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TOPOLOGY OF 3-COSYMPLECTIC MANIFOLDS

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ABSTRACT: We find a quaternionic module structure on the odd cohomology spaces of compact 3-cosymplectic manifolds. This gives rise to some topological obstructions to the existence of such structures, expressed by stronger bounds on the Betti numbers compared to those known for the hyper-Kähler case. Nevertheless, we present a nontrivial example of compact 3-cosymplectic manifold which is not the global product of a hyper-Kähler manifold and a flat 3-torus. We also show that there is an action of the Lie algebra $so(4,1)$ on the cohomology spaces of a compact 3-cosymplectic manifold which is the odd-dimensional counterpart of the result of Verbitsky for hyper-Kähler manifolds.

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1. Introduction

Cosymplectic geometry is considered to be the closest odd-dimensional analogue of Kähler geometry (see e.g. [2, Section 6.5], [9, Section 14.5]). This becomes even more evident when one passes to the setting of 3-structures. Indeed, while both cosymplectic and Sasakian manifolds admit a transversal Kähler structure, only 3-cosymplectic manifolds do admit a transversal hyper-Kähler structure (cf. [6, Theorem 3.8]).

In the fundamental paper [8], Chinea, De León and Marrero studied the topology of cosymplectic manifolds, refining the previous results of Blair and Goldberg ([3]). They proved a monotonicity result for the Betti numbers of a compact cosymplectic manifold M^{2n+1} up to the middle dimension. Next, the differences $b_{2p+1} - b_{2p}$ (with $0 \leq p \leq n$) were shown to be even integers (in particular, b_1 is odd). Moreover, they found an example of a compact cosymplectic manifold which is not the global product of a Kähler manifold and the circle. Later on, other nontrivial examples were provided (cf. [18, 11]). More recently, Li ([17]) gave an alternative proof of the monotonicity

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property of the Betti numbers of cosymplectic manifolds (which he prefers to call co-Kähler) by using topological techniques.

A 3-cosymplectic manifold (see e.g. [5, Section 13.1]) is a smooth manifold endowed with an almost contact metric 3-structure such that each structure is cosymplectic. This class of manifolds is contained in the wider class of 3-quasi Sasakian manifolds. Every 3-cosymplectic manifold is in particular cosymplectic hence all the previously mentioned results still hold. A natural problem is whether the quaternionic-like conditions which relate the structure tensors of 3-cosymplectic manifolds can induce additional rigidity to the underlying topological structure. The aim of this paper is to give an answer to this question. First of all, we find a suitable decomposition of the cohomology spaces of any compact 3-cosymplectic manifold M, as well as a family of isomorphisms relating some of the components. This leads to the following key relation between the Betti numbers of the de Rham cohomology of the manifold and the dimensions b_p^h of the spaces $\Omega^p_{H,000}(M)$ of horizontal harmonic p-forms

$$b_p = b_p^h + 3b_{p-1}^h + 3b_{p-2}^h + b_{p-3}^h. \tag{1.1}$$

Moreover, we prove that the graded vector space $\Omega^*_{H,000}(M)$ of horizontal harmonic forms admits an action of the Lie algebra $so(4,1)$. This is the odd dimensional counterpart of the remarkable result obtained by Verbitsky in [26] for the cohomology ring of a compact hyper-Kähler manifold.

In Theorem 5.2 we show that the spaces $\Omega^p_{H,000}(M)$ also admit an $\mathbb{H}$-module structure for odd p. As a consequence the odd horizontal Betti numbers b^h_{2p+1} are divisible by four. Combining this with (1.1) we get that

$$b_{2p} + b_{2p+1} = 4k$$

for some integer k. We also recover the lower bound of Wakakuwa [27] on the even Betti numbers of compact hyper-Kähler manifolds for the horizontal Betti numbers of compact 3-cosymplectic manifolds

$$b^h_{2p} \geq \binom{p+2}{2} \qquad \text{for } 0 \leq p \leq n.$$

Furthermore, for the Betti numbers of a compact 3-cosymplectic manifold we obtain the following stronger lower bound

$$b_p \geq \binom{p+2}{2} \qquad \text{for } 0 \leq p \leq 2n+1.$$

All results on the Betti numbers could also be derived from the existence of a $so(4,1)$-action on the space $\Omega^*_{H,000}(M)$ by the use of representation theory. However, we chose a more elementary approach in this article.

From the above considerations one can see that there are strong obstructions to the existence of compact 3-cosymplectic manifolds. On the other hand, every compact 3-cosymplectic manifold is a local Riemannian product of a hyper-Kähler factor and an abelian three dimensional Lie group. This probably explains why so far the only known examples of such manifolds in the compact case were global Riemannian products of compact hyper-Kähler manifolds with the flat 3-torus. However, we provide a method for constructing compact 3-cosymplectic manifolds exhibiting at least one example which is not the global product of a compact hyper-Kähler manifold with the flat 3-torus.

2. Preliminaries

An *almost contact manifold* is an odd-dimensional manifold M which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

where $I\colon TM \to TM$ is the identity mapping. From the definition it follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and that the $(1,1)$-tensor field ϕ has constant rank $2n$ (cf. [2]). An almost contact manifold (M, ϕ, ξ, η) is said to be *normal* when the tensor field $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. It is known that any almost contact manifold (M, ϕ, ξ, η) admits a Riemannian metric g such that

$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F) \qquad (2.1)$$

holds for all $E, F \in \Gamma(TM)$. This metric g is called a *compatible metric* and the manifold M together with the structure (ϕ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.1), one has $\eta = g(\cdot, \xi)$ and $g(\phi E, F) = -g(E, \phi F)$. Hence $\Phi(E, F) = g(E, \phi F)$ defines a 2-form, which is called the *fundamental* 2*-form* of M. Almost contact metric manifolds such that both η and Φ are closed are called *almost cosymplectic manifolds* and those for which $d\eta = \Phi$ are called *contact metric manifolds*. Finally, a normal almost cosymplectic manifold is called a *cosymplectic manifold*, and a normal contact metric manifold is said to be a *Sasakian manifold*.

In terms of the covariant derivative of ϕ, the cosymplectic and the Sasakian conditions can be expressed respectively by

$$\nabla\phi = 0$$

and

$$(\nabla_E \phi)\, F = g\,(E,F)\,\xi - \eta\,(F)\,E,$$

for all $E, F \in \Gamma\,(TM)$.

It should be noted that both in Sasakian and in cosymplectic manifolds ξ is a Killing vector field. The Sasakian and the cosymplectic manifolds represents the two extremal cases of the larger class of quasi-Sasakian manifolds (cf. [1]). Recently, a study of geometrical structures of odd dimensional manifolds generalizing cosymplectic and quasi-Sasakian structures from the non-metric point of view has been presented in [14].

An *almost contact* 3*-structure* on a $(4n+3)$-dimensional smooth manifold M is given by three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for every $\alpha, \beta \in \{1, 2, 3\}$,

$$\phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\phi_\gamma - \delta_{\alpha\beta} I, \tag{2.2}$$

$$\phi_\alpha \xi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\xi_\gamma, \quad \eta_\alpha \circ \phi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\eta_\gamma, \tag{2.3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric symbol. This notion was introduced by Kuo ([16]) and, independently, by Udriste ([25]). In [16] Kuo proved that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, there exists a Riemannian metric g compatible with each of the structures and hence we can speak of *almost contact metric* 3*-structure*. It is well known that in any almost 3-contact manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to any compatible metric g and that the structural group of the tangent bundle is reducible to $Sp\,(n) \times \{I_3\}$. Moreover, the tangent bundle of any almost 3-contact metric manifold splits up as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where the $4n$-dimensional subbundle $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ is called the *horizontal distribution* and $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ is called the *vertical* (or *Reeb*) *distribution*. An almost 3-contact manifold M is said to be *normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal. Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost contact metric 3-structure. When each structure is Sasakian M is called a 3*-Sasakian manifold*.

By an *almost* 3-*cosymplectic manifold* we mean an almost 3-contact metric manifold M such that each almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is almost cosymplectic. The almost cosymplectic 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is called *cosymplectic* if it is normal. In this case M is said to be a 3-*cosymplectic manifold*. However it has been proved recently that these two notions are the same:

Theorem 2.1. ([10, Theorem 4.13]) *Every almost* 3*-cosymplectic manifold is* 3*-cosymplectic.*

Just as in the case of a single structure, the 3-Sasakian and the 3-cosymplectic manifolds represents the two extremal cases of the larger class of 3-quasi-Sasakian manifolds (cf. [7]).

In any 3-cosymplectic manifold the forms η_α and Φ_α are harmonic ([12, Lemma 3]). Moreover, we have that ξ_α, η_α, ϕ_α and Φ_α are ∇-parallel. In particular

$$[\xi_\alpha, \xi_\beta] = \nabla_{\xi_\alpha}\xi_\beta - \nabla_{\xi_\beta}\xi_\alpha = 0 \tag{2.4}$$

for all $\alpha, \beta \in \{1, 2, 3\}$, so that $\mathcal{V}$ defines a 3-dimensional foliation $\mathcal{F}_3$ of M^{4n+3}. Since each Reeb vector field is Killing and is parallel, such a foliation turns out to be Riemannian with totally geodesic leaves. Concerning this foliated structure we recall the following result.

Theorem 2.2. ([6, Corollary 3.10]) *Let* $\left(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g\right)$ *be a* 3*-cosymplectic manifold. If the foliation* $\mathcal{F}_3$ *is regular (cf.* [22]*), then the space of leaves* $M^{4n+3}/\mathcal{F}_3$ *is a hyper-Kähler manifold of dimension* $4n$*. Consequently, every* 3*-cosymplectic manifold is Ricci-flat.*

Remark 2.3. If we drop the assumption of regularity in Theorem 2.2 and we assume instead that the vertical foliation has compact leaves, then the space of leaves is a hyper-Kähler orbifold, i.e. a second countable Hausdorff space locally modeled on finite quotients of $\mathbb{R}^m$. We refer to [21] for the formal definition and properties of orbifolds and to [23] for the generalization of geometric objects to the orbifold category.

Concerning the horizontal subbundle, note that — unlike the case of 3-Sasakian geometry — in any 3-cosymplectic manifold $\mathcal{H}$ is integrable. Indeed, for all $X, Y \in \Gamma(\mathcal{H})$, $\eta_\alpha([X, Y]) = -2d\eta_\alpha(X, Y) = 0$ since $d\eta_\alpha = 0$.

3. Decomposition of the cohomology of 3-cosymplectic manifolds

In this section we investigate some algebraic properties of the de Rham cohomology $H^*_{dR}(M)$ of a 3-cosymplectic manifold M^{4n+3}. By the Hodge-de Rham theory the vector space $H^k_{dR}(M)$ can be identified with the vector space $\Omega^k_H(M)$ of harmonic k-forms on M.

For $\alpha \in \{1,2,3\}$ we define linear operators λ_α and l_α by

$$l_\alpha \colon \Omega^k(M) \to \Omega^{k+1}(M) \qquad \lambda_\alpha \colon \Omega^{k+1}(M) \to \Omega^k(M)$$
$$\omega \mapsto \eta_\alpha \wedge \omega \qquad \omega \mapsto i_{\xi_\alpha}\omega.$$

We denote by $\{A, B\}$ the anticommutator $AB + BA$ of two linear operators A and B. From $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ it follows that

$$\{\lambda_\alpha, l_\beta\} = \delta_{\alpha\beta}. \tag{3.1}$$

Moreover

$$\{\lambda_\alpha, \lambda_\beta\} = \{l_\alpha, l_\beta\} = 0. \tag{3.2}$$

Define $e_\alpha = l_\alpha\lambda_\alpha$. Then it follows from (3.1) that e_α are idempotents. In fact

$$e_\alpha e_\alpha = l_\alpha\lambda_\alpha l_\alpha\lambda_\alpha = -l_\alpha l_\alpha\lambda_\alpha\lambda_\alpha + l_\alpha\lambda_\alpha = e_\alpha.$$

Moreover from (3.1) and (3.2) it follows that $[e_\alpha, e_\beta] = 0$, for $\alpha \neq \beta$. Thus $\{e_1, e_2, e_3\}$ are pairwise commuting idempotents.

By [8, Proposition 1] all operators l_α, λ_α, and thus e_α, preserve harmonic forms. Now we fix $k \in \{0, \dots, 4n+3\}$ and consider the restrictions of the operators e_α on $\Omega^k_H(M)$, $\alpha \in \{1,2,3\}$. Note that $\Omega^k_H(M)$ is a finite dimensional vector space over $\mathbb{R}$. As e_α is idempotent, its minimal polynomial $m_\alpha(x)$ is a divisor of $x(x-1)$. Therefore the only possible eigenvalues of e_α are 0 and 1. Moreover, since $m_\alpha(x)$ does not have multiple roots, the operator e_α is diagonalizable with 0 and 1 on the diagonal. As the operators $\{e_1, e_2, e_3\}$ commute with each other, by [4, Proposition VII.13] they can be simultaneously diagonalized. Define for all triples ε_1, ε_2, $\varepsilon_3 \in \{0,1\}$

$$\Omega^k_{H,\varepsilon_1\varepsilon_2\varepsilon_3}(M) = \left\{ \omega \in \Omega^k_H(M) \,\middle|\, e_\alpha\omega = \varepsilon_\alpha\omega,\ \alpha = 1,2,3 \right\}.$$

Since e_1, e_2, e_3 can be simultaneously diagonalized on $\Omega^k_H(M)$ we get that

$$\Omega^k_H(M) = \bigoplus_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{0,1\}} \Omega^k_{H,\varepsilon_1\varepsilon_2\varepsilon_3}(M). \tag{3.3}$$

Now let $\omega \in \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$. Then $l_1\omega \in \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}$. In fact

$$\begin{aligned} e_1 l_1 \omega &= l_1 \lambda_1 l_1 \omega = -\lambda_1 l_1 l_1 \omega + l_1 \omega = l_1 \\ e_\alpha l_1 \omega &= l_1 e_\alpha \omega = \varepsilon_\alpha l_1 \omega, \qquad \alpha = 2, 3. \end{aligned}$$

Similarly if $\omega \in \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M)$, then $\lambda_1\omega \in \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$. Therefore, we get maps of vector spaces

$$\begin{aligned} l_1^{\varepsilon_2\varepsilon_3} &\colon \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M) \to \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M), \\ \lambda_1^{\varepsilon_2\varepsilon_3} &\colon \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M) \to \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M). \end{aligned}$$

Now $l_1^{\varepsilon_2\varepsilon_3}\lambda_1^{\varepsilon_2\varepsilon_3}$ is the restriction of e_1 on $\Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M)$ and thus $l_1^{\varepsilon_2\varepsilon_3}\lambda_1^{\varepsilon_2\varepsilon_3} = \mathrm{id}$. Analogously the composition $\lambda_1^{\varepsilon_2\varepsilon_3} l_1^{\varepsilon_2\varepsilon_3}$ is the restriction of

$$l_1\lambda_1 = \mathrm{id} - \lambda_1 l_1 = \mathrm{id} - e_1$$

on $\Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$ and thus $\lambda_1^{\varepsilon_2\varepsilon_3} l_1^{\varepsilon_2\varepsilon_3} = 1$. Thus $\lambda_1^{\varepsilon_2\varepsilon_3}$ and $l_1^{\varepsilon_2\varepsilon_3}$ are inverse isomorphisms between the vector spaces $\Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$ and $\Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M)$. Replacing 1 with 2, 3, and putting all together we get for every $0 \le k \le 4n$ the cube

$$\begin{tikzcd}
 & \Omega^{k+1}_{H,100}(M) \arrow[rr, "l_2"] \arrow[dd, "l_3"] & & \Omega^{k+2}_{H,110}(M) \arrow[dd, "l_3"] \\
\Omega^{k}_{H,000}(M) \arrow[ur, "l_1"] \arrow[rr, "l_2"] \arrow[dd, "l_3"] & & \Omega^{k+1}_{H,010}(M) \arrow[ur, "l_1"] \arrow[dd, "l_3"] & \\
 & \Omega^{k+2}_{H,101}(M) \arrow[rr, "l_2"] & & \Omega^{k+3}_{H,111}(M) \\
\Omega^{k+1}_{H,001}(M) \arrow[ur, "l_1"] \arrow[rr, "l_2"] & & \Omega^{k+2}_{H,011}(M) \arrow[ur, "l_1"] &
\end{tikzcd}$$

whose faces are anti-commutative and edge arrows are isomorphisms of vector spaces. Therefore the whole information about cohomology groups of M is contained in the vector spaces $\Omega^k_{H,000}(M)$, $0 \le k \le 4n$.

Remark 3.1. It is easy to see that $\Omega^k_{H,000}(M)$ is precisely the space of all basic harmonic forms with respect to the Reeb foliation on M.

Denote by b_k^h the dimension of $\Omega^k_{H,000}(M)$. Then

$$\begin{aligned}
&\dim\Omega^k_{H,100} = \dim\Omega^k_{H,010} = \dim\Omega^k_{H,001} = \dim\Omega^{k-1}_{H,000} = b^h_{k-1} && k\geq 1\\
&\dim\Omega^k_{H,110} = \dim\Omega^k_{H,101} = \dim\Omega^k_{H,011} = \dim\Omega^{k-2}_{H,000} = b^h_{k-2} && k\geq 2\\
&\dim\Omega^k_{H,111} = \dim\Omega^{k-3}_{H,000} = b^h_{k-3} && k\geq 3.
\end{aligned}$$

Therefore, from the decomposition (3.3) we get

$$\begin{aligned}
b_0 &= b_0^h\\
b_1 &= b_1^h + 3b_0^h\\
b_2 &= b_2^h + 3b_1^h + 3b_0^h\\
b_k &= b_k^h + 3b^h_{k-1} + 3b^h_{k-2} + b^h_{k-3} \qquad 3\leq k\leq 4n+3.
\end{aligned} \tag{3.4}$$

4. Action of $so\,(4,1)$ on the cohomology of 3-cosymplectic manifolds

In this section we will show that $\Omega^k_{H,000}(M)$ admits an action of the Lie algebra $so\,(4,1)$. This result is the odd-dimensional analogous of the one obtained by Verbitsky in [26] about the action of $so\,(4,1)$ on the cohomology groups of a hyper-Kähler manifold M^{4n}. In fact, intuitively the space $\bigoplus_{k=0}^{4n}\Omega^k_{H,000}(M)$ can be thought of as a cohomology ring of the hyper-Kähler orbifold obtained from M^{4n+3} by taking the quotient under the action of the three Reeb vector fields.

For every cyclic permutation (α,β,γ) of $(1,2,3)$ we denote by Ξ_α the 2-form

$$\Xi_\alpha := \frac{1}{2}\left(\Phi_\alpha + 2\eta_\beta\wedge\eta_\gamma\right). \tag{4.1}$$

Define the operators $L_\alpha\colon \Omega^k(M)\to\Omega^{k+2}(M)$ and $\Lambda_\alpha\colon\Omega^{k+2}(M)\to\Omega^k(M)$ by $L_\alpha\omega = \Xi_\alpha\wedge\omega$ and $\Lambda_\alpha := *L_\alpha *$.

We will give now a local description of these operators. Let

$$\{X_1,\phi_1X_1,\phi_2X_1,\phi_3X_1,\dots,X_n,\phi_1X_n,\phi_2X_n,\phi_3X_n,\xi_1,\xi_2,\xi_3\}$$

be an orthonormal basis of vector fields in some open subset U of M. Denote by ζ_s the 1-form dual to X_s, that is $\zeta_s = g(X_s,-)$. Then

$$i_{\phi_\alpha X_s}(\phi_\alpha^*\zeta_t) = g(X_s,\phi_\alpha(\phi_\alpha X_t)) = g\left(X_s,\phi_\alpha^2X_t\right) = -\delta_{st}, \tag{4.2}$$

for $1\leq s,t\leq n$. Therefore the set

$$\{\zeta_1,\phi_1^*\zeta_1,\phi_2^*\zeta_1,\phi_3^*\zeta_1,\dots,\zeta_n,\phi_1^*\zeta_n,\phi_2^*\zeta_n,\phi_3^*\zeta_n,\eta_1,\eta_2,\eta_3\} \tag{4.3}$$

is a basis of 1-forms on U.

Proposition 4.1. *Let* (α,β,γ) *be a cyclic permutation of* $(1,2,3)$. *Then*

$$\Phi_\alpha = 2\sum_{s=1}^{n}\left(\zeta_s\wedge\phi_\alpha^*\zeta_s - \phi_\beta^*\zeta_s\wedge\phi_\gamma^*\zeta_s\right) - 2\eta_\beta\wedge\eta_\gamma \tag{4.4}$$

and therefore

$$\Xi_\alpha = \sum_{s=1}^{n}\left(\zeta_s\wedge\phi_\alpha^*\zeta_s - \phi_\beta^*\zeta_s\wedge\phi_\gamma^*\zeta_s\right). \tag{4.5}$$

Proof: Let us denote by $\langle\ ,\ \rangle$ the natural pairing between k-forms and k-vector fields. By definition of Φ_α we have

$$\begin{aligned}\langle\Phi_\alpha, X_s\wedge\phi_\alpha X_s\rangle &= g\left(X_s,\phi_\alpha^2 X_s\right) = -1\\ \langle\Phi_\alpha,\phi_\beta X_s\wedge\phi_\gamma X_s\rangle &= g\left(\phi_\beta X_s,\phi_\alpha\phi_\gamma X_s\right) = g\left(\phi_\beta X_s,-\phi_\beta X_s\right) = -1\\ \langle\Phi_\alpha,\eta_\beta\wedge\eta_\gamma\rangle &= g\left(\eta_\beta,\phi_\alpha\eta_\gamma\right) = g\left(\eta_\beta,-\eta_\beta\right) = -1,\end{aligned}$$

and $\langle\Phi_\alpha, V\rangle = 0$ for any other element V of the basis of the space of bivector fields on U. On the other hand,

$$\begin{aligned}\langle\zeta_s\wedge\phi_\alpha^*\zeta_s, X_s\wedge\phi_\alpha X_s\rangle &= \frac{1}{2}\zeta_s\left(X_s\right)\phi_\alpha^*\zeta_s\left(\phi_\alpha X_s\right) = -\frac{1}{2}\\ \left\langle\phi_\beta^*\zeta_s\wedge\phi_\gamma^*\zeta_s,\phi_\beta X_s\wedge\phi_\gamma X_s\right\rangle &= \frac{1}{2}\phi_\beta^*\zeta_s\left(\phi_\beta X_s\right)\phi_\gamma^*\zeta_s\left(\phi_\gamma X_s\right) = \frac{1}{2}\\ \langle\eta_\beta\wedge\eta_\gamma,\xi_\beta\wedge\xi_\gamma\rangle &= \frac{1}{2}\eta_\beta\left(\xi_\beta\right)\eta_\gamma\left(\xi_\gamma\right) = \frac{1}{2}.\end{aligned}$$

■

Note that for any k-form ω on M, any vector field Y of unit norm, and ρ the dual 1-form such that $\rho(Y) = 1$, we have

$$*\left(\rho\wedge *\omega\right) = (-1)^{(4n+3-k)(k-1)}\, i_Y\omega. \tag{4.6}$$

From (4.2), (4.5), (4.6), and the fact that $*^2 = *$ for odd dimensional manifolds, it is easy to obtain the formula

$$\Lambda_\alpha = \sum_{s=1}^{n}\left(i_{X_s} i_{\phi_\alpha X_s} + i_{\phi_\beta X_s} i_{\phi_\gamma X_s}\right).$$

From the explicit formulas for Ξ_α and Λ_α it follows that the operators L_α and Λ_α commute with the operators e_β for any pair $1\leq\alpha,\beta\leq 3$.

Remark 4.2. From [3, Lemma 2.3] it follows that the operators $\omega \mapsto \Phi_\alpha \wedge \omega$ preserve harmonic forms. Since the operator $\omega \mapsto \eta_\beta \wedge \eta_\gamma \wedge \omega$ is equal to $l_\beta l_\gamma$, it also preserves harmonicity. Then, by definition of the operators L_α, they preserve harmonicity as well. Since the Hodge star $*$ preserves harmonic forms we get that also Λ_α preserves them. As consequence, we can restrict the operators L_α and Λ_α to $\Omega^*_{H,000}(M)$. From now on, we will consider L_α and Λ_α as endomorphisms of $\Omega^*_{H,000}(M)$.

Define the operator $H\colon \Omega^k_{H,000}(M) \to \Omega^k_{H,000}(M)$ by $H\omega = (2n-k)\,\omega$.

Proposition 4.3. *We have* $[L_\alpha, \Lambda_\alpha] = -H$ *on* $\Omega^*_{H,000}(M)$.

Proof: Every element of $\Omega^k_{H,000}$ can be locally written as a linear combination of wedges of elements in

$$\left\{\zeta_1, \phi_1^*\zeta_1, \phi_2^*\zeta_1, \phi_3^*\zeta_1, \dots, \zeta_n, \phi_1^*\zeta_n, \phi_2^*\zeta_n, \phi_3^*\zeta_n\right\}. \tag{4.7}$$

We have

$$\begin{aligned} [L_\alpha, \Lambda_\alpha] = \sum_{s=1}^{n} \Big(& [\zeta_s \wedge \phi_\alpha^*\zeta_s \wedge -, i_{X_s} i_{\phi_\alpha X_s}] \\ & - \left[\phi_\beta^*\zeta_s \wedge \phi_\gamma^*\zeta_s \wedge -, i_{\phi_\beta X_s} i_{\phi_\gamma X_s}\right]\Big), \end{aligned} \tag{4.8}$$

where (α, β, γ) is a cyclic permutation of $(1,2,3)$. For any linear operators a, b, c, d, we have

$$\begin{aligned} [ab, cd] &= [ab, c]\, d + c\, [ab, d] \\ &= (a\{b,c\} - \{a,c\}\, b)\, d + c\,(a\{b,d\} - \{a,d\}\, b) \\ &= a\{b,c\}\, d - \{a,c\}\, bd + ca\{b,d\} - c\{a,d\}\, b \\ &= a\{b,c\}\, d - \{a,c\}\, bd - ac\{b,d\} - c\{a,d\}\, b \\ &\quad + \{a,c\}\{b,d\}. \end{aligned} \tag{4.9}$$

It is also obvious that for arbitrary α, $\beta \neq \gamma$:

$$\begin{aligned} \{\zeta_s \wedge -, i_{\phi_\alpha X_s}\} &= 0 & \{\zeta_s \wedge -, i_{X_s}\} &= 1 \\ \left\{\phi_\beta^*\zeta_s \wedge -, i_{\phi_\gamma X_s}\right\} &= 0 & \left\{\phi_\beta^*\zeta_s \wedge -, i_{\phi_\beta X_s}\right\} &= -1 \\ \{\phi_\alpha^*\zeta_s \wedge -, i_{X_s}\} &= 0. \end{aligned} \tag{4.10}$$

Therefore, using (4.9) we get

$$\begin{aligned}[L_\alpha, \Lambda_\alpha] = & \sum_{s=1}^{n} \big(-\phi_\alpha^* \zeta_s \wedge i_{\phi_\alpha X_s} + \zeta_s \wedge i_{X_s} - 1 \\ & - \left(\phi_\gamma^* \zeta_s \wedge i_{\phi_\gamma X_s} + \phi_\beta^* \zeta_s \wedge i_{\phi_\beta X_s}\right) + 1\big) \\ = & - 2n + \sum_{s=1}^{n} \big(\zeta_s \wedge i_{X_s} - \phi_\alpha^* \zeta_s \wedge i_{\phi_\alpha X_s} \\ & - \phi_\beta^* \zeta_s \wedge i_{\phi_\beta X_s} - \phi_\gamma^* \zeta_s \wedge i_{\phi_\gamma X_s}\big) .\end{aligned}$$

Now the sum in the last row operates on any fixed-degree form involving only elements in (4.7) by multiplying the form by its degree. Hence

$$[L_\alpha, \Lambda_\alpha]\, \omega = -H\omega$$

for all $\omega \in \Omega^*_{H,000}$. ■

For every cyclic permutation (α, β, γ) of $(1, 2, 3)$ we define the operator

$$K_\alpha = \sum_{s=1}^{n} \left(\phi_\alpha^* \zeta_s \wedge i_{X_s} + \zeta_s \wedge i_{\phi_\alpha X_s} + \phi_\gamma^* \zeta_s \wedge i_{\phi_\beta X_s} - \phi_\beta^* \zeta_s \wedge i_{\phi_\gamma X_s}\right) .$$

Let $\rho_1, \ldots, \rho_k$ be a sequence of elements in (4.7). Then from (4.2) and

$$\phi_\alpha^* \phi_\beta^* = -\phi_\gamma^*, \qquad \phi_\beta^* \phi_\alpha^* = \phi_\gamma^*,$$

it follows that

$$K_\alpha \left(\rho_1 \wedge \cdots \wedge \rho_k\right) = \sum_{j=1}^{k} (-1)^{j+1} \rho_1 \wedge \cdots \wedge \phi_\alpha^* \rho_j \wedge \cdots \wedge \rho_k .$$

Proposition 4.4. *For any cyclic permutation (α, β, γ) of $(1, 2, 3)$ we have on $\Omega^*_{H,000}(M)$*

$$[L_\alpha, \Lambda_\beta] = K_\gamma \tag{4.11}$$

$$[L_\alpha, \Lambda_\gamma] = -K_\beta . \tag{4.12}$$

In particular K_α is globally defined, for each $\alpha \in \{1, 2, 3\}$.

Proof: We have

$$
\begin{aligned}
[L_\alpha, \Lambda_\beta] = \sum_{s=1}^{n} \Big(& \left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{X_s} i_{\phi_\beta X_s}\right] \\
& + \left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{\phi_\gamma X_s} i_{\phi_\alpha X_s}\right] \\
& - \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{X_s} i_{\phi_\beta X_s}\right] \\
& - \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{\phi_\gamma X_s} i_{\phi_\alpha X_s}\right] \Big) .
\end{aligned}
$$

Now, by (4.9) and (4.10) we get

$$
\begin{aligned}
[L_\alpha, \Lambda_\beta] &= \sum_{s=1}^{n} \Big(-\phi_\alpha^* \zeta_s \wedge i_{\phi_\beta X_s} + \zeta_s \wedge i_{\phi_\gamma X_s} \\
&\qquad -i_{X_s} \left(\phi_\gamma^* \zeta_s \wedge -\right) + \phi_\beta^* \zeta_s \wedge i_{\phi_\alpha X_s} \Big) \\
&= \sum_{s=1}^{n} \Big(\zeta_s \wedge i_{\phi_\gamma X_s} + \phi_\gamma^* \zeta_s \wedge i_{X_s} \\
&\qquad +\phi_\beta^* \zeta_s \wedge i_{\phi_\alpha X_s} - \phi_\alpha^* \zeta_s \wedge i_{\phi_\beta X_s} \Big) \\
&= K_\gamma .
\end{aligned}
$$

Equation (4.12) is proved as follows. We have

$$
\begin{aligned}
[L_\alpha, \Lambda_\gamma] = \sum_{s=1}^{n} \Big(& \left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{X_s} i_{\phi_\gamma X_s}\right] \\
& + \left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{\phi_\alpha X_s} i_{\phi_\beta X_s}\right] \\
& - \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{X_s} i_{\phi_\gamma X_s}\right] \\
& - \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{\phi_\alpha X_s} i_{\phi_\beta X_s}\right] \Big) .
\end{aligned}
$$

Again by (4.9) we get

$$\begin{aligned}
[L_\alpha, \Lambda_\gamma] &= \sum_{s=1}^{n} \big(-\phi_\alpha^*\zeta_s \wedge i_{\phi_\gamma X_s} - \zeta_s \wedge i_{\phi_\beta X_s} \\
&\qquad -\phi_\beta^*\zeta_s \wedge i_{X_s} - i_{\phi_\alpha X_s}\left(\phi_\gamma^*\zeta_s \wedge -\right)\big) \\
&= -\sum_{s=1}^{n} \big(\zeta_s \wedge i_{\phi_\beta X_s} + \phi_\beta^*\zeta_s \wedge i_{X_s} \\
&\qquad +\phi_\alpha^*\zeta_s \wedge i_{\phi_\gamma X_s} - \phi_\gamma^*\zeta_s \wedge i_{\phi_\alpha X_s}.\big) \\
&= -K_\beta.
\end{aligned}$$

■

Theorem 4.5. *The linear span $\mathfrak{g}$ of the operators*

$$\{\, L_\alpha, \Lambda_\alpha, K_\alpha, H \,|\, \alpha = 1, 2, 3\}$$

on $\Omega^*_{H,000}(M)$ is a Lie algebra.

Proof: We have to check that $\mathfrak{g}$ is closed under taking commutators. Clearly it is enough to check that the commutator of any two operators from the set $\{\, L_\alpha, \Lambda_\alpha, K_\alpha, H \,|\, \alpha = 1, 2, 3\}$ lies in $\mathfrak{g}$. It is obvious that $[L_\alpha, L_\beta] = 0$ and $[\Lambda_\alpha, \Lambda_\beta] = 0$ for any pair $1 \le \alpha, \beta \le 3$. Since K_α does not change the degree of forms, L_α raises the degree by 2 and Λ_α decreases the degree by 2, we get

$$[K_\alpha, H] = 0 \qquad [L_\alpha, H] = 2L_\alpha \qquad [\Lambda_\alpha, H] = -2\Lambda_\alpha. \tag{4.13}$$

Furthermore, by Proposition 4.3 we know that $[L_\alpha, \Lambda_\alpha] = -H$, and by Proposition 4.4 that $[L_\alpha, \Lambda_\beta] = K_\gamma$ for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. Therefore it is left to check that the commutators $[K_\alpha, L_\alpha]$, $[K_\alpha, L_\beta]$, $[K_\alpha, \Lambda_\alpha]$, $[K_\alpha, L_\beta]$ and $[K_\alpha, K_\beta]$ for all pairs $1 \le \alpha, \beta \le 3$ lie in $\mathfrak{g}$.

For any cyclic permutation (α, β, γ) of $(1, 2, 3)$ we have

$$\begin{aligned}
[K_\alpha, L_\alpha] &\overset{(4.12)}{=\!=} [[L_\beta, \Lambda_\gamma], L_\alpha] = [[L_\beta, L_\alpha], \Lambda_\gamma] + [L_\beta, [\Lambda_\gamma, L_\alpha]] \\
&= [L_\beta, K_\beta] \\
&= -[K_\beta, L_\beta].
\end{aligned}$$

As (α, β, γ) is an arbitrary cyclic permutation of $(1, 2, 3)$ we get also

$$[K_\beta, L_\beta] = -[K_\gamma, L_\gamma] \qquad\qquad [K_\gamma, L_\gamma] = -[K_\alpha, L_\alpha]$$

and combining we obtain $[K_\alpha, L_\alpha] = -[K_\alpha, L_\alpha]$, which implies $[K_\alpha, L_\alpha] = 0$ for all $1 \leq \alpha \leq 3$. Similarly, we have $[K_\alpha, \Lambda_\alpha] = 0$.

Now for any cyclical permutation (α, β, γ) of $(1, 2, 3)$ we have

$$\begin{aligned}
[K_\alpha, L_\beta] &= -\left[[L_\gamma, \Lambda_\beta], L_\beta\right] = -\left[L_\gamma, [\Lambda_\beta, L_\beta]\right] = -[L_\gamma, H] = -2L_\gamma,\\
[K_\alpha, L_\gamma] &= \left[[L_\beta, \Lambda_\gamma], L_\gamma\right] = \left[L_\beta, [\Lambda_\gamma, L_\gamma]\right] = [L_\beta, H] = 2L_\beta,\\
[K_\alpha, \Lambda_\beta] &= \left[[L_\beta, \Lambda_\gamma], \Lambda_\beta\right] = \left[[L_\beta, \Lambda_\beta], \Lambda_\gamma\right] = [-H, \Lambda_\gamma] = -2\Lambda_\gamma,\\
[K_\alpha, \Lambda_\gamma] &= -\left[[L_\gamma, \Lambda_\beta], \Lambda_\gamma\right] = -\left[[L_\gamma, \Lambda_\gamma], \Lambda_\beta\right] = [H, \Lambda_\beta] = 2\Lambda_\beta,\\
[K_\alpha, K_\beta] &= \left[[L_\beta, \Lambda_\gamma], K_\beta\right] = \left[L_\beta, [\Lambda_\gamma, K_\beta]\right] = [L_\beta, 2\Lambda_\alpha] = -2K_\gamma.
\end{aligned}$$

■

Now we prove that the Lie algebra $\mathfrak{g}$ can be identified with the Lie algebra $so\,(4,1)$. Let us recall the definition of $so\,(4,1)$. We denote by E_1 the matrix

$$\mathrm{diag}\,(1,1,1,1,-1)\,.$$

Then

$$so\,(4,1) := \left\{\, A \in M_5\,(\mathbb{R}) \,\middle|\, AE_1 = -E_1A^t \right\}$$

as a set. The Lie bracket on $so\,(4,1)$ is given by the usual commutator of matrices. We denote by e_{ij} the matrix with 1 at the place (i,j) and zeros elsewhere. Define for $1 \leq i < j \leq 5$

$$t_{ij} = \begin{cases} e_{i5} + e_{5i} & j = 5 \\ e_{ij} - e_{ji} & \text{otherwise.} \end{cases}$$

Then the set $\{\, t_{ij} \,|\, 1 \leq i < j \leq 5\}$ is a basis of $so\,(4,1)$. A direct computation shows that

$$\begin{array}{llll}
[t_{ij}, t_{ik}] = -t_{jk} & [t_{ij}, t_{jk}] = t_{ik} & [t_{ik}, t_{jk}] = -t_{ij} & i < j < k < 5 \\
[t_{ij}, t_{i5}] = -t_{j5} & [t_{ij}, t_{j5}] = t_{i5} & [t_{i5}, t_{j5}] = t_{ij} & i < j < 5
\end{array}$$

We will also use t_{ji} to denote $-t_{ij}$ for $1 \leq i < j \leq 4$. Now for any cyclic permutation (α, β, γ) of $(1,2,3)$ we have

$$\begin{aligned}
[t_{\alpha 5} + t_{\alpha 4}, t_{\alpha 5} - t_{\alpha 4}] &= [t_{\alpha 5}, -t_{\alpha 4}] + [t_{\alpha 4}, t_{\alpha 5}] = -2t_{45} \\
[t_{\alpha 5} + t_{\alpha 4}, 2t_{45}] &= 2\,(t_{\alpha 4} + t_{\alpha 5}) \\
[t_{\alpha 5} - t_{\alpha 4}, 2t_{45}] &= 2\,(t_{\alpha 4} - t_{\alpha 5}) = -2\,(t_{\alpha 5} - t_{\alpha 4}) \\
[t_{\alpha 5} + t_{\alpha 4}, t_{\beta 5} + t_{\beta 4}] &= t_{\alpha,\beta} - t_{\alpha,\beta} = 0 \\
[t_{\alpha 5} + t_{\alpha 4}, t_{\beta 5} - t_{\beta 4}] &= t_{\alpha,\beta} + t_{\alpha,\beta} = 2t_{\alpha,\beta} \\
[t_{\alpha 5} + t_{\alpha 4}, t_{\gamma 5} - t_{\gamma 4}] &= -2t_{\gamma,\alpha} \\
[2t_{\beta,\gamma}, t_{\beta 5} + t_{\beta 4}] &= -2\left(t_{\gamma 5 + t_{\gamma 4}}\right) \\
[2t_{\beta,\gamma}, t_{\gamma 5} + t_{\gamma 4}] &= 2\,(t_{\beta,5} + t_{\beta 4}) \\
[2t_{\beta,\gamma}, t_{\beta 5} - t_{\beta 4}] &= -2\,(t_{\gamma 5} - t_{\gamma 4}) \\
[2t_{\beta,\gamma}, t_{\gamma 5} - t_{\gamma 4}] &= 2\,(t_{\beta 5} - t_{\beta 4})\,.
\end{aligned}$$

Therefore the assignment

$$H \mapsto 2t_{45} \qquad L_\alpha \mapsto t_{\alpha 5} + t_{\alpha 4} \qquad \Lambda_\alpha \mapsto t_{\alpha 5} - t_{\alpha 4} \qquad K_\alpha \mapsto 2t_{\beta,\gamma}$$

induces an isomorphism of Lie algebras $so\,(4,1) \to \mathfrak{g}$. Thus we have proved the following result.

Theorem 4.6. *The operators L_α, Λ_α, $\alpha \in \{1,2,3\}$, give a structure of $so\,(4,1)$-module on $\Omega^*_{H,000}\,(M)$.*

5. Action of $\mathbb{H}$ on $\Omega^{2k+1}_{H,000}\,(M)$ and Betti numbers of compact 3-cosymplectic manifolds

Let $U \subset M$ be an open subset and

$$\{\zeta_1, \phi_1^*\zeta_1, \phi_2^*\zeta_1, \phi_3^*\zeta_1, \dots, \zeta_n, \phi_1^*\zeta_n, \phi_2^*\zeta_n\phi_3^*\zeta_n, \eta_1, \eta_2, \eta_3\}$$

an orthonormal basis of 1-forms on U. Define $\Omega^*_{000}\,(U)$ as a linear span with coefficients in $C^\infty\,(U)$ of the set

$$Y := \{\zeta_1, \phi_1^*\zeta_1, \phi_2^*\zeta_1, \phi_3^*\zeta_1, \dots, \zeta_n, \phi_1^*\zeta_n, \phi_2^*\zeta_n, \phi_3^*\zeta_n\}\,.$$

Then $\Omega^*_{H,000}\,(U)$ is a subspace of $\Omega^*_{000}\,(U)$. Define the operator I_α on $\Omega^*_{000}\,(U)$ extending by linearity the map

$$\rho_1 \wedge \cdots \wedge \rho_k \mapsto \phi_\alpha^*\rho_1 \wedge \cdots \wedge \phi_\alpha^*\rho_k \qquad\qquad \rho_1, \dots, \rho_k \in Y.$$

Proposition 5.1. *The operators I_α, $\alpha \in \{1,2,3\}$, are well-defined on $\Omega^*_{000}(M)$. Moreover, they preserve harmonic forms. In particular, we can consider I_α as an endomorphism of $\Omega^*_{H,000}(M)$.*

Proof: For $1 \le s \le k$, we define the operators $K_{\alpha,s}$ on $\Omega^k_{000}(U)$ extending by linearity the map

$$\rho_1 \wedge \cdots \wedge \rho_k \mapsto \sum_{1 \le j_1 < \cdots < j_s \le k} (-1)^{j_1+\cdots+j_s+s} \rho_1 \wedge \cdots \wedge \phi^*_\alpha \rho_{j_1} \wedge \cdots \wedge \phi^*_\alpha \rho_{j_s} \wedge \cdots \wedge \rho_k,$$

where $\rho_1, \ldots, \rho_k \in Y$. We also denote the identity operator by $K_{\alpha,0}$. Then $K_{\alpha,1} = K_\alpha$ and $K_{\alpha,k} = (-1)^{\binom{k+1}{2}} I_\alpha$. It is easy to check in local coordinates that

$$K_\alpha K_{\alpha,s} = (s+1) K_{\alpha,s+1} - (k-s+1) K_{\alpha,s-1}.$$

These formulae can be used to show that $K_{\alpha,s}$ is a polynomial in K_α with constant coefficients which do not depend on the used local chart. Since K_α are globally defined and preserve harmonic forms we get that the operators $K_{\alpha,s}$ are globally defined and preserve harmonic forms for all s. In particular, I_α is a well-defined operator on $\Omega^*_{000}(M)$ and preserves harmonic forms. ∎

It is straightforward to see that the operators I_α, $\alpha \in \{1,2,3\}$, restricted to $\Omega^{odd}_{H,000}(M)$ satisfy the same relations as the units of the quaternion algebra $\mathbb{H}$. Therefore we get

Theorem 5.2. *Let k be odd. Then $\Omega^k_{H,000}(M)$ is an $\mathbb{H}$-module.*

Corollary 5.3. *Let k be odd. Then b^h_k is divisible by 4.*

Proof: Every finite dimensional module over $\mathbb{H}$ is a direct sum of regular modules. As the dimension of the regular module is 4, the result follows. ∎

We denote by (d) the principal ideal in $\mathbb{Z}$ generated by d. In other words, (d) will be the set of the integers divisible by d.

Corollary 5.4. *Let M be a compact 3-cosymplectic manifold. For any odd k we have $b_{k-1} + b_k \in (4)$.*

Proof: Using (3.4) we get for $k = 1$

$$b_0 + b_1 = b^h_0 + b^h_1 + 3b^h_0 = b^h_1 + 4b^h_0 \in (4).$$

Similarly, for $k=3$ we get

$$\begin{aligned} b_2+b_3 &= b_2^h+3b_1^h+3b_0^h+b_3^h+3b_2^h+3b_1^h+b_0^h \\ &= b_3^h+4b_2^h+6b_1^h+4b_0^h \in (4)\,. \end{aligned}$$

Finally, for odd $k \geq 5$ we have

$$\begin{aligned} b_{k-1}+b_k &= b_{k-1}^h+3b_{k-2}^h+3b_{k-3}^h+b_{k-4}^h+b_k^h+3b_{k-1}^h+3b_{k-2}^h+b_{k-3}^h \\ &= b_k^h+4b_{k-1}^h+6b_{k-2}^h+4b_{k-3}^h+b_{k-4}^h \in (4)\,. \end{aligned}$$

■

6. Inequalities on Betti numbers

In this section we give a bound from below on the Betti numbers of a compact 3-cosymplectic manifold. We start with the following statement about horizontal Betti numbers, which is a generalization of Wakakuwa's Theorem 9.1 in [27].

Proposition 6.1. *Let M be a compact 3-cosymplectic manifold of dimension $4n+3$. Then for $0 \leq k \leq n$*

$$b_{2k}^h \geq \binom{k+2}{2}.$$

Proof: Recall the definition (4.1) of the 2-forms Ξ_α. Let us fix $0 \leq k \leq n$. We consider the set

$$S_k := \left\{ \Xi_1^{k_1} \wedge \Xi_2^{k_2} \wedge \Xi_3^{k_3} \,\middle|\, k_1+k_2+k_3=k \right\}.$$

All the elements of S_k can be obtained from the constant 0-form 1 on M by successive applications of operators L_α, $\alpha \in \{1,2,3\}$. Therefore by Remark 4.2 we get $S_k \subset \Omega^{2k}_{H,000}(M)$. Thus to prove the proposition, it is enough to show that S_k contains $\binom{k+2}{2}$ linearly independent elements. This can be checked locally. Let U be a trivializing neighbourhood like in Section 4. We order the elements of the basis (4.3) of 1-forms on U by

$$\begin{aligned} &\zeta_1 < \zeta_2 < \cdots < \zeta_n < \phi_1^*\zeta_1 < \phi_1^*\zeta_2 < \cdots < \phi_1^*\zeta_n \\ &\qquad < \phi_2^*\zeta_1 < \cdots < \phi_2^*\zeta_n < \phi_3^*\zeta_1 < \cdots < \phi_3^*\zeta_n < \eta_1 < \eta_2 < \eta_3. \end{aligned}$$

Then we get an induced lexicographical ordering on the basis of $\Omega^k(U)$. By using the local expression (4.5) of Ξ_α, $\alpha \in \{1,2,3\}$, we see that the first basis element with respect to this ordering that enters $\Xi_1^{k_1} \wedge \Xi_2^{k_2} \wedge \Xi_3^{k_3}$ with non-zero coefficient is

$$\zeta_1 \wedge \phi_1^*\zeta_1 \wedge \zeta_2 \wedge \phi_1^*\zeta_2 \wedge \cdots \wedge \zeta_{k_1} \wedge \phi_1^*\zeta_{k_1} \wedge \zeta_{k_1+1} \wedge \phi_2^*\zeta_{k_1+1} \wedge \cdots \wedge \zeta_{k_1+k_2} \\ \wedge \phi_2^*\zeta_{k_1+k_2} \wedge \zeta_{k_1+k_2+1} \wedge \phi_3^*\zeta_{k_1+k_2+1} \wedge \cdots \wedge \zeta_{k_1+k_2+k_3} \wedge \phi_3^*\zeta_{k_1+k_2+k_3}.$$

Since for different triples (k_1, k_2, k_3) such that $k_1 + k_2 + k_3 = k$ the above basis elements are different, we get that S_k contains $\binom{k+2}{2}$ elements and they are linearly independent. ■

As a consequence we get the following lower bound on the Betti numbers of a compact 3-cosymplectic manifold.

Theorem 6.2. *Let M be a compact 3-cosymplectic manifold of dimension $4n+3$. Then for $0 \leq k \leq 2n+1$*

$$b_k \geq \binom{k+2}{2}.$$

Proof: For $k = 0$ we have obviously $b_0 = 1 = \binom{2}{2}$. First we consider the case $k = 2l$, $1 \leq l \leq n$. Then by (3.4) and Proposition 6.1

$$\begin{aligned} b_k &= b^h_{2l} + 3b^h_{2l-1} + 3b^h_{2l-2} + b^h_{2l-3} \\ &\geq \binom{l+2}{2} + 3\cdot 0 + 3\binom{l-1+2}{2} + 0 \\ &= \frac{(l+2)\,(l+1)}{2} + 3\frac{(l+1)\,l}{2} = \frac{(l+1)\,(l+2+3l)}{2} \\ &= \frac{(2l+2)\,(2l+1)}{2} = \binom{k+2}{2}. \end{aligned}$$

Now, suppose that $k = 2l+1$, $0 \leq l \leq n$. Then, again by (3.4) and Proposition 6.1

$$\begin{aligned} b_k &= b^h_{2l+1} + 3b^h_{2l} + 3b^h_{2l-1} + b^h_{2l-2} \\ &\geq 0 + 3\binom{l+2}{2} + 3\cdot 0 + \binom{l-1+2}{2} \\ &= 3\frac{(l+2)\,(l+1)}{2} + \frac{(l+1)\,l}{2} = \frac{(l+1)\,(3l+6+l)}{2} \\ &= \frac{(2l+2)\,(2l+3)}{2} = \binom{2l+3}{2} = \binom{k+2}{2}. \end{aligned}$$

■

7. Nontrivial examples of compact 3-cosymplectic manifolds

The standard example of a compact 3-cosymplectic manifold is given by the torus $\mathbb{T}^{4n+3}$ with the following structure (cf. [19, p.561]). Let $\{\theta_1, \dots, \theta_{4n+3}\}$ be a basis of 1-forms such that each θ_i is integral and closed. Let us define a Riemannian metric g on $\mathbb{T}^{4n+3}$ by

$$g := \sum_{i=1}^{4n+3} \theta_i \otimes \theta_i.$$

For each $\alpha \in \{1, 2, 3\}$ we define a tensor field ϕ_α of type $(1, 1)$ by

$$\phi_\alpha = \sum_{i=1}^{n} (E_{\alpha n+i} \otimes \theta_i - E_i \otimes \theta_{\alpha+i} + E_{\gamma n+i} \otimes \theta_{\beta n+i} \\ -E_{\beta n+i} \otimes \theta_{\gamma n+i}) + E_{4n+\gamma} \otimes \theta_{4n+\beta} - E_{4n+\beta} \otimes \theta_{4n+\gamma},$$

where $\{E_1, \dots, E_{4n+3}\}$ is the dual (orthonormal) basis of $\{\theta_1, \dots, \theta_{4n+3}\}$ and (α, β, γ) is a cyclic permutation of $\{1, 2, 3\}$. Setting, for each $\alpha \in \{1, 2, 3\}$, $\xi_\alpha := E_{4n+\alpha}$ and $\eta_\alpha := \theta_{4n+\alpha}$, one can easily check that the torus $\mathbb{T}^{4n+3}$ endowed with the structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is 3-cosymplectic.

On the other hand, the standard example of a noncompact 3-cosymplectic manifold is given by $\mathbb{R}^{4n+3}$ with the structure described in [6, Theorem 4.4].

Both the above examples are the global product of a hyper-Kähler manifold with a 3-dimensional flat abelian Lie group. In fact, locally this is always true.

Proposition 7.1. *Any 3-cosymplectic manifold M^{4n+3} is locally the Riemannian product of a hyper-Kähler manifold N^{4n} and a 3-dimensional flat abelian Lie group G^3.*

Proof: The tangent bundle of M^{4n+3} splits up as the orthogonal sum of the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, which define Riemannian foliations with totally geodesic leaves. Therefore, by the de Rham decomposition theorem the manifold M is locally the Riemannian product of a leaf N^{4n} of $\mathcal{H}$ and a leaf G^3 of $\mathcal{V}$. The structure tensors ϕ_1, ϕ_2, ϕ_3 induce an almost hyper-complex structure (J_1, J_2, J_3) on N^{4n}. Furthermore, for each

$\alpha \in \{1, 2, 3\}$ and for all $X, X' \in \Gamma(TN^{4n}) = \Gamma(\mathcal{H})$,

$$[J_\alpha, J_\alpha](X, X') = N^{(1)}_{\phi_\alpha}(X, X') - 2d\eta_\alpha(X, X')\xi_\alpha = 0,$$

as M^{4n+3} is normal and η_α is closed. Consequently, the structure is hypercomplex. Finally, the induced metric is clearly compatible with such a hypercomplex structure, so that N^{4n} is hyper-Kähler. On the other hand, from Lie group theory (see e.g. [24, p. 10]) it follows that G^3 is an abelian Lie group. Since the Reeb vector fields are parallel, we get

$$R(\xi_\alpha, \xi_\beta)\xi_\gamma = \nabla_{\xi_\alpha}\nabla_{\xi_\beta}\xi_\gamma - \nabla_{\xi_\beta}\nabla_{\xi_\alpha}\xi_\gamma - \nabla_{[\xi_\alpha,\xi_\beta]}\xi_\gamma = 0, \tag{7.1}$$

Therefore G^3 is flat. ■

When $n = 0$ of course we have no splitting, and M is necessarily a 3-torus in the compact case, as it is shown in the following proposition.

Proposition 7.2. *Suppose M^3 is a compact three dimensional 3-cosymplectic manifold. Then M^3 is a three dimensional torus.*

Proof: First of all M^3 is clearly flat. Indeed, in this case the three Reeb vector fields span all the vector fields over the ring of smooth functions. Furthermore, they commute with each other and are parallel. Thus, similarly to (7.1) we get $R(\xi_\alpha, \xi_\beta)\xi_\gamma = 0$ for any triple of indices $1 \leq \alpha, \beta, \gamma \leq 3$.

The manifold M^3 is orientable, since $\eta_1 \wedge \eta_2 \wedge \eta_3 \neq 0$ is a volume form on M^3. Moreover η_1, η_2, η_3 are three linear independent harmonic forms of degree 1, so that $b_1\left(M^3\right) \geq 3$. The complete list of compact orientable Euclidean three-dimensional manifolds was obtained in §2-3 of [13]. The unique manifold with $b_1 \geq 3$ in this list is the three dimensional torus. ■

Due to Proposition 7.1, it is natural to ask whether there are examples of 3-cosymplectic manifolds which are not the global product of a hyper-Kähler manifold with an abelian Lie group. We will give an example of a compact 3-cosymplectic manifold in dimension seven that is not a product of a hyper-Kähler manifold and a three-dimensional torus. Before describing the construction, we remind the following well-known result.

Proposition 7.3. *If M^4 is a compact four-dimensional hyper-Kähler manifold, then M^4 is either a K3 surface or a four dimensional torus.*

Proof: From [27, Theorem 8.1] it follows that $b_1(M^4)$ is even. Moreover, since every hyper-Kähler manifold is Calabi-Yau, M^4 has a trivial canonical bundle. Therefore, by the Kodaira classification (cf. [15, Section 6A]) M^4 is either a K3 surface or a 4-torus. ∎

Let (M^{4n}, J_α, G) be a compact hyper-Kähler manifold, where (J_1, J_2, J_3) is the hyper-complex structure of M^{4n} and G is the compatible Riemannian metric. Let $f : M^{4n} \longrightarrow M^{4n}$ be a hyper-Kählerian isometry, that is f is an isometry such that

$$f_* \circ J_\alpha = J_\alpha \circ f_* \tag{7.2}$$

for each $\alpha \in \{1, 2, 3\}$. Let us define the action φ of $\mathbb{Z}^3$ on the product manifold $M^{4n} \times \mathbb{R}^3$ by

$$\varphi\left(\left(k_1, k_2, k_3\right), \left(x, t_1, t_2, t_3\right)\right) = \left(f^{k_1+k_2+k_3}(x), t_1 + k_1, t_2 + k_2, t_3 + k_3\right).$$

Note that the action φ is free and properly discontinuous, hence the orbit space $M_f^{4n+3} := (M^{4n} \times \mathbb{R}^3)/\mathbb{Z}^3$ is a smooth manifold. We define a 3-cosymplectic structure on M_f^{4n+3} in the following way. Let $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3$ be the vector fields on $M^{4n} \times \mathbb{R}^3$ given by $\hat{\xi}_\alpha := \frac{\partial}{\partial t_\alpha}$, and let $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ be the 1-forms defined by $\hat{\eta}_\alpha := \hat{g}(\cdot, \hat{\xi}_\alpha)$, where

$$\hat{g} = G + dt_1 \otimes dt_1 + dt_2 \otimes dt_2 + dt_3 \otimes dt_3.$$

Let $\hat{\phi}_\alpha$ be the tensor field of type $(1, 1)$ on $M^{4n} \times \mathbb{R}^3$ defined as follows. Let E be a vector field on M. We can uniquely decompose E into the sum of a vector field X tangent to M^{4n} and its vertical part $\sum_{\beta=1}^{3} \hat{\eta}_\beta(E)\hat{\xi}_\beta$. Then we set

$$\hat{\phi}_\alpha E := J_\alpha X + \sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\hat{\eta}_\beta(E)\hat{\xi}_\gamma.$$

By a straightforward computation one can check that $(\hat{\phi}_\alpha, \hat{\xi}_\alpha, \hat{\eta}_\alpha, \hat{g})$ defines an almost cosymplectic 3-structure on $M^{4n} \times \mathbb{R}^3$. Then, normality is granted by Theorem 2.1.

Since f is an isometry, $\hat{g}$ descends to a Riemannian metric on the quotient manifold M_f^{4n+3}. Furthermore, the vector fields $\hat{\xi}_1$, $\hat{\xi}_2$, $\hat{\xi}_3$, together with their dual 1-forms $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$, are clearly invariant under the action φ. Finally, because of (7.2), also the endomorphisms $\hat{\phi}_\alpha$ induce three endomorphisms on the tangent spaces of M_f^{4n+3}. We denote the induced structure by $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, 2, 3\}$. Thus, $(M_f^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-cosymplectic manifold. Moreover, M_f^{4n+3} is not in general a global product of a hyper-Kähler manifold by the torus $\mathbb{T}^3$. To see this we will consider the following more specific seven-dimensional example.

Let $\mathbb{H}$ be the algebra of quaternions. We consider $\mathbb{H}$ as a hyper-Kähler four-dimensional manifold with a hyper-complex structure given by left multiplication by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Define the action of $\mathbb{Z}^4$ on $\mathbb{H}$ by

$$\begin{aligned} \mathbb{Z}^4 \times \mathbb{H} &\to \mathbb{H} \\ ((a,b,c,d),q) &\mapsto q + a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \end{aligned}$$

By distributivity of multiplication in $\mathbb{H}$ this action commutes with the left multiplication by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Furthermore, the Euclidean metric on $\mathbb{H}$ is translation invariant. Thus the quotient space $\mathbb{H}/\mathbb{Z}^4$ is diffeomorphic to $\mathbb{T}^4$ and inherits a hyper-Kähler structure from $\mathbb{H}$.

Let $\bar{f}\colon \mathbb{H} \to \mathbb{H}$ be the map given by the right multiplication by $\mathbf{i}$. Then from associativity of multiplication on $\mathbb{H}$ it follows that $\bar{f}$ commutes with the hyper-complex structure maps on $\mathbb{H}$. Moreover, from the distributivity of multiplication in $\mathbb{H}$ it follows that $\bar{f}$ induces a hyper-Kählerian isometry f on $\mathbb{H}/\mathbb{Z}^4$.

Proposition 7.4. *Let $M^4 = \mathbb{T}^4$ and f be as above. Then M_f^7 is not a global product of a compact hyper-Kähler four-manifold and the torus $\mathbb{T}^3$.*

Proof: The idea of the proof is to compare the cohomology groups of M_f^7 with the cohomology groups of spaces $K^4 \times \mathbb{T}^3$, where K^4 is a four dimensional hyper-Kähler manifold. Note, that by Proposition 7.3 there are only two possibilities for K^4: either $K^4 \cong \mathbb{H}/\mathbb{Z}^4$ or K^4 is a complex K3 surface. In the first case the Hilbert-Poincaré series of $K^4 \times \mathbb{T}^3$ is $(1+t)^7 = 1 + 7t + 21t^2 + \dots$, in the second case it equals

$$(1 + 22t^2 + t^4)(1+t)^3 = 1 + 3t + 25t^2 + \dots$$

We will show in Proposition 7.7 that $b_2(M_f^7) < 21$. This will imply that M_f^4 does not fail in either of two classes described above. ∎

To get an estimate on $b_2(M_f^7)$ we will define a structure of CW-complex on M_f^7.

Recall the definition of CW-complex (cf. [20, Definition 7.3.1]). We will modify it by replacing the balls in $\mathbb{R}^n$ by cubes

$$Q^k = \left\{ x \in \mathbb{R}^k \,\middle|\, 0 \le x_i \le 1,\ i = 1, \dots, k \right\}.$$

Definition 7.5. A *CW-complex* is a Hausdorff space X, together with an indexing set I_k for each integer $k \geq 0$ and maps $\phi^k_\alpha \colon Q^k \to X$, $k \geq 0$, $\alpha \in I_k$ such that

(1) $X = \bigcup_{k\geq 0} \bigcup_{\alpha\in I_k} \phi^k_\alpha(\mathring{Q}^k)$;
(2) $\phi^k_\alpha(\mathring{Q}^k) \cap \phi^l_\beta(\mathring{Q}^l) = \emptyset$ unless $k = l$ and $\alpha = \beta$;
(3) $\phi^k_\alpha|_{\mathring{Q}^k}$ is one-to-one;
(4) Let $X^k = \bigcup_{j\leq k} \bigcup_{\alpha\in I_j} \phi^j_\alpha(\mathring{Q}^j)$. Then $\phi^k_\alpha(\partial Q^k) \subset X^{k-1}$ for each $k \geq 1$ and $\alpha \in A_n$.
(5) A subset Z of X is closed if and only if $\left(\phi^k_\alpha\right)^{-1}(Z)$ is closed in Q^k for each $k \geq 0$ and $\alpha \in I_k$.
(6) For each $k \geq 0$ and $\alpha \in I_k$ the set $\phi^k_\alpha(Q^k)$ is contained in the union of a finite number of sets of the form $\phi^l_\beta(\mathring{Q}^l)$.

Let X be a CW-complex. Then we have the induced maps

$$\overline{\phi^k_\alpha} \colon S^k \cong Q^k \big/ \partial Q^k \to X^k \big/ X^{k-1}.$$

We will denote the image of this map by S^k_α. By [20, Example 7.3.15] we get a homeomorphism of topological spaces

$$\overline{\phi^k} = \bigvee_{\alpha\in I_k} \overline{\phi^k_\alpha} \colon \bigvee_{\alpha\in I_k} Q^k \big/ \partial Q^k \to \bigvee_{\alpha\in I_k} S^k_\alpha = X^k \big/ X^{k-1},$$

where $\bigvee_{\alpha\in I_k} S^k_\alpha$ denote the one point union (see e.g. [20], page 205). We denote by q_β the map from $\bigvee_{\alpha\in I_k} S^k_\alpha$ to S^k_β that acts as the identity on S^k_β and collapses all the other spheres to the basic point.

Now we explain how the homology groups of a CW-complex can be computed. We define $C_k(X)$ to be the free abelian group generated by I_k. Now for every pair $\alpha \in I_k$ and $\beta \in I_{k-1}$ we define the map $d_{\alpha,\beta}$ to be the composition

$$S^{k-1} \cong \partial Q^k \xrightarrow{\phi^k_\alpha} X^{k-1} \xrightarrow{\pi} X^{k-1} \big/ X^{k-2} = \bigvee_{\gamma\in I_k} S^{k-1}_\gamma \xrightarrow{q_\beta} S^{k-1}_\beta.$$

We denote by $[d_{\alpha,\beta}]$ the degree of the map $d_{\alpha,\beta}$. Now define the differential $\partial \colon C_k(X) \to C_{k-1}(X)$ by $\partial(\alpha) = \sum_{\beta\in I_{k-1}} [d_{\alpha,\beta}]\, \beta$. It is proved in Chapter 8 of [20] that the homology groups of the complex $(C_*(X), \partial)$ are isomorphic to the integral homology groups of the space X.

Since $\mathbb{R}$ is torsion free and $C_k(X)$ are free $\mathbb{Z}$-modules, it follows from the universal coefficient theorem that

$$H_k(C_*(X) \otimes_{\mathbb{Z}} \mathbb{R}) \cong H_k(C_*(X)) \otimes_{\mathbb{Z}} \mathbb{R} = H_k^{\mathbb{R}}(X), \qquad k \geq 0.$$

If X is an m-dimensional compact Riemannian manifold then we have by the Poincaré duality

$$H_k^{\mathbb{R}}(X) \cong H_{dR}^{m-k}(X) \cong \Omega_H^{m-k}(X), \qquad 0 \leq k \leq m.$$

Define $\pi\colon \mathbb{H} \times \mathbb{R}^3 \to M_f^7$ to be the composition

$$\mathbb{H} \times \mathbb{R}^3 \xrightarrow{\pi_1} \left(\mathbb{H}/\mathbb{Z}^4 \right) \times \mathbb{R}^3 \xrightarrow{\pi_2} M_f^7,$$

where π_1 and π_2 are the natural projections.

Now we describe the cellular structure on M_f^7. For every $k = 0, \ldots, 7$ we denote by I_k the set of k-subsets in $\{1, \ldots, 7\}$. For every $S \in I_k$ and $x \in Q^k$ define $\theta_S(x)$ to be the element of $\mathbb{R}^7 \equiv \mathbb{H} \times \mathbb{R}^3$, obtained from x by order preserving placing of coordinates of x into the places $s \in S$ and putting at all other places 0. Now we define $\phi_S^k\colon Q^k \to M_f^7$ to be the composition $\pi \circ \theta_S$.

Proposition 7.6. *The maps* $\left\{ \phi_S^k \,\middle|\, S \in I_k,\ k = 0, \ldots, 7 \right\}$ *give a CW-complex structure on* M_f^7.

Proof: The topological space M_f^7 is Hausdorff since it is a manifold. Now we show that the restriction of π to $[0,1)^7$ is a bijection. For any $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the integral part of x and by $\{x\}$ the fractional part $x - \lfloor x \rfloor$ of x.

Let $[[q], \vec{x}] \in M_f^7$. Then

$$[[q], \vec{x}] = \left[\left[q i^{-(\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \lfloor x_3 \rfloor)} \right], \{x_1\}, \{x_2\}, \{x_3\} \right]$$

in M_f^7 by definition of the action of $\mathbb{Z}^3$ on $\mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3$. Thus the restriction of π_2 to $\mathbb{H}/\mathbb{Z}^4 \times [0,1)^3$ is a surjection. To see that π_2 is a bijection we note that the map

$$\begin{aligned} f\colon \mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3 &\to \mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3 \\ ([q], \vec{x}) &\mapsto \left(\left[q\mathbf{i}^{-(\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \lfloor x_3 \rfloor)} \right], \{x_1\}, \{x_2\}, \{x_3\} \right) \end{aligned}$$

is $\mathbb{Z}^3$-invariant. Since for different points of $\mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3$ the values of f are obviously different we see that the restriction of π_2 to $\mathbb{H}/\mathbb{Z}^4 \times [0,1)^3$ is injective. Similarly we can show that the restriction of π_1 to $[0,1)^7$ gives a bijection between $[0,1)^7$ and $H/\mathbb{Z}^4 \times [0,1)^3$. Thus we get that the restriction of π on $[0,1)^7$ gives a bijection between $[0,1)^7$ and M^7_f.

Now we check that the maps ϕ^k_S satisfy the properties of CW-structure.

(1) We have

$$\bigcup_{k=0}^{7} \bigcup_{S \in I_k} \theta^k_S \left(\mathring{Q}^k \right) = [0,1)^7 ,$$

which implies that the similar union with ϕ^k_S in place of θ^k_S gives M^7_f.

(2) Let $S \in I_k$ and $T \in I_l$. Then the points of $\theta^k_S \left(\mathring{Q}^k \right)$ have non-integer coordinates at places $s \in S$ and integer coordinates in all other places. Similarly for the points of $\theta^l_T \left(\mathring{Q}^l \right)$. This implies that if $S \neq T$ then there are no common points in the sets $\theta^k_S \left(\mathring{Q}^k \right)$ and $\theta^l_T \left(\mathring{Q}^l \right)$. As the restriction of π on $[0,1)^7$ is a bijection the same property holds for $\phi^k_S \left(\mathring{Q}^k \right)$ and $\phi^l_T \left(\mathring{Q}^l \right)$.

(3) As $\theta^k_S \left(\mathring{Q}^k \right) \subset [0,1)^7$ we see that the restriction of ϕ^k_S to $\mathring{Q}^k$ is one-to-one, for any $0 \leq k \leq 7$, $S \in I_k$.

(4) From the considerations at the beginning of the proof we can see that if two points (q,x), $(q',x') \in \mathbb{H} \times \mathbb{R}^3$ are representatives of the same point in M^7_f then the number of integer coordinates in (q,x) and (q',x') is the same. Now $X^k \subset M^7_f$ can be identified with those points $[[q], x] \in M^7_f$ such that (q,x) has at most k fractional coordinates. Now every point ∂Q^k contains at least one integral coordinate. Therefore for $S \in I_k$, $\theta^k_S \left(\partial Q^k \right)$ contains at least $7 - k + 1 = 8 - k$ integral coordinates, or, in other words, at most $k-1$ non-integral coordinates. Thus $\phi^k_S \left(\partial Q^k \right) = \pi \circ \theta^k_S \left(\partial Q^k \right)$ is a subset of X^{k-1}.

(5) If $Z \in M^7_f$ is closed then for any $0 \leq k \leq 7$ and $S \in I_k$ the sets $\left(\phi^k_S \right)^{-1} (Z)$ are obviously closed, as the maps ϕ^k_α are continuous. Suppose now that for every $0 \leq k \leq 7$ and $S \in I_k$ the sets $\left(\phi^k_S \right)^{-1} (Z)$ are closed. As M^7_f has the quotient topology under the projection π, we have to show that $\pi^{-1}(Z)$ is a closed subset in $\mathbb{H} \times \mathbb{R}^3$. Let $(q_n, \vec{x}_n)$ be a sequence in $\pi^{-1}(Z)$ that converges to $(q, \vec{x}) \in \mathbb{H} \times \mathbb{R}^3$. We have

to show that $(q, \vec{x}) \in \pi^{-1}(Z)$. Let $i \in \{1, 2, 3\}$. If x^i is fractional, then starting from some n we have $\lfloor x_n^i \rfloor = \lfloor x^i \rfloor$. If x^i is integer then for infinitely many n we have $x_n^i < \lfloor x^i \rfloor$ or $\lfloor x^i \rfloor \leq x_n^i$. By passing to an appropriate subsequence we can assume that for all n either $\lfloor x_n^i \rfloor = \lfloor x^i \rfloor - 1$ or $\lfloor x_n^i \rfloor = \lfloor x^i \rfloor$. We denote the common integer part of x_n^i by $\widetilde{x}^i$. Define

$$q'_n = q_n i^{-\widetilde{x}^1 - \widetilde{x}^2 - \widetilde{x}^3} \qquad (x'_n)^i = x_n^i - \widetilde{x}^i$$
$$q' = q i^{-\widetilde{x}^1 - \widetilde{x}^2 - \widetilde{x}^3} \qquad (x')^i = x^i - \widetilde{x}^i.$$

Then (q'_n, x'_n) is a sequence of points in $\pi^{-1}(Z)$ that converges to (q', x'). Moreover $(q', x') \in \pi^{-1}(Z)$ if and only if $(q, x) \in \pi^{-1}(Z)$. We also have $0 \leq (x'_n)^i < 1$ and $x^i \in [0, 1]$.

Now, similarly to the considerations above, by passing to an appropriate subsequence we can assume that the integer parts of the coefficients of q'_n does not depend on n. Denote by $\widetilde{q}$ the quaternion with coefficients equal to the integer parts of q'_n. Define $q''_n = q'_n - \widetilde{q}$ and $q'' = q' - \widetilde{q}$. Then $(q''_n, x'_n) \in \pi^{-1}(Z)$ converges to (q'', x'). Moreover, $(q'', x') \in \pi^{-1}(Z)$ if and only if $(q', x') \in \pi^{-1}(Z)$ if and only if $(q, x) \in \pi^{-1}(Z)$.

Let $S = \{1, \ldots, 7\}$. Note that $\theta_S^7 \colon Q^7 \to \mathbb{R}^7$ is the identity map on Q^7. Therefore $\left(\phi_S^7\right)^{-1}(Z) = Q^7 \cap \pi^{-1}(Z)$. Thus the intersection $Q^7 \cap \pi^{-1}(Z)$ is closed in Q^7 and thus in $\mathbb{R}^7$. Since the sequence (q''_n, x''_n) lies in $Q^7 \cap \pi^{-1}(Z)$ we get that also its limit (q'', x'') is an element of $Q^7 \cap \pi^{-1}(Z) \subset \pi^{-1}(Z)$.

(6) Obvious, as we have only finitely many cells at every dimension.

■

With the cellular structure on M_f^7 given in Proposition 7.6 we get

Proposition 7.7. *The degree of the map* $d_{\{3,5\},\{3\}}$ *is* 1. *Therefore* $\partial_2(\{3, 5\}) \neq 0$. *In particular,*

$$b_2\left(M_f^7\right) = \dim\left(H_2^{\mathbb{R}}\left(M_f^7\right)\right) \leq \dim\left(\ker\left(\partial_2\right)\right) < 21.$$

Proof: Below we identify $\mathbb{R}^7$ with $\mathbb{H} \times \mathbb{R}^3$. Note that X^0 consists of one point $[[\mathbf{0}], 0, 0, 0]$. Therefore $X^1 \big/ X^0 = X^1$. Now we describe the image of ∂Q^2

in X^1 under $\phi_{\{3,5\}}$. We have

$$\begin{aligned}\partial Q^2 = &\{\,(0,x)\,|\,0\le x\le 1\}\cup\{\,(x,1)\,|\,0\le x\le 1\}\\ &\cup\{\,(1,x)\,|\,0\le x\le 1\}\cup\{\,(x,0)\,|\,0\le x\le 1\}\end{aligned}$$

in $\mathbb{R}^2$. Now for all $0\le x\le 1$

$$\begin{aligned}\phi^2_{\{3,5\}}(0,x) &= [[\mathbf{0}]\,,x,0,0] = \phi^1_{\{5\}}(x)\in S^1_{\{5\}}\\ \phi^2_{\{3,5\}}(x,1) &= [[x\mathbf{j}]\,,1,0,0] = [[x\mathbf{j}\,(\mathbf{i})]\,,0,0,0] = [[x\mathbf{k}]\,,0,0,0]\\ &= \phi^1_{\{4\}}(x)\in S^1_{\{4\}}\\ \phi^2_{\{3,5\}}(1,x) &= [[\mathbf{j}]\,,x,0,0] = [[\mathbf{0}]\,,x,0,0] = \phi^1_{\{5\}}(x)\in S^1_{\{5\}}\\ \phi^2_{\{3,5\}}(x,0) &= [[x\mathbf{j}]\,,0,0,0] = \phi^1_{\{3\}}(x)\in S^1_{\{3\}}.\end{aligned}$$

Therefore after composing $\phi_{\{3,5\}}$ with $q_{\{3\}}$ we get that for $0\le x\le 1$

$$\begin{aligned}&d_{\{3,5\}\{3\}}(0,x) = d_{\{3,5\}\{3\}}(x,1) = d_{\{3,5\}\{3\}}(1,x) = [[\mathbf{0}],0,0,0]\in S^1_{\{3\}}\\ &d_{\{3,5\}\{3\}}(x,0) = [[x\mathbf{j}]\,,0,0,0]\in S^1_{\{3\}}.\end{aligned}$$

Now it is obvious that the degree of $d_{\{3,5\},\{3\}}$ is one. ■

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