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# REDUCTION OF POISSON-NIJENHUIS LIE ALGEBROIDS TO SYMPLECTIC-NIJENHUIS LIE ALGEBROIDS WITH NONDEGENERATE NIJENHUIS TENSOR

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Abstract: We show how to reduce, under certain regularities conditions, a Poisson-Nijenhuis Lie algebroid to a symplectic-Nijenhuis Lie algebroid with nondegenerate Nijenhuis tensor. We generalize the work done by Magri and Morosi for the reduction of Poisson-Nijenhuis manifolds. The choice of the more general framework of Lie algebroids is motivated by the geometrical study of some reduced bi-Hamiltonian systems. An explicit example of reduction of a Poisson-Nijenhuis Lie algebroid is also provided.

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**Keywords:** Poisson-Nijenhuis Lie algebroid, Reduction, bi-Hamiltonian system.

## 1. Introduction

Poisson-Nijenhuis structures on manifolds were introduced by Magri and Morosi [16] and then intensively studied by many authors [9, 12, 19, 21, 22]. Recall that a Poisson-Nijenhuis manifold consists of a triple $(M, \Lambda, N)$, where $M$ is a manifold endowed with a Poisson bivector field $\Lambda$ and a $(1, 1)$-tensor $N$ whose Nijenhuis torsion vanishes, together with some compatibility conditions between $\Lambda$ and $N$. Poisson-Nijenhuis manifolds are very important in the study of integrable systems since they produce bi-Hamiltonian systems [9, 12, 16]. In particular, Magri and Morosi showed how to reduce a Poisson-Nijenhuis manifold to a nondegenerate one, i.e., one where the Poisson structure is actually symplectic and the Nijenhuis tensor is kernel-free. In this paper we show how to perform the same process of reduction in the more general framework of Lie algebroids. This type of structures have deserved a lot of interest in relation with the formulation of the Mechanics on disparate situations as systems with symmetry, systems evolving on semidirect products, Lagrangian and Hamiltonian systems on Lie algebras, and field theory equations (see, for instance, [3, 11] and the references therein).

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More precisely, in this paper we will see how to reduce a Poisson-Nijenhuis Lie algebroid to a symplectic-Nijenhuis Lie algebroid with nondegenerate Nijenhuis tensor. One could wonder about the interest of such a generalization. However, we show that working in the framework of Poisson-Nijenhuis Lie algebroids one may understand the geometrical structure of some physical examples related with bi-Hamiltonian systems and hence it is not a mere academic exercise. Indeed we present, as a motivating example, the study of the classic Toda lattice which, as is well known, admits a Poisson-Nijenhuis structure on $\mathbb{R}^{2n}$. Nevertheless, when switching to the more convenient Flaschka coordinates, one sees that the Poisson-Nijenhuis structure is lost, since there is no more a recursion operator connecting the hierarchy of Poisson structures. Nevertheless, the Poisson-Nijenhuis structure can be recovered if the system is described as a Lie algebroid (see also [2]).

The paper is organized as follows. In Section 2 we recall the notion of Poisson-Nijenhuis manifolds, then we describe the example of the Toda lattice as a motivation for the introduction of Poisson-Nijenhuis Lie algebroids. Next, we present the reduced Toda lattice as a Poisson-Nijenhuis Lie algebroid (see also [2]). Moreover, we show how this example can be framed in a more general case by considering a $G$-invariant Poisson-Nijenhuis structure on the total space $M$ of a $G$-principal bundle. Such a structure, in general does not induce a Poisson-Nijenhuis structure on $M/G$. Nevertheless, it gives rise to a Poisson-Nijenhuis Lie algebroid on the associated Atiyah bundle, which allows to build the bi-Hamiltonian system in the reduced space $M/G$. In the following sections we present the reduction of Poisson-Nijenhuis Lie algebroids. The reduction process is carried on in two steps. The first step, described in Section 3, consists in selecting a generalized foliation $D = \rho_A(P^\sharp A^*)$ on the given Poisson-Nijenhuis Lie algebroid $(A, [\cdot,\cdot]_A, \rho_A, P, N)$ and then showing that restricting on each leaf $L$ of $D$ one obtains a symplectic-Nijenhuis Lie algebroid structure. The leaves of the foliation $D$ are generally larger than those of the symplectic foliation of the induced Poisson structure on the base manifold. In Section 4 we deal with Lie algebroid epimorphisms introducing the notion of projectability of Poisson-Nijenhuis structures. We prove that given a projectable Poisson-Nijenhuis structure on a Lie algebroid and a Lie algebroid epimorphism we obtain a Poisson-Nijenhuis structure on the target Lie algebroid. Finally, we introduce the notion of Poisson-Nijenhuis Lie algebroid morphism. In Section 5 we study the reduction of a Lie algebroid by the foliation generated by the vertical and complete lifts of the sections of a Lie subalgebroid using an epimorphism of Lie algebroids. In Section 6, we use the previous constructions to obtain a reduced symplectic-Nijenhuis Lie algebroid with nondegenerate Nijenhuis tensor from an arbitrary symplectic-Nijenhuis Lie algebroid, under suitable conditions. In this way we complete the second and final step of the process of reduction. By putting together the two steps, we obtain our main result, which is the following one.

**Theorem.** *Let* $(A, [\cdot,\cdot]_A, \rho_A, P, N)$ *be a Poisson-Nijenhuis Lie algebroid such that*

*i) The Poisson structure* $P$ *has constant rank in the leaves of the foliation* $D = \rho_A(P^\sharp(A^*))$.

*If* $L$ *is a leaf of* $D$*, then, we have a symplectic-Nijenhuis Lie algebroid structure* $([\cdot,\cdot]_{A_L}, \rho_{A_L}, \Omega_L, N_L)$ *on* $A_L = P^\sharp(A^*)_{|L} \to L$.

*Assume, moreover, that*

*ii) The induced Nijenhuis tensor* $N_L : A_L \to A_L$ *has constant Riesz index* $k$*;*

*iii) The dimension of the subspace* $B_x = \ker N_x^k$ *is constant, for all* $x \in L$ *(thus,* $B = \ker N_L^k$ *is a vector subbundle of* $A$*);*

*iii) The foliations* $\rho_A(B)$ *and* $\mathcal{F}^B$ *are regular, where*

$$(\mathcal{F}^B)_a = \{X^c(a) + Y^v(a)/X, Y \in \Gamma(B)\}, \text{ for } a \in A_L$$

*iv)* (condition $\mathcal{F}^B$) *For all* $x \in L$, $a_x - a'_x \in B_x$ *if* $a_x$ *and* $a'_x$ *belong to the same leaf of the foliation* $\mathcal{F}^B$.

*Then, we obtain a symplectic-Nijenhuis Lie algebroid structure* $([\cdot,\cdot]_{\widetilde{A_L}}, \rho_{\widetilde{A_L}}, \widetilde{\Omega_L}, \widetilde{N_L})$ *on the vector bundle* $\widetilde{A_L} = A_L/\mathcal{F}^B \to \widetilde{L} = L/\rho_{A_L}(B)$ *with* $\widetilde{N_L}$ *nondegenerate.*

The last section of the paper contains an explicit example of reduction of a Poisson-Nijenhuis Lie algebroid which illustrates our theory. This is obtained by considering a Lie group $G$ which is the semidirect product of two Lie groups. We construct a $G$-invariant Poisson-Nijenhuis structure on the cotangent bundle $T^*G$ and then we obtain a Poisson-Nijenhuis structure on the associated Atiyah Lie algebroid which is degenerate. Thus, it may be effectively reduced, according to our main theorem.

## 2. Poisson-Nijenhuis Lie algebroids: a motivating example

In this section we will motivate the introduction of the notion of Poisson-Nijenhuis Lie algebroids with a simple example: the Toda lattice. Firstly, we will recall some definitions and results on Poisson-Nijenhuis manifolds.

**2.1. Poisson-Nijenhuis manifolds.** Let $\Lambda \in \Gamma(\wedge^2 TM)$ be a bivector field on a manifold $M$. We denote by $\Lambda^\sharp$ the usual bundle map

$$\Lambda^\sharp : \quad T^*M \longrightarrow TM, \quad \alpha \longmapsto \Lambda^\sharp(\alpha) = i_\alpha \Lambda. \tag{1}$$

Recall that $\Lambda$ defines a *Poisson structure on* $M$ if the Schouten bracket $[\Lambda, \Lambda]$ vanishes. In this case, one defines a *Poisson bracket* by

$$\{f, g\}_\Lambda := \Lambda(df, dg), \quad f, g \in C^\infty(M)$$

which makes $C^\infty(M)$ into a Lie algebra, and which is a derivation if either $f$ or $g$ is fixed. The Poisson bracket on $C^\infty(M)$ extends to a Lie bracket on the space $\Omega^1(M)$ of 1-forms on $M$ defined by

$$[\alpha, \beta]_\Lambda = \mathcal{L}_{\Lambda^\sharp \alpha}\beta - \mathcal{L}_{\Lambda^\sharp \beta}\alpha - \mathrm{d}\left(\Lambda(\alpha, \beta)\right), \quad \alpha, \beta \in \Omega^1(M), \tag{2}$$

such that on exact 1-forms one has $[df, dg]_\Lambda = d\{f, g\}_\Lambda$.

If a $(1,1)$-tensor field $N : TM \to TM$ is given on a manifold $M$, then its torsion $\mathcal{T}_N \in \Gamma(\Lambda^2 T^*M \otimes TM)$ is defined by

$$\mathcal{T}_N(X, Y) := [NX, NY] - N[X, Y]_N, \quad X, Y \in \mathfrak{X}(M), \tag{3}$$

where $[\cdot, \cdot]_N$ is given by

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \mathfrak{X}(M). \tag{4}$$

When $\mathcal{T}_N = 0$, the tensor field $N$ is called a *Nijenhuis tensor*.

Now, if $\Lambda \in \Gamma(\wedge^2 TM)$ is a Poisson structure on $M$, we say that a bundle map $N : TM \to TM$ is *compatible* with $\Lambda$ if $N\Lambda^\sharp = \Lambda^\sharp N^*$ and the *Magri-Morosi concomitant* vanishes:

$$\mathcal{C}(\Lambda, N)(\alpha, \beta) = [\alpha, \beta]_{N\Lambda} - [\alpha, \beta]_\Lambda^{N^*} = 0,$$

where $[\cdot, \cdot]_{N\Lambda}$ is the bracket defined by the section $N\Lambda \in \Gamma(\wedge^2 A)$ in a similar way as in (2), and $[\cdot, \cdot]_\Lambda^{N^*}$ is the Lie bracket obtained from the Lie bracket $[\cdot, \cdot]_\Lambda$ by deformation along the dual map $N^* : T^*M \to T^*M$ in a similar way as in (4), i.e.

$$[\alpha, \beta]_\Lambda^{N^*} = [N^*\alpha, \beta]_\Lambda + [\alpha, N^*\beta]_\Lambda - N^*\left[\alpha, \beta\right]_\Lambda.$$

**Definition 2.1.** ([16]) A *Poisson-Nijenhuis manifold* $(M, \Lambda, N)$ is a manifold $M$ equipped with a Poisson structure $\Lambda$ and a Nijenhuis tensor $N : TM \to TM$ compatible with $\Lambda$.

In such a case, one may obtain a hierarchy of compatible Poisson structures on $M$

$$\Lambda,\ N\Lambda,\ N^2\Lambda, \dots, N^k\Lambda, \dots$$

We recall that two Poisson bi-vectors $\Lambda$ and $\Lambda'$ on $M$ are compatible if $\Lambda+\Lambda'$ is again a Poisson structure or equivalently if $[\Lambda, \Lambda'] = 0$.

An example of Poisson-Nijenhuis manifold is given by a manifold $M$ endowed with two compatible Poisson structures $\Lambda_1$ and $\Lambda_2$, such that the first one is nondegenerate. Thus, $(M, \Lambda_1, N)$ is a Poisson-Nijenhuis manifold where

$$N := \Lambda_2^\sharp \circ (\Lambda_1^\sharp)^{-1}.$$

If additionally $N$ is nondegenerate then we have a hierarchy of compatible symplectic structures on $M$. Moreover, if $X$ is a bi-Hamiltonian vector field (i.e. a Hamiltonian vector field with respect to both Poisson structures, $\Lambda_1$ and $\Lambda_2$) and the first de Rham cohomology group of $M$ is trivial, we obtain a sequence of integrals of motion in involution (see [16]).

**Example 2.2. The Toda lattice.** The finite, non-periodic Toda lattice (see, for instance, [2, 12, 17]) is a system of $n$ particles on the line under exponential interaction with nearby particles. Its phase space is $\mathbb{R}^{2n}$ with canonical coordinates $(q^i, p_i)$ where $q^i$ is the displacement of the $i$-th particle from its equilibrium position and $p_i$ is the corresponding momentum. This system is particularly interesting when we consider exponential forces. Then the Hamiltonian function associated with the equations of motion is

$$H_1 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{(q^i - q^{i+1})}.$$

Now, we consider the following two compatible Poisson structures on $\mathbb{R}^{2n}$

$$\Lambda_0 = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

$$\Lambda_1 = -\sum_{i<j} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n-1} e^{(q^i - q^{i+1})} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i}.$$

Note that $\Lambda_0$ is the Poisson bivector corresponding to the canonical symplectic structure of $\mathbb{R}^{2n}$. Furthermore, the Hamiltonian vector field $\mathcal{H}^{\Lambda_0}_{H_1}$ is bi-Hamiltonian. In fact,

$$\mathcal{H}^{\Lambda_0}_{H_1} = \Lambda_0^\sharp(dH_1) = \Lambda_1^\sharp(dH_0),$$

with $H_0 = \sum_{i=1}^n p_i$.

In what follows, we will reduce the bi-Hamiltonian structure of the Toda lattice using the action of $\mathbb{R}$ over $\mathbb{R}^{2n}$ given by

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n} \\ (t, (q^i, p_i)) &\longmapsto (q^i + t, p_i) \end{aligned}$$

which induces the principal bundle

$$\pi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}/\mathbb{R}.$$

Note that $\mathbb{R}^{2n}/\mathbb{R}$ may be identified with $(\mathbb{R}^+)^{n-1} \times \mathbb{R}^n$ by

$$\mathbb{R}^{2n}/\mathbb{R} \longrightarrow (\mathbb{R}^+)^{n-1} \times \mathbb{R}^n, \quad (q^i, p_i) \longmapsto (e^{(q^i - q^{i+1})}, p_i). \tag{5}$$

This identification corresponds to the choice of the so called *Flaschka coordinates* which are actually global coordinates on $\mathbb{R}^{2n}/\mathbb{R}$, usually denoted by $(a_1, \ldots, a_{n-1},\, b_1,\, \ldots, b_n)$. The Poisson structures $\Lambda_0$ and $\Lambda_1$ are $\mathbb{R}$-invariant so that they descend to the quotient $\mathbb{R}^{2n}/\mathbb{R} \cong (\mathbb{R}^+)^{n-1} \times \mathbb{R}^n$. The reduced Poisson structures are

$$\begin{aligned} \bar\Lambda_0 &= \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial a_i} \wedge \left( \frac{\partial}{\partial b_i} - \frac{\partial}{\partial b_{i+1}} \right), \\ \bar\Lambda_1 &= \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial a_i} \wedge \left( b_i \frac{\partial}{\partial b_i} - b_{i+1} \frac{\partial}{\partial b_{i+1}} \right) \\ &\quad + \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial b_{i+1}} \wedge \frac{\partial}{\partial b_i} + \sum_{i=1}^{n-2} a_i a_{i+1} \frac{\partial}{\partial a_{i+1}} \wedge \frac{\partial}{\partial a_i}. \end{aligned} \tag{6}$$

These bivectors are again compatible and moreover we obtain by projection a hierarchy of compatible Poisson structures on the reduced space. However, they cannot be related through a recursion tensor $\bar N$. Indeed, if this were the case, then

$$\bar\Lambda_1^\sharp = \bar N \circ \bar\Lambda_0^\sharp.$$

Thus, using that $\bar{\Lambda}_0^\sharp(\sum_{i=1}^{n} db_i) = 0$, we deduce that $\bar{\Lambda}_1^\sharp(\sum_{i=1}^{n} db_i) = 0$ which is not true.

The problem is that if we want to induce a tensor $\bar{N} : T(\mathbb{R}^{2n}/\mathbb{R}) \to T(\mathbb{R}^{2n}/\mathbb{R})$ it is necessary that $N$ sends vertical vectors with respect to $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{R}$ into vertical vectors. Note that $\Lambda_0$ and $N$ are $\mathbb{R}$-invariant but $N(\ker T\pi) \nsubseteq \ker T\pi$.

Furthermore, the Hamiltonian vector field $\mathcal{H}^{\Lambda_0}_{H_1}$ projects just in $\mathcal{H}^{\bar{\Lambda}_0}_{\bar{H}_1}$ and

$$\bar{\Lambda}_0^\sharp d\bar{H}_1 = \mathcal{H}^{\bar{\Lambda}_0}_{\bar{H}_1} = \bar{\Lambda}_1^\sharp d\bar{H}_0$$

with $\bar{H}_1 = \frac{1}{2}\sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n-1} a_i$ and $\bar{H}_0 = \sum_{i=1}^{n} b_i$.

These facts suggest that, although the structure of Poisson-Nijenhuis can not be reduced, perhaps there exists another structure in a different space from which we may induce the above structures on the reduced space $\mathbb{R}^{2n}/\mathbb{R}$. The answer to this question is associated with the notion of a Poisson-Nijenhuis Lie algebroid. $\diamondsuit$

**2.2. Poisson-Nijenhuis Lie algebroids.** A *Lie algebroid* is a vector bundle $\tau_A \colon A \to M$ endowed with

(i) an *anchor*, i.e. a vector bundle morphism $\rho_A \colon A \to TM$

(ii) a Lie bracket $[\cdot,\cdot]_A$ on the space of the sections of $A$, $\Gamma(A)$, such that the *Leibniz rule*,

$$[X, fY]_A = f\,[X,Y]_A + \rho_A(X)(f)Y,$$

is satisfied for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

We denote such a Lie algebroid by $(A, [\cdot,\cdot]_A, \rho_A)$ or simply by $A$.

In such a case the map $\rho_A$ induces a morphism of Lie algebras from $(\Gamma(A), [\cdot,\cdot]_A)$ to $(\mathfrak{X}(M), [\cdot,\cdot])$ which we denote by the same symbol, i.e.

$$\rho_A\left([X,Y]_A\right) = \left[\rho_A(X), \rho_A(Y)\right].$$

Now, we will describe an interesting example of a Lie algebroid. For further details about Lie algebroids and other examples see e.g. [13].

**Example 2.3. The Atiyah algebroid associated with a principal $G$-bundle.** Let $p : M \to M/G$ be a principal $G$-bundle. It is well-known that the tangent lift of the principal action of $G$ on $M$ induces a principal action of $G$ on $TM$ and the space of orbits $TM/G$ of this action is a vector bundle over $M/G$ with vector bundle projection $\tau_{TM/G} : TM/G \to M/G$ given by

$$\tau_{TM/G}([v_x]) = p(x), \quad \forall v_x \in T_xM.$$

Furthermore, the space of sections $\Gamma(TM/G)$ may be identified with the set of $G$-invariant vector fields on $M$ and the Lie bracket of two $G$-invariant vector fields on $M$ is still $G$-invariant. Thus, the standard Lie bracket of vector fields induces a Lie bracket $[\cdot,\cdot]_{TM/G}$ on the space $\Gamma(TM/G)$ in a natural way.

On the other hand, the anchor map $\rho_{TM/G} : TM/G \to T(M/G)$ is given by

$$\rho_{TM/G}([v_x]) = (T_xp)(v_x), \text{ for } v_x \in T_xM,$$

where $Tp : TM \to T(M/G)$ is the tangent map to the principal bundle projection $p : M \to M/G$.

The resultant Lie algebroid $(TM/G, [\cdot,\cdot]_{TM/G}, \rho_{TM/G})$ is called *the Atiyah algebroid* associated with the principal $G$-bundle $p : M \to M/G$. $\diamondsuit$

Associated to a given Lie algebroid $(A, [\cdot,\cdot]_A, \rho_A)$ there is a *Lie algebroid differential* $\mathrm{d}^A \colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1}A^*)$ defined by

$$\begin{aligned}(\mathrm{d}^A\omega)(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i\rho_A(X_i)\Big(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)\Big)\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j]_A, X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k),\end{aligned}$$

for $\omega \in \Gamma(\wedge^k A^*)$, $X_0,\dots,X_k \in \Gamma(A)$. We have that $(\mathrm{d}^A)^2 = 0$, which implies that $d^A$ is a cohomology operator. Moreover, if $X$ is a section of $A$, one may introduce, in a natural way, the *Lie derivative with respect to* $X$ as the operator $\mathcal{L}^A_X \colon \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$ given by

$$\mathcal{L}^A_X = i_X \circ \mathrm{d}^A + \mathrm{d}^A \circ i_X. \tag{7}$$

It is easy to prove that the Lie derivative $\mathcal{L}^A_X$ and the Lie bracket $[\cdot,\cdot]_A$ are related by

$$\mathcal{L}^A_X i_Y = i_Y\mathcal{L}^A_X + i_{[X,Y]_A}, \text{ with } X, Y \in \Gamma(A). \tag{8}$$

The Lie algebra bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ can be extended to the exterior algebra $(\Gamma(\wedge^\bullet A),\wedge)$ using the properties

$$\begin{aligned} [P,Q]_A &\in \Gamma(\wedge^{p+q-1}A) \\ [P,Q]_A &= -(-1)^{(p-1)(q-1)}\,[Q,P]_A\,, \\ [P,Q\wedge R]_A &= [P,Q]_A\wedge R+(-1)^{(p-1)q}Q\wedge [P,R]_A\,, \end{aligned} \tag{9}$$

with $P\in\Gamma\left(\wedge^p A\right), Q\in\Gamma\left(\wedge^q A\right)$ and $R\in\Gamma\left(\wedge^r A\right)$.

The resulting bracket is called *Schouten bracket* (see e.g. [13]). Note that

$$[X,P]_A(\alpha_1,\ldots,\alpha_p)=\rho_A(X)(P(\alpha_1,\ldots,\alpha_p))-\sum_{i=1}^{p}P(\alpha_1,\ldots,\mathcal{L}^A_X\alpha_i,\ldots,\alpha_p) \tag{10}$$

for $X\in\Gamma(A), P\in\Gamma(\wedge^p A)$ and $\alpha_1,\ldots,\alpha_p\in\Gamma(A^*)$.

Let $(A,[\cdot,\cdot]_A,\rho_A)$ be a Lie algebroid over a manifold $M$ and $P$ be a section of the vector bundle $\wedge^2 A\to M$. We denote by $P^\sharp$ the usual bundle map

$$P^\sharp\colon A^*\longrightarrow A,\quad \alpha\longmapsto P^\sharp(\alpha)=i_\alpha P. \tag{11}$$

We say that $P$ defines a *Poisson structure on* $A$ if $[P,P]_A=0$. In this case, the bracket on the sections of $A^*$ defined by

$$[\alpha,\beta]_P=\mathcal{L}^A_{P^\sharp\alpha}\beta-\mathcal{L}^A_{P^\sharp\beta}\alpha-\mathrm{d}^A\left(P(\alpha,\beta)\right),\quad \alpha,\beta\in\Gamma(A^*), \tag{12}$$

is a Lie bracket, $P^\sharp:(\Gamma(A^*),[\cdot,\cdot]_P)\to(\Gamma(A),[\cdot,\cdot]_A)$ is a Lie algebra morphism and the triple $A^*_P=(A^*,[\cdot,\cdot]_P,\rho_A\circ P^\sharp)$ is a Lie algebroid [14]. In fact, the pair $(A,A^*_P)$ is a special kind of a Lie bialgebroid called a *triangular Lie bialgebroid* [14]. A Poisson structure $P\in\Gamma(\wedge^2 A)$ on a Lie algebroid $(A,[\cdot,\cdot]_A,\rho_A)$ induces a Poisson structure $\Lambda\in\Gamma(\wedge^2 TM)$ on the base manifold $M$, defined by

$$\Lambda^\sharp=\rho_A\circ P^\sharp\circ\rho_A^*. \tag{13}$$

An *almost symplectic structure* on the Lie algebroid $(A,[\cdot,\cdot]_A,\rho_A)$ is a section $\Omega_A$ of the vector bundle $\wedge^2A^*\to M$ such that $\Omega_A$ is nondegenerate. In such a case, the map $\Omega_A^\flat:\Gamma(A)\to\Gamma(A^*)$ given by

$$\Omega^\flat_A(X)=i_X\Omega_A,\qquad \text{for } X\in\Gamma(A),$$

is an isomorphism of $C^\infty(M)$-modules. Thus, one can define from $\Omega_A$ a nondegenerate section of the vector bundle $\wedge^2A\to M$ as follows

$$P_{\Omega_A}(\alpha,\beta)=\Omega_A((\Omega^\flat_A)^{-1}(\alpha),(\Omega^\flat_A)^{-1}(\beta)),\qquad \text{for } \alpha,\beta\in\Gamma(A^*). \tag{14}$$

An almost symplectic structure $\Omega_A$ is called *symplectic* if $d^A\Omega_A = 0$. In this case, $P_{\Omega_A} \in \Gamma(\wedge^2 A)$ is a Poisson structure on $A$. Conversely, if $P$ is a nondegenerate Poisson structure on $A$, then

$$\Omega_P(X,Y) = P((P^\sharp)^{-1}(X),(P^\sharp)^{-1}(Y)), \qquad \text{for } X,Y \in \Gamma(A),$$

defines a symplectic structure $\Omega_P$ on $A$ (see [1]).

Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over a manifold $M$. The torsion of a bundle map $N : A \to A$ (over the identity) is defined by

$$\mathcal{T}_N(X,Y) := [NX,NY]_A - N[X,Y]_N, \quad X,Y \in \Gamma(A), \tag{15}$$

where $[\cdot,\cdot]_N$ is given by

$$[X,Y]_N := [NX,Y]_A + [X,NY]_A - N[X,Y]_A, \quad X,Y \in \Gamma(A). \tag{16}$$

When $\mathcal{T}_N = 0$, the bundle map $N$ is called a *Nijenhuis operator*, the triple $A_N = (A, [\cdot,\cdot]_N, \rho_N = \rho\circ N)$ is a new Lie algebroid and $N : A_N \to A$ is a Lie algebroid morphism (see [4, 9]).

Now, if $P \in \Gamma(\wedge^2 A)$ is a Poisson structure on $A$, we say that a bundle map $N : A \to A$ is *compatible* with $P$ if $N\circ P^\sharp = P^\sharp \circ N^*$ and the *Magri-Morosi concomitant*

$$\mathcal{C}(P,N)(\alpha,\beta) = [\alpha,\beta]_{NP} - [\alpha,\beta]_P^{N^*}, \quad \text{for } \alpha,\beta \in \Gamma(A^*) \tag{17}$$

vanishes, where $[\cdot,\cdot]_{NP}$ is the bracket defined by the section $NP \in \Gamma(\wedge^2 A)$ in a similar way as in (12), and $[\cdot,\cdot]_P^{N^*}$ is the Lie bracket obtained from the Lie bracket $[\cdot,\cdot]_P$ by deformation along the dual map $N^* : A^* \to A^*$, i.e.,

$$[\alpha,\beta]_P^{N^*} = [N^*\alpha,\beta]_P + [\alpha,N^*\beta]_P - N^*[\alpha,\beta]_P. \tag{18}$$

**Definition 2.4.** ([4]) A *Poisson-Nijenhuis Lie algebroid* $(A,P,N)$ is a Lie algebroid $A$ equipped with a Poisson structure $P$ and a Nijenhuis operator $N : A \to A$ compatible with $P$.

If, in particular, the Poisson tensor $P$ in Definition 2.4 is nondegenerate, i.e. it comes from a symplectic structure $\Omega_A$ on $A$ like in (14), then $(A,\Omega_A,N)$ is said to be a *symplectic-Nijenhuis Lie algebroid.* This is the case of two compatible Poisson 2-sections $P_0$ and $P_1$, where $P_0$ is associated with a symplectic structure.

**Example 2.5. The Poisson-Nijenhuis Lie algebroid associated with the Toda lattice (see [2]).** We will describe the Poisson-Nijenhuis Lie algebroid associated to the reduction of the Toda lattice presented in Example

2.2. Consider the Atiyah algebroid $\tau_A : A = (T\mathbb{R}^{2n})/\mathbb{R} \to \mathbb{R}^{2n}/\mathbb{R}$ associated with the principal bundle $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{R}$.

A global basis of $\mathbb{R}$-invariant vector fields on $\mathbb{R}^{2n}$ is

$$\{e_i = e^{(q^{i+1}-q^i)}\sum_{k=1}^{i}\frac{\partial}{\partial q^k},\quad e_n = \sum_{k=1}^{n}\frac{\partial}{\partial q^k},\quad f_j = \frac{\partial}{\partial p_j}\}_{\substack{i=1,..,n-1\\ j=1,..,n}}$$

Note that

$$[e_i, e_j] = [f_i, f_j] = [e_i, f_j] = 0$$

for $i, j \in \{1, \dots, n\}$. Moreover, the vector field $e_k$, with $k \in \{1, \dots, n-1\}$ (respectively, $f_l$, with $l \in \{1, \dots, n\}$) is $\pi$-projectable over the vector field $\dfrac{\partial}{\partial a^k}$ (respectively, $\dfrac{\partial}{\partial b_l}$) on $(\mathbb{R}^+)^{n-1} \times \mathbb{R}^n$. In addition, the vertical bundle of $\pi$ is generated by the vector field $e_n$.

Thus, the Lie algebroid structure $([\cdot,\cdot]_A, \rho_A)$ on $A$ is characterized by the following conditions

$$[e_i, e_j]_A = [f_i, f_j]_A = [e_i, f_j]_A = 0,$$

and

$$\rho_A(e_i) = \frac{\partial}{\partial a_i}\quad (i = 1, \dots, n-1),\qquad \rho_A(e_n) = 0,\qquad \rho_A(f_j) = \frac{\partial}{\partial b_j}\quad (j = 1, \dots, n).$$

We may define the following two Poisson structures on $A$

$$\begin{aligned}
\pi_0 &= \sum_{i=1}^{n-1} a_i e_i \wedge (f_i - f_{i+1}) + e_n \wedge f_n\\
\pi_1 &= -\sum_{i=1}^{n-2} a_i a_{i+1} e_i \wedge e_{i+1} - a_{n-1}e_{n-1}\wedge e_n + \sum_{i=1}^{n-1} a_i e_i \wedge (b_i f_i - b_{i+1}f_{i+1})\\
&\quad + b_n e_n \wedge f_n - \sum_{i=1}^{n-1} a_i f_i \wedge f_{i+1}.
\end{aligned}$$

These Poisson structures cover ordinary Poisson tensors on the base manifold $\mathbb{R}^{2n}/\mathbb{R}$ which are just the Poisson structures $\bar{\Lambda}_0$ and $\bar{\Lambda}_1$ given by (6). Since $\pi_0$ is symplectic, the Poisson structures on $A$ are related by the recursion operator $N = \pi_1^\sharp \circ (\pi_0^\sharp)^{-1}$ and $(A, \pi_0, N)$ is a symplectic-Nijenhuis Lie algebroid. $\Diamond$

This example may be framed within a more general framework as follows.

Let $p : M \to \bar{M} = M/G$ be a principal $G$-bundle. If a $G$-invariant Poisson-Nijenhuis structure $(\Lambda, N)$ is given on $M$, then in general we cannot induce a Poisson-Nijenhuis structure on $M/G$ since the condition $N(\ker Tp) \nsubseteq \ker Tp$ might not be satisfied. Nevertheless, we obtain a reduced Poisson-Nijenhuis Lie algebroid. In fact, as we know, the space of sections of $\tilde{p} : TM/G \to \bar{M} = M/G$ (respectively, $\tilde{p}^* : (TM/G)^* \cong T^*M/G \to \bar{M} = M/G$) may be identified with the set of $G$-invariant vector fields $\mathfrak{X}^G(M)$ (respectively, $G$-invariant 1-forms $\Omega^1(M)^G$) on $M$.

Now, since $\Lambda$ and $N$ are $G$-invariant, we deduce that

$$\Lambda(\alpha, \beta) \text{ is a } p\text{-basic function, for } \alpha, \beta \in \Omega^1(M)^G$$

and

$$NX \in \mathfrak{X}^G(M), \text{ for } X \in \mathfrak{X}^G(M).$$

Thus, $\Lambda$ (respectively, $N$) induces a section $\tilde{\Lambda}$ (respectively, $\widetilde{N}$) on the vector bundle $\wedge^2(TM/G) \to \bar{M} = M/G$ (respectively, $TM/G \otimes T^*M/G \to \bar{M} = M/G$) in such a way that

$$\begin{array}{lll} \widetilde{\Lambda}(\alpha, \beta) \circ p = \Lambda(\alpha, \beta) & \text{for} & \alpha, \beta \in \Omega^1(M)^G, \\ \tilde{N}X = NX, & \text{for} & X \in \mathfrak{X}^G(M). \end{array}$$

Moreover, using the definition of the Lie algebroid structure on the Atiyah algebroid $p : TM/G \to \bar{M} = M/G$ and the fact that $(\Lambda, N)$ is a Poisson-Nijenhuis structure on $M$, we may prove the following result

**Proposition 2.6.** *Let $p : M \to \bar{M} = M/G$ be a principal $G$-bundle and $(\Lambda, N)$ be a $G$-invariant Poisson-Nijenhuis structure on $M$. Then:*

- *i) $(\Lambda, N)$ induces a Poisson-Nijenhuis Lie algebroid structure $(\tilde{\Lambda}, \tilde{N})$ on the Atiyah algebroid $\tilde{p} : TM/G \to \bar{M} = M/G$*
- *ii) The Poisson structures $\Lambda$ and $N\Lambda$ on $M$ are $p$-projectable to two compatible Poisson structures $\bar{\Lambda}$ and $\overline{N\Lambda}$ on $\bar{M} = M/G$.*
- *iii) The Poisson structures on $\bar{M} = M/G$ which are induced by the Poisson bi-sections $\tilde{\Lambda}$ and $\tilde{N}\tilde{\Lambda}$ on the Atiyah algebroid $p : M \to \bar{M} = M/G$ are just $\bar{\Lambda}$ and $\overline{N\Lambda}$, respectively.*

## 3. Reduction of Poisson-Nijenhuis Lie algebroids by restriction

We consider the Poisson-Nijenhuis Lie algebroid $A = (T\mathbb{R}^{2n})/\mathbb{R}$ associated with the Toda lattice. It is easy to prove that if we restrict to a suitable open subset of the base manifold $\mathbb{R}^{2n}/\mathbb{R}$ then $A = (T\mathbb{R}^{2n})/\mathbb{R}$ is a symplectic-Nijenhuis Lie algebroid with nondegenerate Nijenhuis tensor. The main result of this paper is prove that, under regularities conditions, every Poisson-Nijenhuis algebroid may be reduced to a nondegenerate symplectic-Nijenhuis Lie algebroid. This reduction has two steps. In the first step we obtain a symplectic-Nijenhuis Lie algebroid, and then we will reduce it to a symplectic-Nijenhuis Lie algebroid with nondegenerate Nijenhuis tensor using a general theory about the projectability of a Poisson-Nijenhuis structure with respect to a Poisson-Nijenhuis Lie algebroid epimorphism. In this section we will describe the first step which is a reduction by restriction. Previously, we recall some notions about Lie algebroid morphisms which will be useful in the sequel.

**3.1. Lie algebroid morphisms and subalgebroids.** Let $\tau_A\colon A \to M$ and $\tau_{\widetilde{A}}\colon \widetilde{A} \to \widetilde{M}$ be vector bundles. Suppose that we have a morphism of vector bundles $(F, f)$ from $A$ to $\widetilde{A}$:

$$\begin{array}{ccc} A & \xrightarrow{F} & \widetilde{A} \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_{\widetilde{A}}} \\ M & \xrightarrow{f} & \widetilde{M} \end{array}$$

A section of $A$, $X : M \to A$, is said to be $F$-*projectable* if there is $\widetilde{X} \in \Gamma(\widetilde{A})$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{F} & \widetilde{A} \\ \uparrow{\scriptstyle X} & & \uparrow{\scriptstyle \widetilde{X}} \\ M & \xrightarrow{f} & \widetilde{M} \end{array}$$

A section $\alpha\colon M \to \wedge^k A^*$ of $\tau^k_{A^*}\colon \wedge^k A^* \to M$ is said to be $F$-*projectable* if there is $\widetilde{\alpha} \in \Gamma(\wedge^k \widetilde{A}^*)$ such that $\alpha = F^*\widetilde{\alpha}$, where $F^*\widetilde{\alpha} \in \Gamma(\wedge^k A^*)$ is defined by

$$(F^*\widetilde{\alpha})(x)(a_1, \ldots, a_k) = \widetilde{\alpha}(f(x))(F(a_1), \ldots, F(a_k)) \tag{19}$$

with $x \in M$ and $a_1, \dots, a_k \in A_x$.

Now, we consider Lie algebroid structures $([\cdot,\cdot]_A, \rho_A)$ and $([\cdot,\cdot]_{\widetilde{A}}, \rho_{\widetilde{A}})$ on $A$ and $\widetilde{A}$, respectively. We say that $(F, f)$ is a *Lie algebroid morphism* if

$$\mathrm{d}^A(F^*\widetilde{\alpha}) = F^*(\mathrm{d}^{\widetilde{A}}\widetilde{\alpha}) \quad \text{for all } \widetilde{\alpha} \in \Gamma(\wedge^k \widetilde{A}^*) \text{ and all } k. \tag{20}$$

Any Lie algebroid morphism preserves the anchor, i.e.,

$$\rho_{\widetilde{A}} \circ F = Tf \circ \rho_A. \tag{21}$$

Moreover, if $X$ and $Y$ are $F$-projectable sections on $\widetilde{X}$ and $\widetilde{Y}$, respectively, it follows that $[X, Y]_A$ is a $F$-projectable section on $[\widetilde{X}, \widetilde{Y}]_{\widetilde{A}}$.

In addition, if $X \in \Gamma(A)$ is $F$-projectable and $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$, then $\mathcal{L}^A_X(F^*\widetilde{\alpha})$ is a $F$-projectable section of $A^*$. In fact, using (7) and (20), we have that

$$\mathcal{L}^A_X(F^*\widetilde{\alpha}) = F^*(\mathcal{L}^{\widetilde{A}}_{\widetilde{X}}\widetilde{\alpha}), \tag{22}$$

where $\widetilde{X} \in \Gamma(\widetilde{A})$ satisfies $F \circ X = \widetilde{X} \circ f$.

Note that if $M = \widetilde{M}$ and $f$ is the identity map for $M$, then $F : A \to \widetilde{A}$ is a Lie algebroid morphism if and only if

$$F[X, Y]_A = [FX, FY]_{\widetilde{A}}, \qquad \rho_{\widetilde{A}}(FX) = \rho_A(X) \tag{23}$$

for $X, Y \in \Gamma(A)$.

A *Lie subalgebroid* is a morphism of Lie algebroids $I : B \to A$ over $\iota : N \to M$ such that $\iota$ is an injective immersion and $I_{|B_x} : B_x \to A_{\iota(x)}$ is a monomorphism, for all $x \in N$ (see [7]).

**3.2. The first step of the reduction: Reduction of Poisson-Nijenhuis Lie algebroids by restriction.** Let $(A, P)$ be a Poisson Lie algebroid. In order to reduce $A$ to a symplectic Lie algebroid, let us consider the generalized distribution $D \subset TM$ defined as follows: for each $x \in M$,

$$D(x) := \rho_A(P^\sharp(A^*_x)) \subset T_xM.$$

Since $P^\sharp$ and $\rho_A$ are Lie algebroid morphisms over the identity $id_M : M \to M$, we have

$$\left[\rho_A(P^\sharp\alpha), \rho_A(P^\sharp\beta)\right] = \rho_A(P^\sharp[\alpha, \beta]_P),$$

for any $\alpha, \beta \in \Gamma(A^*)$, i.e. $D$ is involutive. Furthermore, $D$ is locally finitely generated as a $C^\infty(M)$-module. As a consequence $D$ defines a generalized foliation of $M$ in the sense of Sussmann [20]. Note that, due to (13), the tangent distribution $S = \Lambda^\sharp(T^*M)$ of the symplectic foliation of the induced

Poisson structure $\Lambda \in \Gamma(\wedge^2 TM)$ on the base manifold $M$ is a subset of $D = \rho_A(P^\sharp(A^*))$.

Let $L \subset M$ be a leaf of the foliation $D$ and consider the subset $A_L := P^\sharp(A^*)_{|L} \subset A$. We assume that the Poisson structure $P^\sharp : A^* \to A$ has constant rank on each leaf $L$. Then, $A_L \to L$ is a vector subbundle of the vector bundle $A \to M$ and, since that $\rho_A(A_L) \subseteq TL$, we deduce that the Lie algebroid structure $([\cdot,\cdot]_A, \rho_A)$ on $A$ induces a Lie algebroid structure $([\cdot,\cdot]_{A_L}, \rho_{A_L})$ on $A_L$. In fact, $\rho_{A_L} = (\rho_A)_{|A_L}$ and the Lie bracket $[\cdot,\cdot]_{A_L}$ is characterized by the condition

$$\left[P^\sharp\alpha_{|L}, P^\sharp\beta_{|L}\right]_{A_L} = (\left[P^\sharp\alpha, P^\sharp\beta\right]_A)_{|L} = (P^\sharp\left[\alpha,\beta\right]_P)_{|L}$$

for all $\alpha, \beta \in \Gamma(A^*)$. Note that if $\alpha, \alpha' \in \Gamma(A^*)$ and $P^\sharp(\alpha)_{|L} = P^\sharp(\alpha')_{|L}$ then, using that the restriction to $L$ of $\rho_A(P^\sharp(\beta))$ is tangent to $L$, we obtain that

$$(\left[P^\sharp\alpha, P^\sharp\beta\right]_A)_{|L} = (\left[P^\sharp\alpha', P^\sharp\beta\right]_A)_{|L}.$$

Furthermore, if we denote by $I : A_L \to A$ and $\iota : L \to M$, respectively, the inclusion mappings of $A_L$ in $A$ and of $L$ in $M$, then $I$ is a monomorphism of Lie algebroids from $A_L$ to $A$ over $\iota : L \to M$ so that $A_L$ is a Lie subalgebroid of $A$.

Now, we will prove that the Lie algebroid $A_L$ is symplectic.

Note that for any $X_L \in \Gamma(A_L)$ there exists a section $\alpha \in \Gamma(A^*)$ such that $X_L$ $I$-projects on $P^\sharp\alpha$, i.e., $I \circ X_L = P^\sharp\alpha \circ \iota$.

Let us define a section $\Omega_L\colon L \to \wedge^2 A_L^*$ by setting

$$\Omega_L(X_L, Y_L) = P(\alpha,\beta)\circ\iota, \qquad \text{for any } X_L, Y_L \in \Gamma(A_L) \tag{24}$$

$\alpha, \beta$ being sections of $A^*$ such that $X_L$ and $Y_L$ $I$-project on $P^\sharp\alpha$ and $P^\sharp\beta$, respectively. Clearly, $\Omega_L$ is well defined. Indeed, if $P^\sharp\alpha\circ\iota = P^\sharp\alpha'\circ\iota$ then $P(\alpha,\beta)\circ\iota = P(\alpha',\beta)\circ\iota$, for all $\beta \in \Gamma(A^*)$.

Moreover, $\Omega_L$ is nondegenerate. Note that if $X_L \in \Gamma(A_L)$,

$$I \circ X_L = (P^\sharp\alpha)\circ\iota$$

and $\Omega_L(X_L, Y_L) = 0$, for all $Y_L \in \Gamma(A_L)$, then $P^\sharp\alpha\circ\iota = 0$ and therefore $X_L = 0$. Hence, $\Omega_L$ is an almost symplectic structure on $A_L$.

In order to show that $\Omega_L$ is symplectic, we will prove the following Lemma.

**Lemma 3.1.** *Let $X_L, Y_L$ be sections of $A_L$ and $\alpha, \beta \in \Gamma(A^*)$ such that $I \circ X_L = P^\sharp\alpha\circ\iota$ and $I\circ Y_L = P^\sharp\beta\circ\iota$. Then:*

*(i)* $\Omega_L^\flat(X_L) = -I^*\alpha$,

*(ii)* $i_{[X_L,Y_L]_{A_L}}\Omega_L = \mathcal{L}^{A_L}_{X_L}\beta_L - \mathcal{L}^{A_L}_{Y_L}\alpha_L + \mathrm{d}^{A_L}(P(\alpha,\beta)\circ\iota)$,
*where* $\alpha_L = i_{X_L}\Omega_L$ *and* $\beta_L = i_{X_L}\Omega_L$.

*Proof*: *(i)* If $Y_L \in \Gamma(A_L)$ is a section of $A_L$ which $I$-projects on $P^\sharp\beta$, for some $\beta \in \Gamma(A^*)$, then

$$\Omega_L^\flat(X_L)(Y_L) = (\beta\circ\iota)(P^\sharp\alpha\circ\iota) = -(\alpha\circ\iota)(P^\sharp\beta\circ\iota) = -(\alpha\circ\iota)(I\circ Y_L) = -I^*\alpha(Y_L).$$

*(ii)* Note that, since $(I,\iota)$ and $P^\sharp$ are Lie algebroid morphisms, we have

$$I\circ[X_L,Y_L]_{A_L} = \left[P^\sharp\alpha, P^\sharp\beta\right]_A\circ\iota = P^\sharp\,[\alpha,\beta]_P\circ\iota.$$

So, by *(i)* we obtain

$$i_{[X_L,Y_L]_{A_L}}\Omega_L = -I^*\,[\alpha,\beta]_P\,. \tag{25}$$

Now, from (12), (22) and (25) we obtain the claim. ■

**Proposition 3.2.** *The* 2*-section* $\Omega_L$ *on* $A_L$ *defined by* (24) *is symplectic.*

*Proof*: We have only to prove that $\Omega_L$ is closed. In fact, for any $X_L, Y_L \in \Gamma(A_L)$, we have

$$\begin{aligned} i_{X_L}i_{Y_L}\mathrm{d}^{A_L}\Omega_L &= i_{X_L}\mathcal{L}^{A_L}_{Y_L}\Omega_L - i_{X_L}\mathrm{d}^{A_L}i_{Y_L}\Omega_L \\ &= \mathcal{L}^{A_L}_{Y_L}i_{X_L}\Omega_L + i_{[X_L,Y_L]_{A_L}}\Omega_L - i_{X_L}\mathrm{d}^{A_L}i_{Y_L}\Omega_L, \end{aligned} \tag{26}$$

where we have used (7) and (8).

By applying Lemma 3.1, from (26) we get

$$i_{X_L}i_{Y_L}\mathrm{d}^{A_L}\Omega_L = \mathrm{d}^{A_L}i_{X_L}\beta_L + \mathrm{d}^{A_L}(P(\alpha,\beta)\circ\iota) = 0.$$

■

Now, we consider a Nijenhuis operator $N : A \to A$ on the Lie algebroid $A$ which is compatible with the Poisson structure $P$. Using the compatibility condition $N\circ P^\sharp = P^\sharp\circ N^*$, we may induce by restriction a new operator $N_L : A_L \to A_L$ on $A_L$ such that

$$I\circ N_L(X_L) = N(P^\sharp\alpha)\circ\iota, \text{ for all } X_L \in \Gamma(A_L) \tag{27}$$

where $\alpha \in \Gamma(A^*)$ is a section of $A^*$ such that $X_L$ $I$-projects on $P^\sharp\alpha$.

Note that, from (27), we deduce that

$$I\circ N_L = N\circ I \tag{28}$$

which implies that

$$N_L^*(I^*\alpha) = I^*(N^*\alpha), \text{ for } \alpha \in \Gamma(A^*). \tag{29}$$

**Theorem 3.3.** *Let $(A, P, N)$ be a Poisson-Nijenhuis Lie algebroid such that the Poisson structure has constant rank in the leaves of the foliation $D = \rho_A(P^\sharp(A^*))$. Then, we have a symplectic-Nijenhuis Lie algebroid $(A_L, \Omega_L, N_L)$ on each leaf $L$ of $D$.*

*Proof*: From Proposition 3.2, we deduce that $(A_L, \Omega_L)$ is a symplectic Lie algebroid. Denote by $P_L$ the Poisson structure corresponding to $\Omega_L$, defined by $P_L^\sharp := -(\Omega_L^\flat)^{-1}$. Note that, using Lemma 3.1, we have that

$$P_L(I^*\alpha, I^*\beta) = P(\alpha, \beta) \circ \iota, \quad \text{for all } \alpha, \beta \in \Gamma(A^*). \tag{30}$$

Next, we prove that $N_L$ is a Nijenhuis operator compatible with $P_L$. Indeed, firstly consider $X_L, Y_L$ sections of $A_L$. Then, there are $\alpha$ and $\beta$ sections of $A^*$ such that $X_L$ and $Y_L$ $I$-project on $P^\sharp\alpha$ and $P^\sharp\beta$, respectively. Thus, using (28) and the fact that $(I, \iota)$ is a monomorphism of Lie algebroids, we deduce that

$$I \circ \mathcal{T}_{N_L}(X_L, Y_L) = \mathcal{T}_N(P^\sharp\alpha, P^\sharp\beta) \circ \iota = 0. \tag{31}$$

On the other hand, for $\alpha \in \Gamma(A^*)$, we consider the section $X_L \in \Gamma(A_L)$ defined by

$$I \circ X_L = P^\sharp\alpha \circ \iota.$$

Using Lemma 3.1 we deduce that

$$P_L^\sharp(I^*\alpha) = X_L. \tag{32}$$

Now, from (27) and since $N \circ P^\sharp = P^\sharp \circ N^*$, it follows that

$$I(N_L(X_L)) = P^\sharp(N^*\alpha) \circ \iota.$$

Therefore, using again Lemma 3.1, we obtain that

$$P_L^\sharp(I^*(N^*\alpha)) = N_L(X_L) = N_L(P_L^\sharp(I^*\alpha))$$

which implies that (see (29))

$$P_L^\sharp(N_L^*(I^*\alpha)) = N_L(P_L^\sharp(I^*\alpha)).$$

This proves that $P_L^\sharp \circ N_L^* = N_L \circ P_L^\sharp$.

Finally, from (17), (18), (22), (27), (30) and using that $N \circ P^\sharp = P^\sharp \circ N^*$ and the fact that $(I, \iota)$ is a Lie algebroid monomorphism, we conclude that

$$0 = I^*(C(P, N)(\alpha, \beta)) = C(P_L, N_L)(I^*\alpha, I^*\beta) \circ \iota,$$

for $\alpha, \beta \in \Gamma(A^*)$.

This ends the proof of the result.

■

## 4. Reduction of Poisson-Nijenhuis Lie algebroids by epimorphisms of Lie algebroids

In order to complete the process of reduction, we now deal with the general problem of the projectability of a Poisson-Nijenhuis structure on a Lie algebroid with respect to a vector bundle epimorphism.

Let $\tau_A \colon A \to M$ and $\tau_{\widetilde{A}} \colon \widetilde{A} \to \widetilde{M}$ be vector bundles on the manifolds $M$ and $\widetilde{M}$, respectively, and let $(\Pi, \pi)$ be an epimorphism of vector bundles,

$$\begin{tikzcd} A \arrow[r, "\Pi"] \arrow[d, "\tau_A"'] & \widetilde{A} \arrow[d, "\tau_{\widetilde{A}}"] \\ M \arrow[r, "\pi"] & \widetilde{M} \end{tikzcd}$$

i.e., the map $\pi \colon M \to \widetilde{M}$ is a surjective submersion and, for each $x \in M$, $\Pi_x \colon A_x \to \widetilde{A}_{\pi(x)}$ is an epimorphism of vector spaces.

Denote by $\Gamma_p(A)$ (respectively, $\Gamma_p(A^*)$) the space of the $\Pi$-projectable sections of $A$ (respectively, of $A^*$). In [8] a characterization is found to establish when a vector bundle epimorphism is a Lie algebroid epimorphism.

**Proposition 4.1.** *(see [8]) Let $(\Pi, \pi) : A \to \widetilde{A}$ be a vector bundle epimorphism. Suppose that $([\cdot,\cdot]_A, \rho_A)$ is a Lie algebroid structure over $A$. Then, there exists a unique Lie algebroid structure on $\widetilde{A}$ such that $(\Pi, \pi)$ is a Lie algebroid epimorphism if and only if the following conditions hold:*

*i) The space $\Gamma_p(A)$ of the $\Pi$-projectable sections of $A$ is a Lie subalgebra of $(\Gamma(A), [\cdot,\cdot]_A)$ and*

*ii) $\Gamma(\ker \Pi)$ is an ideal of $\Gamma_p(A)$.*

In such a case, the structure of Lie algebroid over $\widetilde{A}$ is characterized by

$$[\widetilde{X}, \widetilde{Y}]_{\widetilde{A}} \circ \pi = \Pi \circ [X, Y]_A, \qquad \rho_{\widetilde{A}}(\widetilde{X})(\widetilde{f}) \circ \pi = \rho_A(X)(\widetilde{f} \circ \pi), \tag{33}$$

where $\widetilde{X}, \widetilde{Y} \in \Gamma(\widetilde{A})$, $\widetilde{f} \in C^\infty(\widetilde{M})$ and $X, Y \in \Gamma(A)$ are such that

$$\widetilde{X} \circ \pi = \Pi \circ X, \qquad \widetilde{Y} \circ \pi = \Pi \circ Y.$$

Note that the real function $\rho_A(X)(\widetilde{f}\circ\pi)$ on $M$ is basic with respect to $\pi$ (see [8]).

Let $(A, [\cdot,\cdot]_A, \rho_A)$ and $(\widetilde{A}, [\cdot,\cdot]_{\widetilde{A}}, \rho_{\widetilde{A}})$ be Lie algebroids over $M$ and $\widetilde{M}$, respectively, and let $(\Pi,\pi) : A \to \widetilde{A}$ be an epimorphism of Lie algebroids. We denote by $V\pi$ the vertical subbundle of $\pi\colon M \to \widetilde{M}$. Then, $\rho_A(Ker\Pi) \subseteq V\pi$ (see (21)).

We can always find a local basis $\{\xi_i, X_a\}$ of sections of $A$ such that $\xi_i \in \Gamma(Ker\Pi)$, for all $i$, and $X_a$ is a $\Pi$-projectable section, for all $a$. Indeed, to obtain such a base we choose a bundle metric on $A$ which gives us the decomposition $A = Ker\Pi \oplus (Ker\Pi)^\perp$ where $(Ker\Pi)^\perp$ is the orthogonal complement defined by the chosen metric. Then we consider a local basis $\{\xi_i\}$ of sections of $Ker\Pi$ and a local basis $\{\widetilde{X}_a\}$ of sections of $\widetilde{A}$. It follows that $\{\xi_i, X_a = \widetilde{X}_a^H\}$, where $\widetilde{X}_a^H$ is the horizontal lift of $\widetilde{X}_a$, is a local basis of sections of $A$. Furthermore, note that if $\{\eta_i, \alpha_a\}$ is the dual basis of $\{\xi_i, X_a\}$, then $\alpha_a = \Pi^*\widetilde{\alpha}_a$, where $\{\widetilde{\alpha}_a\}$ is the dual basis of $\{\widetilde{X}_a\}$ in $\widetilde{A}$. By using these tools we can prove the following results about projectable sections of $A$ and $A^*$.

**Proposition 4.2.** *Let $(\Pi,\pi) : A \to \widetilde{A}$ be an epimorphism of Lie algebroids and suppose that $X \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$. Then,*

*i) If $X$ is a $\Pi$-projectable section of $A$, then $[\xi, X]_A \in \Gamma(Ker\Pi)$ for any $\xi \in \Gamma(Ker\Pi)$. Moreover, if $\alpha$ is a $\Pi$-projectable section of $A^*$, then $\alpha(\xi) = 0$ and $\mathcal{L}^A_\xi\alpha = 0$, for any $\xi \in \Gamma(Ker\Pi)$.*

*ii) Assume that $\rho_A(Ker\Pi) = V\pi$. Then,*

*a) $X$ is a $\Pi$-projectable section of $A$ if and only if $[\xi, X]_A \in \Gamma(Ker\Pi)$, for any $\xi \in \Gamma(Ker\Pi)$.*

*b) $\alpha$ is a $\Pi$-projectable section of $A^*$ if and only if $\alpha(\xi) = 0$ and $\mathcal{L}^A_\xi\alpha = 0$, for any $\xi \in \Gamma(Ker\Pi)$.*

*Proof*: The first part of $i)$ is a consequence of Proposition 4.1.

Assume that there exists $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$ such that $\alpha = \Pi^*\widetilde{\alpha}$. If $\xi \in \Gamma(Ker\Pi)$ then $\alpha(\xi) = \Pi^*\widetilde{\alpha}(\xi) = 0$ and, by using (22),

$$\mathcal{L}^A_\xi\alpha = \mathcal{L}^A_\xi\Pi^*\widetilde{\alpha} = 0.$$

To prove $ii)$ we proceed as follows. Let $\{\xi_i, X_a\}$ be a local basis of sections of $A$ such that $\xi_i \in \Gamma(Ker\Pi)$, for all $i$, and $X_a$ is a $\Pi$-projectable section over $\widetilde{X}_a \in \Gamma(\widetilde{A})$ for all $a$.

*a*) Suppose that $X \in \Gamma(A)$ is such that $[\xi, X]_A \in \Gamma(\ker \Pi)$, for any $\xi \in \Gamma(\ker \Pi)$. If

$$X = f^i \xi_i + F^a X_a \qquad \text{with } f^i, F^a \text{ local } C^\infty\text{-functions on } M$$

then, by using Proposition 4.1, we have that

$$0 = \Pi \circ [\xi_i, X]_A = \Pi \circ (\rho_A(\xi_i)(F^a) X_a) = \rho_A(\xi_i)(F^a)(\widetilde{X}_a \circ \pi).$$

So, if $Z \in V_x \pi$, with $x \in M$, then there exists $\xi \in \Gamma(Ker\Pi)$ such that $Z = \rho_A(\xi)(x)$ and therefore

$$Z(F^a) = \rho_A(\xi)(F^a)(x) = 0.$$

We conclude that there exists $\widetilde{F}^a \in C^\infty(\widetilde{M})$ such that

$$F^a = \widetilde{F}^a \circ \pi,$$

and $X$ is a $\Pi$-projectable section of $A$.

*b*) Assume that $\alpha$ is a section of $A^*$ such that $\alpha(\xi) = 0$ and $\mathcal{L}^A_\xi \alpha = 0$, for any $\xi \in \Gamma(Ker\Pi)$. Let $\{\eta_i, \Pi^* \widetilde{\alpha}_a\}$ be the dual basis of $\{\xi_i, X_a\}$. Thus,

$$\alpha = g^i \eta_i + \sigma^a \Pi^* \widetilde{\alpha}_a, \qquad \text{with } g^i, \sigma^a \in C^\infty(M).$$

As $\alpha(\xi_i) = 0$, we deduce that $g^i = 0$. On the other hand, using (8) and Proposition 4.1,

$$0 = \mathcal{L}^A_{\xi_i} \alpha(X_a) = \rho_A(\xi_i)(\sigma^a) - \alpha([\xi_i, X_a]_A) = \rho_A(\xi_i)(\sigma^a).$$

As before, this implies that $\sigma^a = \widetilde{\sigma}^a \circ \pi$ for some function $\widetilde{\sigma}^a \in C^\infty(\widetilde{M})$. Hence, $\alpha$ is $\Pi$-projectable.

■

We consider now a section $P$ of the vector bundle $\wedge^2 A \to M$. $P$ is said to be $\Pi$-*projectable* if, for each $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$, we have $P^\sharp \Pi^* \widetilde{\alpha} \in \Gamma_p(A)$.

**Proposition 4.3.** *Let* $(\Pi, \pi) : A \to \widetilde{A}$ *be an epimorphism of Lie algebroids. If* $P \in \Gamma(\wedge^2 A)$ *is* $\Pi$*-projectable, then*

$$([\xi, P]_A)^\sharp(\Gamma_p(A^*)) \subseteq \Gamma(Ker\Pi) \tag{34}$$

*for any* $\xi \in \Gamma(Ker\Pi)$. *Moreover, if* $\rho_A(Ker\Pi) = V\pi$, *then* $P$ *is* $\Pi$*-projectable if and only if* (34) *holds.*

*Proof*: Assume that $P$ is $\Pi$-projectable. Then, for any $\alpha \in \Gamma_p(A^*)$ and $\xi \in \Gamma(Ker\Pi)$, by using (9), (11) and Proposition 4.2 we have

$$\begin{aligned}([\xi, P]_A)^\sharp(\alpha) &= [\xi, P^\sharp\alpha]_A - P^\sharp\mathcal{L}^A_\xi\alpha \\ &= [\xi, P^\sharp\alpha]_A \in \Gamma(Ker\Pi).\end{aligned}$$

Now, we suppose that $P$ satisfies (34) and $\rho_A(Ker\Pi) = V\pi$. Consider a local basis of sections $\{\xi_i, X_a\}$ of $A$ such that $\xi_i \in \Gamma(Ker\Pi)$ and $X_a \in \Gamma_p(A)$. Let $\{\eta_i, \Pi^*\widetilde{\alpha}_a\}$ be the dual basis of $\{\xi_i, X_a\}$. We have

$$P^\sharp\Pi^*\widetilde{\alpha}_a = f^i_a\xi_i + F^b_aX_b, \qquad \text{with } f^i_a, F^b_a \text{ local real } C^\infty\text{-functions on } M.$$

Note that $F^b_a = -F^a_b$.

By using (9) and Proposition 4.2 we have

$$\begin{aligned}0 =& \Pi\circ(([\xi, P]_A)^\sharp(\Pi^*\widetilde{\alpha}_a))(\widetilde{\alpha}_b) = ([\xi, P]_A)(\Pi^*\widetilde{\alpha}_a, \Pi^*\widetilde{\alpha}_b) \\ &= \rho_A(\xi)(P(\Pi^*\widetilde{\alpha}_a, \Pi^*\widetilde{\alpha}_b)) = \rho_A(\xi)(F^b_a),\end{aligned}$$

for any $\xi \in \Gamma(Ker\Pi)$.

So, if $Z \in V_x\pi$, then there exists $\xi \in \Gamma(Ker\Pi)$ such that $Z = \rho_A(\xi)(x)$ and therefore

$$Z(F^b_a) = 0.$$

Hence, there exists a local real $C^\infty$-function $\widetilde{F^b_a}$ on $\widetilde{M}$ such that

$$F^b_a = \widetilde{F^b_a}\circ\pi.$$

■

If $P$ is a $\Pi$-projectable Poisson structure on $A$, then we may construct the 2-section $\widetilde{P} \in \Gamma(\wedge^2\widetilde{A})$ of $\widetilde{A}$ characterized by

$$(\widetilde{P}^\sharp\widetilde{\alpha})\circ\pi = \Pi(P^\sharp(\Pi^*\widetilde{\alpha})), \text{ for any } \widetilde{\alpha} \in \Gamma(\widetilde{A}^*) \tag{35}$$

or equivalently,

$$\widetilde{P}(\widetilde{\alpha}, \widetilde{\beta})\circ\pi = P(\Pi^*\widetilde{\alpha}, \Pi^*\widetilde{\beta}), \text{ for any } \widetilde{\alpha}, \widetilde{\beta} \in \Gamma(\widetilde{A}^*). \tag{36}$$

**Proposition 4.4.** *Let $(\Pi, \pi) : A \to \widetilde{A}$ be an epimorphism of Lie algebroids. If $P$ is a $\Pi$-projectable Poisson structure on $A$, then $\widetilde{P}$ is a Poisson structure on $\widetilde{A}$.*

*Proof*: Let $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$. Then, by using (9) and (11) one may prove that

$$\frac{1}{2} i_{\widetilde{\alpha}}[\widetilde{P}, \widetilde{P}]_{\widetilde{A}} = -\widetilde{P}^\sharp(\mathrm{d}^{\widetilde{A}}\widetilde{\alpha}) + [\widetilde{P}^\sharp \widetilde{\alpha}, \widetilde{P}]_{\widetilde{A}} \tag{37}$$

where $\widetilde{P}^\sharp(\mathrm{d}^{\widetilde{A}}\widetilde{\alpha})$ is the section of the vector bundle $\wedge^2 \widetilde{A} \to \widetilde{M}$ defined by

$$\widetilde{P}^\sharp(\mathrm{d}^{\widetilde{A}}\widetilde{\alpha})(\widetilde{\beta}_1, \widetilde{\beta}_2) = \mathrm{d}^{\widetilde{A}}\widetilde{\alpha}(\widetilde{P}^\sharp \widetilde{\beta}_1, \widetilde{P}^\sharp \widetilde{\beta}_2),$$

for any $\widetilde{\beta}_1, \widetilde{\beta}_2 \in \Gamma(\widetilde{A}^*)$.

From the equality (37) for the Poisson structure $P$ and the 1-section $\Pi^*\widetilde{\alpha}$ of $A$, we deduce that

$$P^\sharp(\mathrm{d}^A \Pi^* \widetilde{\alpha}) = [P^\sharp(\Pi^*\widetilde{\alpha}), P]_A. \tag{38}$$

On the other hand, from (20) and (35) we deduce that

$$\wedge^2\, \Pi \circ P^\sharp \mathrm{d}^A \Pi^* \widetilde{\alpha} = \widetilde{P}^\sharp \mathrm{d}^{\widetilde{A}} \widetilde{\alpha} \circ \pi. \tag{39}$$

Projecting by $\Pi$, the equation (38) and using (39) we get

$$\widetilde{P}^\sharp \mathrm{d}^{\widetilde{A}} \widetilde{\alpha} \circ \pi = \wedge^2 \Pi \circ [P^\sharp \Pi^* \widetilde{\alpha}, P]_A. \tag{40}$$

Since $(\Pi, \pi)$ is an epimorphism of Lie algebroids, from (22) and (35) we also obtain

$$\mathcal{L}^A_{P^\sharp(\Pi^*\widetilde{\alpha})}(\Pi^* \widetilde{\beta}) = \Pi^*(\mathcal{L}^{\widetilde{A}}_{\widetilde{P}^\sharp \widetilde{\alpha}} \widetilde{\beta}) \qquad \text{for any } \widetilde{\beta} \in \Gamma(\widetilde{A}^*). \tag{41}$$

$$\begin{aligned} \rho_A(P^\sharp(\Pi^*\widetilde{\alpha}))(\widetilde{f} \circ \pi) &= \mathcal{L}^A_{P^\sharp(\Pi^*\widetilde{\alpha})}(\widetilde{f} \circ \pi) = (\mathcal{L}^{\widetilde{A}}_{\widetilde{P}^\sharp \widetilde{\alpha}} \widetilde{f}) \circ \pi \\ &= \rho_{\widetilde{A}}(\widetilde{P}^\sharp \widetilde{\alpha})(\widetilde{f}) \circ \pi, \qquad \text{with } \widetilde{f} \in C^\infty(\widetilde{M}). \end{aligned} \tag{42}$$

This fact allows us to prove that

$$\wedge^2\, \Pi \circ [P^\sharp(\Pi^*\widetilde{\alpha}), P]_A = [\widetilde{P}^\sharp \widetilde{\alpha}, \widetilde{P}]_{\widetilde{A}} \circ \pi, \tag{43}$$

by using (10).

From (37), (40) and (43) we deduce that

$$i_{\widetilde{\alpha}}[\widetilde{P}, \widetilde{P}]_{\widetilde{A}} = 0,$$

for any $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$. In conclusion $\widetilde{P}$ is a Poisson structure. ■

Assume that $N\colon A \to A$ is a Nijenhuis operator on $A$. $N$ is said to be $\Pi$-*projectable* if

$$N(\Gamma_p(A)) \subseteq \Gamma_p(A) \quad \text{and} \quad N(\Gamma(Ker\Pi)) \subseteq \Gamma(Ker\Pi).$$

**Proposition 4.5.** *Let* $(\Pi, \pi) : A \to \widetilde{A}$ *be an epimorphism of Lie algebroids. If* $N$ *is a* $\Pi$*-projectable Nijenhuis operator on* $A$*, then*

$$\mathcal{L}^A_\xi N(\Gamma_p(A)) \subseteq \Gamma(Ker\Pi) \quad \textit{and} \quad N(\Gamma(Ker\Pi)) \subseteq \Gamma(Ker\Pi), \tag{44}$$

*for any* $\xi \in \Gamma(Ker\Pi)$*. Moreover, if* $\rho_A(Ker\Pi) = V\pi$*, then* $N$ *is* $\Pi$*-projectable if and only if* (44) *holds.*

*Proof*: Assume that $N$ is $\Pi$-projectable. For any $X \in \Gamma_p(A)$

$$(\mathcal{L}^A_\xi N)(X) = [\xi, NX]_A - N([\xi, X]_A).$$

Since $NX \in \Gamma_p(A)$, $[\xi, NX]_A \in \Gamma(Ker\Pi)$ and $[\xi, X]_A \in \Gamma(Ker\Pi)$ (see Proposition 4.1), then

$$\mathcal{L}^A_\xi N(X) \in \Gamma(Ker\Pi).$$

Now we suppose that $\rho_A(Ker\Pi) = V\pi$ and that (44) holds. Consider a local basis of sections $\{\xi_i, X_a\}$ of $A$ such that $\xi_i \in \Gamma(Ker\Pi)$ and $X_a \in \Gamma_p(A)$. Then, $X \in \Gamma_p(A)$, implies that $N(X) \in \Gamma_p(A)$. Indeed,

$$N(X) = f^i\xi_i + F^a X_a \qquad \text{with } f^i, F^a \text{ local real } C^\infty\text{-functions on } M.$$

Hence, keeping in account that $N[\xi, X]_A \in \Gamma(Ker\Pi)$, we have

$$\begin{aligned} 0 &= \Pi \circ (\mathcal{L}^A_\xi N(X)) = \Pi \circ ([\xi, NX]_A - N([\xi, X]_A)) \\ &= \Pi \circ ([\xi, NX]_A) = \Pi \circ (\rho_A(\xi)(F^a)X_a). \end{aligned}$$

Therefore, $\rho_A(\xi)(F^a) = 0$ for any $\xi \in \Gamma(Ker\Pi)$.

Let $Z \in V_x\pi$, with $x \in M$. Hence, there exists $\xi \in \Gamma(Ker\Pi)$ such that

$$Z = \rho_A(\xi)(x).$$

Thus, we can conclude that $Z(F^a) = 0$, i.e. there exists a local $C^\infty-$function $\widetilde{F_a}$ on $\widetilde{M}$ such that

$$F_a = \widetilde{F_a} \circ \pi.$$

∎

If $N$ is a $\Pi$-projectable Nijenhuis operator on $A$, then we can construct a new operator $\widetilde{N}\colon \widetilde{A} \to \widetilde{A}$ as follows.

$$(\widetilde{N}\widetilde{X}) \circ \pi = \Pi \circ (NX) \qquad \text{for any } \widetilde{X} \in \Gamma(\widetilde{A}), \tag{45}$$

where $X \in \Gamma_p(A)$ is a projectable section such that $\Pi \circ X = \widetilde{X} \circ \pi$. Note that $\widetilde{N}$ is well defined since $X \in \Gamma_p(A)$ and therefore $NX \in \Gamma_p(A)$. Moreover, if $X'$ is another section of $A$ such that $\Pi \circ X' = \Pi \circ X$ then $X' - X \in \Gamma(Ker\Pi)$ and $NX = NX'$.

From previous results, we give conditions for obtaining a Poisson-Nijenhuis structure on the Lie algebroid image of a Lie algebroid epimorphism.

**Theorem 4.6.** *Let* $(\Pi, \pi) : A \to \widetilde{A}$ *be a Lie algebroid epimorphism. Assume that* $(P, N)$ *is a Poisson-Nijenhuis structure on* $A$ *such that* $P$ *and* $N$ *are* $\Pi$*-projectable. Then,* $(\widetilde{P}, \widetilde{N})$ *is a Poisson-Nijenhuis structure on* $\widetilde{A}$.

*Proof*: We will show that $\widetilde{N}$ is compatible with the Poisson structure $\widetilde{P}$. Indeed, firstly we show that

$$\widetilde{N} \circ \widetilde{P}^\sharp = \widetilde{P}^\sharp \circ \widetilde{N}^*. \tag{46}$$

From (35) and (45) it follows that for any $\widetilde{\alpha} \in \Gamma(\widetilde{A^*})$

$$\widetilde{N}(\widetilde{P}^\sharp\widetilde{\alpha}) \circ \pi = \Pi \circ N(P^\sharp\Pi^*\widetilde{\alpha}) = \Pi \circ P^\sharp N^*(\Pi^*\widetilde{\alpha}) = \Pi \circ P^\sharp(\Pi^*\widetilde{N}^*\widetilde{\alpha}) = \widetilde{P}^\sharp(\widetilde{N}^*\widetilde{\alpha}) \circ \pi.$$

On the other hand, using (15), (16) and the fact that $(\Pi, \pi)$ is a Lie algebroid morphism, we get

$$\mathcal{T}_{\widetilde{N}}(\widetilde{X}, \widetilde{Y}) \circ \pi = \Pi \circ \mathcal{T}_N(X, Y) \qquad \text{for any } \widetilde{X}, \widetilde{Y} \in \Gamma(\widetilde{A}), \tag{47}$$

where $X, Y \in \Gamma(A)$ are such that $\widetilde{X} \circ \pi = \Pi \circ X$, $\widetilde{Y} \circ \pi = \Pi \circ Y$.

Finally, by using (23), (22), (35) and (45), we can prove that

$$\Pi^*[\widetilde{\alpha}, \widetilde{\beta}]_{\widetilde{P}} = [\Pi^*\widetilde{\alpha}, \Pi^*\widetilde{\beta}]_P, \quad \Pi^*[\widetilde{\alpha}, \widetilde{\beta}]_{\widetilde{N}\widetilde{P}} = [\Pi^*\widetilde{\alpha}, \Pi^*\widetilde{\beta}]_{NP}$$

and

$$\Pi^*[\widetilde{\alpha}, \widetilde{\beta}]_{\widetilde{P}}^{\widetilde{N}^*} = [\Pi^*\widetilde{\alpha}, \Pi^*\widetilde{\beta}]_P^{N^*}.$$

As a consequence,

$$\Pi^*(\mathcal{C}(\widetilde{P}, \widetilde{N})(\widetilde{\alpha}, \widetilde{\beta})) = \mathcal{C}(P, N)(\Pi^*\widetilde{\alpha}, \Pi^*\widetilde{\beta}), \tag{48}$$

for any $\widetilde{\alpha}, \widetilde{\beta} \in \Gamma(\widetilde{A}^*)$.

From (46), (47) and (48) we obtain that $(\widetilde{P}, \widetilde{N})$ is a Poisson-Nijenhuis structure on $\widetilde{A}$. ∎

The above result suggests us to introduce the following definition.

**Definition 4.7.** Let $(\Pi, \pi) : A \to \widetilde{A}$ be a Lie algebroid morphism. We say that $(\Pi, \pi)$ is a *Poisson-Nijenhuis Lie algebroid morphism* if we have Poisson-Nijenhuis structures $(P, N)$, $(\widetilde{P}, \widetilde{N})$ on $A$ and $\widetilde{A}$, respectively, such that

$$(\widetilde{P}^\sharp\widetilde{\alpha}) \circ \pi = \Pi \circ (P^\sharp(\Pi^*\widetilde{\alpha})),$$
$$(\widetilde{N}\widetilde{X}) \circ \pi = \Pi \circ (NX),$$

for all $\widetilde{\alpha} \in \Gamma(\widetilde{A}^*)$, $X \in \Gamma(A)$ and $\widetilde{X} \in \Gamma(\widetilde{A})$ such that $\widetilde{X} \circ \pi = \Pi \circ X$.

The following result follows easily from Proposition 4.1 and Theorem 4.6.

**Theorem 4.8.** *Let $(\Pi, \pi) : A \to \widetilde{A}$ be a vector bundle epimorphism. Suppose that $([\cdot,\cdot]_A, \rho_A, P, N)$ is a Poisson-Nijenhuis Lie algebroid structure over $A$. Then, there exists a unique Poisson-Nijenhuis Lie algebroid structure on $\widetilde{A}$ such that $(\Pi, \pi)$ is a Poisson-Nijenhuis Lie algebroid epimorphism if and only if the following conditions hold:*

*i) The space $\Gamma_p(A)$ of the $\Pi$-projectable sections of $A$ is a Lie subalgebra of $(\Gamma(A), [\cdot,\cdot]_A)$;*
*ii) $\Gamma(\ker \Pi)$ is an ideal of $\Gamma_p(A)$ and*
*iii) $P$ and $N$ are $\Pi$-projectable.*

## 5. Reduction of a Lie algebroid induced by a Lie subalgebroid

In this section we will describe, using the above results about reduction by epimorphisms of Lie algebroids, the reduction of a Lie algebroid by a certain foliation associated with a given Lie subalgebroid. In the next section, we will use this construction for obtaining, under suitable regularity conditions, a reduced nondegenerate Poisson-Nijenhuis Lie algebroid from an arbitrary Poisson-Nijenhuis Lie algebroid through a suitable choice of the Lie subalgebroid.

In this reduction procedure of a Lie algebroid, fundamental tools are the complete and vertical lifts of sections associated with a Lie algebroid. Firstly, we recall these notions and some properties about them.

**5.1. Complete and vertical lifts in a Lie algebroid.** Let $(A, [\cdot,\cdot]_A, \rho_A)$ be a Lie algebroid over a manifold $M$ and $\tau_A : A \to M$ be the corresponding vector bundle projection.

Given $f \in C^\infty(M)$, we will denote by $f^c$ and $f^v$ *the complete and vertical lift* to $A$ of $f$. Here $f^c$ and $f^v$ are the real functions on $A$ defined by

$$f^c(a) = \rho_A(a)(f), \quad f^v(a) = f(\tau_A(a)), \tag{49}$$

for all $a \in A$.

Now, let $X$ be a section of $A$. Then, we can consider *the vertical lift of $X$ to $A$* as the vector field $X^v$ on $A$ given by

$$X^v(a) = X(\tau_A(a))^v_a, \quad \text{for } a \in A,$$

where $\overset{v}{\underset{a}{}} : A_{\tau_A(a)} \to T_a(A_{\tau_A(a)})$ is the canonical isomorphism between the vector spaces $A_{\tau_A(a)}$ and $T_a(A_{\tau_A(a)})$.

On the other hand, there exists a unique vector field $X^c$ on $A$, *the complete lift of $X$ to $A$,* characterized by the two following conditions:

(i) $X^c$ is $\tau$-projectable on $\rho_A(X)$ and

(ii) $X^c(\hat{\alpha}) = \widehat{\mathcal{L}^A_X \alpha}$,

for all $\alpha \in \Gamma(A^*)$ (see [4]). Here, if $\beta \in \Gamma(A^*)$ then $\hat{\beta}$ is the linear function on $A$ defined by

$$\hat{\beta}(a) = \beta(\tau_A(a))(a), \quad \text{for all } a \in A.$$

Complete and vertical lifts may be extended to associate to any section $Q\colon M \to \wedge^q A$ of the bundle $\wedge^q A \to M$ a pair of $q$-multivectors $Q^c, Q^v : A \to \wedge^q(TA)$ on $A$. These extensions are uniquely determined by the following equalities (see [4]):

$$(Q \wedge R)^c = Q^c \wedge R^v + Q^v \wedge R^c, \tag{50}$$

and

$$(Q \wedge R)^v = Q^v \wedge R^v, \tag{51}$$

which are satisfied by any pair of sections $Q : M \to \wedge^q A$, $R : M \to \wedge^r A$.

A direct computation proves that (see [4])

$$[Q^c, R^c] = [Q, R]^c_A, \quad [Q^c, R^v] = [Q, R]^v_A, \quad [Q^v, R^v] = 0. \tag{52}$$

Given $X \in \Gamma(A)$, we can also define the complete lift of $X$ to $A^*$ as the vector field $X^{*c}$ over $A^*$ such that it is $\tau_{A^*}$-projectable on $\rho_A(X)$ and

$$X^{*c}(\widehat{Y}) = \widehat{[X, Y]_A}, \tag{53}$$

for all $Y \in \Gamma(A)$ (see [5]). Here $\widehat{Z}$, with $Z \in \Gamma(A)$, is the linear map over $A^*$ induced by $Z$. In fact, the complete lifts of a section $X \in \Gamma(A)$ to $A$ and $A^*$ are related by the following formula

$$X^{*c}(\widehat{Y}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\widehat{Y} \circ \varphi_t^*), \qquad \text{for any } Y \in \Gamma(A) \tag{54}$$

where $\varphi_t\colon A \to A$ is the flow of $X^c \in \mathfrak{X}(A)$ (see [15, 18]).

Suppose that $(x^i)$ are coordinates on an open subset $U$ of $M$, $\{e_\alpha\}$ is a basis of sections of $\tau_A^{-1}(U) \to U$ and $\{e^\alpha\}$ is the dual basis of sections of $\tau_{A^*}^{-1}(U) \to U$. Denote by $(x^i, y^\alpha)$ the corresponding local coordinates on

$\tau_A^{-1}(U)$ and by $(x^i, y_\alpha)$ the local coordinates on $\tau_{A^*}^{-1}(U)$. Finally, let $\rho_\alpha^i$ and $C_{\alpha\beta}^\gamma$ be the corresponding local structure functions of $A$, defined by

$$\rho_A(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i} \quad \text{and} \quad [e_\alpha, e_\beta]_A = C_{\alpha\beta}^\gamma e_\gamma.$$

If $X$ is a section of $A$ and on $U$ we have

$$X = X^\alpha e_\alpha,$$

then the coordinate expressions of the lifts are given by

$$\begin{aligned} X^v &= X^\alpha \frac{\partial}{\partial y^\alpha}, \\ X^c &= X^\alpha \rho_\alpha^i \frac{\partial}{\partial x^i} + \left( \rho_\beta^i \frac{\partial X^\alpha}{\partial x^i} - X^\gamma C_{\gamma\beta}^\alpha \right) y^\beta \frac{\partial}{\partial y^\alpha}, \\ X^{*c} &= X^\alpha \rho_\alpha^i \frac{\partial}{\partial x^i} - \left( \rho_\alpha^i \frac{\partial X^\beta}{\partial x^i} y_\beta + C_{\alpha\beta}^\gamma y_\gamma X^\beta \right) \frac{\partial}{\partial y_\alpha}. \end{aligned} \tag{55}$$

In particular,

$$e_\alpha^v = \frac{\partial}{\partial y^\alpha}, \quad e_\alpha^c = \rho_\alpha^i \frac{\partial}{\partial x^i} - C_{\alpha\beta}^\gamma y^\beta \frac{\partial}{\partial y^\gamma}, \quad e_\alpha^{*c} = \rho_\alpha^i \frac{\partial}{\partial x^i} + C_{\alpha\beta}^\gamma y_\gamma \frac{\partial}{\partial y_\beta}. \tag{56}$$

**5.2. Reduction procedure of a Lie algebroid induced by a Lie subalgebroid.** Before describing this procedure, we prove the following general lemma on vector bundles which will be useful in the sequel.

**Lemma 5.1.** *Let $\pi_A\colon A \to M$ a vector bundle of rank $k$ and $\pi_B\colon B \to M'$ be a surjective submersion. Assume that there exist two smooth maps $\Phi\colon A \to B$ and $\phi\colon M \to M'$ in such a way that the following diagram*

$$\begin{tikzcd} A \arrow[r, "\pi_A"] \arrow[d, "\Phi"] & M \arrow[d, "\phi"] \\ B \arrow[r, "\pi_B"] & M' \end{tikzcd}$$

*is commutative and such that*

1) *$\phi$ is a submersion;*
2) *$\forall x \in M$, $\Phi_x\colon \pi_A^{-1}(x) \to \pi_B^{-1}(\phi(x))$ is a diffeomorphism and*

3) *$\forall x, y \in M$ such that $\phi(x) = \phi(y)$,*

$$\Phi_y^{-1} \circ \Phi_x \colon \pi_A^{-1}(x) \to \pi_A^{-1}(y)$$

*is an isomorphism of vector spaces.*

*Then, $\pi_B \colon B \to M'$ is a vector bundle of rank $k$ and $(\Phi, \phi)$ is a vector bundle epimorphism.*

*Proof*: Let $x' = \phi(x) \in M'$. Then, there exists a unique structure of vector space on the fiber $\pi_B^{-1}(x')$ in such a way that the diffeomorphism $\Phi_x : \pi_A^{-1}(x) \to \pi_B^{-1}(x')$ is an isomorphism of vector spaces. Moreover, this structure doesn't depend on the chosen point $x \in M$. In fact, if $y \in M$ and $\phi(y) = \phi(x) = x'$ then, from (3), we deduce that the map $\Phi_y^{-1} \circ \Phi_x : \pi_A^{-1}(x) \to \pi_A^{-1}(y)$ is an isomorphism of vector spaces.

On the other hand, using that $\phi$ is a submersion and the fact that $\pi_A : A \to M$ is a vector bundle, we have that there exists an open neighbourhood $U \subset M$ of $x \in M$, an open neighbourhood $U' \subseteq M'$ of $x' \in M'$ and two smooth maps $s \colon U' \to U$ and $\psi \colon U \times \mathbb{R}^k \to \pi_A^{-1}(U)$ such that

1) $\phi \circ s = 1_{U'}$ and $s(x') = x$.
2) $\psi$ is a diffeomorphism, $\pi_A \circ \psi = pr_1$ and for each $y \in U$, $\psi_y \colon \mathbb{R}^k \to \pi_A^{-1}(y)$ is a vector space isomorphism.

Therefore we can construct a diffeomorphism

$$\overline{\psi} \colon U' \times \mathbb{R}^k \to \pi_B^{-1}(U')$$

as follows: $\overline{\psi}(y', g) = (\Phi \circ \psi_{s(y')})(g)$ for $(y', g) \in U' \times \mathbb{R}^k$. Note that $\overline{\psi}^{-1}(b) = \Big(\pi_B(b), (\psi^{-1}_{s(\pi_B(b))} \circ \Phi^{-1}_{s(\pi_B(b))})(b)\Big)$. Moreover, if $y' \in U'$ it is easy to prove that $\overline{\psi}_{y'} \colon \mathbb{R}^k \to \pi_B^{-1}(y')$ is an isomorphism of vector spaces. ■

Let $\tau_A \colon A \to M$ be a vector bundle and $([\cdot, \cdot]_A, \rho_A)$ be a Lie algebroid structure on $A$. Consider a Lie subalgebroid $\tau_B \colon B \to M$ of $A$. Then, we have the following result.

**Proposition 5.2.**

1) *The generalized distribution $\rho_A(B)$ on $M$ defined by*

$$\rho_A(B)_x = \rho_A(B_x) \subseteq T_xM, \text{ for every } x \in M,$$

*is a generalized foliation. Moreover,*

$$\dim(\rho_A(B)_x) \leq \operatorname{rank} B, \text{ for every } x \in M.$$

2) *The generalized distribution* $\mathcal{F}^B$ *on* $A$ *defined by*

$$\mathcal{F}^B_a = \{X^c(a) + Y^v(a) \mid X, Y \in \Gamma(B)\} \subseteq T_aA, \quad \textit{for } a \in A$$

*is a generalized foliation. Furthermore,* $\dim \mathcal{F}^B_b = \dim(\rho_A(B))_{\tau_B(b)} + \operatorname{rank} B$*, for* $b \in B$*. Thus, if* $\mathcal{F}^B$ *has constant rank then* $\rho_A(B)$ *also has constant rank.*

*Proof*:

1) It is clear that $\rho_A(B)$ is a finitely generated distribution. Moreover, if $X, Y \in \Gamma(B)$ then, using that $[X, Y]_B = [X, Y]_A$, we deduce that

$$[\rho_A(X), \rho_B(Y)] = \rho_A[X, Y]_B$$

which implies that $\rho_A(B)$ is an involutive distribution. Thus, $\rho_A(B)$ is a generalized foliation.

On the other hand, if $x \in M$, we have that

$$\dim(\rho_A(B)_x) \leq \dim B_x = \operatorname{rank} B.$$

2) $\mathcal{F}^B$ is a finitely generated distribution. In fact, let $U$ be an open subset of $M$ and $\{X_i\}$ be a basis of $\Gamma(\tau_B^{-1}(U))$. Then, $\{X_i^c(a), X_i^v(a)\}$ is a generator system of $\mathcal{F}^B_a$, for all $a \in \tau_A^{-1}(U)$.

Moreover, since $B$ is a Lie subalgebroid of $A$, we deduce that $\mathcal{F}^B$ is an involutive distribution (see (52))).

Now, let $b$ be a point of $B_x$, with $x \in M$, and suppose that $\{v_a, v_\beta\}$ is a basis of $B_x$, such that $\{\rho_A(v_a)\}$ (respectively, $\{v_\beta\}$) is a basis of $\rho_A(B_x)$ (respectively, $Ker(\rho_{A|B_x})$). Then, we can choose an open subset $U$ of $M$, with $x \in U$, and a basis $\{X_a, X_\beta\}$ of $\Gamma(\tau_B^{-1}(U))$ satisfying

$$X_a(x) = v_a, \quad X_\beta(x) = v_\beta.$$

We complete the basis $\{X_a, X_\beta\}$ to a basis of $\Gamma(\tau_A^{-1}(U))$

$$\{X_a, X_\beta, X_{\bar{a}}\}$$

Next, we will assume, without the loss of generality, that on $U$ we have a system of local coordinates $(x^i)$. Thus, we can consider the corresponding local coordinates $(x^i, y^a, y^\beta, y^{\bar{a}})$ on $\tau_A^{-1}(U)$.

Using (55), we deduce that

$$X_a^v(b) = \frac{\partial}{\partial y^a}_{|b}, \quad X_\beta^v(b) = \frac{\partial}{\partial y^\beta}_{|b}$$

$$(T\tau_B)(X_a^c(b)) = \rho_A(v_a), \quad X_\beta^c(b) \in V_b\tau_B =< \frac{\partial}{\partial y^a}_{|b}, \frac{\partial}{\partial y^\beta}_{|b} >$$

for all $a$ and $\beta$.

Therefore,

$$\dim \mathcal{F}_b^B = \dim(\rho_A(B)_{\tau_B(b)}) + \operatorname{rank} B$$

■

*Remark* 5.3. Note that if $a \in A_x$ then $(T_a\tau_A)(\mathcal{F}_a^B) = \rho_A(B_x)$.

Assume that $\rho_A(B)$ and $\mathcal{F}^B$ are regular foliations, i.e., they have finite constant rank, $M/\rho_A(B)$ and $A/\mathcal{F}^B$ are differentiable manifolds, and

$$\pi\colon M \to M/\rho_A(B) = \widetilde{M} \qquad \text{and} \qquad \Pi\colon A \to A/\mathcal{F}^B = \widetilde{A}$$

are submersions.

We define $\tau_{\widetilde{A}}\colon \widetilde{A} = A/\mathcal{F}^B \to \widetilde{M} = M/\rho_A(B)$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\Pi} & \widetilde{A} = A/\mathcal{F}^B \\ \big\downarrow{\scriptstyle \tau_A} & & \big\downarrow{\scriptstyle \tau_{\widetilde{A}}} \\ M & \xrightarrow{\pi} & \widetilde{M} = M/\rho_A(B) \end{array}$$

The map $\tau_{\widetilde{A}}$ is well defined. Indeed, if $\Pi(a_x) = \Pi(a_{x'}) \in A/\mathcal{F}^B$, with $a_x \in A_x$, $a_{x'} \in A_{x'}$, $x, x' \in M$, then there exists a curve $\sigma_A\colon [0,1] \to A$ continuous, piecewise differentiable, tangent to $\mathcal{F}^B$, such that $\sigma_A(0) = a_x$ and $\sigma_A(1) = a_{x'}$. Consider now the curve $\sigma_M = \tau_A \circ \sigma_A\colon [0,1] \to M$ which results to be continuous, piecewise differentiable, tangent to $\rho_A(B)$ (see Remark 5.3), such that

$$\sigma_M(0) = x \qquad \text{and} \qquad \sigma_M(1) = x'.$$

Hence, $\pi(x) = \pi(x')$.

Note that $\tau_{\widetilde{A}}$ is a submersion since $\tau_A, \Pi$ and $\pi$ are submersions. On the other hand, we have a vector bundle $\tau_{\overline{A}}\colon \overline{A} = A/B \to M$ such that the following diagram is commutative

$$
\begin{tikzcd}
A \arrow[r, "\overline{\Pi}"] \arrow[dr, "\tau_A"'] & A/B = \overline{A} \arrow[d, "\tau_{\overline{A}}"] \\
 & M
\end{tikzcd}
$$

In fact, $\overline{\Pi}$ is a vector bundle epimorphism. Therefore, we can induce a smooth map $\widetilde{\Pi}\colon A/B \to A/\mathcal{F}^B$ making commutative the following diagram

$$
\begin{tikzcd}
\overline{A} = A/B \arrow[r, "\widetilde{\Pi}"] \arrow[d, "\tau_{\overline{A}}"] & \widetilde{A} = A/\mathcal{F}^B \arrow[d, "\tau_{\widetilde{A}}"] \\
M \arrow[r, "\pi"] & \widetilde{M} = M/\rho_A(B)
\end{tikzcd}
$$

Indeed, if $a'_x - a_x \in B_x$, then we can consider the curve $\sigma\colon [0,1] \to A_x$ defined by

$$\sigma(t) = a_x + t(a'_x - a_x).$$

Note that $\sigma(0) = a_x$ and $\sigma(1) = a_{x'}$. Moreover, $\dot{\sigma}(t_0) = (a'_x - a_x)^v_{\sigma(t_0)} \in \mathcal{F}^B_{\sigma(t_0)}$. Hence, $\Pi(a_x) = \Pi(a'_x)$. $\widetilde{\Pi}$ is a smooth map since $\overline{\Pi} : A \to \bar{A} = A/B$ is a submersion.

In order to guarantee that $\tau_{\widetilde{A}}$ is a vector bundle, we suppose that $B$ satisfies *the condition* $\mathcal{F}^B$ , i.e.

$$a_x, a'_x \in A_x \text{ are in the same leaf of } \mathcal{F}^B \qquad \Longleftrightarrow \qquad a'_x - a_x \in B_x.$$

for any $x \in M$.

**Proposition 5.4.** *Assume that $\rho_A(B)$ and $\mathcal{F}^B$ are regular foliations and that $B$ satisfies the condition $\mathcal{F}^B$. Then, $\tau_{\widetilde{A}}\colon \widetilde{A} = A/\mathcal{F}^B \to \widetilde{M} = M/\rho_A(B)$ is a vector bundle, the fiber of $\widetilde{A}$ over the point $\pi(x) \in \widetilde{M}$ is isomorphic to the* *quotient vector space* $A_x/B_x$ *and the diagram*

$$\begin{tikzcd} \overline{A} = A/B \arrow[r, "\widetilde{\Pi}"] \arrow[d, "\tau_{\overline{A}}"] & \widetilde{A} = A/\mathcal{F}^B \arrow[d, "\tau_{\widetilde{A}}"] \\ M \arrow[r, "\pi"] & \widetilde{M} = M/\rho_A(B) \end{tikzcd}$$

*is a vector bundle epimorphism. In fact, the restriction of* $\widetilde{\Pi}$ *to the fiber* $\bar{A}_x = A_x/B_x$ *is a linear isomorphism on* $\widetilde{A}_{\pi(x)}$.

*Proof*: We apply Lemma 5.1 to the following diagram

$$\begin{tikzcd} \overline{A} = A/B \arrow[r, "\tau_{\overline{A}}"] \arrow[d, "\widetilde{\Pi}"] & M \arrow[d, "\pi"] \\ \widetilde{A} = A/\mathcal{F}^B \arrow[r, "\tau_{\widetilde{A}}"] & \widetilde{M} = M/\rho_A(B) \end{tikzcd}$$

Note that $\pi$ and $\tau_{\widetilde{A}}$ are submersions. Then, for all $x \in M$, $\tau_{\widetilde{A}}^{-1}(\pi(x))$ is a regular submanifold of $\widetilde{A}$ and $T_{\Pi(a_x)}\tau_{\widetilde{A}}^{-1}(\pi(x)) = \ker T_{\Pi(a_x)}\tau_{\widetilde{A}}$ for all $a_x \in A_x$.

We will see that

$$\widetilde{\Pi}_x \colon A_x/B_x \to \tau_{\widetilde{A}}^{-1}(\pi(x))$$

is a surjective submersion.

Indeed, let $\Pi(a_{x'}) \in \tau_{\widetilde{A}}^{-1}(\pi(x))$. Then, $\tau_{\widetilde{A}}(\Pi(a_{x'})) = \pi(x)$. Hence, $\pi(x) = \pi(x')$. Therefore, there exists a continuous, piecewise differentiable path $\sigma \colon [0, 1] \to M$ tangent to $\rho_A(B)$ such that $\sigma(0) = x$ and $\sigma(1) = x'$. In each differentiable piece we can find $X \in \Gamma(B)$ such that $\sigma$ is an integral curve of $\rho_A(X)$. Assume, without the loss of generality, that the curve $\sigma$ is smooth and let $\psi \colon \mathbb{R} \times M \to M$ be the flow of $\rho_A(X)$. Then, $\psi_x(0) = x$,

$$\frac{\mathrm{d}\psi_x}{\mathrm{d}t} = \rho_A(X)(\psi_x(t))$$

and there exists $t_0 \in \mathbb{R}$ such that $\psi_{t_0}(x) = x'$. Let $\varphi \colon \mathbb{R} \times A \to A$ be the flow of $X^c \in \mathfrak{X}(A)$. Since $X^c$ projects on $\rho_A(X)$ we have that the following diagram is commutative for any $t$

$$\begin{tikzcd} A \arrow[r, "\varphi_t"] \arrow[d, "\tau_A"] & A \arrow[d, "\tau_A"] \\ M \arrow[r, "\psi_t"] & M \end{tikzcd}$$

Hence $\varphi_{-t_0}(a_{x'}) \in A_x$ and hence we have a curve $\varphi_{a_{x'}}$ on $A$ such that

$$\frac{\mathrm{d}\varphi_{a_{x'}}}{\mathrm{d}t}(t) = X^c(\varphi_{a_{x'}}(t)) \in \mathcal{F}^B_{\varphi_{a_{x'}}(t)},$$

$\varphi_{a_{x'}}(0) = a_{x'}$ and $\varphi_{a_{x'}}(-t_0) = \varphi_{-t_0}(a_{x'})$. Thus,

$$\widetilde{\Pi}_x\overline{\Pi}_x(\varphi_{-t_0}(a_{x'})) = \Pi_{x'}(\varphi_{-t_0}(a_{x'})) = \Pi_{x'}(a_{x'}),$$

where $\Pi_x$ and $\Pi_{x'}$ (respectively, $\overline{\Pi}_x$) are the restrictions of $\Pi$ (respectively, $\overline{\Pi}$) to the fiber over $x$ and $x'$. So, $\widetilde{\Pi}_x$ is surjective. Moreover, using that the following diagram

$$\begin{tikzcd} A_x/B_x \arrow[r, "\widetilde{\Pi}_x"] & \tau_{\widetilde{A}}^{-1}(\pi(x)) \\ A_x \arrow[u, "\overline{\Pi}_x"] \arrow[ur, "\Pi_x"'] & \end{tikzcd}$$

is commutative, we deduce that $\widetilde{\Pi}_x : A_x/B_x \to \tau_{\widetilde{A}}^{-1}(\pi(x))$ is smooth.

As a matter of fact $\widetilde{\Pi}_x$ is a submersion, i.e.,

$$T_{\overline{\Pi}(a_x)}\widetilde{\Pi}_x \colon T_{\overline{\Pi}(a_x)}(A_x/B_x) \to T_{\Pi(a_x)}(\tau_{\widetilde{A}}^{-1}(\pi(x)))$$

is surjective. Indeed, let $\widetilde{X} \in T_{\Pi(a_x)}(\tau_{\widetilde{A}}^{-1}(\pi(x))) = \ker T_{\Pi(a_x)}\tau_{\widetilde{A}}$. Then, since $\Pi \colon A \to \widetilde{A} = A/\mathcal{F}^B$ is a submersion, there exists $X \in T_{a_x}A$ such that

$$\widetilde{X} = T_{a_x}\Pi(X). \tag{57}$$

Hence,

$$0 = T_{\Pi(a_x)}\tau_{\widetilde{A}}(\widetilde{X}) = T_{\Pi(a_x)}(\tau_{\widetilde{A}} \circ \Pi)(X) = T_{a_x}(\pi \circ \tau_A)(X),$$

i.e. $T_{a_x}\tau_A(X) \in \ker T_x\pi = V_x\pi = \rho_A(B_x)$.

From Remark 5.3, we deduce that there exists $Y \in \mathcal{F}^B_{a_x}$ such that

$$T_{a_x}\tau_A(Y) = T_{a_x}\tau_A(X),$$

or equivalently $X - Y \in \ker T_{a_x}\tau_A = V_{a_x}(\tau_A) = T_{a_x}A_x$.

Consider now $\overline{\Pi}_x\colon A_x \to A_x/B_x$. Then,

$$W_x = T_{a_x}\overline{\Pi}_x(X - Y) \in T_{\overline{\Pi}(a_x)}(A_x/B_x).$$

We will see now that $T_{\overline{\Pi}(a_x)}\widetilde{\Pi}_x(W_x) = \widetilde{X}$. In fact,

$$T_{\overline{\Pi}(a_x)}\widetilde{\Pi}_x(W_x) = T_{a_x}(\widetilde{\Pi}_x \circ \overline{\Pi}_x)(X - Y) = T_{a_x}\Pi_x(X - Y).$$

On the other hand, from (57) and since $Y \in \mathcal{F}^B_{a_x}$,

$$\widetilde{X} = T_{a_x}\Pi(X) = T_{a_x}\Pi(X - Y).$$

Therefore, $T_{\bar{\Pi}(a_x)}\widetilde{\Pi}_x$ is surjective. Indeed, $T_{\bar{\Pi}(a_x)}\widetilde{\Pi}_x$ is a linear isomorphism since

$$\dim T_{\overline{\Pi}(a_x)}(A_x/B_x) = \dim A_x - \dim B_x$$

and by using Proposition 5.2

$$\begin{aligned} \dim T_{\widetilde{\Pi}(a_x)}\tau_{\widetilde{A}}^{-1}(\pi(x)) &= \dim \widetilde{A} - \dim \widetilde{M} \\ &= \dim A - \operatorname{rank} \mathcal{F}^B - \dim \widetilde{M} \\ &= \dim A_x - \dim B_x. \end{aligned}$$

Thus,

$$\widetilde{\Pi}_x\colon A_x/B_x \to \tau_{\widetilde{A}}^{-1}(\pi(x))$$

is a local diffeomorphism. Therefore (using that $\widetilde{\Pi}_x$ is bijective), $\widetilde{\Pi}_x$ is a global diffeomorphism.

Finally, if $\pi(x) = \pi(x')$, it is clear that

$$\widetilde{\Pi}_{x'}^{-1} \circ \widetilde{\Pi}_x\colon A_x/B_x \to A_{x'}/B_{x'}$$

is a linear isomorphism. ∎

**Proposition 5.5.** *Under the same conditions as in Proposition 5.4, we can define a Lie algebroid structure on the vector bundle*

$$\tau_{\widetilde{A}}\colon \widetilde{A} = A/\mathcal{F}^B \to \widetilde{M} = M/\rho_A(B)$$

*such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\Pi"] \arrow[d, "\tau_A"'] & \widetilde{A} = A/\mathcal{F}^B \arrow[d, "\tau_{\widetilde{A}}"] \\ M \arrow[r, "\pi"] & \widetilde{M} = M/\rho_A(B) \end{tikzcd}$$

*is an epimorphism of Lie algebroids.*

*Proof*: Due to Proposition 4.1, it is enough to prove the following facts

$i)$ If $X, Y \in \Gamma_p^{\Pi}(A)$ then $[X, Y]_A \in \Gamma_p^{\Pi}(A)$ and

$ii)$ If $X \in \Gamma_p^{\Pi}(A)$ and $Y \in \Gamma(\ker \Pi)$ then $[X, Y]_A \in \Gamma(\ker \Pi)$.

Here $\Gamma_p^{\Pi}(A)$ is the space of $\Pi$-projectable sections.

Note firstly that $\ker \Pi = B$ (see Proposition 5.4 ). Then, we will prove that

$$\Gamma_p^{\Pi}(A) = \{X \in \Gamma(A) \mid [X, Y] \in \Gamma(B), \ \ \forall Y \in \Gamma(B)\}.$$

Once we prove that, condition $i)$ above follows by the Jacobi identity and $ii)$ is a direct consequence.

Let $X \in \Gamma_p^{\Pi}(A)$ and $Y \in \Gamma(B)$. We denote by $\psi \colon \mathbb{R} \times M \to M$ the flow of $\rho_A(Y)$ and by $\varphi \colon \mathbb{R} \times A \to A$ the flow of $Y^c \in \mathfrak{X}(A)$. Using that $Y^c$ is $\tau_A$-projectable over $\rho_A(Y)$, we deduce that the following diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_t"] \arrow[d, "\tau_A"'] & A \arrow[d, "\tau_A"] \\ M \arrow[r, "\psi_t"] & M \end{tikzcd}$$

is commutative and that the couple $(\varphi_t, \psi_t)$ is a Lie algebroid morphism (see [15]). Note that

$$\Pi \circ \varphi_t(a_x) = \Pi \circ \varphi_{a_x}(t) = \Pi(a_x) \quad \text{ and } \quad \pi \circ \psi_t(x) = \pi \circ \psi_x(t) = \pi(x). \tag{58}$$

On the other hand, $X$ is $\Pi$-projectable, thus there exists $\widetilde{X} \in \Gamma(\widetilde{A})$ such that

$$\widetilde{X} \circ \pi = \Pi \circ X.$$

Therefore, by using (58)

$$\Pi(X(\psi_t(x)) - \varphi_t(X(x))) = \widetilde{X}(\pi(\psi_t(x))) - \Pi(X(x)) = 0.$$

In consequence, there exists $Z_t \in \Gamma(B)$ such that

$$X(\psi_t(x)) - \varphi_t(X(x)) = Z_t(\psi_t(x)),$$

i.e.,

$$(X - Z_t)(\psi_t(x)) = \varphi_t(X(x)).$$

Thus, if $\varphi_t^*\colon A^* \to A^*$ is the dual map of $\varphi_t\colon A \to A$, it follows that

$$\widehat{X} - \widehat{Z}_t = \widehat{X} \circ \varphi_t^*$$

By derivation and using (54) we obtain that

$$Y^{*c}(\widehat{X}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\widehat{X} \circ \varphi_t^*) = -\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widehat{Z}_t.$$

We denote by $Z$ the section of $B$ characterized by $\widehat{Z} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widehat{Z}_t$. By using (53) we deduce

$$\widehat{[X,Y]} = \widehat{Z}$$

so that $[X, Y] \in \Gamma(B)$.

Conversely, let $X \in \Gamma(A)$ such that for all $Y \in \Gamma(B)$,

$$[X, Y] \in \Gamma(B).$$

We will see that $X \in \Gamma_p^\Pi(A)$. In order to prove it, we introduce the map

$$\widetilde{X}\colon \widetilde{M} = M/\rho_A(B) \to \widetilde{A} = A/\mathcal{F}^B$$

given by $\widetilde{X}(\pi(x)) = \Pi(X(x))$, which is well defined.

In fact, suppose that $x, x' \in M$ with $\pi(x) = \pi(x')$. Then there exists a map $\sigma\colon [0,1] \to M$ continuous, piecewise differentiable, tangent to $\rho_A(B)$ such that $\sigma(0) = x$ and $\sigma(1) = x'$. So, in each piece there exists $Y \in \Gamma(B)$ such that $\sigma$ is the integral curve of $\rho_A(Y)$. Assume, without the loss of generality, that $\sigma$ is smooth and denote by $\psi_t : \mathbb{R} \times M \to M$ the flow of the vector field $\rho_A(Y)$. We have that there exists $t_0 \in \mathbb{R}$ such that

$$\psi_{t_0}(x) = x'.$$

Now, let $\varphi : \mathbb{R} \times A \to A$ be the flow of the vector field $Y^c$. Then, for each $t \in \mathbb{R}$, the following diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_t"] \arrow[d, "\tau_A"'] & A \arrow[d, "\tau_A"] \\ M \arrow[r, "\psi_t"] & M \end{tikzcd}$$

is commutative.

On the other hand, using that $Y \in \Gamma(B)$, we deduce that there exists $Z \in \Gamma(B)$ such that

$$[X, Y] = Z.$$

This implies the corresponding relation between the linear maps

$$(Y^*)^c(\widehat{X}) = \widehat{[Y, X]} = -\widehat{Z}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\widehat{X} \circ \varphi_t^*) = -\widehat{Z}.$$

So, for each $t_1$

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=t_1}(\widehat{X} \circ \varphi_t^*) = \widehat{Z}_{t_1}, \tag{59}$$

where we have denoted by $Z_{t_1}$ the section of the vector bundle $\tau_B : B \to M$ which is characterized by $\widehat{Z}_{t_1} = -\widehat{Z} \circ \varphi_{t_1}^*$. Since $\widehat{X} \circ \varphi_0^* = \widehat{X}$, by integrating (59) we have

$$\widehat{X} \circ \varphi_{t_1}^* = \widehat{X} + \widehat{W}_{t_1}$$

for each $t_1 \in \mathbb{R}$ with $W_{t_1} \in \Gamma(B)$. Hence, we get the relation

$$\varphi_t \circ X - X \circ \psi_t = W_t \circ \psi_t$$

By applying $\Pi$, we get

$$\Pi \circ \varphi_t \circ X = \Pi \circ X \circ \psi_t$$

Now, since the vector field $Y^c$ on $A$ is tangent to the foliation $\mathcal{F}^B$, it follows that $\Pi \circ \varphi_t = \Pi$. Therefore,

$$\Pi \circ X = \Pi \circ X \circ \psi_t$$

which implies that

$$\Pi \circ X(x) = \Pi \circ X \circ \psi_{t_0}(x) = \Pi \circ X(x').$$

In conclusion, $\widetilde{X}$ is well defined and $X$ is $\Pi$-projectable. ■

# 6. The Reduced nondegenerate symplectic-Nijenhuis Lie algebroid

First of all, we will prove a result which will be useful in the sequel

**Proposition 6.1.** *Let $(A, P, N)$ be a Poisson-Nijenhuis Lie algebroid. If $l$ is a positive integer then the couple $(P, N^l)$ is a Poisson-Nijenhuis structure on the Lie algebroid $A$.*

*Proof*: It is well-known that $N^l$ is a Nijenhuis operator (see, for instance, note 5 in [9]).

On the other hand, it is clear that

$$P^\sharp \circ N^{*l} = N^l \circ P^\sharp. \tag{60}$$

In addition, a long straightforward computation, using (60), proves that

$$C(P, N^l)(\beta, \beta') = 0, \text{ for } \beta, \beta' \in \Gamma(A^*)$$

if and only if

$$(\mathcal{L}^A_{P^\sharp(\beta)} N^l)(X) = P^\sharp(\mathcal{L}^A_X N^{*l}\beta - \mathcal{L}^A_{N^l X}\beta), \quad \text{for } \beta \in \Gamma(A^*) \text{ and } X \in \Gamma(A). \tag{61}$$

Thus, we must prove (61).

We will proceed by induction on $l$. Note that

$$(\mathcal{L}^A_{P^\sharp(\beta)} N^l)(X) = (\mathcal{L}^A_{P^\sharp(\beta)} N^{l-1})(NX) + N^{l-1}[P^\sharp(\beta), NX]_A - N^l[P^\sharp(\beta), X]_A$$

Therefore,

$$\begin{aligned}(\mathcal{L}^A_{P^\sharp(\beta)} N^l)(X) \;&=\; P^\sharp(\mathcal{L}^A_{NX} N^{*l-1}\beta) - P^\sharp(\mathcal{L}^A_{N^l X}\beta) + N^{l-1}[P^\sharp(\beta), NX]_A \\ &\quad -N^l[P^\sharp\beta, X]_A\end{aligned}$$

Now, since

$$P^\sharp(\mathcal{L}^A_{NX}\gamma) = P^\sharp(\mathcal{L}^A_X N^*\gamma) - (\mathcal{L}^A_{P^\sharp\gamma} N)(X), \quad \text{for } \gamma \in \Gamma(A^*),$$

we deduce that

$$\begin{aligned}(\mathcal{L}^A_{P^\sharp(\beta)} N^l)(X) \;&=\; P^\sharp(\mathcal{L}^A_X N^{*l}\beta - \mathcal{L}^A_{N^l X}\beta) - (\mathcal{L}^A_{P^\sharp(N^{*l-1}\beta)}, N)(X) \\ &\quad +N^{l-1}[P^\sharp\beta, NX]_A - N^l[P^\sharp\beta, X]_A\end{aligned}$$

which implies that

$$\begin{aligned}(\mathcal{L}^A_{P^\sharp(\beta)}N^l)(X) &= P^\sharp(\mathcal{L}^A_X N^{*l}\beta - \mathcal{L}^A_{N^lX}\beta) - [P^\sharp(N^{*l-1}\beta), NX]_A \\ &\quad + N[P^\sharp(N^{*l-1}\beta), X]_A + N^{l-1}[P^\sharp(\beta), NX]_A \\ &\quad - N^l[P^\sharp\beta, X]_A \\ &= P^\sharp(\mathcal{L}^A_X N^{*l}\beta - \mathcal{L}^A_{N^lX}\beta) - [N^{l-1}(P^\sharp\beta), NX]_A \\ &\quad + N[N^{l-1}(P^\sharp\beta), X]_A + N^{l-1}[P^\sharp\beta, NX]_A - N^l[P^\sharp\beta, X]_A\end{aligned}$$

On the other hand, using that $0 = \mathcal{T}_N(N^{l-r}(P^\sharp\beta), X)$ for $2 \leq r \leq l$, it follows that

$$-[N^{l-1}(P^\sharp\beta), NX]_A + N[N^{l-1}(P^\sharp\beta), X]_A + N^{l-1}[P^\sharp\beta, NX]_A - N^l[P^\sharp\beta, X]_A = 0.$$

This ends the proof of the result. ■

Let $(A, P, N)$ be a Poisson-Nijenhuis Lie algebroid. Consider now for any fixed $x \in M$ the endomorphism $N_x \colon A_x \to A_x$. Recall [6, 9] that there exists a smallest positive integer $k$ such that the sequences of nested subspaces

$$\operatorname{Im} N_x \supseteq \operatorname{Im} N_x^2 \supseteq \dots$$

and

$$\ker N_x \subseteq \ker N_x^2 \subseteq \dots$$

both stabilize at rank $k$. That is,

$$\operatorname{Im} N_x^k = \operatorname{Im} N_x^{k+1} = \dots,$$

and

$$\ker N_x^k = \ker N_x^{k+1} = \dots.$$

The integer $k$ is called the *Riesz index* of $N$ at $x$.

**Lemma 6.2.** *If the Riesz index of $N$ at $x$ is $k$ then*

$$A_x = ImN_x^k \oplus KerN_x^k$$

*Proof*: It is clear that

$$\dim(ImN_x^k) + \dim(\ker N_x^k) = \dim A_x$$

■

Next, we will prove the following result

**Proposition 6.3.** *Let $(A, P, N)$ be a Poisson-Nijenhuis Lie algebroid with constant Riesz index $k$ and such that the dimension of the subspace $\ker N_x^k$ (respectively, $\operatorname{Im} N_x^k$) is constant, for all $x \in M$. Then:*

*i) The dimension of the subspace* $\operatorname{Im} N_x^k$ *(respectively,* $\ker N_x^k$*) is constant, for all* $x \in M$.

*ii) The vector subbundles* $\ker N^k$ *and* $\operatorname{Im} N^k$ *are Lie subalgebroids of* $A$.

*Proof*: *i)* it follows from Lemma 6.2.

Since $N^k$ is a Nijenhuis operator, we have that

$$[N^k X, N^k Y]_A = N^k [X, Y]_{N^k}, \qquad \text{for any } X, Y \in \Gamma(A), \tag{62}$$

where $[\cdot,\cdot]_{N^k}$ is the bracket defined as in (16). Thus, $\operatorname{Im} N^k$ is a Lie subalgebroid of $A$.

Now, suppose that $X, Y \in \Gamma(A)$ are sections of $A$ satisfying $N^k X = N^k Y = 0$. Then, using (16) and (62), we deduce that

$$0 = [N^k X, N^k Y]_A = N^k [X, Y]_{N^k} = -N^{k+1}[X, Y]_A.$$

Hence, $[X, Y]_A \in \Gamma(\ker N^{k+1}) = \Gamma(\ker N^k)$. This implies that $\ker N^k$ is a Lie subalgebroid of $A$.

■

Let $(A, P, N)$ be a Poisson-Nijenhuis Lie algebroid with constant Riesz index $k$. Suppose that the dimension of the subspace $\ker N_x^k$ is constant, for all $x \in M$. Then, we may consider the Lie subalgebroid $\ker N^k$ of $A$ and the corresponding generalized foliations $\rho_A(\ker N^k)$ on $M$ and $\mathcal{F}^{\ker N^k}$ on $A$.

As in Section 5, we will assume that these foliations are regular and that the condition $\mathcal{F}^{\ker N^k}$ holds, that is, if $a_x, a'_x \in A_x$ we have that

$$a'_x - a_x \in \ker N_x^k \Leftrightarrow a'_x \text{ and } a_x \text{ belong to the same leaf of the foliation } \mathcal{F}^{\ker N^k}.$$

Under these conditions, the space $\widetilde{A} = A/\mathcal{F}^{\ker N^k}$ of the leaves of the foliation $\mathcal{F}^{\ker N^k}$ is a Lie algebroid over the quotient manifold $\widetilde{M} = M/\rho_A(\ker N^k)$ and the canonical projections $\Pi : A \to \widetilde{A} = A/\mathcal{F}^{\ker N^k}$ and $\pi : M \to \widetilde{M} = M/\rho_A(\ker N^k)$ define a Lie algebroid epimorphism

$$\begin{array}{ccc} A & \xrightarrow{\Pi} & \widetilde{A} = A/\mathcal{F}^{\ker N^k} \\ \Big\downarrow{\scriptstyle \tau_A} & & \Big\downarrow{\scriptstyle \tau_{\widetilde{A}}} \\ M & \xrightarrow{\pi} & \widetilde{M} = M/\rho_A(\ker N^k) \end{array}$$

Note that $\ker \Pi = \ker N^k$ and thus,

$$V\pi = \rho_A(\ker N^k) = \rho_A(\ker \Pi).$$

Next, we will prove that $P$ and $N$ are $\Pi$-projectable. Indeed, we have that

$$N(\Gamma(\ker N^k)) \subseteq \Gamma(\ker N^k).$$

Moreover, if $\xi \in \Gamma(\ker N^k)$ one may see that $\mathcal{L}^A_\xi N(\Gamma_p(A)) \subseteq \Gamma(\ker N^k)$. In order to prove this relation, we recall that

$$\Gamma_p(A) = \{X \in \Gamma(A) \mid [X,\xi]_A \in \Gamma(\ker N^k) \quad \forall \xi \in \Gamma(\ker N^k)\}.$$

Now, if $X \in \Gamma_p(A)$ and $\xi \in \Gamma(\ker N^k)$, we get

$$N^k(\mathcal{L}^A_\xi N(X)) = N^k([\xi, NX]_A - N[\xi, X]_A) = N^k[\xi, NX]_A.$$

By using the fact that $N$ has zero torsion, it follows that

$$0 = N^{k-1}(\mathcal{T}_N(N^{k-1}\xi, X)) = -N^k[N^{k-1}\xi, NX]_A \tag{63}$$

Thus, from (63), we deduce that

$$0 = N^k(\mathcal{T}_N(N^{k-2}\xi, X)) = -N^{k+1}[N^{k-2}\xi, NX]_A$$

and, since $\ker N^{k+1} = \ker N^k$, we obtain that

$$N^k[N^{k-2}\xi, NX]_A = 0$$

Proceeding in a similar way, we may prove that

$$N^k[N^{k-3}\xi, NX]_A = N^k[N^{k-4}\xi, NX] = \cdots = N^k[\xi, NX] = 0.$$

Therefore, $\mathcal{L}^A_\xi N(X) \in \Gamma(\ker N^k)$ and $N$ is $\Pi$-projectable (see Proposition 4.5).

To see that $P$ is $\Pi$-projectable, we have to prove that (see Proposition 4.3)

$$[\xi, P]^\sharp_A(\Gamma_p(A^*)) \subseteq \Gamma(\ker N^k) \qquad \forall \xi \in \Gamma(\ker N^k). \tag{64}$$

From Proposition 4.2,

$$\Gamma_p(A^*) = \{\alpha \in \Gamma(A^*) \mid \mathcal{L}^A_\xi \alpha = 0, \quad \alpha(\xi) = 0, \quad \forall \xi \in \Gamma(\ker N^k)\}.$$

If $\alpha \in \Gamma_p(A^*)$ then

$$\begin{aligned} N^k(([\xi,P]_A)^\sharp(\alpha)) &= N^k(i_\alpha[\xi,P]_A) = N^k([\xi, i_\alpha P]_A - i_{\mathcal{L}^A_\xi \alpha} P) \\ &= N^k[\xi, P^\sharp\alpha]_A = -[P^\sharp\alpha, N^k\xi]_A + (\mathcal{L}^A_{P^\sharp\alpha} N^k)(\xi) = (\mathcal{L}^A_{P^\sharp\alpha} N^k)(\xi). \end{aligned}$$

Hence, using Proposition 6.1, we deduce that

$$N^k([\xi, P]^\sharp_A(\alpha)) = P^\sharp(\mathcal{L}^A_\xi N^{*k}\alpha).$$

On the other hand, since $\alpha \in \Gamma_p(A^*)$, we get

$$\begin{aligned}\mathcal{L}^A_\xi(N^{*k}\alpha)(X) &= \rho_A(\xi)(\alpha(N^kX)) - \alpha(N^k[\xi, X]_A)\\ &= \mathcal{L}^A_\xi\alpha(N^kX) + \alpha([\xi, N^kX]_A - N^k[\xi, X]_A)\\ &= \alpha([\xi, N^kX]_A - N^k[\xi, X]_A).\end{aligned}$$

Moreover, since the torsion of $N^k$ is zero, we have that

$$0 = \mathcal{T}_{N^k}(\xi, X) = -N^k[\xi, N^kX]_A + N^{2k}[\xi, X]_A,$$

that is,

$$[\xi, N^kX]_A - N^k[\xi, X]_A \in \Gamma(\ker N^k).$$

This implies that

$$\alpha([\xi, N^kX]_A - N^k[\xi, X]_A) = 0$$

Hence,

$$N^k([\xi, P]^\sharp_A(\alpha)) = P^\sharp(\mathcal{L}^A_\xi N^{*k}\alpha) = 0$$

and (64) holds.

Therefore, using Theorem 4.6, we have the following result.

**Theorem 6.4.** *Let $(A, [\cdot,\cdot]_A, \rho_A, P, N)$ be a Poisson-Nijenhuis Lie algebroid such that*

*i) $N$ has constant Riesz index $k$;*
*ii) The dimension of the subspace $\ker N^k_x$ is constant, for all $x \in M$ (thus, $B = \ker N^k$ is a vector subbundle of $A$) and*
*iii) $\rho_A(B)$ and $\mathcal{F}^B$ are regular foliations and the condition $\mathcal{F}^B$ is satisfied for $B = \ker N^k$.*

*Then, we may induce a Poisson-Nijenhuis Lie algebroid structure $([\cdot,\cdot]_{\widetilde{A}}, \rho_{\widetilde{A}}, \widetilde{P}, \widetilde{N})$ on $\widetilde{A} = A/\mathcal{F}^B$ such that $\Pi : A \to \widetilde{A} = A/\mathcal{F}^B$ is a Poisson-Nijenhuis Lie algebroid epimorphism over $\pi : M \to \widetilde{M} = M/\rho_A(B)$.*

In the particular case of symplectic-Nijenhuis Lie algebroids, we may prove the following result

**Theorem 6.5.** *Let $(A, [\cdot,\cdot]_A, \rho_A, \Omega, N)$ be a symplectic-Nijenhuis Lie algebroid on the manifold $M$ such that*

*i) $N$ has constant Riesz index $k$;*
*ii) The dimension of the subspace $\ker N^k_x$ is constant, for all $x \in M$ (thus, $B = \ker N^k$ is a vector subbundle of $A$) and*

*iii)* $\rho_A(B)$ *and* $\mathcal{F}^B$ *are regular foliations and the condition* $\mathcal{F}^B$ *is satisfied for* $B = \ker N^k$*.*

*Then, we may induce a symplectic-Nijenhuis Lie algebroid structure on* $\widetilde{A}$ *with nondegenerate Nijenhuis tensor, such that the couple* $\Pi : A \to \widetilde{A} = A/\mathcal{F}^B$ *and* $\pi : M \to \widetilde{M} = M/\rho_A(B)$ *is a Poisson-Nijenhuis Lie algebroid epimorphism.*

*Proof*: Denote by $(\widetilde{P}, \widetilde{N})$ the Poisson-Nijenhuis structure which is defined in the previous theorem. It remains to prove that $\widetilde{P}$ and $\widetilde{N}$ are nondegenerate.

Firstly, we show that $\widetilde{N}$ is nondegenerate, i.e. that $\widetilde{N}_{\pi(x)} \colon \tau_{\widetilde{A}}^{-1}(\pi(x)) \to \tau_{\widetilde{A}}^{-1}(\pi(x))$ is an isomorphism, for all $x \in M$. Consider the following diagram

$$\begin{tikzcd}
A_x \arrow[r, "N_x"] \arrow[d, "\overline{\Pi}_x"'] \arrow[dd, bend right=60, "\Pi_x"'] & A_x \arrow[d, "\overline{\Pi}_x"] \arrow[dd, bend left=60, "\Pi_x"] \\
A_x/\ker N_x^k \arrow[r, "\overline{N}_x"] \arrow[d, "\widetilde{\Pi}_x"'] & A_x/\ker N_x^k \arrow[d, "\widetilde{\Pi}_x"] \\
\tau_{\widetilde{A}}^{-1}(\pi(x)) \arrow[r, "\widetilde{N}_{\pi(x)}"] & \tau_{\widetilde{A}}^{-1}(\pi(x))
\end{tikzcd}$$

where $\overline{\Pi}_x$ and $\widetilde{\Pi}_x$ are defined as in Section 5. Assume that $\widetilde{N}_{\pi(x)}(\Pi_x(a_x)) = 0$. Then,

$$\widetilde{\Pi}_x \overline{N}_x(\overline{\Pi}_x(a_x)) = 0.$$

Since $\widetilde{\Pi}_x \colon A_x/\ker N_x^k \to \tau_{\widetilde{A}}^{-1}(\pi(x))$ is an isomorphism, we deduce that

$$\overline{N}_x(\overline{\Pi}_x(a_x)) = 0$$

or, equivalently $\overline{\Pi}_x N_x(a_x) = 0$, i.e.

$$a_x \in \ker N_x^{k+1} = \ker N_x^k.$$

It follows that

$$\Pi_x(a_x) = \widetilde{\Pi}_x \overline{\Pi}_x(a_x) = 0.$$

In consequence, $\widetilde{N}_{\pi(x)}$ is injective and thus, it is bijective.

Now we show that $\widetilde{P}$ is nondegenerate. Denote by $P$ the Poisson bisection associated with $\Omega$. Let $\widetilde{\alpha}_{\pi(x)} \in \widetilde{A}^*_{\pi(x)}$ be such that

$$0 = \widetilde{P}^\sharp_{\pi(x)}(\widetilde{\alpha}_{\pi(x)}) = \Pi_x P^\sharp_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}).$$

Using that $\widetilde{\Pi}_x \colon A_x/\ker N_x \to \tau^{-1}_{\widetilde{A}}(\pi(x))$ is an isomorphism, we deduce that

$$\overline{\Pi}_x P^\sharp_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}) = 0,$$

i.e. $P^\sharp_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}) \in \ker N^k_x$. It follows that

$$0 = N^k_x P^\sharp_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}) = P^\sharp_x N^{*k}_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}).$$

Since $P_x$ is nondegenerate,

$$N^{*k}_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}) = 0.$$

Note that $N^{*k}_x(\Pi^*_x \widetilde{\alpha}_{\pi(x)}) = \widetilde{N}^{*k}_{\pi(x)}(\widetilde{\alpha}_{\pi(x)})$ and that $\widetilde{N}$ is nondegenerate. Hence, we deduce that $\widetilde{\alpha}_{\pi(x)} = 0$. ■

Under the hypotheses of the previous theorem, we will denote by $\widetilde{\Omega}$ the symplectic section defined by $\widetilde{\Omega}^\flat = -(\widetilde{P^\sharp})^{-1}$.

We summarize the two steps of the reduction procedure given in Theorems 3.3 and 6.5 in the following theorem.

**Theorem 6.6.** *Let $(A, [\cdot,\cdot]_A, \rho_A, P, N)$ be a Poisson-Nijenhuis Lie algebroid such that*

*i) The Poisson structure $P$ has constant rank in the leaves of the foliation $D = \rho_A(P^\sharp(A^*))$.*

*If $L$ is a leaf of $D$, then, we have a symplectic-Nijenhuis Lie algebroid structure $([\cdot,\cdot]_{A_L}, \rho_{A_L}, \Omega_L, N_L)$ on $A_L = P^\sharp(A^*)_{|L} \to L$.*

*Assume, moreover, that*

*ii) The induced Nijenhuis tensor $N_L : A_L \to A_L$ has constant Riesz index $k$;*

*iii) The dimension of the subspace $B_x = \ker N^k_x$ is constant, for all $x \in L$ (thus, $B = \ker N^k_L$ is a vector subbundle of $A$);*

*iii) The foliations $\rho_A(B)$ and $\mathcal{F}^B$ are regular, where*

$$(\mathcal{F}^B)_a = \{X^c(a) + Y^v(a)/X, Y \in \Gamma(B)\}, \text{ for } a \in A_L$$

*iv) (condition $\mathcal{F}^B$) For all $x \in L$, $a_x - a'_x \in B_x$ if $a_x$ and $a'_x$ belong to the same leaf of the foliation $\mathcal{F}^B$.*

*Then, we obtain a symplectic-Nijenhuis Lie algebroid structure* $([\cdot,\cdot]_{\widetilde{A_L}}, \rho_{\widetilde{A_L}}, \widetilde{\Omega_L}, \widetilde{N_L})$ *on the vector bundle* $\widetilde{A_L} = A_L/\mathcal{F}^B \to \widetilde{L} = L/\rho_{A_L}(B)$ *with* $\widetilde{N_L}$ *nondegenerate.*

## 7. An explicit example of reduction of a Poisson-Nijenhuis Lie algebroid

**7.1. A $G$-invariant Poisson-Nijenhuis structure on the cotangent bundle of a semidirect product of Lie groups.** Let $H_1$ and $H_2$ be two Lie groups with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Assume that there is an action $\phi\colon H_1 \times H_2 \to H_2$ of $H_1$ on $H_2$ by Lie group isomorphisms and consider the semidirect product $G = H_1 \times_\phi H_2$ whose operation is defined by

$$(h_1, h_2) \cdot (h_1', h_2') = (h_1 \cdot h_1', h_2 \cdot \phi(h_1, h_2')).$$

Note that $H_2$ is a normal subgroup of $G$. The Lie algebra associated to $G = H_1 \times_\phi H_2$ is $\mathfrak{g} = \mathfrak{h}_1 \times_\Phi h_2$ with the bracket

$$[(\xi_1, \xi_2), (\eta_1, \eta_2)]_{\mathfrak{g}} = \left([\xi_1, \eta_1]_{\mathfrak{h}_1}, \Phi(\xi_1, \eta_2) - \Phi(\eta_1, \xi_2) + [\xi_2, \eta_2]_{\mathfrak{h}_2}\right),$$

for all $\xi_1, \eta_1 \in \mathfrak{h}_1$, $\xi_2, \eta_2 \in \mathfrak{h}_2$, where $\Phi = T_{(e_1,e_2)}\phi\colon \mathfrak{h}_1 \times \mathfrak{h}_2 \to \mathfrak{h}_2$ is the action induced by $\phi\colon H_1 \times H_2 \to H_2$. We remark that $\mathfrak{h}_1$ is a Lie subalgebra and $\mathfrak{h}_2$ is an ideal of $\mathfrak{g}$. Consider now $M = T^*G$. It may be identified with $G \times \mathfrak{g}^*$ as follows:

$$M = T^*G \longrightarrow G \times \mathfrak{g}^*, \qquad \alpha_g \in T_g^*G \longmapsto (g, T_e^* l_g(\alpha_g)) \in G \times \mathfrak{g}^*,$$

where $l_g\colon G \to G$ denotes the left translation by $g \in G$. Under the identification $T^*G \cong G \times \mathfrak{g}^*$, the canonical symplectic structure of $T^*G$

$$\Omega\colon G \times \mathfrak{g}^* \longrightarrow (\mathfrak{g} \times \mathfrak{g}^*)^* \times (\mathfrak{g} \times \mathfrak{g}^*)^*$$

is defined by

$$\Omega_{(g,\mu)}\left((\xi, \pi), (\xi', \pi')\right) = \pi'(\xi) - \pi(\xi') + \mu([\xi, \xi']_{\mathfrak{g}}),$$

for all $\xi, \xi' \in \mathfrak{g}$, $\pi, \pi' \in \mathfrak{g}^*$. Note that $\Omega$ is $G$-invariant.

We define now on $T^*G$ a singular Poisson structure compatible with $\Omega$. Let

$$\mathcal{P}_{\mathfrak{g}}\colon \mathfrak{g} = \mathfrak{h}_1 \times_\Phi \mathfrak{h}_2 \longrightarrow \mathfrak{h}_1$$

be the canonical projection on the first factor. Then we have that $\mathfrak{h}_1 \times \mathcal{P}_{\mathfrak{g}}^*(\mathfrak{h}_1^*) \hookrightarrow \mathfrak{g} \times \mathfrak{g}^*$ is a symplectic subspace of $T_{(e,\mu)}(G \times \mathfrak{g}^*) \cong \mathfrak{g} \times \mathfrak{g}^*$. Indeed, let $\xi \in \mathfrak{h}_1$, $\alpha \in \mathfrak{h}_1^*$ be such that

$$\Omega_{(e,\mu)}\left((\xi, \mathcal{P}_{\mathfrak{g}}^*(\alpha)), (\xi', \mathcal{P}_{\mathfrak{g}}^*(\beta))\right) = \mathcal{P}_{\mathfrak{g}}^*(\beta)(\xi) - \mathcal{P}_{\mathfrak{g}}^*(\alpha)(\xi') + \mu([\xi, \xi']_{\mathfrak{g}}) = 0,$$

for all $\xi' \in \mathfrak{h}_1$, $\beta \in \mathfrak{h}_1^*$. Hence, we have

$$\Omega_{(e,\mu)}\left((\xi, \mathcal{P}_{\mathfrak{g}}^*(\alpha)), (0, \mathcal{P}_{\mathfrak{g}}^*(\beta))\right) = \beta(\xi) = 0 \Rightarrow \xi = 0$$

and

$$\Omega_{(e,\mu)}\left((0, \mathcal{P}_{\mathfrak{g}}^*(\alpha)), (\xi', \mathcal{P}_{\mathfrak{g}}^*(\beta))\right) = -\alpha(\xi') = 0 \Rightarrow \alpha = 0.$$

We consider now the symplectic subbundle

$$\begin{array}{rcl}\mathcal{F}_{\mathfrak{h}_1}^B \colon (g,\mu) \in G \times \mathfrak{g}^* \longmapsto (T_e l_g)(\mathfrak{h}_1) \times \mathcal{P}_{\mathfrak{g}}^*(\mathfrak{h}_1^*) & = & T_{(e,\mu)}(l_g, id)(\mathfrak{h}_1 \times \mathcal{P}_{\mathfrak{g}}^*(\mathfrak{h}_1^*)) \\ & & \subset T_{(g,\mu)}(G \times \mathfrak{g}^*).\end{array}$$

We show that it is integrable. A basis of sections of this subbundle is

$$\{(\overleftarrow{\xi}, C_\alpha) \mid \xi \in \mathfrak{h}_1, \, \alpha \in P^*(\mathfrak{h}_1^*)\},$$

where $\overleftarrow{\xi}$ is the left invariant vector field associated to $\xi$ and $C_\alpha$ is the vector field constant at $\alpha$. The bracket of these basic elements is given by

$$[(\overleftarrow{\xi}, C_\alpha), (\overleftarrow{\xi}', C_\beta)] = ([\overleftarrow{\xi}, \overleftarrow{\xi}'], 0) = (\overleftarrow{[\xi, \xi']_{\mathfrak{g}}}, 0).$$

Since $\mathcal{F}_{\mathfrak{h}_1}^B$ is symplectic, we have the decomposition

$$T(T^*G) \cong T(G \times \mathfrak{g}^*) = \mathcal{F}_{\mathfrak{h}_1}^B \oplus (\mathcal{F}_{\mathfrak{h}_1}^B)^\perp,$$

where $(\mathcal{F}_{\mathfrak{h}_1}^B)^\perp$ is the orthogonal to $\mathcal{F}_{\mathfrak{h}_1}^B$ with respect to the symplectic form $\Omega$.

Now, we define a Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{h}_1}$ in $T^*G$ as follows. For $f, g \in C^\infty(T^*G)$,

$$\{f, g\}_{\mathfrak{h}_1} = \Omega(\mathcal{P}(\mathcal{H}_f^\Omega), \mathcal{P}(\mathcal{H}_g^\Omega)) = \mathcal{P}(\mathcal{H}_g^\Omega(f)) = \{f, g\}_\Omega - \mathcal{Q}(\mathcal{H}_g^\Omega)(f),$$

where $\mathcal{P} \colon T(T^*G) \to \mathcal{F}_{\mathfrak{h}_1}^B$ and $\mathcal{Q} \colon T(T^*G) \to (\mathcal{F}_{\mathfrak{h}_1}^B)^\perp$ are the symplectic projectors and $\{\cdot,\cdot\}_\Omega$ is the Poisson bracket on $T^*G$ associated with the canonical symplectic structure of $T^*G$. Here $\mathcal{H}_s^\Omega$ denotes the hamiltonian vector field of $s \in C^\infty(T^*G)$ with respect to the canonical symplectic structure of $T^*G$.

The symplectic foliation of $\{\cdot,\cdot\}_{\mathfrak{h}_1}$ is $\mathcal{F}_{\mathfrak{h}_1}^B$ since

$$\mathcal{H}_g^{\{\cdot,\cdot\}_{\mathfrak{h}_1}} = \mathcal{P}(\mathcal{H}_g^\Omega),$$

where $\mathcal{H}_g^{\{\cdot,\cdot\}_{\mathfrak{h}_1}}$ is the hamiltonian vector field of $g$ with respect to the Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{h}_1}$.

Keep into account that if $\theta \in T^*G$ and $L_\theta$ is the leaf of $\mathcal{F}_{\mathfrak{h}_1}^B$ passing through $\theta$, then we have

$$\mathcal{H}_{f \circ \iota_\theta}^{\iota_\theta^*\Omega} = \mathcal{P}(\mathcal{H}_f^\Omega)_{|L_\theta},$$

where $\iota_\theta\colon L_\theta \hookrightarrow T^*G$ is the canonical inclusion. Note that

$$(\iota(\mathcal{P}(\mathcal{H}^{\Omega}_{f})_{|L_\theta})\iota_\theta^*\Omega)(\mathcal{P}X) = \Omega_{|L_\theta}(\mathcal{H}^{\Omega}_{f|L_\theta}, \mathcal{P}X) = d(f\circ\iota_\theta)(\mathcal{P}X).$$

Thus,

$$\{f\circ\iota_\theta, g\circ\iota_\theta\}_{\iota_\theta^*\Omega} = (\{f,g\}_{\mathfrak{h}_1})_{|L_\theta} \qquad \text{for all } f,g\in C^\infty(T^*G).$$

Therefore $L_\theta$ is the leaf of the symplectic foliation of $\{\cdot,\cdot\}_{\mathfrak{h}_1}$ through the point $\theta$.

It is clear that the Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{h}_1}$ is $G$-invariant.

We now study the compatibility between $\{\cdot,\cdot\}_{\mathfrak{h}_1}$ and $\{\cdot,\cdot\}_{\Omega}$. We know that

$$\{f,g\}_{\mathfrak{h}_1} = \{f,g\}_\Omega - \mathcal{Q}(\mathcal{H}^\Omega_g)(f).$$

Next, we check that $\{f,g\}_{\mathfrak{h}_2} = \mathcal{Q}(\mathcal{H}^\Omega_g)(f)$ is Poisson.

Since $\mathcal{F}^B_{\mathfrak{h}_1}$ and $\Omega$ are $G$-invariant, then $(\mathcal{F}^B_{\mathfrak{h}_1})^\perp$ is $G$-invariant. Therefore for describing $(\mathcal{F}^B_{\mathfrak{h}_1})^\perp$ is enough to know $(\mathcal{F}^B_{\mathfrak{h}_1})^\perp(e,\mu)$. A direct computation proves that

$$(\mathcal{F}^B_{\mathfrak{h}_1})^\perp(e,\mu) = \{(\xi,\pi)\in\mathfrak{g}\times\mathfrak{g}^* \mid \xi\in\ker\mathcal{P},\ \pi_{|\mathfrak{h}_1} = -\xi_{\mathfrak{g}^*}(\mu)_{|\mathfrak{h}_1}\},$$

where $\xi_{\mathfrak{g}^*}(\mu) = -ad^*_\xi\mu$. Hence

$$\begin{aligned}(\mathcal{F}^B_{\mathfrak{h}_1})^\perp(g,\mu) &= T_{(e,\mu)}(l_g, id)\left((\mathcal{F}^B_{\mathfrak{h}_1})^\perp(e,\mu)\right)\\ &= \{(v_g,\pi)\in T_gG^*\times\mathfrak{g}^* \mid (T_g l_{g^{-1}})(v_g)\in\ker\mathcal{P},\ \pi_{|\mathfrak{h}_1} = -\xi_{\mathfrak{g}^*}(\mu)_{|\mathfrak{h}_1}\}.\end{aligned}$$

The sections of $(\mathcal{F}^B_{\mathfrak{h}_1})^\perp$ are of the form

$$\{(\overleftarrow{\xi}, X) \mid \xi\in\ker\mathcal{P},\ X\in\mathfrak{X}(\mathfrak{g}^*),\ X(\mu)_{|\mathfrak{h}_1} = -\xi_{\mathfrak{g}^*}(\mu)_{|\mathfrak{h}_1}, \forall\mu\in\mathfrak{g}^*\}\subseteq\mathfrak{X}(G)\times\mathfrak{X}(\mathfrak{g}^*).$$

and the brackets of them

$$[(\overleftarrow{\xi}, X), (\overleftarrow{\xi'}, Y)] = ([(\overleftarrow{\xi}, \overleftarrow{\xi'}], [X,Y]) = (\overleftarrow{[\xi,\xi']_{\mathfrak{g}}}, [X,Y]).$$

with $\xi,\xi'\in\ker\mathcal{P}$, $X,Y\in\mathfrak{X}(\mathfrak{g}^*)$ such that $X(\mu)(\hat\eta) = \mu([\xi,\eta])$ and $Y(\mu)(\hat\eta) = \mu([\xi',\eta])$, for all $\mu\in\mathfrak{g}^*, \eta\in\mathfrak{h}_1$. Here $\hat\eta:\mathfrak{g}^*\to\mathbb{R}$ is the linear function induced by $\eta$.

Since $\xi,\xi'\in\mathfrak{h}_2$ and $\mathfrak{h}_2$ is a Lie subalgebra of $\mathfrak{g}$, it follows that $[\xi,\xi']_{\mathfrak{g}}\in\mathfrak{h}_2$. If $\mu\in\mathfrak{g}^*$ and $\eta\in\mathfrak{h}_1$, then

$$[X,Y](\mu)(\hat\eta) = X(\mu)(Y(\hat\eta)) - Y(\mu)(X(\hat\eta)).$$

Moreover, $Y(\hat{v})(\mu') = Y(\mu')(\hat{\eta}) = \mu'([\xi', \eta]_{\mathfrak{g}})$, for all $\mu' \in \mathfrak{g}^*$. Therefore, keeping in account that $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}$ we get

$$X(\mu)(Y(\hat{\eta})) = X(\mu)(\widehat{[\xi', \eta]_{\mathfrak{g}}}) = \mu([\xi, [\xi', \eta]_{\mathfrak{g}}]_{\mathfrak{g}}),$$
$$Y(\mu)(X(\hat{\eta})) = \mu([\xi', [\xi, \eta]_{\mathfrak{g}}]_{\mathfrak{g}}).$$

Hence

$$[X, Y](\mu)(\hat{v}) = \mu([\xi, [\xi', \eta]_{\mathfrak{g}}]_{\mathfrak{g}} + [\xi', [\eta, \xi]_{\mathfrak{g}}]_{\mathfrak{g}}) = -\mu([\eta, [\xi, \xi']_{\mathfrak{g}}]_{\mathfrak{g}}) = \mu([[\xi, \xi']_{\mathfrak{g}}, \eta]_{\mathfrak{g}}).$$

Therefore $(\mathcal{F}^B_{\mathfrak{h}_1})^\perp$ is a symplectic foliation, so that we can consider the Poisson bracket $\{\cdot, \cdot\}_{\mathfrak{h}_2}$ associated to $(\mathcal{F}^B_{h_1})^\perp$, given by

$$\{f, g\}_{\mathfrak{h}_2} = \mathcal{Q}(\mathcal{H}^\Omega_g)(f).$$

Thus $\{\cdot, \cdot\}_\Omega$ and $-\{\cdot, \cdot\}_{\mathfrak{h}_1}$ are compatible, since

$$\{\cdot, \cdot\}_\Omega + (-\{\cdot, \cdot\}_{\mathfrak{h}_1}) = \{\cdot, \cdot\}_{\mathfrak{h}_2}.$$

Consequently, we can consider the Poisson-Nijenhuis manifold $(T^*G, \Omega, N)$, where $N = \Lambda^\sharp_{\mathfrak{h}_1} \circ \Omega^\flat$ and $\Lambda^\sharp_{\mathfrak{h}_1} : T^*(TG) \to T(T^*G)$ is the morphism induced by the Poisson bracket $\{\cdot, \cdot\}_{\mathfrak{h}_1}$. Using that $\{\cdot, \cdot\}_{\mathfrak{h}_1}$ is $G$-invariant, it follows that $N$ is $G$-invariant.

**7.2. The Poisson-Nijenhuis Lie algebroid and its reduction.** We consider the action of $G$ on $T^*G \cong G \times \mathfrak{g}^*$ by left translations, that is

$$\begin{aligned} G \times (G \times \mathfrak{g}^*) &\longrightarrow G \times \mathfrak{g}^* \\ (g', (g, \eta)) &\longmapsto (g' \cdot g, \eta). \end{aligned}$$

and let $\pi \colon T^*G \to T^*G/G$ be the corresponding principal $G$-bundle. Since $\Omega$ and $N$ are $G$-invariant, we can consider the corresponding Atiyah algebroid on

$$\tilde{\pi} \colon (T(T^*G))/G \longrightarrow T^*G/G.$$

We denote by $([\cdot, \cdot], \rho)$ the Lie algebroid structure on $\tilde{\pi} : (T(T^*G))/G \to (T^*G)/G$.

Note that $\Gamma(\widetilde{\pi})$ may be identified with the set $\mathfrak{X}^G(T^*G)$ of $G$-invariant vector fields on $T^*G$ and that if $X \in \mathfrak{X}^G(T^*G)$ then $X$ is $\pi$-projectable. In fact, $\rho(X) = (T\pi)(X)$. Using Proposition 2.6, we obtain a Poisson-Nijenhuis structure on $\tilde{\pi}$ which we denote by $(\tilde{\Lambda}, \tilde{N})$. The foliation defined by the distribution $D = \rho(\tilde{\Lambda}^\sharp((T^*(T^*G))/G))$ has just one leaf which is the whole $(T^*G)/G$, since $\Omega^\sharp((\Omega^1(T^*G))^G) = \mathfrak{X}^G(T^*G)$ and these vector fields generate all the vector fields in $T^*G$. In fact, $\tilde{\Lambda}$ is nondegenerate on $\tilde{\pi} : (T(T^*G))/G \to T^*G/G$.

Next, we compute $\ker N$.

Let $X \in \mathfrak{X}^G(T^*G)$ be such that $N(X) = 0$. Then, we have $\Lambda^\sharp_{\mathfrak{h}_1} \circ \Omega^\flat(X) = 0$ and hence $\Omega^\flat(X) \in \ker \Lambda^\sharp_{\mathfrak{h}_1}$. Now,

$$\alpha \in \ker \Lambda^\sharp_{\mathfrak{h}_1} \Longleftrightarrow \Lambda^\sharp_{\mathfrak{h}_1}(\alpha) = \mathcal{P}(\Omega^\sharp(\alpha)) = 0 \Longleftrightarrow \alpha \in \Omega^\flat(\ker \mathcal{P}).$$

Therefore, $\ker \tilde{N} = (\mathcal{F}^B_{\mathfrak{h}_1})^\perp$. Note that $(\mathcal{F}^B_{\mathfrak{h}_1})^\perp$ is a $G$-invariant foliation and hence it is regular.

Let $X \in \mathfrak{X}^G(T^*G)$ be such that $\tilde{N}^2(X) = 0$. Then, we have $\tilde{N}(X) \in \ker \tilde{N} = \ker \mathcal{P}$. That is, $\Omega^\flat(X) \in (\Lambda^\sharp_{\mathfrak{h}_1})^{-1}(\ker \mathcal{P})$. Now,

$$\alpha \in (\Lambda^\sharp_{\mathfrak{h}_1})^{-1}(\ker \mathcal{P}) \Longleftrightarrow \Lambda^\sharp_{\mathfrak{h}_1}(\alpha) \in \ker \mathcal{P} \Longleftrightarrow \Omega^\sharp(\alpha) \in \ker \mathcal{P},$$

since $\Omega^\sharp = \Lambda^\sharp_{\mathfrak{h}_1} + \Lambda^\sharp_{\mathfrak{h}_2}$ and $\Lambda^\sharp_{\mathfrak{h}_2}(\alpha) \in \ker \mathcal{P}$. Hence $\ker \tilde{N}^2 = (\mathcal{F}^B_{\mathfrak{h}_1})^\perp$. Therefore, the Riesz index is 1.

We study now the foliation $\mathcal{F}^{\ker N}$. The complete and vertical lifts of the sections of $\ker N$ are complete and vertical lifts of $G$-invariant vector fields in $T^*G$. Since $\ker N$ is regular, then $\mathcal{F}^{\ker N}$ is regular.

Then, if $L_\theta$ is the leaf of $\ker N$ passing through $\theta$, we have that the leaf of $\mathcal{F}^{\ker N}$ passing through $v_\theta$ is

$$v_\theta + TL_\theta = v_\theta + (\bigcup_{x \in L_\theta} T_x L_\theta) = v_\theta + (\bigcup_{x \in L_\theta} \ker N(x)).$$

Note that the condition $\mathcal{F}^{\ker N}$ is therefore also satisfied and hence Theorem 6.6 can be applied.

Finally, note that this example can be generalized if we consider a Lie group $G$ with Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$ and $\mathcal{P}_\mathfrak{g} \colon \mathfrak{g} \longrightarrow \mathfrak{h}$ a projector ($\mathcal{P}_{\mathfrak{g}|\mathfrak{h}} = 1_\mathfrak{h}$) such that $\ker \mathcal{P}_\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and $\mathcal{P}$ is linear. Thus, on $T^*G$ we can define two compatible Poisson structures (one of them being the canonical symplectic structure on $T^*G$) and hence we can induce a Poisson-Nijenhuis structure on $T^*G$.

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