H}. \end{cases} \tag{55}$$

We remark that $u_H^n \in W_H$ is also solution of the fully discrete finite difference problem

$$\begin{cases} D_{-t}u_H^{n+1} + A_Hu_H^{n+1} = \Delta t\sum_{\ell=0}^{n} B_H(t_\ell, t_{n+1})u_H^\ell + f_H^{n+1} \text{ in } \Omega_H \\ \qquad n = 0,\ldots,N-1, \\ u_H^n = R_H\psi(t_n) \text{ on } \partial\Omega_H, n = 1,\ldots,N, \\ u_H^0 = u_{0,H}. \end{cases} \tag{56}$$

This method is of implicit-explicit type and it can be established combining the spatial discretization introduced before with the left rectangular rule to

provided $u \in L^1(0,T;H^2(\Omega))$, $\frac{du}{dt}(t) \in H^2(\Omega), t \in (0,T)$, *and*

$$\begin{aligned}\tau^{(2)}(u(t)) \leq\ & C\Big(\Big(\sum_{\Delta\in T_H}(\mathrm{diam}\Delta)^4\|u(t)\|^2_{H^3(\Delta)}\Big)^{1/2} \\ & + \Big(\sum_{\Delta\in T_H}(\mathrm{diam}\Delta)^4\|\frac{du}{dt}(t)\|^2_{H^2(\Delta)}\Big)^{1/2} \\ & + \int_0^t\Big(\sum_{\Delta\in T_H}(\mathrm{diam}\Delta)^4\|u(s)\|^2_{H^3(\Delta)}\Big)^{1/2}ds\Big) \\ & + \sigma_{mix}\Big(\Big(\sum_{\Delta\in T_H^{obl}}(\mathrm{diam}\Delta)^{4(1-1/p)}|u(t)|^2_{W^{2,p}(\Delta)}\Big)^{1/2} \\ & + \int_0^t\Big(\sum_{\Delta\in T_H^{obl}}(\mathrm{diam}\Delta)^{4(1-1/p)}|u(s)|^2_{W^{2,p}(\Delta)}\Big)^{1/2}ds\Big) \\ \leq\ & CH^2_{max}\Big(\|u(t)\|_3 + \|\frac{du}{dt}(t)\|_2 + \int_0^t\|u(s)\|_3 ds\Big) \\ & + C\sigma_{mix}H_{max}^{3/2-1/p}\Big(|u(t)|_{W^{2,p}(\Omega_H^{obl})} + \int_0^t|u(s)|_{W^{2,p}(\Omega_H^{obl})}ds\Big),\end{aligned} \tag{48}$$

provided $u \in L^1(0,T;H^3(\Omega))$, $\frac{du}{dt}(t) \in H^2(\Omega), t \in (0,T)$.

In (48), $\sigma_{mix} = 1$ *if* Ω *has an oblique side and* $a_m \neq 0$ *or* $b_m \neq 0$, $\sigma_{mix} = 0$ *if* Ω *is a rectangle or* $a_m = b_m = 0$.

Theorem 3. *Let the grids* $\overline{\Omega}_H$, $H \in \Lambda$, *satisfy condition (Geom) and let* $\mu \in \{1,2\}$ *and assume that the coefficients of* A *are in* $W^{\mu,\infty}(\Omega)$ *and the coefficients of* B *are in* $W^{\mu,\infty}(\Omega)$ *for* $t, s \in [0,T]$. *If* $a_H(.,.)$ *and* $b_H(s,t,.,.)$ *satisfy (30) and (31), respectively, then*

$$\begin{aligned}&\|e_H(t)\|^2_H + \int_0^t\|P_He_H(s)\|_1^2\,ds \\ &\leq \frac{1}{\min\{1, 2(a_e - \epsilon^2 - \eta^2)\}}e^{\tilde{C}t}\Big(\|e_H(0)\|^2_H + \frac{1}{2\eta^2}\int_0^t e^{-\tilde{C}t}\tau_c^{(\mu)}(\varsigma)^2(s)\,ds\Big),\end{aligned} \tag{49}$$

where ϵ and η are non zero constants such that

$$a_e - \epsilon^2 - \eta^2 > 0, \tag{50}$$

$$\tilde{C} = \frac{\max\{2\lambda, \frac{Tb_c^2}{2\epsilon^2}\}}{\min\{1, 2(a_e - \epsilon^2 - \eta^2)\}} \tag{51}$$

and

$$\begin{aligned}\tau_c^{(1)}(t)^2 = C\Big(\sum_{\Delta\in \mathrm{T}_H} (\mathrm{diam}\Delta)^2 \Big(\|u\|^2_{L^\infty(0,T;H^2(\Delta))} \\ + \|\frac{du}{dt}\|^2_{L^\infty(0,T;H^2(\Delta))} + \|u\|^2_{L^2(0,t;H^2(\Delta))} \Big) \\ \le CH^2_{max}\Big(\|u\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\frac{du}{dt}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|u\|^2_{L^2(0,T;H^2(\Omega))} \Big),\end{aligned} \tag{52}$$

provided that $u \in W^{1,\infty}(0,T;H^2(\Omega))$, or

$$\begin{aligned}\tau_c^{(2)}(t)^2 = C\Big(\sum_{\Delta\in \mathrm{T}_H} (\mathrm{diam}\Delta)^4 \Big(\|u\|^2_{L^\infty(0,T;H^3(\Delta))} \\ + \|\frac{du}{dt}\|^2_{L^\infty(0,T;H^2(\Delta))} + \|u\|^2_{L^2(0,t;H^3(\Delta))} \Big) \\ + \sigma_{mix} \sum_{\Delta\in \mathrm{T}_H^{obl}} (\mathrm{diam}\Delta)^{4(1-1/p)} \Big(|u|^2_{L^\infty(0,T;W^{2,p}(\Delta))} + |u|^2_{L^2(0,t;W^{2,p}(\Delta))} \Big)\Big) \\ \le CH^4_{max}\Big(\|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \|\frac{du}{dt}\|^2_{L^\infty(0,T;H^2(\Omega)} + \|u\|^2_{L^2(0,t;H^3(\Omega))} \Big) \\ + C\sigma_{mix} H^{3-2/p}_{max} \Big(|u|^2_{L^\infty(0,T;W^{2,p}(\Omega_H^{obl}))} + |u|^2_{L^2(0,t;W^{2,p}(\Omega_H^{obl}))} \Big),\end{aligned} \tag{53}$$

provided that $u \in L^\infty(0,T;H^3(\Omega)) \cap W^{1,\infty}(0,T;H^2(\Omega))$.
In (53) $\sigma_{mix} = 1$ if Ω has an oblique side and $a_m \neq 0$ or $b_m \neq 0$, $\sigma_{mix} = 0$ if Ω is a rectangle or $a_m = b_m = 0$.

Proof: Considering in (42) the assumptions (30) and (31) for $a_H(.,.)$ and $b_H(s,t,.,.)$, respectively, we deduce

$$\begin{aligned}&\frac{d}{dt}\|e_H(t)\|^2_H + 2(a_e - \epsilon^2 - \eta^2)\|P_H e_H(t)\|^2_1 \\ &\le \frac{Tb_c^2}{2\epsilon^2}\int_0^t \|P_H e_H(s)\|_1^2\,ds + 2\lambda\|e_H(t)\|^2_H + \frac{1}{2\eta^2}\tau^{(\mu)}(u(t))^2\end{aligned}$$

and consequently, with $Y(t) = \|e_H(t)\|^2_H + 2(a_e - \epsilon^2 - \eta^2)\|P_H e_H(t)\|^2_1$, we have

$$\frac{d}{dt}\Big(Y(t)e^{-\tilde{C}t} - \frac{1}{2\eta^2}\int_0^t e^{-\tilde{C}s}\tau^{(\mu)}(u(s))^2\,ds \Big) \le 0, \tag{54}$$

for ϵ and η satisfying (50) and with $\tilde{C}$ defined by (51). Inequality (54) leads to (49).

□

Remark 1. *Considering Corollary 6.2 of* [15], *under the assumptions of Theorem 3, if $u \in L^\infty(0,T;C^2(\overline{\Omega}\cup\Omega_0))$, where Ω_0 is a neighborhood of the oblique part of $\partial\Omega$, we can state for $\tau_c^{(2)}(t)$ the following estimate*

$$\begin{aligned}\tau_c^{(2)}(t)^2 \le C\Big(\sum_{\Delta\in \mathrm{T}_H} (\mathrm{diam}\Delta)^4 \Big(\|u(t)\|^2_{H^3(\Delta)} + \|\frac{du}{dt}(t)\|^2_{H^2(\Delta)} \\ + \int_0^t \|u(s)\|^2_{H^3(\Delta)}\,ds \Big) \\ + \sigma_{mix} \sum_{\Delta\in \mathrm{T}_H^{obl}} (\mathrm{diam}\Delta)^4 \Big(\|u(t)\|^2_{C^2(\overline{\Delta})} + \int_0^t \|u(s)\|^2_{C^2(\overline{\Delta})}\,ds \Big)\Big) \\ \le CH^4_{max}\Big(\|u(t)\|^2_3 + \|\frac{du}{dt}(t)\|^2_2 + \int_0^t \|u(s)\|^2_3\,ds \Big) \\ + \sigma_{mix} H^3_{max} \Big(\sum_{\Delta\in \mathrm{T}_H^{obl}} \mathrm{diam}\Delta \Big) \Big(\|u(t)\|^2_{C^2(\Omega_H^{obl})} + \int_0^t \|u(s)\|^2_{C^2(\Omega_H^{obl})}\,ds \Big)\Big)\end{aligned}$$