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# STRUCTURE RELATIONS FOR ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract: Structure relations for orthogonal polynomials on the unit circle are studied. We begin by proving that semi-classical orthogonal polynomials on the unit circle satisfy structure relations of the following type: $\sum_{k=0}^{s_1} \beta_{n+s,k} P_{n+s-k} + \sum_{k=0}^{s_2} \gamma_{n-1,k} z^k P^*_{n-1-k} = \sum_{k=0}^{r_1} \alpha_{n+s,k} P^{[1]}_{n+s-k} + \sum_{k=0}^{r_2} \eta_{n+s,k} \left(P^*_{n+s-s'-k}\right)'$, where $s_1, s_2, r_1, r_2$ are integers (specified in the text), the polynomials $P_n^*$ are the reversed polynomials and $P_n^{[1]}$ denotes the monic polynomial of degree $n$ corresponding to $P_n$, $P_n^{[1]} = P'_{n+1}/(n+1)$. Then, we study the semi-classsical character of sequences of orthogonal polynomials on the unit circle $\{R_n\}, \{P_n\}$, connected through a structure relation of the following type: $\sum_{k=0}^{s_1} \beta_{n,k} R_{n+s_1-k} + \sum_{k=0}^{s_2} \gamma_{n,k} R^*_{n+s_2-k}$ $= \sum_{k=0}^{r_1} \alpha_{n,k} P^{[1]}_{n+r_1-k} + \sum_{k=0}^{r_2} \eta_{n,k} \left(P^*_{n+r_2-k}\right)'$, where the integers $s_1, s_2, r_1, r_2$ satisfy some natural conditions.

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## 1. Introduction

The so-called structure relations for orthogonal polynomials, that is, finite-type relations involving sequences of orthogonal polynomials and its derivatives, have been widely studied in the literature of orthogonal polynomials (see, for example, [2, 3, 15] and its list of references). Such type of relations appear in the framework of Sobolev orthogonal polynomials, within the study of coherence of measures (see [4, 11, 16]). They also appear in problems concerning quasi-orthogonality and formulas of quadrature (see [15, 17]), where well-known connections to linear combinations of orthogonal polynomials emerge.

Structure relations and linear combinations of orthogonal polynomials were studied in a vast list of papers, and we refer the reader to [2, 3, 5, 7, 13]. The subject matter is the modification of measures. In [13] it was proven that

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the measures of orthogonality of two sequences of orthogonal polynomials on the unit circle (OPUC), say $\{P_n\}$, $\{R_n\}$, related through

$$\sum_{j=0}^{k(n)} \alpha_{n,j} P_{n-j} = \sum_{j=0}^{l(n)} \beta_{n,j} R_{n-j} \,,$$

where $\alpha_{n,j}, \beta_{n,j} \in \mathbb{C}$ and $k(n), l(n)$ depend on $n$, are a rational modification of each other. Relations of the above type arise in problems of signal analysis and linear prediction of stochastic processes (cf. [13, Section 1]).

Concerning orthogonality on the real line, a well-known topic of research in the literature is to establish the semi-classical character of sequences of orthogonal polynomials in terms of structure relations (see [3, 6, 14, 15] and their list of references). Unlike the real case, this topic has not been getting much attention for OPUC. As far as the authors of the present manuscript know, the references where some relations between the semi-classical character of OPUC and structure relations were studied are [4, 20].

In general terms, in the present paper we are interested in the study of the semi-classical character of sequences of OPUC satisfying some structure relations (these will be made precise throughout the text). We begin by proving that semi-classical OPUC satisfy structure relations of the following type:

$$\sum_{k=0}^{s_1} \beta_{n+s,k} P_{n+s-k} + \sum_{k=0}^{s_2} \gamma_{n-1,k} z^k P^*_{n-1-k} = \sum_{k=0}^{r_1} \alpha_{n+s,k} P^{[1]}_{n+s-k} + \sum_{k=0}^{r_2} \eta_{n+s,k} \left(P^*_{n+s-s'-k}\right)' \,,$$

where the polynomials $P^*_n$ are the reversed polynomials (see its definition in section 2), the integers $s_1, s_2$ and $r_1, r_2$ depend on the degrees of the polynomials involved in the corresponding distributional equation satisfied by the functional of orthogonality (see section 3). Furthermore, we study sequences of OPUC, $\{R_n\}, \{P_n\}$, related through

$$\sum_{k=0}^{s_1} \beta_{n,k} R_{n+s_1-k} + \sum_{k=0}^{s_2} \gamma_{n,k} R^*_{n+s_2-k} = \sum_{k=0}^{r_1} \alpha_{n,k} P^{[1]}_{n+r_1-k} + \sum_{k=0}^{r_2} \eta_{n,k} \left(P^*_{n+r_2-k}\right)' \tag{1}$$

where, for the sake of compatibility, the integers $s_1, s_2, r_1, r_2$ satisfy some natural conditions (these shall be specified throughout the text). Under some conditions, stated in Lemma 2, we deduce the semi-classical character of

$\{P_n\}$ and $\{R_n\}$. Further, we obtain a connecting formula between $\{R_n\}$ and $\{P_n\}$ (see Lemma 2).

Note that when $s_1 = r_1 = 0$ and the $\gamma$'s and the $\eta$'s are all zero in (1), we get $R_n = \frac{P'_{n+1}}{n+1}$. In this case it is known [12] that $P_n = z^n, \forall n \geq 1$, thus $R_n = P_n, \forall n \geq 1$. When $s_1 = 0, r_1 = 2$ and the $\gamma$'s and the $\eta$'s are all zero in (1), we get

$$R_n = \frac{P'_{n+1}}{n+1} + \alpha_{n,1}\frac{P'_n}{n} + \alpha_{n,2}\frac{P'_{n-1}}{n-1} \tag{2}$$

that is, $(P_n, R_n)$ is a generalized coherent pair on the unit circle (according to the definition of generalized pair on the real line introduced and studied in [11]). When $s_1 = 0, r_1 = 1$ and the $\gamma$'s and the $\eta$'s are all zero, we get

$$R_n = \frac{P'_{n+1}}{n+1} + \alpha_{n,1}\frac{P'_n}{n} \tag{3}$$

that is, $(P_n, R_n)$ constitute a coherent pair on the unit circle. In such a case both, $\{P_n\}$ and $\{R_n\}$, are semi-classical (see [4, theorem 4]). Furthermore, the measures of orthogonality of $\{R_n\}$ and $\{P_n\}$ are a rational modification of each other. We remark that such relationship between the measures follows from the application of the techniques on quasi-orthogonality [1] to a connecting formula between $\{R_n\}$ and $\{P_n\}$,

$$zA\psi_n = \mathcal{M}_n\varphi_n, \quad \varphi_n = \begin{bmatrix} P_n \\ P_n^* \end{bmatrix}, \quad \psi_n = \begin{bmatrix} R_n \\ R_n^* \end{bmatrix},$$

where $A$ is a polynomial that does not depend on $n$, and $\mathcal{M}_n$ is a matrix whose entries are bounded degree polynomials. Unlike the cases of coherency (2) and (3), to study the semi-classical character and to analyze the relation between the measures of orthogonality becomes much more difficult when in (1) one considers $s_1 \geq 1$, since that implies that $\{P_n\}$ and $\{R_n\}$ are related through

$$zAK_n\psi_n = \mathcal{T}_n\varphi_n\,, \; n \geq 1\,, \tag{4}$$

where there is a dependence on $n$ in the polynomial $K_n$. Note that relations such as (4) are more general than the ones studied in [7, Section 4.1].

The structure of the paper is as follows. In section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In section 3 we deduce structure relations for semi-classical OPUC. In section 4 we study sequences of OPUC connected through structure relations such as (1). In section 5 we present some examples of OPUC related through (1).

## 2. Preliminary results and notations

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, let $\Lambda = \text{span}\,\{z^k : k \in \mathbb{Z}\}$ be the linear space of Laurent polynomials with complex coefficients, and let $\mathbb{P} = \text{span}\,\{z^k : k \geq 0\}$ be the space of polynomials with complex coefficients. Given a linear functional $u : \Lambda \to \mathbb{C}$, and the sequence of moments $(c_n)_{n\in\mathbb{Z}}$ of $u$, $c_n = \langle u, \xi^n\rangle,\ n \in \mathbb{Z},\ c_0 = 1$, define the minors of the Toeplitz matrix $\Delta = (c_{k-j})$, by

$$\Delta_{-1} = 1,\ \Delta_0 = c_0,\ \Delta_k = \begin{vmatrix} c_0 & \dots & c_k \\ \vdots & \ddots & \vdots \\ c_{-k} & \dots & c_0 \end{vmatrix},\ k \in \mathbb{N}.$$

$u$ is said to be Hermitian if $c_{-n} = \bar{c}_n, \forall n \geq 0$, and quasi-definite (respectively, positive definite) if $\Delta_n \neq 0, \forall n \geq 0$ (respectively, $\Delta_n > 0$), $\forall n \geq 0$ (see [19]). We will denote by $\mathcal{H}$ the set of Hermitian linear functionals defined on $\Lambda$.

**Definition 1.** Let $\{P_n\}$ be a sequence of complex polynomials with deg $(P_n)$ $= n$, and let $u \in \mathcal{H}$ be quasi-definite. $\{P_n\}$ is said to be a sequence of orthogonal polynomials with respect to $u$ if

$$\begin{cases} \langle u, P_n(z)z^{-k}\rangle = 0\,,\ k = 0, \dots, n-1\,,\ n \geq 1\,, \\ \langle u, P_n(z)z^{-n}\rangle = e_n\,,\ e_n \neq 0,\ n = 0, 1, \dots . \end{cases} \tag{5}$$

If $P_n(z) = z^n$+lower degree terms, $\{P_n\}$ will be called a sequence of monic orthogonal polynomials, and we will denote it by MOPS.

If the linear functional $u$ is positive-definite, then it has an integral representation given in terms of a nontrivial probability measure $\mu$ with infinite support on the unit circle,

$$\langle u, e^{in\theta}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} d\mu(\theta),\ n \in \mathbb{Z}\,,$$

and the corresponding MOPS will also be called the MOPS with respect to $\mu$.

We consider the Carathéodory function of $u \in \mathcal{H}$,

$$F_u(z) = 1 + 2\sum_{k=1}^{+\infty} \bar{c}_k z^k\,. \tag{6}$$

If $u \in \mathcal{H}$ is positive definite, associated to a measure whose absolutely continuous part is $w$, then $\lim_{r\uparrow 1} \Re e\, F(re^{i\theta}) = \Re e\, F(e^{i\theta})$ exists a.e. for $\theta \in [0, 2\pi]$, and the inversion formula $w(\theta) = \Re e\, F(e^{i\theta})$ holds (see [9, 18, 19]).

Given a polynomial $B(z)=\sum_{k=0}^{m} b_k z^k$ and $p\in\mathbb{N}$, the polynomial $B^{*_p}$ is defined by $B^{*_p}(z)=z^p\overline{B}(1/z)$, that is, $B^{*_p}(z)=\sum_{k=0}^{m}\overline{b}_k z^{-k+p}$. Throughout the paper we will omit the index $p$ in $B^{*_p}$ if, and only if, the degree of $B$ is exactly $p$.

The following properties will be useful throughout the text (see [20]).

**Lemma 1.** *Let $P\in\mathbb{P}$ and $m\in\mathbb{N}$. The following equalities hold:*

$$
\begin{aligned}
&(P^{*_m}(z))^{*_m}=P(z)\,,\\
&(zP'(z))^{*_m}=(P'(z))^{*_{m-1}}\,,\\
&z(P^{*_m}(z))'=mP^{*_m}(z)-(P'(z))^{*_{m-1}}\,.
\end{aligned}
$$

Note that the reversed polynomials of a sequence of orthogonal polynomials satisfy

$$\langle u,P_n^*(z)\,z^{-k}\rangle=0,\ k=1,\dots,n,\ \ \langle u,P_n^*(z)\rangle=e_n\,,\ n=0,1,\dots\,,$$

where $e_n$ is the same as in (5).

In the sequel we will use the vectors defined by

$$\varphi_n(z)=[P_n(z)\ \ P_n^*(z)]^T\,,\ n\geq 0\,,$$

where $T$ denotes the transpose operator.

It is well-known that MOPS on the unit circle satisfy the following recurrence relations, known as Szegő recurrence relations (see [10]):

$$\varphi_n=\mathcal{A}_n\varphi_{n-1}\,,\ \ \mathcal{A}_n=\begin{bmatrix} z & a_n\\ \bar{a}_n z & 1\end{bmatrix},\ n\in\mathbb{N}\,,\ \ a_n=P_n(0)\,.$$

Therefore, for all $k\geq 1$,

$$\varphi_{n+k}(z)=\prod_{l=0}^{k-1}\mathcal{A}_{n+k-l}\,\varphi_n(z)\,. \tag{7}$$

For $u\in\mathcal{H}$ and $A\in\mathbb{P}$, we define

$$
\begin{aligned}
&\langle Au,f\rangle=\langle u,A(z)f(z)\rangle,\ \ f\in\Lambda\,,\\
&\langle (A+\bar{A})u,f\rangle=\langle u,\big(A(z)+\bar{A}(1/z)\big)\,f(z)\rangle,\ \ f\in\Lambda\\
&\langle \mathcal{D}u,f\rangle=-i\langle u,zf'(z)\rangle\,,\ \ f\in\Lambda\,.
\end{aligned}
$$

Notice that if $u\in\mathcal{H}$, then $(A+\bar{A})u$, as well as $\mathcal{D}u$, belong to $\mathcal{H}$. We will use the notation $u^A=(A(z)+\bar{A}(1/z))u$.

**Definition 2** (see [1]). Let $v \in \mathcal{H}$, $p \in \mathbb{N}$, and let $\{P_n\}$ be a sequence of monic polynomials. $\{P_n\}$ is said to be $\mathbb{T}$-*quasi-orthogonal* of order $p$ with respect to $v$ if:
(i) $\langle v, P_n(z)\ z^{-k}\rangle = 0$, for every $k$ with $p \leq k \leq n-p-1$ and for every $n \geq 2p+1$;
(ii) There exists $n_0 \geq 2p$ such that $\langle v, P_{n_0}(z)\ z^{-n_0+p}\rangle \neq 0$.

**Definition 3** (see [20]). Let $u \in \mathcal{H}$. $u$ is said to be *semi-classical* if there exists $A, B \in \mathbb{P}$, $A \not\equiv 0$, such that $\mathcal{D}(Au) = Bu$. The corresponding $\{P_n\}$ orthogonal with respect to $u$ is said to be *semi-classical*.

Taking into account Theorem 4.1 of [1], in [4] the following was proved.

**Theorem 1** ([4]). *Let $\{P_n\}$ be a MOPS on the unit circle and $\{P_n^*\}$ be the sequence of reversed polynomials. Let $\{P_n\}$ satisfy a structure relation with bounded degree polynomials, $n \geq 1$,*

$$z\Pi_n(z)P_n'(z) = G_n(z)P_n(z) + H_n(z)P_n^*(z) \tag{8}$$
$$z\Pi_n(z)(P_n^*)'(z) = S_n(z)P_n(z) + T_n(z)P_n^*(z)\,. \tag{9}$$

*Then $\Pi_n$ does not depend on $n$. Let $p = \max\{\deg(G_n), \deg(H_n)+1, \deg(S_n), \deg(\Pi_1 - T_n)\}$, $\forall n \geq 1$. If there exists $n_0 \geq 2p$ such that $\deg(\Pi_1 - T_{n_0}) = p$, then $\{P_n\}$ is semi-classical.*

## 3. Structure relations for semi-classical OPUC

In what follows we shall adopt the convention $\sum_l^m \cdot = 0$ whenever $m < l$.

**Theorem 2.** *Let $\{P_n\}$ be a MOPS with respect to a semiclassical linear functional $u$ that satisfies $\mathcal{D}(Au) = Bu$ Let $\deg(A) = s, \deg(B) = s'$, $s \geq 1$. Assume that $(iB + kA)(0) \neq 0$, $k = s', \ldots, n$, $n \in \mathbb{N}$, in the case $s < s'$, and $A(0) \neq 0$ in the case $s \geq s'$. Then, there exist sequences $(\beta_{n,k}), (\gamma_{n-1,k})$, $(\alpha_{n,k})$ and $(\eta_{n,k})$ such that forall $n \geq \max\{s, s'\} + 1$*

$$\sum_{k=0}^{s} \beta_{n+s,k} P_{n+s-k} + \sum_{k=0}^{s'-s-2} \gamma_{n-1,k} z^k P_{n-1-k}^* \\ = \sum_{k=0}^{\min(s,s'-1)} \alpha_{n+s,k} P_{n+s-k}^{[1]} + \sum_{k=0}^{s-s'} \eta_{n+s,k} \left(P_{n+s-s'-k}^*\right)'\,. \tag{10}$$

*Proof*: Case I: $\deg(A) < \deg(B)$.
Let $p = s' - s$. Then, $\forall n \geq s'$, (10) reads as

$$\sum_{k=0}^{s} \beta_{n+s,k} P_{n+s-k} + \sum_{k=0}^{p-2} \gamma_{n-1,k} z^k P^*_{n-1-k} = \sum_{k=0}^{s} \alpha_{n+s,k} P^{[1]}_{n+s-k} \,. \tag{11}$$

We write

$$P_{n+l} = \sum_{j=0}^{n+l} c_{n+l,j} P^{[1]}_{n+l-j} \,, \; l = 0, \ldots, s \,,$$

$$z^k P^*_{n-1-k} = \sum_{j=0}^{n-1} c^k_{n-1,j} P^{[1]}_{n-1-j} \,, \; k = 0, \ldots, p-2 \,.$$

Thus,

$$\sum_{k=0}^{s} \beta_{n+s,k} P_{n+s-k} + \sum_{k=0}^{p-2} \gamma_{n-1,k} z^k P^*_{n-1-k} = P^{[1]}_{n+s} + \sum_{j=0}^{n+s-1} \mu_{n+s-1,j} P^{[1]}_{n+s-1-j} \tag{12}$$

where

$$\mu_{n+s-1,j} = \begin{cases} c_{n+s,j+1} + \displaystyle\sum_{k=1}^{j+1} \beta_{n+s,k} c_{n+s-k,j+1-k} \,, \; j = 0, \ldots, s-1 \,, \\ c_{n+s,j+1} + \displaystyle\sum_{k=1}^{s} \beta_{n+s,k} \, c_{n+s-k,j+1-k} \\ \qquad\qquad + \displaystyle\sum_{k=0}^{p-2} \gamma_{n-1,k} c^k_{n-1,j-s} \,, \; j = s, \ldots, n+s-1 \,. \end{cases} \tag{13}$$

Let us multiply (12) by $z^{-k+1}$ and apply $Au$. Then, the left-hand side gives us, for $k = s+p, \ldots, n$,

$$\langle u, (\sum_{k=0}^{s} \beta_{n+s,k} P_{n+s-k} + \sum_{k=0}^{p-2} \gamma_{n-1,k} z^k P^*_{n-1-k}) \, A \, z^{-k+1} \rangle = 0 \,, \tag{14}$$

where $p \geq 1$.

The right-hand side gives us, after using the definition of the operator $\mathcal{D}$ as well as $\mathcal{D}(Au) = Bu$,

$$\langle A \, u, P^{[1]}_{n+s-1-j} \, z^{-k+1} \rangle = \frac{1}{n+s-j} \langle (iB + kA) u, P_{n+s-j} \, z^{-k} \rangle \,.$$

Therefore, taking into account $\max\{\deg(iB+kA),\ k\geq 1\}=\deg(B)$, as $\deg(A)<\deg(B)$, there follows

$$\langle A\,u, P^{[1]}_{n+s-1-j}\,z^{-k+1}\rangle = 0,\ k=s',\dots,n+s-j-1\,. \tag{15}$$

Taking into account (14) and (15) (note that $s'=s+p$), we obtain that the coefficients $\mu_{n+s-1,j}$ in (12) satisfy

$$0=\sum_{j=n+s-k}^{n+s-1}\mu_{n+s-1,j}\,\xi_{k,j},\ k=s',\dots,n\,, \tag{16}$$

where $\xi_{k,j}=\dfrac{\langle (iB+kA)u, P_{n+s-j}\,z^{-k}\rangle}{n+s-j}$.

Our goal is to prove that there exist $\beta_{n+s,k}$, $k=1,\dots,s$, $\gamma_{n-1,k}$, $k=0,\dots,p-2$, such that in (12) one has $\mu_{n+s-1,s}=\mu_{n+s-1,s+1}=\cdots=\mu_{n+s-1,n+s-1}=0$.

We expand (16), thus getting

$$\begin{bmatrix}\xi_{n,s} & \cdots & \xi_{n,n-p}\\ & \ddots & \vdots\\ & & \xi_{s+p,n-p}\end{bmatrix}\begin{bmatrix}\mu_{n+s-1,s}\\ \vdots\\ \mu_{n+s-1,n-p}\end{bmatrix}$$
$$+\begin{bmatrix}\xi_{n,n-p+1} & \cdots & \xi_{n,n+s-1}\\ \vdots & & \vdots\\ \xi_{s+p,n-p+1} & \cdots & \xi_{s+p,n+s-1}\end{bmatrix}\begin{bmatrix}\mu_{n+s-1,n-p+1}\\ \vdots\\ \mu_{n+s-1,n+s-1}\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}. \tag{17}$$

We remark that

$$\mu_{n+s-1,n-p+1}=\mu_{n+s-1,n-p+2}=\cdots=\mu_{n+s-1,n+s-1}=0 \tag{18}$$

implies $\mu_{n+s-1,s}=\mu_{n+s-1,s+1}=\cdots=\mu_{n+s-1,n-p}=0$, because if (18) holds then (17) becomes

$$\begin{bmatrix}\xi_{n,s} & \cdots & \xi_{n,n-p}\\ & \ddots & \vdots\\ & & \xi_{s+p,n-p}\end{bmatrix}\begin{bmatrix}\mu_{n+s-1,s}\\ \vdots\\ \mu_{n+s-1,n-p}\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}, \tag{19}$$

where the matrix of the system (19) is nonsingular (upper triangular), as the elements of the diagonal are given by

$$\xi_{n-l,s+l}=\frac{\langle u,(iB+(n-l)A)P_{n-l}z^{-n+l}\rangle}{n-l},\ l=0,1,\dots,n-(p+s)\,,$$

and since $(iB+(n-l)A)(0)\neq 0,\ l=0,1,\dots,n-(s+p)$ , there follows that $\xi_{n-l,s+l}\neq 0,\ \forall n\geq s'$ .
Let us return to (18). Taking into account (13) one expands (18) as

$$\mathcal{G}^p_{n,s-1}\,\mathcal{B}^p_{n,s}=-\left[c_{n+s,n-p+2}\ \cdots\ c_{n+s,n+s}\right]^T , \tag{20}$$

where

$$\mathcal{G}^p_{n,s-1}=\begin{bmatrix} c_{n+s-1,n-p+1} & \cdots & c_{n,n-p-s+2} & c^0_{n-1,n-p+1-s} & \cdots & c^{p-2}_{n-1,n-p+1-s}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ c_{n+s-1,n+s-1} & \cdots & c_{n,n} & c^0_{n-1,n-1} & \cdots & c^{p-2}_{n-1,n-1}\end{bmatrix}$$

$$\mathcal{B}^p_{n,s}=\left[\beta_{n+s,1}\ \cdots\ \beta_{n+s,s}\ \gamma_{n-1,0}\ \cdots\ \gamma_{n-1,p-2}\right]^T .$$

Note that $\mathcal{G}^p_{n,s-1}$ is a $(s+p-1)\times(s+p-1)$ matrix.

Let us discuss the system (20). We denote by $\tilde{\mathcal{G}}^p_{n,s-1}$ the $(s+p-1)\times(s+p)$ matrix given by

$$\left[\begin{array}{c|c} & -c_{n+s,n-p+2}\\ \mathcal{G}^p_{n,s-1} & \vdots\\ & -c_{n+s,n+s}\end{array}\right] .$$

If $\det(\mathcal{G}^p_{n,s-1})\neq 0$, then (20) has a solution, and this means that there exist $\beta_{n+s,1},\dots,\beta_{n+s,s}$ such that $\mu_{n+s-1,n-p+1}=\mu_{n+s-1,n+1}=\cdots=\mu_{n+s-1,n+s-1}=0$ , thus, from our previous discussion, there follows $\mu_{n+s,s}=\mu_{n+s,s+1}=\cdots=\mu_{n+s,n-p}=0$, and (11) holds.

If $\det(\mathcal{G}^p_{n,s-1})=0$ , then (20) is possible if, and only if, the matrices $\mathcal{G}^p_{n,s-1},\tilde{\mathcal{G}}^p_{n,s-1}$ have precisely the same number of independent rows.

Let us assume, without loss of generality, that the $i$-th and the $j$-th rows of $\mathcal{G}_{n,s-1}$ are linearly dependent, that is,

$$\begin{aligned}\frac{c_{n+s-1,n+i}}{c_{n+s-1,n+j}}=\frac{c_{n+s-2,n+i-1}}{c_{n+s-2,n+j-1}}&=\cdots=\frac{c_{n,n+i-(s-1)}}{c_{n,n+j-(s-1)}}\\ &=\cdots=\frac{c^0_{n-1,n-p+1-i}}{c^0_{n-1,n-p+1-j}}=\cdots=\frac{c^{p-2}_{n-1,n-p+1-i}}{c^{p-2}_{n-1,n-p+1-j}} .\end{aligned} \tag{21}$$

Note that $n$ is arbitrary and the algorithm described above can be carried out to $n+1$, thus we get the same proportion as above using the matrix $\mathcal{G}^p_{n+1,s-1}$, thus we can take $n+1$ in (21), and we conclude that the $i$-th and the $j$-th rows of $\tilde{\mathcal{G}}^p_{n,s-1}$, are linearly dependent. With a similar reasoning one concludes that $\mathcal{G}^p_{n,s-1},\tilde{\mathcal{G}}^p_{n,s-1}$ have precisely the same number of independent rows. Consequently, (20) is possible and, similarly to the previous discussion in the case $\det(\mathcal{G}^p_{n,s-1}) \neq 0$, we conclude that (11) holds.

Case II: $\deg(A) \geq \deg(B)$.

Let $q = s - s'$. Then, $\forall n \geq s+1$, (10) reads as

$$\sum_{k=0}^{s} \beta_{n+s,k} P_{n+s-k} = \sum_{k=0}^{s-q-1} \alpha_{n+s,k} P^{[1]}_{n+s-k} + \sum_{k=0}^{q} \eta_{n+s,k} \left(P^*_{n+q-k}\right)' . \tag{22}$$

Let us write

$$P^{[1]}_{n+s-l} = \sum_{j=0}^{n+s-l} c_{n+s-l,j} P_{n+s-l-j} \, , \; l = 0, \dots, s-q-1 \, ,$$
$$(P^*_{n+q-l})' = \sum_{j=0}^{n+q-l-1} c^*_{n+q-l,j} P_{n+q-l-1-j} \, , \; l = 0, \dots, q \, .$$

Thus,

$$\sum_{k=0}^{s-q-1} \alpha_{n+s,k} P^{[1]}_{n+s-k} + \sum_{k=0}^{q} \eta_{n+s,k} \left(P^*_{n+q-k}\right)' = P_{n+s} + \sum_{j=0}^{n+s-1} \mu_{n+s-1,j} P_{n+s-1-j} \tag{23}$$

where, for $j = n, \dots, n+s-1$, the $\mu_{n+s-1,j}$'s are given by

$$\mu_{n+s-1,j} = c_{n+s,j+1} + \sum_{k=1}^{s-q-1} \alpha_{n+s,k} \, c_{n+s-k,j+1-k} + \sum_{k=1}^{1+q} \eta_{n+s,k} \, c^*_{n+q-k,j+q-s-k+1} \, . \tag{24}$$

Let us multiply (23) by $z^{-k+1}$ and apply $Au$. Then, the left-hand side gives us

$$\langle (iB + kA)u, \big( \sum_{k=0}^{s-q-1} \alpha_{n+s,k} \frac{P_{n+s-k+1}}{n+s-k+1} + \sum_{k=0}^{q} \eta_{n+s,k} P^*_{n+q-k} \big) \, z^{-k} \rangle \, ,$$

where we used the definition of $\mathcal{D}$, as well as $\mathcal{D}(Au) = Bu$. Therefore, taking into account $\max\{\deg(iB + kA), \; k \geq 1\} = \deg(A)$, as $\deg(A) \geq \deg(B)$, there follows, for $k = s+1, \dots, n$,

$$\langle (iB + kA)u, (\sum_{k=0}^{s-q-1} \alpha_{n+s,k} \frac{P_{n+s-k+1}}{n+s-k+1} + \sum_{k=0}^{q} \eta_{n+s,k} P^*_{n+q-k}) z^{-k} \rangle = 0 \, . \quad (25)$$

From the right-hand side there follows

$$\langle A\, u, P_{n+s-1-j}\, z^{-k+1} \rangle = 0, \;\; k = s+1, \dots, n+s-j-1 \, . \quad (26)$$

Thus, taking into account (25) and (26) we obtain that the coefficients $\mu_{n+s-1,j}$ in (23) satisfy

$$0 = \sum_{j=n+s-k}^{n+s-1} \mu_{n+s-1,j}\, \xi_{k,j}, \;\; k = s+1, \dots, n \, , \quad (27)$$

where $\xi_{k,j} = \langle Au, P_{n+s-1-j}\, z^{-k+1} \rangle$.

As in the previous case, one can prove that there exist $\alpha_{n+s,k}$, $k = 1, \dots, s-q-1$, $\eta_{n+s,k}$, $k = 0, \dots, q$, such that in (23) one has $\mu_{n+s-1,s} = \mu_{n+s-1,s+1} = \cdots = \mu_{n+s-1,n+s-1} = 0$.

We expand (27), thus getting

$$\begin{bmatrix} \xi_{n,s} & \cdots & \xi_{n,n-1} \\ & \ddots & \vdots \\ & & \xi_{s+1,n-1} \end{bmatrix} \begin{bmatrix} \mu_{n+s-1,s} \\ \vdots \\ \mu_{n+s-1,n-1} \end{bmatrix} + \begin{bmatrix} \xi_{n,n} & \cdots & \xi_{n,n+s-1} \\ \vdots & & \vdots \\ \xi_{s+1,n} & \cdots & \xi_{s+1,n+s-1} \end{bmatrix} \begin{bmatrix} \mu_{n+s-1,n} \\ \vdots \\ \mu_{n+s-1,n+s-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} . \quad (28)$$

We remark that

$$\mu_{n+s-1,n} = \mu_{n+s-1,n+1} = \cdots = \mu_{n+s-1,n+s-1} = 0 \quad (29)$$

implies $\mu_{n+s-1,s} = \mu_{n+s-1,s+1} = \cdots = \mu_{n+s-1,n-1} = 0$, because if (29) holds then (28) becomes

$$\begin{bmatrix} \xi_{n,s} & \cdots & \xi_{n,n-1} \\ & \ddots & \vdots \\ & & \xi_{s+1,n-1} \end{bmatrix} \begin{bmatrix} \mu_{n+s-1,s} \\ \vdots \\ \mu_{n+s-1,n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} , \quad (30)$$

where the matrix of the system (30) is nonsingular (upper triangular), as the elements of the diagonal are given by

$$\xi_{n-l,s+l} = \langle u, AP_{n-l-1} z^{-n+l+1} \rangle \, , \;\; l = 0, 1, \dots, n-(s+1) \, ,$$

and since $A(0) \neq 0$ , there follows that

$$\xi_{n-l,s+l} \neq 0,\ \forall n \geq s+1\,.$$

Let us return to (29). Taking into account (24) one expands (29) as

$$\mathcal{G}_{n,s-1}\,\mathcal{B}_{n,s} = -\begin{bmatrix} c_{n+s,n+1} & \cdots & c_{n+s,n+s}\end{bmatrix}^T , \tag{31}$$

where

$$\mathcal{G}_{n,s-1} = \begin{bmatrix} c_{n+s-1,n} & \cdots & c_{n+q+1,n+q-s+2} & c^*_{n+q-1,n+q-s} & \cdots & c^*_{n-1,n-s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n+s-1,n+s-1} & \cdots & c_{n+q+1,n+q+1} & c^*_{n+q-1,n+q-1} & \cdots & c^*_{n-1,n-1} \end{bmatrix}$$

$$\mathcal{B}_{n,s} = \begin{bmatrix} \alpha_{n+s,1} & \cdots & \alpha_{n+s,s-q-1} & \eta_{n+s,0} & \cdots & \eta_{n+s,q}\end{bmatrix}^T .$$

Note that $\mathcal{G}_{n,s-1}$ is a $s \times s$ matrix.

The discussion of the system (31) is similar to the one in case I, thus we conclude that in (23) one has $\mu_{n+s-1,s} = \mu_{n+s-1,s+1} = \cdots = \mu_{n+s-1,n+s-1} = 0$, hence (22) holds. ■

*Remark* . If $\deg(B) = \deg(A) + 1$, that is, $s' = s+1$, then (10) becomes

$$\sum_{k=0}^{s} \beta_{n+s,k} P_{n+s-k} = \sum_{k=0}^{s} \alpha_{n+s,k} P^{[1]}_{n+s-k}\,,\ \forall n \geq s'+1\,. \tag{32}$$

## 4. OPUC related through (1)

Given the MOPS $\{P_n\}$ and $\{R_n\}$, we will consider the following notations:

$$\varphi_n = \begin{bmatrix} P_n \\ P^*_n \end{bmatrix} ,\quad \psi_n = \begin{bmatrix} R_n \\ R^*_n \end{bmatrix} ,\ n \in \mathbb{N}.$$

Note that $\varphi_n$ and $\psi_n$ satisfy the Szegő recurrence relations in the matrix form

$$\varphi_n = \mathcal{A}_n \varphi_{n-1},\ \psi_n = \mathcal{B}_n \psi_{n-1},\ n \in \mathbb{N},$$

where

$$\mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix},\quad \mathcal{B}_n = \begin{bmatrix} z & b_n \\ \overline{b}_n z & 1 \end{bmatrix},$$

with $a_n = P_n(0)$ and $b_n = R_n(0)$.

In what follows we will denote by $X^{(i,j)}$ the element of a matrix $X$ in the position $(i,j)$.

**Lemma 2.** *Let $\{R_n\}, \{P_n\}$ be two sequences of MOPS on the unit circle related through (1), $\forall n \geq 1$,*

$$\sum_{k=0}^{s_1} \beta_{n,k} R_{n+s_1-k} + \sum_{k=0}^{s_2} \gamma_{n,k} R^*_{n+s_2-k} = \sum_{k=0}^{r_1} \alpha_{n,k} P^{[1]}_{n+r_1-k} + \sum_{k=0}^{r_2} \eta_{n,k} \left(P^*_{n+r_2-k}\right)' ,$$

*where the integers $s_1, s_2, r_1, r_2$ satisfy $s_1 = r_1$, $\max\{s_2, r_2\} < s_1$. Let $\varphi_n = \begin{bmatrix} P_n \\ P^*_n \end{bmatrix}$, $\psi_n = \begin{bmatrix} R_n \\ R^*_n \end{bmatrix}$, $n \geq 1$. Then, the following assertions hold:*
*(a) there exists $A \in \mathbb{P}$ and matrices $\mathcal{X}_n$ such that*

$$zA\varphi'_n = \mathcal{X}_n \varphi_n\,, \ n \geq 1\,. \tag{33}$$

*Further, let*

$$p = \max\{\deg(\mathcal{X}_n^{(1,1)}), \deg(\mathcal{X}_n^{(1,2)}) + 1, \deg(\mathcal{X}_n^{(2,1)}), \deg(A - \mathcal{X}_n^{(2,2)})\}, \ n \geq 1\,.$$

*If $\exists\, n_0 \geq 2p$ such that $\deg(A - \mathcal{X}_{n_0}^{(2,2)}) = p$, then $\{P_n\}$ is semi-classical.*
*(b) there exists $K_n \in \mathbb{P}$ and non-singular matrices $\mathcal{T}_n$ such that $\psi_n$ and $\varphi_n$ are related trough*

$$zAK_n\psi_n = \mathcal{T}_n \varphi_n\,, \ n \geq 1\,. \tag{34}$$

*(c) there exists $A_1 \in \mathbb{P}$ and matrices $\mathcal{U}_n$ such that*

$$zA_1\psi'_n = \mathcal{U}_n \psi_n\,, \ n \geq 1\,. \tag{35}$$

*Further, let*

$$p = \max\{\deg(\mathcal{U}_n^{(1,1)}), \deg(\mathcal{U}_n^{(1,2)}) + 1, \deg(\mathcal{U}_n^{(2,1)}), \deg(A_1 - \mathcal{U}_n^{(2,2)})\}, \ n \geq 1\,.$$

*If $\exists\, n_0 \geq 2p$ such that $\deg(A_1 - \mathcal{U}_{n_0}^{(2,2)}) = p$, then $\{R_n\}$ is semi-classical.*

*Proof*: If we apply the $*_{n+s_1}$ operator to (1) and then write the resulting equation (after using the relations from Lemma 1) together with (1) in the matrix form, we get

$$\sum_{k=0}^{s_1} \mathcal{E}_{n,k}\psi_{n+k} = \sum_{k=0}^{r_1+1} \mathcal{F}_{n,k}\varphi_{n+k} + \sum_{k=0}^{r_1+1} \mathcal{G}_{n,k}\varphi'_{n+k} \tag{36}$$

with

$$\mathcal{E}_{n,k}=\begin{cases}\begin{bmatrix}\beta_{n,s_1-k} & \gamma_{n,s_2-k}\\ \overline{\gamma}_{n,s_2-k}\,z^{s_1-k} & \overline{\beta}_{n,s_1-k}\,z^{s_1-k}\end{bmatrix}, & k=0,\ldots,s_2\\[2ex] \begin{bmatrix}\beta_{n,s_1-k} & 0\\ 0 & \overline{\beta}_{n,s_1-k}\,z^{s_1-k}\end{bmatrix}, & k=s_2+1,\ldots,s_1\,,\end{cases}$$

$$\mathcal{F}_{n,k}=\begin{cases}\begin{bmatrix}0 & 0\\ \overline{\eta}_{n,r_2-k}(n+k)\,z^{r_1+1-k} & \overline{\alpha}_{n,r_1+1-k}\,z^{r_1+1-k}\end{bmatrix}, & k=0,\ldots,r_2\\[2ex] \begin{bmatrix}0 & 0\\ 0 & \overline{\alpha}_{n,r_1+1-k}\,z^{r_1+1-k}\end{bmatrix}, & k=r_2+1,\ldots,r_1+1\,,\end{cases}$$

$$\mathcal{G}_{n,k}=\begin{cases}\begin{bmatrix}\frac{\alpha_{n,r_1+1-k}}{n+k} & \eta_{n,r_2-k}\\ -\overline{\eta}_{r_2-k}\,z^{r_1+2-k} & -\frac{\overline{\alpha}_{n,r_1+1-k}}{n+k}z^{r_1+2-k}\end{bmatrix}, & k=0,\ldots,r_2\\[2ex] \begin{bmatrix}\frac{\alpha_{n,r_1+1-k}}{n+k} & 0\\ 0 & -\frac{\alpha_{n,r_1+1-k}}{n+k}z^{r_1+2-k}\end{bmatrix}, & k=r_2+1,\ldots,r_1+1\,,\end{cases}$$

where, by convention, $\alpha_{n,r_1+1}=0$.

Using the recurrence relations (7) for $\varphi_n$ as well as for $\psi_n$, with the convention that $\prod_{l=m}^{n}\cdots=I$ whenever $m>n$, there follows

$$\mathcal{H}_n\varphi_n'=\mathcal{J}_n\varphi_n+\mathcal{K}_n\psi_n\,,\ \forall n\in\mathbb{N}\,. \tag{37}$$

with

$$\mathcal{H}_n=\sum_{k=0}^{s_1+1}\mathcal{G}_{n,k}\prod_{l=0}^{k-1}\mathcal{A}_{n+k-l}\,,$$

$$\mathcal{J}_n=-\sum_{k=0}^{s_1+1}\mathcal{F}_{n,k}\prod_{l=0}^{k-1}\mathcal{A}_{n+k-l}-\sum_{k=0}^{s_1+1}\mathcal{G}_{n,k}\left(\prod_{l=0}^{k-1}\mathcal{A}_{n+k-l}\right)'\,,$$

$$\mathcal{K}_n=\sum_{k=0}^{s_1}\mathcal{E}_{n,k}\prod_{l=0}^{k-1}\mathcal{B}_{n+k-l}\,.$$

If we multiply (37) by $\operatorname{adj}(\mathcal{K}_n)$ there follows

$$\mathcal{L}_n\varphi_n' = \mathcal{M}_n\varphi_n + K_n\psi_n\,,\ n \geq 1\,. \tag{38}$$

with non-singular matrices $\mathcal{L}_n, \mathcal{M}_n$, and a polynomial $K_n$ given by

$$\mathcal{L}_n = \operatorname{adj}(\mathcal{K}_n)\mathcal{H}_n,\ \mathcal{M}_n = \operatorname{adj}(\mathcal{K}_n)\mathcal{J}_n,\ K_n = \det(\mathcal{K}_n)\,.$$

To deduce (33) we start by writing (38) to $n+1$ and use the recurrence relations, thus getting

$$\mathcal{L}_{n,1}\varphi_n' = \mathcal{M}_{n,1}\varphi_n + \xi_n\psi_n \tag{39}$$

with

$$\begin{aligned} \mathcal{L}_{n,1} &= \operatorname{adj}(\mathcal{B}_{n+1})\mathcal{L}_{n+1}\mathcal{A}_{n+1}\,, \\ \mathcal{M}_{n,1} &= \operatorname{adj}(\mathcal{B}_{n+1})(\mathcal{M}_{n+1}\mathcal{A}_{n+1} - \mathcal{L}_{n+1}\mathcal{A}_{n+1}')\,, \\ \xi_n &= K_{n+1}\det(\mathcal{B}_{n+1})\,. \end{aligned}$$

The elimination of $\psi_n$ between (38) and (39) gives us

$$\mathcal{N}_n\varphi_n' = (\xi_n\mathcal{M}_n - K_n\mathcal{M}_{n,1})\varphi_n\,,\ \ \mathcal{N}_n = \xi_n\mathcal{L}_n - K_n\mathcal{L}_{n,1}\,. \tag{40}$$

After multiplying (40) by $\operatorname{adj}(\mathcal{N}_n)$, we get

$$\det(\mathcal{N}_n)\varphi_n' = \mathcal{S}_n\varphi_n\,,\ \ \mathcal{S}_n = \operatorname{adj}(\mathcal{N}_n)(\xi_n\mathcal{L}_n - K_n\mathcal{L}_{n,1})\,.$$

Taking into account Theorem 1, there follows (33) and the assertion concerning the semi-classical character of $\{P_n\}$.

Eq. (34) follows by eliminating $\varphi_n'$ between (38) and (33),

$$zAK_n\psi_n = \mathcal{T}_n\varphi_n\,,\ \ \mathcal{T}_n = \mathcal{L}_n\mathcal{X}_n - zA\mathcal{M}_n\,.$$

To obtain (35) we take derivatives on (34), then we multiply the resulting equation by $zA$ and use (33), thus getting

$$zA(zAK_n)'\psi_n + (zA)^2K_n\psi_n' = (zA\mathcal{T}_n' + \mathcal{T}_n\mathcal{X}_n)\varphi_n\,. \tag{41}$$

The multiplication of (41) by $\det(\mathcal{T}_n)$ and the use of relation (34) in the equivalent form

$$\det(\mathcal{T}_n)\varphi_n = zAK_n\operatorname{adj}(\mathcal{T}_n)\psi_n$$

yields

$$(zA)^2K_n\det(\mathcal{T}_n)\psi_n' = \mathcal{V}_n\psi_n,$$

with

$$\mathcal{V}_n = (zA\mathcal{T}_n' + \mathcal{T}_n\mathcal{X}_n)zAK_n\operatorname{adj}(\mathcal{T}_n) - zA(zAK_n)'\det(\mathcal{T}_n).$$

Taking into account Theorem 1, there follows (35) and the assertion concerning the semi-classical character of $\{R_n\}$. ■

*Remark* . The Theorem 4 of [4], which asserts the semi-classical character of coherent pairs on the unit circle, is a particular case of the previous Lemma.

**Corollary 1.** *MOPS $\{P_n\}$ satisfying (10) also satisfy*

$$zA\varphi_n' = \mathcal{X}_n\varphi_n\,, \quad \varphi_n = \begin{bmatrix} P_n \\ P_n^* \end{bmatrix}, \tag{42}$$

*with $A \in \mathbb{P}$ and matrices $\mathcal{X}_n$. Thus, for*

$$p = \max\{\deg(\mathcal{X}_n^{(1,1)}), \deg(\mathcal{X}_n^{(1,2)}) + 1, \deg(\mathcal{X}_n^{(2,1)}), \deg(A - \mathcal{X}_n^{(2,2)})\},\ n \geq 1,$$

*if $\exists\, n_0 \geq 2p$ such that $\deg(A - \mathcal{X}_{n_0}^{(2,2)}) = p$, then $\{P_n\}$ is semi-classical.*

## 5. Examples

The case $r_1 \geq 1$, $s_2, r_2 < 0$ in (1), that is,

$$R_{n+r_1} = \sum_{k=0}^{r_1} \alpha_{n,k} P_{n+r_1-k}^{[1]}\,, \quad \alpha_{n,0} = 1\,,\ n \geq 0\,,$$

gives (cf. (34))

$$zA\psi_n = \mathcal{T}_n\varphi_n\,.$$

Notice that, in such a case, the polynomial $K_n$ in (34) satisfies $K_n = 1$. Taking into account the results of [1] on $\mathbb{T}$-quasi-orthogonality, there follows that if $\{P_n\}$ is semi-classical, then $\{R_n\}$ is also semi-classical, since the corresponding linear functionals are a rational modification of each other (the proof follows the same technique as in [4, Theorem 4]). In particular, in [4] we gave some examples of MOPS satisfying structure relations such as

$$R_n = \frac{P_{n+1}'}{n+1} + \alpha_n \frac{P_n'}{n}, \quad n \geq 1\,. \tag{43}$$

The case $s_1 = r_1 = 1$ with $\alpha_{n,0} = 0$ and $s_2, r_2 < 0$ in (1) gives a structure relation of the following type:

$$R_{n+1} + \beta_n R_n = \frac{P_{n+2}'}{n+2}\,, \ \beta_n \neq 0\,,\ n \geq 0\,. \tag{44}$$

In what follows we consider $u, v$ the linear functionals corresponding to the MOPS $\{P_n\}, \{R_n\}$, respectively. Further, we assume that $u$ and $v$ are associated to the measures $d\mu$ and $d\sigma$, respectively.

*Example* 1. Let $\mu$ be the Lebesgue measure on the unit circle, $d\mu = \dfrac{d\theta}{2\pi}$, that is, $P_n = z^n,\ n \geq 0$. Then, (44) becomes

$$R_{n+1} + \beta_n R_n = z^{n+1}\,,\ n \geq 0\,.$$

If we apply the linear functional $v$ to the above equality there follows $\langle v, z^{n+1}\rangle = 0$, $\forall n \geq 1$. Thus, the moments of $v$, which we denote by $v_n$, satisfy $v_n = 0, \forall n \geq 2$. Notice that we assume that $v_0 = 1$. To compute $v_1$ we use $R_1 + \beta_0 R_0 = z$, from which we get, applying $v$, $\beta_0 v_0 = v_1$. Therefore,

$$v_n = \begin{cases} v_0 = 1,\ n = 0 \\ \beta_0,\ n = 1 \\ 0,\ \ n \geq 2 \end{cases}$$

Therefore, the Toeplitz matrix corresponding to $v$ is tridiagonal,

$$\Delta = \begin{bmatrix} 1 & v_1 & & & & \\ \overline{v}_1 & 1 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \overline{v}_1 & 1 & v_1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}\,,\ n \geq 2\,.$$

According to the results of [8], the measure $\sigma$ is a perturbation of the Lebesgue measure,

$$d\sigma = \frac{d\theta}{2\pi} + (z + \overline{z})\frac{d\theta}{2\pi}\,,\ z = e^{i\theta}\,.$$

*Example* 2. Let $\mu$ be the Bernstein-Szegő measure $d\mu = d\theta/(2\pi\,|z+c|^2)$, $|c| < 1$, that is, $P_n = z^{n-1}(z+c)$. Then, (44) becomes

$$R_{n+1} + \beta_n R_n = z^{n+1} + c\,\frac{n+1}{n+2} z^n\,,\ n \geq 0\,.$$

Applying $v$ there follows

$$v_{n+1} + c\frac{n+1}{n+2} v_n = 0\,,\ \forall n \geq 1,$$

thus

$$v_{n+1} = \frac{2}{n+2}(-c)^n v_1\,,\ \forall n \geq 1\,.$$

Notice that we assume $v_0 = 1$. To compte $v_1$ we take $n = 0$ in (44), thus getting $v_1 = \beta_0 - c/2$. Thus, we have

$$v_n = \begin{cases} 1\,, & n = 0 \\ \beta_0 - c/2\,, & n = 1 \\ \frac{2}{n+1}(-c)^{n-1} v_1\,, & n \geq 2\,. \end{cases}$$

The corresponding Carathéodory function, $F_v(z) = 1 + 2\sum_{n=1}^{+\infty} \overline{v}_n z^n$ , becomes

$$F_v(z) = 1 + 2\left(\overline{v}_1 z + 2\overline{v}_1 \sum_{n=2}^{+\infty} \frac{(-1)^{n-1}}{n+1} \overline{c}^{n-1} z^n\right)\,. \tag{45}$$

Taking into account that

$$\sum_{n=2}^{+\infty} \frac{(-1)^{n-1}}{n+1} \overline{c}^{n-1} z^n = \frac{-1}{\overline{c}^2 z}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} (\overline{c}z)^{n+1} - \overline{c}z + \frac{(\overline{c}z)^2}{2}\right)$$

there follows

$$\sum_{n=2}^{+\infty} \frac{(-1)^{n-1}}{n+1} \overline{c}^{n-1} z^n = \frac{-1}{\overline{c}^2 z} \ln(1 + \overline{c}z) + \frac{1}{\overline{c}} - \frac{z}{2},$$

which we substitute into (45), thus obtaining

$$F_v(z) = 1 + \frac{4\overline{v}_1}{\overline{c}} - \frac{4\overline{v}_1}{\overline{c}^2 z} \ln(1 + \overline{c}z)\,.$$

The inversion formula gives us the absolutely continuous part of $\sigma$,

$$d\sigma = \left(\lim_{r\uparrow 1} \Re e\ F_v(re^{i\theta})\right) d\theta\,. \tag{46}$$

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