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A HETEROGENEOUS CHARACTERIZATION OF SUBDIRECTLY IRREDUCIBLE BIJECTIVE BOOLEAN MODULES

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ABSTRACT: We follow throughout a heterogeneous approach to present some characterizations (Theorem 3.5 and Theorem 3.35) of subdirectly irreducible bijective Boolean modules. The notions of essential element (as modular element) and [1]-closed element (as Boolean element) were implemented to this class of algebras. Our final aim of reaching the class of simple bijective Boolean modules, as a particular case, was also achieved, coinciding with the class previously found.

KEYWORDS: Relation algebras; Boolean modules; modular heterogeneous congruence; modular heterogeneous ideal; simple Boolean module.
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1. Introduction

In classical universal algebra, a well known representation theorem due to Birkhoff states that any member of a variety V is isomorphic to a subdirect product of subdirectly irreducible members of the variety. This implies that $V = SP(Si)$ where Si represents the class of subdirectly irreducible algebras in V. This fact is, in itself, a purpose to the classification of this class within each variety.

Both Rautenberg [6] and Venema [9] characterized subdirectly irreducible and simple Boolean algebras with operators. To achieve their goal they introduced the notion of essential elements in that class of algebras. Our aim is to establish, in the class of bijective Boolean modules, an entity with a similar role to the one expressed by an essential element in Boolean algebras with operators (cf. Definition 3.9). We are able to achieve Theorem 3.23 asserting that a bijective Boolean module is subdirectly irreducible if and only if contains an essential element, a result similar to Theorem 4.16 of [9] for Boolean algebras with operators and Theorem 3.21 of [4] for

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separable dynamic algebras. Moreover, following Sambin [7], in a subdirectly irreducible bijective Boolean module, we are able to highlight, a Boolean element playing a decisive role, the greatest among the [1]-closed elements on a bijective Boolean module. This led us to prove Theorem 3.35 where such an element appears as the principal filter generator of the set of all Boolean parts of essential elements united with the Boolean element 1. Forward, our study leads us to conclude that this element coincides with the zero element of the Boolean part, and therefore that both homogeneous and heterogeneous characterizations of subdirectly irreducible bijective Boolean modules are equivalent.

2. Boolean modules

Boolean modules were introduced by Brink [1] as homogeneous algebras, Boolean algebras with a multiplication (Peircean product) from a relation algebra. A Boolean module is, from a heterogeneous point of view, a two sorted algebra containing a Boolean algebra, a relation algebra and an operator (a heterogeneous operation, the Peircean operator) taking a pair of a relation algebra element and a Boolean algebra element and originating a Boolean algebra element. We present here the standard definition of relation algebras given by Brink (originated from Chin and Tarski [2] and modified in Tarski [8]).

Definition 2.1. A *relation algebra* is an algebra $\mathcal{R} = (R, \vee, \wedge, ', \mathrm{o}, 1, ;, \breve{}, e)$ satisfying the following axioms for each $a, b, c \in R$

R1 $(R, \vee, \wedge, ', \mathrm{o}, 1)$ is a Boolean algebra
R2 $a;(b;c) = (a;b);c$
R3 $a;e = a = e;a$
R4 $\breve{\breve{a}} = a$
R5 $(a \vee b);c = a;c \vee b;c$
R6 $(a \vee b)\breve{} = \breve{a} \vee \breve{b}$
R7 $(a;b)\breve{} = \breve{b};\breve{a}$
R8 $\breve{a};(a;b)' \leq b'$.

Notation. For $a, b \in R$ we also write ab instead of $a;b$.

The standard class of models of relation algebras is the class of proper relation algebras.

Definition 2.2. A *proper relation algebra* over a non-empty set U is a set of binary relations on U that contains the identity relation and is closed with respect to union, intersection, complementation, relational composition and converse. If a proper relation algebra consists of all binary relations defined on U, then this algebra is called the *full relation algebra* and is denoted by $\mathcal{R}(U)$. More precisely, $\mathcal{R}(U)$ is the power set algebra over U^2 endowed with composition ("$;$"), converse ("$\smile$") and identity ("Id") operations defined, for $a, b \subseteq U^2$, by

$a; b = \{(s,t) :$ exists $u \in U$ such that $(s,u) \in a$ and $(u,t) \in b\}$
$a^{\smile} = \{(s,t) : (t,s) \in a\}$
$Id = \{(s,s) : s \in U\}$.

The arithmetic of relation algebras can be described by the facts assembled on the following theorem.

Theorem 2.3. *On any relation algebra $\mathcal{R} = (R, \vee, \wedge, ', o, 1, ;, \breve{}, e)$ the following hold for any $a, b, c, d \in R$*

R9 $\breve{e} = e, \quad \breve{o} = o, \quad \breve{1} = 1$
R10 $a \leq b$ *if and only if* $a^{\smile} \leq b^{\smile}$
R11 $(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}, \quad a'^{\smile} = a^{\smile\prime}$
R12 $a; o = o = o; a, \quad 1; 1 = 1$
R13 $a(b \vee c) = ab \vee ac$
R14 *If* $a \leq b$ *then* $ca \leq cb$ *and* $ac \leq bc$.
R15 $(ab) \wedge c = o$ *if and only if* $(\breve{a}c) \wedge b = o$ *if and only if* $(c\breve{b}) \wedge a = o$
R16 $(ab) \wedge (cd) \leq a\big((\breve{a}c) \wedge (b\breve{d})\big)d$.

Proof: R9-R16 are proved in [2]. ■

Brink introduced the notion of a Boolean $\mathcal{R}$-module $\mathcal{B}$ as a homogeneous algebra. Our attention is now devoted to the heterogeneous approach.

Definition 2.4. A *Boolean module* is a two-sorted algebra $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ where $\mathcal{B}$ is a Boolean algebra, $\mathcal{R}$ is a relation algebra and $:$ is a mapping $\mathcal{R} \times \mathcal{B} \longrightarrow \mathcal{B}$ (written $a : p$) such that for any $a, b \in R$ and $p, q \in B$, the following assertions are satisfied.

M1 $a : (p \vee q) = a : p \vee a : q$
M2 $(a \vee b) : p = a : p \vee b : p$
M3 $a : (b : p) = (a; b) : p$
M4 $e : p = p$
M5 $\mathrm{o} : p = 0$
M6 $a^{\smile} : (a : p)' \leq p'$

Notation. For $a, b \in R$ and $p \in B$ we also use ap to represent $a : p$. We define the modal operator $[\,]$, that for each $a \in R$ assigns $[a] : \mathcal{B} \to \mathcal{B}$ defined by $[a]p = \sim (a(\sim p))$, for every $p \in B$.

The standard models of Boolean modules are provided by the class of proper Boolean modules.

Definition 2.5. A *proper Boolean module* on a non-empty set U is a two-sorted algebra of a proper Boolean algebra (a field of sets on U) and a proper relation algebra on U together with *Peirce product* defined on sets and relations. For any relation a over U and any subset p of U, the *Peirce product* $:$ of a and p is defined by

$$a : p = \{s \in U : \text{ there exists } t \in p \text{ such that } (s,t) \in a\}.$$

A *full Boolean module* $\mathcal{M}(U)$ over a non-empty set U is the Boolean module $(\mathcal{B}(U), \mathcal{R}(U), :)$, where $\mathcal{B}(U)$ is the power set algebra over U, $\mathcal{R}(U)$ is the full relation algebra over U and $:$ is the *Peirce product* defined set-theoretically.

Below are some facts valid on Boolean modules.

Theorem 2.6. *On any Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ the following hold for any $a, b \in R$ and $p, q \in B$*

$M7$ *If* $p \leq q$ *then* $ap \leq aq$.
$M8$ *If* $a \leq b$ *then* $ap \leq bp$.
$M9$ $a(p \wedge q) \leq (ap \wedge aq)$
$M10$ $(a \wedge b)p \leq (ap \wedge bp)$
$M11$ $ap \wedge q = 0$ *if and only if* $\breve{a}q \wedge p = 0$
$M12$ *If* $\sum_{i\in I} p_i$ *exists, then so does* $\sum_{i \in I} ap_i$, *and* $a \sum_{i\in I} p_i = \sum_{i\in I} ap_i$.
$M13$ $a0 = 0$
$M14$ $1 : 1 = 1$
$M15$ $(a1)' \leq a'1$
$M16$ $ap \wedge q \leq a(p \wedge \breve{a}q)$
$M17$ $1p \geq p$
$M18$ $[1]p \leq p$
$M19$ *If* $p \leq q$ *then* $[a]p \leq [a]q$.
$M20$ $[1]([1]p) = [1]p$.

Proof: M7-M17 are proved in [1]. M18 and M19 are immediate consequences of M17 and M7, respectively. To prove M20 we use mainly M3 and R12. In fact,

$$
\begin{aligned}
[1]([1]p) &= [1](\sim(1(\sim p))) \\
&= \sim(1(\sim(\sim(1(\sim p))))) \\
&= \sim(1:(1(\sim p))) \\
&= \sim((1;1)(\sim p)) \qquad \text{(by M3)} \\
&= \sim(1(\sim p)) \qquad \text{(by R12)} \\
&= [1]p
\end{aligned}
$$

■

It is a well known fact that the concept of congruence has a fundamental role in universal algebra. Next notion follows, once more, a heterogeneous approach.

Definition 2.7. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a Boolean module. A pair $\theta = (\theta_1, \theta_2)$ is a *(modular) congruence relation* on $\mathcal{M}$ if θ_1 is a congruence relation on $\mathcal{B}$, θ_2 is a congruence relation on $\mathcal{R}$ and $ap\ \theta_1\ bq$ whenever ($p\ \theta_1\ q$ and $a\ \theta_2\ b$).

In a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ we define congruences Δ_B and ∇_B on B and Δ_R and ∇_R on R as expected

$$
\begin{aligned}
\Delta_B &= \{(p,p) : p \in B\}, & \nabla_B &= \{(p,q) : p,q \in B\}, \\
\Delta_R &= \{(a,a) : a \in R\}, & \nabla_R &= \{(a,b) : a,b \in R\}.
\end{aligned}
$$

One can easily show that the pairs $(\Delta_B, \Delta_R), (\nabla_B, \nabla_R)$ and (∇_B, Δ_R) are congruences on $\mathcal{M}$, but in general (Δ_B, ∇_R) is not a congruence on $\mathcal{M}$. In fact, (Δ_B, ∇_R) is a congruence on $\mathcal{M}$ if and only if $\Delta_B = \nabla_B$.

Next we state some results for later use.

On any Boolean module the pair (∇_B, ∇_R) is the unique modular congruence having ∇_R as relational part.

Proposition 2.8. [4] *On a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ the pair (θ_1, ∇_R) is a modular congruence on $\mathcal{M}$ if and only if $\theta_1 = \nabla_B$.*

On an arbitrary Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ (not full) it is possible that, for some relation algebra elements a and b, we may have $ap = bp$ for all $p \in B$ without having $a = b$. Boolean modules for which this situation is forbidden is presented next.

Definition 2.9. A Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is *bijective* if and only if, for all $a, b \in R$ we have $a = b$ whenever $ap = bp$ for all $p \in B$.

On a bijective Boolean module the pair (Δ_B, Δ_R) is the unique modular congruence having Δ_B as Boolean part.

Proposition 2.10. [4] *Let* $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ *be a bijective Boolean module. Then*

$$(\Delta_B, \theta_2) \text{ is a congruence if and only if } \theta_2 = \Delta_R.$$

Corollary 2.11. [4] *On a bijective Boolean module* $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$

$$(\Delta_B, \nabla_R) \text{ is a congruence iff card } R = 1 \quad (\text{ iff } \nabla_R = \Delta_R).$$

3. Subdirectly Irreducible Bijective Boolean Modules

The importance of the class of the subdirectly irreducible algebras is justified on the well known representation theorems due to Birkhoff.

For any algebra $\mathcal{A}$ we denote by $\mathcal{C}ong(\mathcal{A})$ the lattice of all congruences on $\mathcal{A}$.

Definition 3.1. [3] An algebra A is *subdirectly irreducible* if and only if A is trivial or there is a minimum congruence in $\mathcal{C}ong(\mathcal{A}) - \{\Delta\}$ (if and only if there exist two elements x, y in A such that $x\theta y$, for any no trivial congruence θ). In the latter case the minimum element is given by $\cap(\mathcal{C}ong(\mathcal{A}) - \{\Delta\})$ (a principal congruence).

Theorem 3.2. *(Birkhoff) Every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras (homomorphic images of A).*

Theorem 3.3. *(Birkhoff) Every finite algebra is isomorphic to a subdirect product of a finite number of subdirectly irreducible finite algebras.*

The class of the simple algebras is a relevant subclass of the subdirectly irreducible algebras.

Definition 3.4. An algebra A is simple if and only if $\mathcal{C}ong(\mathcal{A}) = \{\Delta, \nabla\}$.

In [5] the class of all simple bijective Boolean modules was described.

Theorem 3.5. [5] *The degenerate Boolean module* $\mathcal{M} = (\{0\}, \{o\}, :)$ *is the unique simple bijective Boolean module.*

We now analyze the degenerate case of the Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ with $B = \{0\}$.

If $R = \{\mathrm{o}\}$, then $\mathcal{C}ong(\mathcal{M}) = \{(\Delta_B, \Delta_R) = (\nabla_B, \nabla_R)\}$ and we consider that $\mathcal{M}$ is both simple and subdirectly irreducible.

If $R = \{\mathrm{o}, 1\}$, then $\mathcal{M}$ is not bijective and $\mathcal{C}ong(\mathcal{M}) = \{(\Delta_B, \Delta_R), (\Delta_B, \nabla_R)\}$ and once more $\mathcal{M}$ is both simple and subdirectly irreducible.

If $R \not\supseteq \{o, 1\}$, then $\mathcal{M}$ is not bijective and $\mathcal{C}ong(\mathcal{M}) = \{(\nabla_B, \theta) : \theta \in \mathcal{C}ong(\mathcal{R})\}$. Therefore $\mathcal{M}$ is simple or subdirectly irreducible if and only if $\mathcal{R}$ is, respectively, simple or subdirectly irreducible.

3.1. Use of Essential Elements. Both Rautenberg [6] and Venema [9] characterized subdirectly irreducible and simple Boolean algebras with operators. In this section, it is our purpose to establish, in the class of bijective Boolean modules, an entity with a similar role to the one expressed by an essential element in Boolean algebras with operators and prove Theorem 3.23 asserting that a bijective Boolean module is subdirectly irreducible if and only if contains an essential element, a result in accordance with Theorem 4.16 of [9] for Boolean algebras with operators and with Theorem 3.21 of [4] for separable dynamic algebras.

Similar to a result obtained by Venema [9] for Boolean algebras with operators, we are now able to establish a lattice isomorphism between the collections of all open filters and all open congruences on a bijective Boolean module.

Definition 3.6. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. A subset F of B is an *open Boolean filter of* $\mathcal{M}$ if

(1) F is a Boolean filter ($p_1 \wedge p_2 \in F$, whenever $p_1, p_2 \in F$; and $q \in F$ whenever $p_1 \leq q$ and $p_1 \in F$);
(2) $[a]p \in F$, for $a \in R$ and $p \in F$.

We denote by $\mathcal{F}_{op}(\mathcal{M})$ the collection of all the open Boolean filters on $\mathcal{M}$.

Definition 3.7. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. A Boolean congruence θ is said to be a *open Boolean congruence on* $\mathcal{M}$ if

(1) θ is a Boolean congruence on $\mathcal{B}$;
(2) $ap\theta aq$ whenever $p\theta q$ and $a \in R$.

We denote by $\mathcal{C}ong_{op}(\mathcal{M})$ the collection of all the open Boolean congruences on $\mathcal{M}$.

Proposition 3.8. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. Then*

(1) *the collection of all open Boolean filters, $\mathcal{F}_{op}(\mathcal{M})$, is closed under taking arbitrary intersections and hence forms a lattice with respect to subset ordering;*

(2) *this lattice is isomorphic to the lattice of the open Boolean congruences on $\mathcal{M}$, $\mathcal{C}ong_{op}(\mathcal{M})$, through the isomorphism*

$$\Pi : \mathcal{F}_{op}(\mathcal{M}) \longrightarrow \mathcal{C}ong_{op}(\mathcal{M})$$

given by

$$\Pi_F := \{(p, q) \in B \times B : p \leftrightarrow q \in F\}$$

and its inverse

$$N : \mathcal{C}ong_{op}(\mathcal{M}) \longrightarrow \mathcal{F}_{op}(\mathcal{M})$$

by

$$N_\theta := \{p \in B : p\theta 1\}.$$

Proof: As in Proposition 3.14 of [4]. ■

The notion of essential element on Boolean algebras with operators was used by Venema [9] to specify the class of subdirectly irreducible algebras. An entity playing a similar role is now required for bijective Boolean modules.

Definition 3.9. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module, $p_e \in B$ and $a \in R$. The pair (p_e, a) is called an *essential element in* $\mathcal{M}$ if

(1) $p_e \neq 1$
(2) $ap \geq p$ for all $p \in B$
(3) $a(\sim p) \geq \sim p_e$ (or, equivalently, $[a]p \leq p_e$) for every $p \neq 1$.

We note that if (p, a) is an essential element on a bijective Boolean module $\mathcal{M}$, then $(p, 1)$ is also an essential element on $\mathcal{M}$.

If, in a bijective Boolean module, the relational element 1 assumes a particular feature, each pair $(p, 1)$ is an essential element, for each Boolean element $p \neq 1$. This will be achieved after the introducing of next definition.

Definition 3.10. [5] Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a Boolean module with relational part containing an element $\exists_s$ satisfying $\exists_s 0 = 0$ and $\exists_s p = 1$, for every $p \neq 0$. We call this element of R a *simple quantifier*.

Proposition 3.11. *If $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is a Boolean module with relational part containing an element $\exists_s$, then $1p = 1$ for every $p \neq 0$. (If $\mathcal{M}$ is bijective, then $\exists_s = 1$.)*

Corollary 3.12. *If $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is a bijective Boolean module with $1q = 1$ for every $q \neq 0$, then for every $p \in B$, with $p \neq 1$, the pair $(p, 1)$ is an essential element in $\mathcal{M}$.*

We examine the existence of essential elements on two bijective Boolean modules.

Example 3.13. Let us consider the Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ where $B = \{0, 1\}$ and $R = \{o, 1\}$.

The pair $(0, 1)$ is the unique essential element on $\mathcal{M}$.

Example 3.14. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be the proper Boolean module where $B = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$, $R = \{\Lambda, a, b, c\}$, Λ is the empty relation, $a = \{(p, p)\}$, $b = \{(q, q)\}$ and $c = \{(p, p), (q, q)\}$.

We have

$\Lambda\breve{}\emptyset = \Lambda\emptyset = \emptyset$	$a\breve{}\emptyset = a\emptyset = \emptyset$	$b\breve{}\emptyset = b\emptyset = \emptyset$	$c\breve{}\emptyset = c\emptyset = \emptyset$
$\Lambda\breve{}\{p\} = \Lambda\{p\} = \emptyset$	$a\breve{}\{p\} = a\{p\} = \{p\}$	$b\breve{}\{p\} = b\{p\} = \emptyset$	$c\breve{}\{p\} = c\{p\} = \{p\}$
$\Lambda\breve{}\{q\} = \Lambda\{q\} = \emptyset$	$a\breve{}\{q\} = a\{q\} = \emptyset$	$b\breve{}\{q\} = b\{q\} = \{q\}$	$c\breve{}\{q\} = c\{q\} = \{q\}$
$\Lambda\breve{}\{p, q\} = \Lambda\{p, q\} = \emptyset$	$a\breve{}\{p, q\} = a\{p, q\} = \{p\}$	$b\breve{}\{p, q\} = b\{p, q\} = \{q\}$	$c\breve{}\{p, q\} = c\{p, q\} = \{p, q\}$

The pairs $(\emptyset, c)$, $(\{p\}, c)$ and $(\{q\}, c)$ are all possible candidates to be essential elements on $\mathcal{M}$ since c is the unique relational element satisfying condition 2. of Definition 3.9. However, we have, $c\{p\} = \{p\} \not\geq \sim \emptyset = \{p, q\}$, $c\{p\} = \{p\} \not\geq \sim \{p\} = \{q\}$ and $c\{q\} = \{q\} \not\geq \sim \{q\} = \{p\}$, contradicting condition 3. of Definition 3.9. Therefore $\mathcal{M}$ has no essential elements.

Proposition 3.15. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module and s any element of B. The set*

$$F_s = \{q \in B : [1]s \leq q\} \qquad (\textit{or } F_s = \{q \in B : 1(\sim s) \geq \sim q\})$$

is the smallest open Boolean filter on $\mathcal{M}$ containing s.

Proof: First we notice that F_s is a Boolean filter. In fact, if $p, q \in F_s$ then $1(\sim s) \geq \sim p$ and $1(\sim s) \geq \sim q$. Therefore $1(\sim s) \vee 1(\sim s) \geq \sim p \vee \sim q$, *i.e.*, $1(\sim s) \geq \sim (p \wedge q)$ and so $p \wedge q \in F_s$. Now let $p \in F_s$ and $q \in B$ such that $p \leq q$. Then $1(\sim s) \geq \sim p$. But since $\sim p \geq \sim q$ we have $1(\sim s) \geq \sim q$ and therefore $q \in F_s$.

Now it has to be proved that for $p \in F_s$ and $a \in R$ then $[a]p \in F_s$. Since $p \in F_s$ we have $1(\sim s) \geq \sim p$ hence $(a; 1)(\sim s) = a(1(\sim s)) \geq a(\sim p)$. But $1(\sim s) \geq (a; 1)(\sim s)$ and so $1(\sim s) \geq a(\sim p) = \sim ([a]p)$ yielding $[a]p \in F_s$.

The proof that $s \in F_s$ is immediate since $1(\sim p) \geq \sim p$ for every $p \in B$.

It remains to be proved that F_s is the smallest filter containing s.

Let F be a filter containing s. We intend to show that $F_s \subseteq F$. For $q \in F_s$ we have $[1]s \leq q$. Since $s \in F$, an open Boolean filter of $\mathcal{M}$, and $1 \in R$ then $[1]s \in F$ so $q \in F$. ∎

Remark 3.16. We notice that $F_s = \{1\}$, the trivial open filter, if and only if $s = 1$.

(In fact, since $s \in F_s$ if $F_s = \{1\}$, then $s = 1$. Conversely $F_1 = \{q \in B : 1(\sim 1) \geq \sim q\} = \{q \in B : 1(0) \geq \sim q\} = \{q \in B : 1 \leq q\} = \{1\}$.)

Moreover, for $\mathcal{M}$ a non-degenerate bijective Boolean module, $F_s = B$ if and only if $1(\sim s) = 1$.

(If $F_s = B$ then $0 \in F_s$ and hence $1(\sim s) \geq \sim 0$ and so $1(\sim s) = 1$.)

Proposition 3.17. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. The set F_{p_e} is the smallest nontrivial open Boolean filter on M if and only if the pair (p_e, a) is an essential element on $\mathcal{M}$ for some $a \in R$ (if and only if the pair $(p_e, 1)$ is an essential element on $\mathcal{M}$).*

Proof: If F is the smallest nontrivial open Boolean filter on $\mathcal{M}$ necessarily $F = F_{p_e}$ for some p_e on B with $p_e \neq 1$. We intend to prove that there exists $a \in R$ such that (p_e, a) is an essential element on $\mathcal{M}$. Let $q \neq 1$ and let us consider the smallest open Boolean filter on $\mathcal{M}$ containing q, F_q. Since F_{p_e} is the smallest nontrivial open Boolean filter on $\mathcal{M}$ then $F_{p_e} \subseteq F_q$, and therefore $p_e \in F_q$. So $1(\sim q) \geq \sim p_e$ implying $(p_e, 1)$ to be an essential element on $\mathcal{M}$.

Now let us admit that there exists an $a \in R$ such that (p_e, a) is an essential element on $\mathcal{M}$. Immediately $(p_e, 1)$ is an essential element on $\mathcal{M}$, i.e., $p_e \neq 1$ and $1(\sim p) \geq \sim p_e$ for every $p \neq 1$. Let F be a nontrivial open Boolean filter on $\mathcal{M}$. Since F is nontrivial then $F \neq \{1\}$, so there exists $p \in F$, $p \neq 1$. But since $(p_e, 1)$ is essential, then $1(\sim p) \geq \sim p_e$ and so $[1]p \leq p_e$. The fact that $p \in F$ implies $[1]p \in F$ and so $p_e \in F$. Therefore $F_{p_e} \subseteq F$ as required. ∎

The existence of Boolean congruences on $\mathcal{B}$ which are not the Boolean part of any modular congruences on a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ was already previously mentioned in [5]. That fact gave rise to the introducing of the following definition.

Definition 3.18. [5] Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a Boolean module. A Boolean congruence θ_1 on $\mathcal{B}$ is called *pro-modular* on $\mathcal{M}$ whenever there exists a congruence θ_2 on $\mathcal{R}$ such that (θ_1, θ_2) is a modular congruence on $\mathcal{M}$.

Proposition 3.19. [5] *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a Boolean module and let θ_1 be a (Boolean) congruence on $\mathcal{B}$. The congruence θ_1 is a pro-modular congruence on $\mathcal{M}$ if and only if the pair $(\theta_1, \Delta_{\mathcal{R}})$ is a modular congruence on $\mathcal{M}$.*

Proposition 3.20. [5] *For θ_1 a pro-modular congruence on a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ the pair $(\theta_1, \Delta_{\mathcal{R}})$ is the smallest modular congruence on $\mathcal{M}$ having θ_1 as Boolean part.*

On a Boolean module the concepts of open Boolean congruence and of pro-modular congruence are equivalents.

Proposition 3.21. *On a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ a congruence $\theta_1 \in \mathcal{C}ong_{op}(\mathcal{M})$ if and only if the pair $(\theta_1, \Delta_R) \in \mathcal{C}ong(\mathcal{M})$ (if and only if θ_1 is a pro-modular congruence on $\mathcal{M}$).*

To prove one of our main assertions, Theorem 3.23, a fundamental result is required. There, the existence of the minimum element on the set $\mathcal{C}ong(\mathcal{M}) - \{(\Delta_B, \Delta_R)\}$ equivalent to the existence of the minimum element on $\mathcal{C}ong_{op}(\mathcal{M}) - \{\Delta_B\}$ is assured.

Proposition 3.22. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. The minimum element on $\mathcal{C}ong(\mathcal{M}) - \{(\Delta_B, \Delta_R)\}$ exists if and only if there exists the minimum element on $\mathcal{C}ong_{op}(\mathcal{M}) - \{\Delta_B\}$.*

Proof: Similar to Proposition 3.20 of [4]. ∎

The class of subdirectly irreducible bijective Boolean modules is characterized by the existence of an essential element on each subdirectly irreducible algebra, in agreement with that for the class of Boolean algebras with operators [9].

Theorem 3.23. *A non-degenerate bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is subdirectly irreducible if and only if there exists an element $p_e \in B$ such that the pair (p_e, a) is an essential element on M, for some $a \in R$.*

Proof: As in Theorem 3.21 of [4]. ∎

Corollary 3.24. *If $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is a bijective Boolean module with $1q = 1$ for every $q \neq 0$, then $\mathcal{M}$ is subdirectly irreducible.*

Proof: Trivial by Corollary 3.12. ∎

Brink on [1] presented, following a homogeneous point of view, a characterization of subdirectly irreducible and simple Boolean modules. There, a

Boolean module is subdirectly irreducible if and only if is simple, if and only if $1p = 1$ for every Boolean element $p \neq 0$. Therefore each subdirectly irreducible bijective Boolean module under the Brink' homogeneous approach is also a subdirectly irreducible bijective Boolean module under our heterogeneous point of view. Later on, we will reach an equivalent characterization that will enable us to obtain the same class on both approaches, as already ensured by Proposition 3.22.

The following examples give some insight to the result established by Theorem 3.23.

Example 3.25. Previously in Example 3.13 we presented a Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ with $B = \{0, 1\}$ and $R = \{\mathrm{o}, 1\}$ containing, as a unique essential element, the pair $(0, 1)$. Therefore by Theorem 3.23 this Boolean module is subdirectly irreducible. In fact, since $\mathcal{C}ong\mathcal{B} = \{\Delta_B, \nabla_B\}$ and $\mathcal{C}ong\mathcal{R} = \{\Delta_R, \nabla_R\}$ we have $\mathcal{C}ong\mathcal{M} = \{(\Delta_B, \Delta_R), (\nabla_B, \Delta_R), (\nabla_B, \nabla_R)\}$. The minimum element of the set $\mathcal{C}ong\mathcal{M} - \{(\Delta_B, \Delta_R)\}$ exists (and is equal to (∇_B, Δ_R)). Therefore $\mathcal{M}$ is subdirectly irreducible.

Example 3.26. In Example 3.14 the Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ with $B = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$, $R = \{\Lambda, a, b, c\}$, Λ is the empty relation, $a = \{(p, p)\}$, $b = \{(q, q)\}$ and $c = \{(p, p), (q, q)\}$ contains no essential elements and so is a non-subdirectly irreducible bijective Boolean module.

We have $\mathcal{C}ong\mathcal{B} = \{\Delta_B, \theta_1, \gamma_1, \nabla_B\}$ where

$$\theta_1 = \Delta_B \cup \{(\emptyset, \{p\}), (\{p\}, \emptyset), (\{q\}, \{p, q\}), (\{p, q\}, \{q\})\}$$

and

$$\gamma_1 = \Delta_B \cup \{(\emptyset, \{q\}), (\{q\}, \emptyset), (\{p\}, \{p, q\}), (\{p, q\}, \{p\})\}.$$

The set $\{\Delta_R, \theta_2, \gamma_2, \nabla_R\}$ where

$$\theta_2 = \Delta_R \cup \{(\Lambda, a), (a, \Lambda), (b, c), (c, b)\}$$

and

$$\gamma_2 = \Delta_R \cup \{(\Lambda, b), (b, \Lambda), (a, c), (c, a)\}$$

contains all the Boolean congruences defined on $\mathcal{R}$. Each congruence is compatible with converse and composition attending to

$\Lambda^\smile\emptyset = \Lambda\emptyset = \emptyset$	$a^\smile\emptyset = a\emptyset = \emptyset$	$b^\smile\emptyset = b\emptyset = \emptyset$	$c^\smile\emptyset = c\emptyset = \emptyset$
$\Lambda^\smile\{p\} = \Lambda\{p\} = \emptyset$	$a^\smile\{p\} = a\{p\} = \{p\}$	$b^\smile\{p\} = b\{p\} = \emptyset$	$c^\smile\{p\} = c\{p\} = \{p\}$
$\Lambda^\smile\{q\} = \Lambda\{q\} = \emptyset$	$a^\smile\{q\} = a\{q\} = \emptyset$	$b^\smile\{q\} = b\{q\} = \{q\}$	$c^\smile\{q\} = c\{q\} = \{q\}$
$\Lambda^\smile\{p, q\} = \Lambda\{p, q\} = \emptyset$	$a^\smile\{p, q\} = a\{p, q\} = \{p\}$	$b^\smile\{p, q\} = b\{p, q\} = \{q\}$	$c^\smile\{p, q\} = c\{p, q\} = \{p, q\}$

and

;	Λ	a	b	c
Λ	Λ	Λ	Λ	Λ
a	Λ	a	Λ	a
b	Λ	Λ	b	b
c	Λ	a	b	c

Therefore $\mathcal{C}ong\mathcal{R} = \{\Delta_R, \theta_2, \gamma_2, \nabla_R\}$.

Among all pairs of congruences we are obliged to choose
$\mathcal{C}ong\mathcal{M} = \{(\Delta_B, \Delta_R), (\theta_1, \Delta_R), (\theta_1, \theta_2), (\gamma_1, \Delta_R), (\gamma_1, \gamma_2), (\nabla_B, \Delta_R), (\nabla_B, \theta_2), (\nabla_B, \gamma_2), (\nabla_B, \nabla_R)\}$
since:

(i) (Δ_B, ϕ) is a modular congruence if and only if $\phi = \Delta_R$ (Prop. 2.10);
(ii) (ϕ, ∇_R) is a modular congruence if and only if $\phi = \nabla_B$ (Prop. 2.8);
(iii) (θ_1, θ_2) and (θ_1, Δ_R) are modular congruences asserted by the table

$\theta_2 \backslash \theta_1$	$(\emptyset,\emptyset)$	$(\{p\},\{p\})$	$(\{q\},\{q\})$	$(\{p,q\},\{p,q\})$	$(\emptyset,\{p\})$	$(\{p\},\emptyset)$	$(\{q\},\{p,q\})$	$(\{p,q\},\{q\})$
(Λ,Λ)	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$
(a,a)	$(\emptyset,\emptyset)$	$(\{p\},\{p\})$	$(\emptyset,\emptyset)$	$(\{p\},\{p\})$	$(\emptyset,\{p\})$	$(\{p\},\emptyset)$	$(\emptyset,\{p\})$	$(\{p\},\emptyset)$
(b,b)	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\{q\},\{q\})$	$(\{q\},\{q\})$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\{q\},\{q\})$	$(\{q\},\{q\})$
(c,c)	$(\emptyset,\emptyset)$	$(\{p\},\{p\})$	$(\{q\},\{q\})$	$(\{p,q\},\{p,q\})$	$(\emptyset,\{p\})$	$(\{p\},\emptyset)$	$(\{q\},\{p,q\})$	$(\{p,q\},\{q\})$
(Λ,a)	$(\emptyset,\emptyset)$	$(\emptyset,\{p\})$	$(\emptyset,\emptyset)$	$(\emptyset,\{p\})$	$(\emptyset,\{p\})$	$(\emptyset,\emptyset)$	$(\emptyset,\{p\})$	$(\emptyset,\emptyset)$
(a,Λ)	$(\emptyset,\emptyset)$	$(\{p\},\emptyset)$	$(\emptyset,\emptyset)$	$(\{p\},\emptyset)$	$(\emptyset,\emptyset)$	$(\{p\},\emptyset)$	$(\emptyset,\emptyset)$	$(\{p\},\emptyset)$
(b,c)	$(\emptyset,\emptyset)$	$(\emptyset,\{p\})$	$(\{q\},\{q\})$	$(\{q\},\{p,q\})$	$(\emptyset,\{p\})$	$(\emptyset,\emptyset)$	$(\{q\},\{p,q\})$	$(\{q\},\{q\})$
(c,b)	$(\emptyset,\emptyset)$	$(\{p\},\emptyset)$	$(\{q\},\{q\})$	$(\{p,q\},\{q\})$	$(\emptyset,\emptyset)$	$(\{p\},\emptyset)$	$(\{q\},\{q\})$	$(\{p,q\},\{q\})$

(iv) similarly to (iii), the pairs (γ_1, γ_2) and (γ_1, Δ_R) are modular congruences;
(v) (θ_1, γ_2) and (γ_1, θ_2) are not modular congruences since we have, respectively,
$\{q\}\theta_1\{q\}$ and $\Lambda\gamma_2 b$ but $\emptyset = \Lambda\{q\} \not\theta_1 b\{q\} = \{q\}$ and
$\{p\}\gamma_1\{p\}$ and $\Lambda\theta_2 a$ but $\emptyset = \Lambda\{p\} \not\gamma_1 a\{p\} = \{p\}$;
(vi) obviously the pairs $(\nabla_B, \Delta_R), (\nabla_B, \theta_2), (\nabla_B, \gamma_2)$ are modular congruences.

Since the minimum element of the set $\mathcal{C}ong\mathcal{M} - \{(\Delta_B, \Delta_R)\}$ does not exist, $\mathcal{M}$ is not subdirectly irreducible.

3.2. Use of [1]**-closed elements.** Some characterizations of subdirectly irreducible bijective Boolean modules is given, agreeing with that of Sambin [7] for modal algebras satisfying K4. We are able to single out, on a subdirectly irreducible bijective Boolean module, a Boolean element, the greatest among the [1]-closed elements on a bijective Boolean module. This led us to prove Theorem 3.35 where such an element is qualified as the principal filter generator, of the set of all Boolean parts of essential elements united with the Boolean element 1. Moreover, we will prove that this element is the zero element of the Boolean part, allowing us to infer that both homogeneous and heterogeneous characterizations of subdirectly irreducible bijective Boolean modules (given, respectively, by Brink and by us) are equivalent.

On a bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ we denote by $E_{\mathcal{M}}$ the set of all $p \in B$ such that, for some $a \in R$, the pair (p, a) is an essential element on $\mathcal{M}$, i.e.,

$$E_{\mathcal{M}} = \{p \in B : (p, a) \text{ is an essential element on } M, \text{for some } a \in R\}.$$

Using Theorem 3.23 we can infer that a bijective Boolean module is subdirectly irreducible if and only if $E_{\mathcal{M}} \neq \emptyset$. By Corollary 3.12 we note that if $\mathcal{M}$ is a bijective Boolean module with $1p = 1$ for every $p \neq 0$, then $E_{\mathcal{M}} \cup \{1\} = B$.

We now prove that if a bijective Boolean module $\mathcal{M}$ is subdirectly irreducible then $E_{\mathcal{M}} \cup \{1\} = F_{p_e}$ for each $p_e \in E_{\mathcal{M}}$.

Theorem 3.27. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. If $E_{\mathcal{M}} \neq \emptyset$, then $E_{\mathcal{M}} \cup \{1\} = F_{p_e}$ for each $p_e \in E_{\mathcal{M}}$.*

Proof: Let $p_e \in E_{\mathcal{M}}$. We have to prove that $E_{\mathcal{M}} \cup \{1\} = F_{p_e}$. If $p_0 \in E_{\mathcal{M}}$, then there exists $b \in R$ with (p_0, b) an essential element. Therefore $b(\sim p) \geq \sim p_0$ for every $p \neq 1$. Since $p_e \neq 1$ we have $b(\sim p_e) \geq \sim p_0$ and then $1(\sim p_e) \geq b(\sim p_e) \geq \sim p_0$. Therefore $p_0 \in F_{p_e}$ and $E_{\mathcal{M}} \cup \{1\} \subseteq F_{p_e}$.

Let $q \in F_{p_e}$ and $q \neq 1$. Then $1(\sim p_e) \geq \sim q$. Since $(p_e, 1)$ is essential we have $1(\sim p) \geq \sim p_e$ for every $p \neq 1$, and then $1(1(\sim p)) \geq 1(\sim p_e)$ for every $p \neq 1$, i.e., $1(\sim p) \geq 1(\sim p_e)$ for every $p \neq 1$. Therefore $1(\sim p) \geq \sim q$ for every $p \neq 1$. So the pair $(q, 1)$ is an essential element on $\mathcal{M}$, i.e., $q \in E_{\mathcal{M}}$. Therefore $F_{p_e} \subseteq E_{\mathcal{M}} \cup \{1\}$. ■

On a subdirectly irreducible bijective Boolean module $\mathcal{M}$ the non-empty set $E_{\mathcal{M}}$ contains an element playing a very special role.

Definition 3.28. Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ a bijective Boolean module. An element $q \in B$ is said to be a $[1]$-*closed element* on $\mathcal{M}$ if $[1]q = q$ (or, equivalently, $1(\sim q) = \sim q$).

In any bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ both elements $0, 1$ of B are $[1]$-closed elements on $\mathcal{M}$.

Proposition 3.29. *If $q \in B$ is a $[1]$-closed element on a bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$, then $\sim q$ is also a $[1]$-closed element on $\mathcal{M}$.*

Proof: Since q is a $[1]$-closed element we have

$$\begin{array}{rcll} 1(\sim q) & = & \sim q & \\ (1(\sim q)) \wedge q & = & (\sim q) \wedge q & \\ \breve{1}q \wedge \sim q & = & 0 & \text{(using } M11) \\ 1q \wedge \sim q & = & 0 & \text{(using } R9) \end{array}$$

so $1q \leq q$. But by $M17$ we have $1q \geq q$ hence $1q = q$. Therefore $[1](\sim q) =\sim q$, as required. ■

Proposition 3.30. *If $\mathcal{M}$ is a bijective Boolean module with $1p = 1$ for every $p \neq 0$, then 0 and 1 are all $[1]$-closed elements on $\mathcal{M}$.*

Proof: If p_0 is a $[1]$-closed element on $\mathcal{M}$, then $1(\sim p_0) =\sim p_0$. But if $\sim p_0 = 0$, then $p_0 = 1$. If $\sim p_0 \neq 0$ then $1(\sim p_0) = 1$, so $1 =\sim p_0$, i.e., $p_0 = 0$. ■

An equivalent condition to the $[1]$-closed element's definition is given below.

Proposition 3.31. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ a bijective Boolean module. An element $p_0 \in B$ is $[1]$-closed element on $\mathcal{M}$ if and only if $[a]p_0 \geq p_0$ (or, equivalently, $a(\sim p_0) \leq\sim p_0$) for every $a \in R$.*

Proof: If $p_0 \in B$ is a $[1]$-closed element on $\mathcal{M}$, then $1(\sim p_0) =\sim p_0$. But, for every $a \in R$ we have $a(\sim p_0) \leq 1(\sim p_0)$ and then $a(\sim p_0) \leq\sim p_0$. Conversely, if $a(\sim p_0) \leq\sim p_0$ for every $a \in R$, then for $a := 1$ we have $1(\sim p_0) \leq\sim p_0$. But, by M17 we have $1(\sim p_0) \geq\sim p_0$, therefore $1(\sim p_0) =\sim p_0$. ■

The two following conditions will be of fundamental importance in the proof of Theorem 3.35.

Proposition 3.32. *On a bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ each element $[1]p$ is a $[1]$-closed element on $\mathcal{M}$ for every $p \in B$.*

Proof: By M20 we have $[1]([1]p) = [1]p$ for every $p \in B$, as required. ■

A Boolean element emerges from next proposition. In fact, on a bijective Boolean module, we have $[1]p_1 = [1]p_2$, for arbitrary essentials elements (p_1, a) and (p_2, b).

Proposition 3.33. *The set* $\{[1]p : (p, a)$ *is an essential element for some* $a\}$ $\subseteq E_{\mathcal{M}}$*, on a bijective Boolean module* $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$*, is either singular or empty.*

Proof: Let (p_1, a) and (p_2, b) be essential elements on $\mathcal{M}$. Then $(p_1, 1)$ is an essential element on $\mathcal{M}$ and so $1(\sim p) \geq \sim p_1$ for every $p \neq 1$, i.e., $[1]p \leq p_1$ for every $p \neq 1$. Since (p_2, b) is an essential element then $p_2 \neq 1$ and then $[1]p_2 \leq p_1$. Using M19 we have $[1]([1]p_2) \leq [1]p_1$ and by M20 we infer that $[1]p_2 \leq [1]p_1$. And similarly.

It remains to be proved that such an element, $([1]p_e, 1)$, for any essential element (p_e, a), is essential on M. We have $1(\sim p) \geq \sim p_e$ for any $p \neq 1$. Hence

$$\begin{aligned} 1(\sim p) &\geq \sim p_e \\ 1 : (1(\sim p)) &\geq 1(\sim p_e) \\ 1(\sim p) &\geq \sim [1]p_e, \qquad \text{for every } p \neq 1. \end{aligned}$$

■

On a subdirectly irreducible bijective Boolean module, the greatest [1]-closed element, distinct from 1, coincides with the element of the singular set described above.

Proposition 3.34. *On a subdirectly irreducible bijective Boolean module* $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$*, for each* $p_e \in E_{\mathcal{M}}$*, the element* $[1]p_e$ *is the greatest* $[1]$*-closed element distinct from 1.*

Proof: Since (p_e, a) is an essential element on M, for some $a \in R$, then $(p_e, 1)$ is an essential element on M. The element $[1]p_e$ is a [1]-closed element (Proposition 3.32). Let p_1 a [1]-closed element with $p_1 \neq 1$. So $[1]p_1 = p_1$. Since $(p_e, 1)$ is an essential element we have $1(\sim p) \geq \sim p_e$ for every $p \neq 1$ and since $p_1 \neq 1$ we get $1(\sim p_1) \geq \sim p_e$, i.e., $[1]p_1 \leq p_e$. Therefore $[1]([1]p_1) \leq [1]p_e$ so $[1]p_1 \leq [1]p_e$. Since $[1]p_1 = p_1$ we obtain $p_1 \leq [1]p_e$. Since (p_e, a) is essential we have $p_e \neq 1$ so $[1]p_e \leq p_e \neq 1$ and immediately $[1]p_e \neq 1$. Therefore $[1]p_e$ is the greatest [1]-closed element distinct from 1. ■

A similar result to the one obtained by Sambin [7] for modal algebras satisfying K4 can now be asserted for bijective Boolean modules. Next Theorem states, among other results, that if $\mathcal{M}$ is a subdirectly irreducible bijective Boolean module then the set $E_{\mathcal{M}} \cup \{1\}$ is a principal filter generated by $[1]p_e$ for any $p_e \in E_{\mathcal{M}}$.

Theorem 3.35. *For a bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ the following assertions are equivalent:*

(1) *$\mathcal{M}$ is subdirectly irreducible;*
(2) *$E_{\mathcal{M}} \cup \{1\}$ is a principal filter distinct from $\{1\}$;*
(3) *$\mathcal{M}$ has a $[1]$-closed element p_0 such that there exists an element $a \in R$ with (p_0, a) an essential element;*
(4) *M has a greatest $[1]$-closed element distinct from 1.*

Proof: We prove that $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4)$.

$(4) \Rightarrow (3)$

Let p_0 the greatest $[1]$-closed element distinct from 1. By Proposition 3.32 the element $[1]p$ is $[1]$-closed for every $p \in B$, so $[1]p \leq p_0$, for every $p \neq 1$. Therefore $(p_0, 1)$ is an essential element on $\mathcal{M}$.

$(3) \Rightarrow (2)$

If $\mathcal{M}$ has a $[1]$-closed element p_0 such that there exist $a \in R$ with (p_0, a) an essential element, then $(p_0, 1)$ is an essential element and $E_{\mathcal{M}} \neq \emptyset$. By Theorem 3.27 we have $E_{\mathcal{M}} \cup \{1\} = F_{p_e}$ for any p_e such that there exists $a \in R$ with (p_e, a) an essential element. We put $p_e := p_0$. So

$$\begin{aligned} E_{\mathcal{M}} \cup \{1\} &= F_{p_0} \\ &= \{q \in B : 1(\sim p_0) \geq \sim q\} \\ &= \{q \in B : \sim p_0 \geq \sim q\} \qquad \text{(since } p_0 \text{ is [1]-closed)} \\ &= \{q \in B : p_0 \leq q\} \end{aligned}$$

and therefore $E_{\mathcal{M}} \cup \{1\}$ is a principal filter. Since $E_{\mathcal{M}} \cup \{1\} = F_{p_0}$ and $p_0 \neq 1$ ($(p_0, 1)$ is an essential element), then $E_{\mathcal{M}} \cup \{1\}$ is a principal filter distinct from $\{1\}$ (Remark 3.16).

$(2) \Rightarrow (1)$

It is trivial by Theorem 3.23.

$(1) \Rightarrow (4)$

If M is subdirectly irreducible, then there exists an element $p_0 \in B$ such that the pair (p_0, a) is an essential element on M for some $a \in R$. Then the required result follows from Proposition 3.34. ∎

Proposition 3.36. *On a bijective Boolean module $\mathcal{M}$, if $p_e \in E_{\mathcal{M}} - \{0\}$, then $1p_e = 1$.*

Proof: For any $p_e \in E_{\mathcal{M}}$, since $[1](\sim p_e)$ is a $[1]$-closed element, we have either $[1](\sim p_e) = 1$ or $[1](\sim p_e) \leq [1]p_e$, by Proposition 3.34. If $[1](\sim p_e) = 1$, then $\sim (1p_e) = 1$, i.e., $1p_e = 0$. Since $p_e \leq 1p_e$ we have $p_e = 0$. Now if $[1](\sim p_e) \leq [1]p_e$, then $\sim (1p_e) \leq \sim (1(\sim p_e))$ and $1p_e \geq 1(\sim p_e)$. Therefore

$1p_e \vee 1(\sim p_e) = 1p_e$, i.e., $1(p_e \vee (\sim p_e)) = 1p_e$. Hence $1 : 1 = 1p_e$, i.e., $1p_e = 1$. ■

Proposition 3.37. *A bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is subdirectly irreducible if and only $E_{\mathcal{M}} \cup \{1\} = B$.*

Proof: If $E_{\mathcal{M}} \cup \{1\} = B$, then $E_{\mathcal{M}} \neq \emptyset$ for every $\mathcal{M}$ distinct from the degenerate Boolean module, $(\{0\}, \{o\}, :)$ that will be assumed as subdirectly irreducible. Then $\mathcal{M}$ is subdirectly irreducible, using Theorem 3.23.

Now, for $\mathcal{M}$ subdirectly irreducible, using Theorem 3.35, $\mathcal{M}$ has a [1]-closed element p_0 such that there exists an element $a \in R$ with (p_0, a) an essential element. Let us admit that $p_0 \neq 0$. Then, using Proposition 3.36, we have $1p_0 = 1$. Since p_0 is a [1]-closed element then $\sim p_0$ is also a [1]-closed element, by Proposition 3.29. Hence $1(\sim\sim p_0) = \sim\sim p_0$, i.e., $1p_0 = p_0$. Therefore $p_0 = 1$ contradicting the definition of (p_0, a) as essential element. So $p_0 = 0$. ■

We note that we have just proved that on a subdirectly irreducible bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$, for each Boolean element $p \neq 1$, there exists $a \in R$ such that (p, a) (and , consequently, $(p, 1)$) is an essential element on $\mathcal{M}$. In particular, since $(0, 1)$ is an essential element on $\mathcal{M}$, then $1(\sim q) \geq \sim 0$ for any $q \neq 1$, i.e., $1(s) = 1$ for any $s \neq 0$. Therefore our characterization becomes equivalent to the one given by Brink, [1], as previously alleged. There, the classes of simple bijective Boolean modules and subdirectly irreducible bijective Boolean modules coincide. But next section contains a heterogeneous characterization for simple bijective Boolean modules that gives rise to a class of simple bijective Boolean modules reduced to the degenerated Boolean module $\mathcal{M} = (\{0\}, \{o\}, :)$ already determined in [5] and differing from the class described by Brink.

4. Simple Bijective Boolean Modules

The definition of essential element used by us on bijective Boolean modules is suitable to obtain a result analogous to Rautenberg's characterization of simple Boolean algebras with operators. Here, the simple bijective Boolean modules are again fully characterized by its essential elements.

Proposition 4.1. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ be a bijective Boolean module. If for some $a \in R$ the pair $(0, a)$ is an essential element on $\mathcal{M}$ then $a = 1$.*

Proof: By definition we have $ap \geq \sim 0 = 1$ for any $p \neq 0$. Immediately we get $ap = 1$ for every $p \neq 0$. The result follows from Proposition 3.11. ∎

The Rautenberg's characterization obtained for Boolean algebras with operators can now be revisited for bijective Boolean modules.

Theorem 4.2. *A bijective Boolean module $\mathcal{M} = (\mathcal{B}, \mathcal{R}, :)$ is simple if and only if, for every element $1 \neq p_e \in B$, and $a \in R$, the pair (p_e, a) is an essential element.*

Proof: The unique simple bijective Boolean module is the degenerated Boolean module $(B = \{0\}, R = \{\mathrm{o}\}, :)$ and the required condition is trivially satisfied.

If, for every Boolean element $p_e \neq 1$ and $a \in R$, the pair (p_e, a) is an essential element, then in particular, on a Boolean module with $B \not\supseteq \{0\}$ the element $(0, \mathrm{o})$ is an essential element asserting that, for every $p \in B$ we have $\mathrm{o}p \geq p$, i.e., $0 \geq p$, and so $B = \{0\}$ a contradiction. Now the fact that $B = \{0\}$ implies that $R = \{\mathrm{o}\}$ since we are assuming that $\mathcal{M}$ is bijective. ∎

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